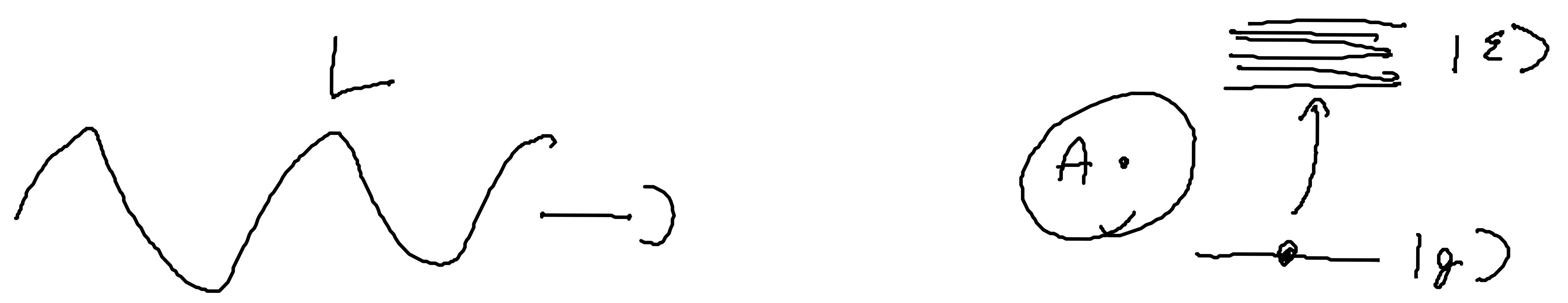
7. Photodetection theory

We try to model photoelectric effect



$$\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_A \qquad \hat H = \underbrace{\hat H_L + \hat H_A}_{\hat H_0} + \hat H_I$$

$$\hat H_I = -\hat{\vec{d}} \circ \hat{\vec{E}}(\vec r)$$

dipole moment operator

$$\left\{ \hat{\vec{d}} = \sum_{ij} \overbrace{\langle e_i | e \hat{\vec q} | e_j\rangle}^{d_{ij}} |e_i\rangle\langle e_j| = e\vec{q} \right.$$

basis in $\mathcal{H}_A$ $\qquad H_A|e_i\rangle = \varepsilon_i |e_i\rangle$

We work in the interaction picture (Dirac)

$$H_I^D = e^{\frac{iH_0t}{\hbar}} H_I e^{-\frac{iH_0 t}{\hbar}} = -\hat{\vec d}(t) \circ \hat{\vec E}(\vec r, t)$$

$$\hat{\vec d}(t) = \sum_{ij} e^{\frac{i(\varepsilon_i - \varepsilon_j)t}{\hbar}} d_{ij} |e_i\rangle\langle e_j|$$

Initially: $|\Psi(t_0)\rangle_{LA} = |\psi\rangle_L \otimes |g\rangle_A$

$$|\Psi^D(t)\rangle_{LA} = \mathcal{T}\left[e^{-\frac{i}{\hbar}\int_{t_0}^{t} H_I^D(t)\,dt'}\right] |\Psi^D(t_0)\rangle_{LA} =$$

$$= \left(\mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, \hat H_I^D(t_1) + \left(\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \hat H_I^D(t_1)\, \hat H_I^D(t_2) + \dots\right) |\psi^D(t_0)\rangle$$

Let $|\varphi\rangle|\varepsilon\rangle$ be some state with $\langle\varepsilon|g\rangle = 0$ $\left\{\right.$ detector "clicked"

Probability amplitude of getting $|\varphi\rangle|\varepsilon\rangle$: after interaction for a time δt:

$\left\{\right.$ $|\varepsilon\rangle$ - energy eigenstate $H_A|\varepsilon\rangle = \varepsilon|\varepsilon\rangle$

$$\langle\varphi|\langle\varepsilon|\, |\Psi^D(t_0+\delta t)\rangle \overset{O(\delta t)}{\simeq} -\frac{i}{\hbar}\int_{t_0}^{t_0+\delta t} \langle\varphi|\langle\varepsilon|\, \hat H_I^D(t')\,|\psi\rangle|g\rangle\, dt'$$

$$= -\frac{i}{\hbar}\langle\varepsilon|\hat{\vec d}|g\rangle \int_{t_0}^{t_0+\Delta t} dt'\, \underline{e^{\frac{i(\varepsilon-\varepsilon_g)t'}{\hbar}}}\, \langle\varphi|\, \underset{\hat{\vec E}^{(+)} + \hat{\vec E}^{(-)}}{\underset{\|}{\hat{\vec E}(\vec r, t)}}\, |\psi\rangle$$

$$\hat{\vec{E}}(\vec r,t) = \underset{\hat a\, e^{-i\omega t}}{\hat{\vec{E}}^{(+)}(\vec r,t)} + \underset{\hat a^{+} e^{i\omega t}}{\hat{\vec{E}}^{(-)}(\vec r,t)}$$

since $\Delta t \gg \frac{1}{\omega_0}$ ← characteristic frequency of light $\sim 10^{15}\,Hz$

this mean that integral will give non-zero contributi if we can cancel this oscillation with $e^{i\frac{(\varepsilon-\varepsilon_g)t}{\hbar}}$

This means $\hat{\vec{E}}^{(-)}(\vec r,t)$ will not contribute.

Rotating wave approximation (RWA):

$$\langle\varphi|\langle\varepsilon|\psi_I^D(t_0+\Delta t)\rangle = -\frac{i}{\hbar}\langle\varepsilon|\hat{\vec d}|g\rangle\cdot\int_{t_0}^{t_0+\Delta t}dt'\, e^{i\frac{(\varepsilon-\varepsilon_g)t}{\hbar}}\langle\varphi|\hat{\vec E}^{(+)}(\vec r,t)|\psi\rangle$$

For simplicity consider linear polarization:

$$\vec E^{(+)}(\vec r,t) = \vec e_\lambda \hat E^{(+)}(\vec r,t) = \vec e_\lambda \underset{\text{central frequency}}{e^{-i\omega_0 t}}\, \underset{\text{slowly varying envelope}}{\hat{\mathcal E}(\vec r,t)}$$

$$= -\frac{i}{\hbar}\underbrace{\langle\varepsilon|\hat{\vec d}\cdot\vec e_\lambda|g\rangle}_{d^\lambda_{\varepsilon g}}\int_{t_0}^{t_0+\Delta t} e^{i\frac{(\varepsilon-\varepsilon_g)t}{\hbar}-i\omega_0 t}\,\underbrace{\langle\varphi|\hat{\mathcal E}(\vec r,t)|\psi\rangle}_{\text{variations are negligible on the time scale }\Delta t}$$

$$\approx -\frac{i}{\hbar}d^\lambda_{\varepsilon g}\langle\varphi|\hat{\mathcal E}(\vec r,t)|\psi\rangle\,\frac{\hbar}{i}\,\frac{e^{\frac{i}{\hbar}(\varepsilon-\varepsilon_g-\hbar\omega_0)(t_0+\frac{\Delta t}{2})}\left(e^{\frac{i}{\hbar}(\varepsilon-\varepsilon_g-\hbar\omega_0)\frac{\Delta t}{2}}-c.c\right)}{(\varepsilon-\varepsilon_g)-\hbar\omega_0}$$

transition probability:

$$P = |\langle\varphi|\langle\varepsilon|\psi_I^D(t_0+\Delta t)\rangle|^2 = |d^\lambda_{\varepsilon g}|^2\langle\psi|\hat{\mathcal E}^*(\vec r,t)|\varphi\rangle\langle\varphi|\hat{\mathcal E}(\vec r,t)|\psi\rangle$$

$$\cdot\left(\frac{\sin\left[\frac{1}{2}\left(\frac{\varepsilon-\varepsilon_g}{\hbar}-\omega_0\right)\Delta t\right]}{\frac{1}{2}\left[\varepsilon-\varepsilon_g-\hbar\omega_0\right]\sqrt{\Delta t}}\right)^2\Delta t$$

$$\Big\{\text{ We use: } \frac{\sin^2\left(\frac{\omega t}{2}\right)}{t\omega^2}\xrightarrow{t\to\infty}\frac{\pi}{2}\delta(\omega)$$

$$p = \frac{2\pi}{\hbar} \left|d^{\alpha}_{\varepsilon g}\right|^2 \Delta t \, \delta\left(\frac{\varepsilon - \varepsilon_g}{\hbar} - \omega_0\right) \langle\psi| \hat{\mathcal{E}}^{*}(\vec r, t)|\varphi\rangle\langle\varphi|\hat{\mathcal{E}}(\vec r, t)|\psi\rangle$$

We sum / integrate over output states $|\varphi\rangle|\varepsilon\rangle$

$$\sum_{\varphi} |\varphi\rangle\langle\varphi| = \mathbb{1}$$

$$\underset{\text{energy band}}{\int d\varepsilon} \, \underset{\substack{\uparrow \\ \text{spectral density} \\ \text{of energy levels}}}{\rho(\varepsilon)} \sum_{\varphi} p = \frac{2\pi}{\hbar} \left|\langle \varepsilon_g + \hbar\omega_0 | \vec d \cdot \vec e_x | \varepsilon_g\rangle\right|^2 \rho(\varepsilon_g + \hbar\omega_0) \cdot \langle\psi| \hat{\mathcal{E}}^{*}(\vec r, t) \underset{\substack{\uparrow \\ e^{i\omega_0 t} \cdot e^{-i\omega_0 t}}}{\hat{\mathcal{E}}(\vec r, t)} |\psi\rangle$$

Finally:

$$\boxed{\begin{gathered} p(\vec r, t, \Delta t) = \bar{S} \cdot \Delta t \langle\psi| \hat E^{(-)}(\vec r, t) \, E^{(+)}(\vec r, t) |\psi\rangle \\ \parallel \\ \frac{2\pi}{\hbar} \left|\langle \varepsilon_g + \hbar\omega_0 | \vec d \cdot \vec e_x | \varepsilon_g\rangle\right|^2 \rho(\varepsilon_g + \hbar\omega_0) \end{gathered}}$$

Rate of detection.

$$\frac{p(\vec r, t, \Delta t)}{\Delta t} \sim \underset{\substack{\uparrow \\ \text{single click detection rate.}}}{w_1(\vec r, t)} = \langle\psi| \underset{\substack{\uparrow \\ \hat a^{\dagger}}}{\hat E^{(-)}(\vec r, t)} \, \underset{\substack{\uparrow \\ \hat a}}{\hat E^{(+)}(\vec r, t)} |\psi\rangle$$

For general mixed state $\rho = \sum_k q_k |\psi_k\rangle\langle\psi_k|$

$$\boxed{\begin{gathered} \text{Rate of detection clicks is proportional to} \\ w_1(\vec r, t) = \mathrm{Tr}\left(\rho \, \hat E^{(-)}(\vec r, t) \, \hat E^{(+)}(\vec r, t)\right) \end{gathered}}$$

Define intensity operator $\hat I(\vec r, t) = \hat E^{(-)}(\vec r, t) \, \hat E^{(+)}(\vec r, t)$

$\Big\{$ Comment: when discussing photon wave function for single photon states (Rieman-Silberstein wave function) $|\psi(\vec r, t)|^2 \sim \langle\psi| \hat E^{(-)}(\vec r, t) \, \hat E^{(+)}(\vec r, t) |\psi\rangle$

Two photon detection

$$\mathcal{H}_L \otimes \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$$

$$\hat{H}_{int} = \hat{H}_{int,1} + \hat{H}_{int,2}$$

We want to calculate the probability that detector 1 clicks within time $[t_1, t_1+\Delta t)$ and detector 2 clicks within $[t_2, t_2+\Delta t]$, Initial state: $|\psi\rangle|g\rangle|g\rangle$

$$\begin{cases} \langle \varepsilon_1 | g\rangle = 0 \\ \langle \varepsilon_2 | g\rangle = 0 \end{cases}$$

$$\langle \varphi | \langle \varepsilon_1 | \langle \varepsilon_2 | \Psi^D(t_1, t_2, \Delta t)\rangle =$$

second order expantion

$$\ldots = \frac{1}{i\hbar} d^2_{\varepsilon_1 g}\, d^2_{\varepsilon_2 g} \int_{t_1}^{t_1+\Delta t} e^{i\left[\frac{\varepsilon_1-\varepsilon_g}{\hbar} - \omega_0\right]t'} dt' \int_{t_2}^{t_2+\Delta t} dt''\, e^{i\left[\frac{\varepsilon_2-\varepsilon_g}{\hbar} - \omega_0\right]t''}$$

$$\cdot \langle \varphi | \hat{\mathcal{E}}(\vec{r}_2, \underset{\underset{t_2}{\approx}}{t''})\, \hat{\mathcal{E}}(\vec{r}_1, \underset{\underset{t_1}{\approx}}{t'}) | \psi\rangle$$

Finally:

$$p(\vec{r}_1, \vec{r}_2, t_1, t_2, \Delta t) = \sum_{\varphi} \int d\varepsilon_1 \int d\varepsilon_2\, \rho(\varepsilon_1)\, \rho(\varepsilon_2)\, |\langle \varphi, \varepsilon_1, \varepsilon_2 | \Psi^D(t_1, t_2, \Delta t)\rangle|^2 =$$

$$= (\bar{S}\, \Delta t)^2 \langle \psi | \hat{E}^{(-)}(\vec{r}_1, t_1)\, \hat{E}^{(-)}(\vec{r}_2, t_2)\, \hat{E}^{(+)}(\vec{r}_2, t_2)\, E^{(+)}(\vec{r}_1, t_1) | \psi\rangle$$

Rate of double detections: $\rho = |\psi\rangle\langle\psi|$, $\rho = \sum_k q_k |\psi_k\rangle\langle\psi_k|$

$$\frac{p(\vec{r}_1, \vec{r}_2, t_1, t_2, \Delta t)}{\Delta t^2} \sim w_2(r_1, r_2, t_1, t_2) = \mathrm{Tr}\Big[\rho\, \overset{a^+}{\hat{E}^{(-)}(\vec{r}_1, t_1) \cdot \hat{E}^{(-)}(\vec{r}_2, t_2)}$$

$$\cdot\, \underset{\hat{a}}{\hat{E}^{(+)}(\vec{r}_2, t_2)\, E^{(+)}(\vec{r}_1, t_1)} \Big]$$

$$w_2(r_1, r_2, t_1, t_2) = \mathrm{Tr}\left[\rho : \hat{I}(\vec{r}_1, t_1)\, \hat{I}(\vec{r}_2, t_2) :\right]$$

$$\{ \neq \mathrm{Tr}\left[\rho\, \hat{I}(\vec{r}_1, t_1)\, \hat{I}(\vec{r}_2, t_2)\right]$$

Coherence functions (for fixed polarization $\vec{E} \sim E\vec{e}_1$)

$$G^{(1)}(\vec{r},t;\vec{r}',t') := \langle \hat{E}^{(-)}(\vec{r},t)\,\hat{E}^{(+)}(\vec{r}',t')\rangle = \mathrm{Tr}(\rho\,\hat{E}^{(-)}(\vec{r},t)\,\hat{E}^{(+)}(\vec{r}',t'))$$

first order coherence function.

$$G^{(n)}(\vec{r}_1,\dots,\vec{r}_n, t_1\dots t_n; \vec{r}_1'\dots\vec{r}_n', t_1'\dots t_n') =$$

$$= \langle \hat{E}^{(-)}(\vec{r}_1,t_1)\dots\hat{E}^{(-)}(\vec{r}_n,t_n)\,\hat{E}^{(+)}(\vec{r}_1',t_1')\dots\hat{E}^{(+)}(\vec{r}_n',t_n')\rangle$$

n-th order coherence function.

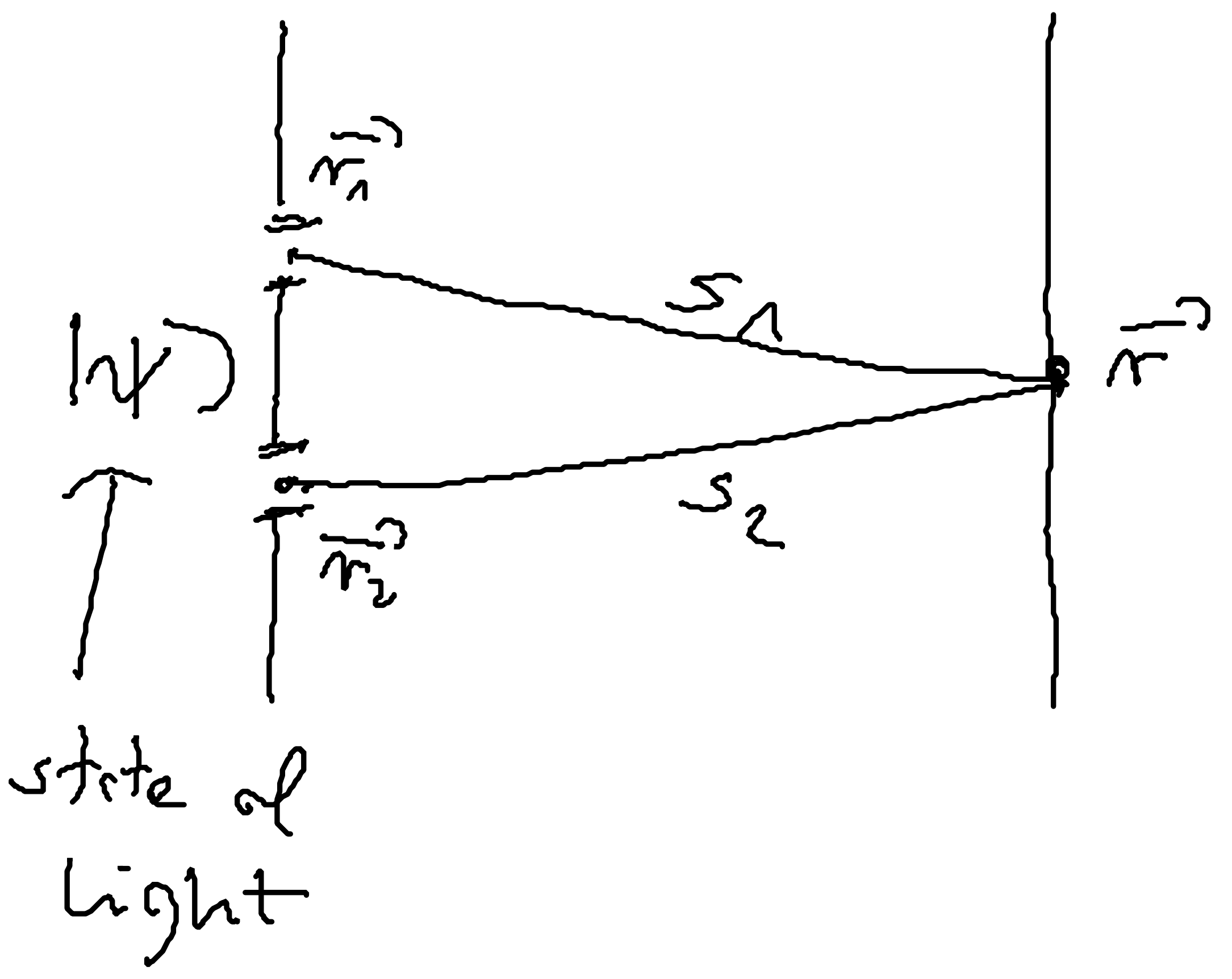
$$g^{(1)}(\vec{r},t;\vec{r}',t') := \frac{G^{(1)}(\vec{r},t;\vec{r}',t')}{\sqrt{G^{(1)}(\vec{r},t;\vec{r},t)\,G^{(1)}(\vec{r}',t';\vec{r}',t')}} = \frac{G^{(1)}(\vec{r},t;\vec{r}',t')}{\sqrt{\langle\hat{I}(\vec{r},t)\rangle\langle\hat{I}(\vec{r}',t')\rangle}}$$

first order normalized coherence function.

$$g^{(n)}(\vec{r}_1,\dots\vec{r}_n, t_1\dots t_n) = \frac{G^{(n)}(\vec{r}_1,\dots\vec{r}_n, t_1\dots t_n; \vec{r}_1\dots\vec{r}_n, t_1\dots t_n)}{G^{(1)}(\vec{r}_1,t_1;\vec{r}_1,t_1)\cdot\,\dots\,\cdot G^{(1)}(\vec{r}_n,t_n;\vec{r}_n,t_n)}$$

n-th order normalized coherence function. $(n \geq 2)$

Young double-slit experiment



Thinking in the Heisenberg picture

$$\vec{E}^{(+)}(\vec{r},t) \approx K_1\vec{E}^{(+)}(\vec{r}_1,t_1) + K_2\vec{E}^{(+)}(\vec{r}_2,t_2)$$

$$\approx \left[K_1E^{(+)}(\vec{r}_1,t_1) + K_2E^{(+)}(\vec{r}_2,t_2)\right]\vec{e}_1$$

$t_1 = t - \frac{s_1}{c}$ $\quad t_2 = t - \frac{s_2}{c}$

K_i – describe classical field propagation takes into account geometry and size of the slits and it should scale like $\frac{1}{s_i}$

Intensity:

$$I(\vec{r},t) = \langle E^{(-)}(\vec{r},t)\,E^{(+)}(\vec{r},t)\rangle =$$

$$= |K_1|^2 \langle \hat{E}^{(-)}(\vec{r}_1, t_1)\, \hat{E}^{(+)}(\vec{r}_1, t_1) \rangle + |K_2|^2 \langle \hat{E}^{(-)}(\vec{r}_2, t_2)\, \hat{E}^{(+)}(\vec{r}_2, t_2) \rangle$$

$$+ \; 2\,\mathrm{Re}\left[K_1^* K_2 \langle \hat{E}^{(-)}(\vec{r}_1, t_1)\, \hat{E}^{(+)}(\vec{r}_2, t_2) \rangle \right] =$$

$$= |K_1|^2 \overbrace{\langle \hat{I}(\vec{r}_1, t_1) \rangle}^{I_1} + |K_2|^2 \overbrace{\langle \hat{I}(\vec{r}_2, t_2) \rangle}^{I_2} +$$

$$+ \; \underbrace{2 \left[\mathrm{Re}\, K_1^* K_2 \; G^{(1)}(\vec{r}_1, t_1, \vec{r}_2, t_2) \right]}_{\text{interference term.}} =$$

$$\Bigg\{ \; g^{(1)} = \frac{G^{(1)}}{\sqrt{I_1 I_2}} \qquad K_i = |K_i| e^{i\varphi_i} \approx |K| e^{i\varphi_i}$$

$$\approx |K|^2 \left[I_1 + I_2 + 2\sqrt{I_1 I_2}\; \mathrm{Re}\left(g^{(1)}(r_1, t_1, r_2, t_2)\, e^{i\underline{(\varphi_1 - \varphi_2)}} \right) \right]$$

- $|\psi\rangle = |\alpha\rangle$ in a plane wave mode $\sim e^{i\vec{k}_0 \vec{r}}$

$$\hat{E}^{(+)} = \sum_{\vec{k}} \underbrace{\sqrt{\frac{\hbar \nu}{2\varepsilon_0 V}}}_{A_k} \hat{a}_{\vec{k}}\, e^{i\vec{k}\vec{r} - \omega t}$$

$$I_1 = \langle \psi | E^{(-)}(r_1, t_1) E^{(+)}(r_1, t_1) | \psi \rangle = |A_{k_0}|^2 |\alpha|^2 \qquad = I_2$$

$$G^{(1)} = \langle \psi | \hat{E}^{(-)}(\vec{r}_1, t_1)\, \hat{E}^{(+)}(\vec{r}_2, t_2) | \psi \rangle =$$

$$= |A_{k_0}|^2 |\alpha|^2 \, e^{i\vec{k}_0(\vec{r}_2 - \vec{r}_1) - i\omega(t_2 - t_1)} \, ,$$

$$I(\vec{r}, t) = 2|K|^2 |A_{k_0}|^2 |\alpha|^2 \left(\underline{1} + \mathrm{Re}\, e^{i\vec{k}_0(\vec{r}_2 - \vec{r}_1) - i\omega(t_2 - t_1)} \right) =$$

$$= 2|K|^2 |A_{k_0}|^2 |\alpha|^2 \left[1 + \cos\left(\vec{k}_0 \cdot (\vec{r}_2 - \vec{r}_1) - i\omega(t_2 - t_1) + \varphi \right) \right]$$

perfect fringe visibility

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \underline{1}$$

• $|\psi\rangle = |n\rangle_{\vec{k}_0}$ in plane wave mode - $e^{i\vec{k}_0\vec{r}}$

$$I_1 = \langle n|\, |A_{k_0}|^2\, \hat{a}^\dagger_{k_0}\hat{a}_{k_0}\, |n\rangle = n\,|A_{k_0}|^2$$

$$\shortparallel$$

$$I_2$$

$$G^{(2)} = |A_{k_0}|^2 \langle n|\, \hat{a}^\dagger_{k_0}\hat{a}_{k_0}\, e^{i\vec{k}_0(\vec{r}_2-\vec{r}_1) - i\omega(t_2-t_1)}|n\rangle =$$

$$= |A_{k_0}|^2\, n\, e^{i\vec{k}_0(\vec{r}_2-\vec{r}_1) - i\omega(t_2-t_1)}$$

$\Downarrow$ we also have perfect int fringes

$$I(r,t) \sim \left(1 + \cos\left(\vec{k}_0(\vec{r}_2-\vec{r}_1) - \omega(t_2-t_1) + \varphi\right)\right)$$

All states with the same average number of photons will give

→ same interference patterns.

$$\left|g^{(1)}(r_1 t_1, r_2 t_2)\right| = 1$$

First order interference does not discriminate between classical and quantum states.

Fact: $\left|g^{(1)}(r_1,t_1,r_2,t_2)\right| \le 1$

Proof: $G^{(1)} = \mathrm{Tr}\big(\underset{\sqrt{\rho}\sqrt{\rho}}{\underset{\shortparallel}{\rho}}\, E^{(-)}(\vec{r}_1,t_1)\, E^{(+)}(\vec{r}_2,t_2)\big) =$

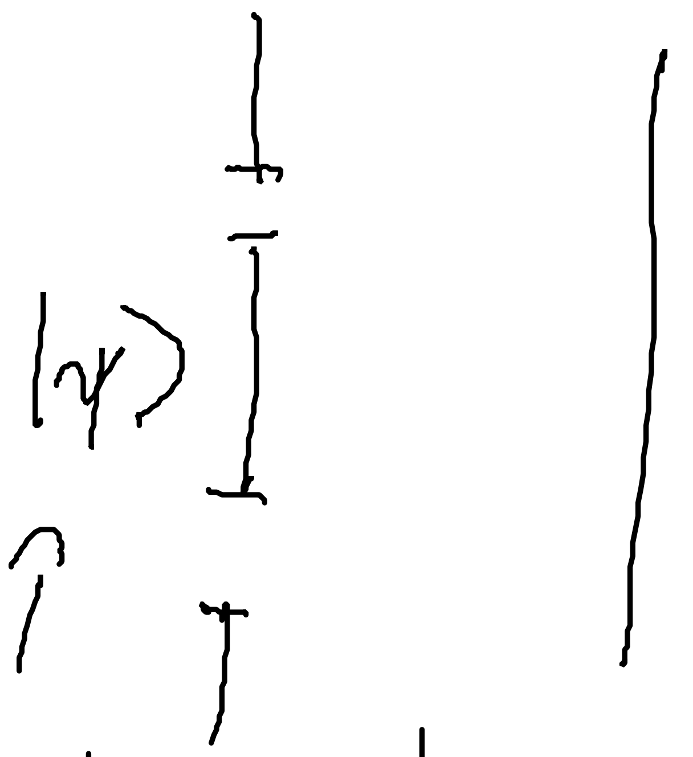
$$= \mathrm{Tr}\big(\underbrace{\sqrt{\rho}\, E^{(-)}(\vec{r}_1,t_1)}_{A^\dagger}\, \underbrace{E^{(+)}(\vec{r}_2,t_2)\sqrt{\rho}}_{B}\big) \le \quad \Big\{ \left|\mathrm{Tr}(A^\dagger B)\right|^2 \le \mathrm{Tr}(\underline{A^\dagger A})\,\mathrm{Tr}(B^\dagger B)$$

$$\le \sqrt{\mathrm{Tr}\big(\rho\, E^{(-)}(\vec{r}_1,t_1)\, E^{(+)}(r_1,t_1)\big)\cdot \mathrm{Tr}\big(\rho\, E^{(-)}(\vec{r}_2,t_2)\, E^{(+)}(\vec{r}_2,t_2)\big)} =$$

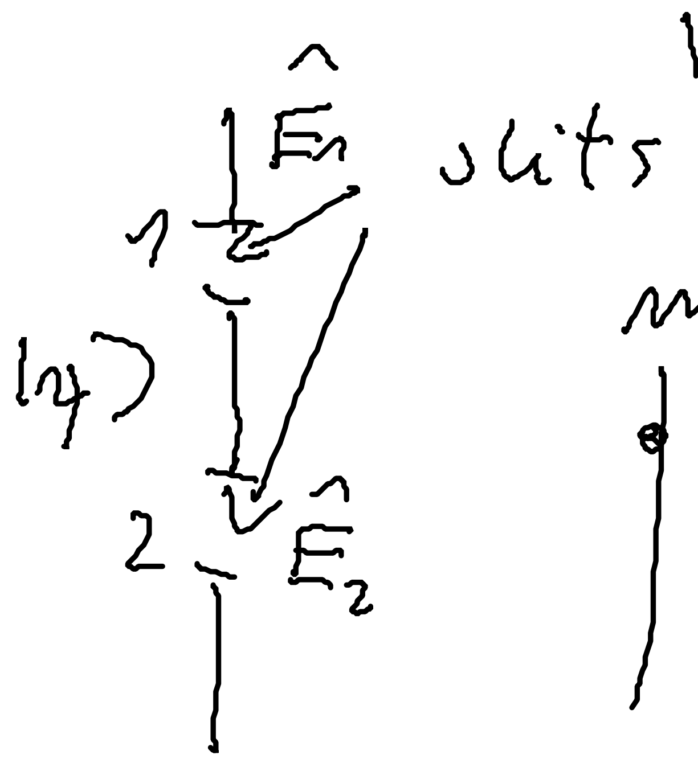
$$= \sqrt{\langle \hat{I}(r_1,t_1)\rangle\, 2\, \hat{I}(r_2,t_2)\rangle}\,, \Rightarrow \frac{G^{(1)}(r_1,t_1,r_2,t_2)}{\sqrt{G^{(1)}(\vec{r}_1 t_1, r_1 t_1)\, G^{(1)}(\vec{r}_2 t_2, \vec{r}_2 t_2)}} \le 1$$

$|g^{(1)}| = 1$ - perfect coherence $\quad |g^{(1)}| = 0$ - no coherence

Alternative look at the Young interference experiment?



described in terms of modes before hitting the screen.



slits can be regarded as localized modes for light

$$\hat{E}(\vec r, t) = K(\hat{E}_1(\vec r_1, t_1) + \hat{E}_2(\vec r_2, t_2))$$

$$\mathcal{H}_1 \otimes \mathcal{H}_2$$

$$|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

$|\psi\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle$ vs. $|\psi\rangle = |n_1\rangle \otimes |n_2\rangle$

↑ coherent states in two slit-modes

interference ?

no interference,?