Higher order coherence functions (mostly second order)

Fact

For a multimode coherent state $|\psi\rangle = \bigotimes_k |\alpha_k\rangle_k =: |\vec{\alpha}\rangle$
$|\{\alpha\}\rangle$

$$g^{(m)}(r_1,\dots,r_m,t_1,\dots t_m) = 1$$

Proof

$$g^{(m)}(r_1,\dots r_m, t_1 \dots t_m) = \frac{G^{(m)}(r_1 \dots r_m, t_1 \dots t_m; r_1 \dots r_m, \dots t_1 \dots t_m)}{G^{(1)}(r_1,t_1;r_1,t_1) \dots \quad G^{(1)}(r_m,t_m;r_m,t_m)} =$$

$$= \frac{\langle : \hat{I}(r_1,t_1) \cdot \dots \cdot \hat{I}(r_m,t_m) : \rangle}{\langle \hat{I}(r_1,t_1)\rangle \cdot \dots \cdot \langle \hat{I}(r_m,t_m)\rangle} \qquad \hat{I}(r,t) = \hat{E}^{(-)}(r,t)\hat{E}^{(+)}(r,t)$$

$$E^{(+)}(\bar{r},t) = \sum_k \overbrace{\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}}^{A_k} \hat{a}_k e^{i(\vec{k}\vec{r} - \omega_k t)}$$

$$\langle\psi| \hat{I}(r,t)|\psi\rangle \underset{\hat{a}_k^\dagger \to \alpha_k^*}{\overset{\hat{a}_k \to \alpha_k}{=}} \left| \sum_k A_k \alpha_k e^{i(\vec{k}\vec{r} - \omega t)} \right|^2$$

$$\langle\psi| : \hat{I}(r_1,t_1) \cdot \dots \cdot \hat{I}(r_m,t_m) : |\psi\rangle = \prod_{i=1}^{m} \left| \sum_k A_k \alpha_k e^{i(\vec{k}\vec{r}_i - \omega_k t_i)} \right|^2$$

$$=$$

Comment: If we have P representation:

$$S = \int d^2\vec{\alpha}\, P(\vec{\alpha}) |\vec{\alpha}\rangle\langle\vec{\alpha}|$$

$$I_{\vec{\alpha}}(\vec{r},t) = \left| \sum_k A_k \alpha_k e^{i(\vec{k}\vec{r} - \omega t)} \right|^2$$

$$G^{(m)}(\dots) = \int d^2\vec{\alpha}\, P(\vec{\alpha})\, G^{(m)}_{|\vec{\alpha}\rangle}(\dots) =$$

$$= \int d^2\vec{\alpha}\, P(\vec{\alpha})\, I_{\vec{\alpha}}(r_1,t_1) \cdot \dots \cdot I_{\vec{\alpha}}(r_m,t_m)$$

Second order temporal coherence function

$$g^{(2)}(\vec{r},\vec{r},t,t+\tau)$$

Consider single plane wave case, $\hat{E}^{(+)} \sim \hat{a}\, e^{i(\vec{k}\vec{r}-\omega t)}$

$$g^{(2)}(0) := g^{(2)}(\vec{r},\vec{r},t,t) = \frac{\langle \hat{a}^{+2}\hat{a}^{2}\rangle}{\langle \hat{a}^{+}\hat{a}\rangle^{2}} =$$

$$= \frac{\langle(\hat{a}^{+}\hat{a})^{2}\rangle - \langle\hat{a}^{+}\hat{a}\rangle}{\langle\hat{a}^{+}\hat{a}\rangle^{2}} = \frac{\langle\hat{n}^{2}\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle^{2}} =$$

$$= 1 + \frac{\overbrace{\langle\hat{n}^{2}\rangle - \langle\hat{n}\rangle^{2}}^{\Delta^{2}n} - \langle\hat{n}\rangle}{\langle\hat{n}\rangle^{2}} = 1 + \frac{\Delta^{2}n - \langle\hat{n}\rangle}{\langle\hat{n}\rangle^{2}}$$

a) $g^{(2)}(0) = 1 \iff \Delta^{2}n = \langle n\rangle$ Poissonian statistics

b) $g^{(2)}(0) > 1 \iff \Delta^{2}n > \langle n\rangle$ super-Poissonian statistics

c) $0 \leq g^{(2)}(0) < 1 \iff \Delta^{2}n < \langle n\rangle$ sub-Poissonian statistics

$|\psi\rangle = |\alpha\rangle \implies g^{(2)}(0) = 1$

$|\psi\rangle = |n\rangle \implies \Delta^{2}n = 0 \quad g^{(2)}(0) = 1 - \frac{n}{n^{2}} = 1 - \frac{1}{n}$

$\rho_T = (1-e^{-\lambda})\sum_n e^{-n\lambda}|n\rangle\langle n| \overset{⇑}{\implies} g^{(2)}(0) = 2$

<u>Fact</u> All classical states have $g^{(2)}(0) \geq 1$
(Poissonian or super-Poissonian statistics)

<u>Proof.</u>

$$\rho = \int d^2\alpha \, P(\alpha) |\alpha\rangle\langle\alpha| \quad , \quad P(\alpha) \geq 0$$

$$g^{(2)}(0) = \frac{\int d^2\alpha \, P(\alpha) |\alpha|^4}{\left[\int d^2\alpha \, P(\alpha)|\alpha|^2\right]^2} = \frac{\int d^2\alpha \, P(\alpha)|\alpha|^4 \int d^2\alpha' \, P(\alpha')}{-//-} =$$

$$\overset{C-S}{\geq} \frac{\left[\int d^2\alpha \, P(\alpha) |\alpha|^2\right]^2}{-//-} = 1$$

$f = \sqrt{P(\alpha)}\, |\alpha|^2$
$g = \sqrt{P(\alpha)}$

<u>Conclusion</u>

whenever you see $g^{(2)}(0) < 1$ we deal with a quantum state of light.

<u>Bunching & Anti-bunching</u>

$$g^{(2)}(\vec{r}_1, \vec{r}_2, t_1, t_2) = \frac{\int d^2\vec{\alpha} \, P(\vec{\alpha}) \, I_{\vec{\alpha}}(\vec{r}_1, t_1) \, I_{\vec{\alpha}}(\vec{r}_2, t_2)}{I(\vec{r}_1, t_1) \quad I(\vec{r}_2, t_2)}$$

If the state is classical $P(\vec{\alpha}) \geq 0$

$$f_1(\vec{\alpha}) = \sqrt{P(\vec{\alpha})}\, I_{\vec{\alpha}}(\vec{r}_1, t_1) \quad , \quad f_2(\vec{\alpha}) = \sqrt{P(\vec{\alpha})}\, I_{\vec{\alpha}}(\vec{r}_2, t_2)$$

$$g^{(2)}(..) \overset{C-S}{\leq} \frac{\sqrt{\int d^2\vec{\alpha} \, P(\vec{\alpha}) I_{\vec{\alpha}}(\vec{r}_1, t_1)^2 \int d^2\vec{\alpha}' \, P(\vec{\alpha}') I_{\vec{\alpha}'}(\vec{r}_2, t_2)^2}}{-//-}$$

$$g^{(2)}(\vec{r}_1, \vec{r}_2, t_1, t_2) \leq \sqrt{g^{(2)}(\vec{r}_1, \vec{r}_1, t_1, t_1) \, g^{(2)}(\vec{r}_2, \vec{r}_2, t_2, t_2)}$$

$$g^{(2)}(r_1,r_2,t_1,t_2) \leq \sqrt{g^{(2)}(r_1,r_1,t_1,t_1)\; g^{(2)}(r_2,r_2,t_2,t_2)}$$

In particular if we choose

$$r_1 = r_2 = r \quad , \quad t_1 = t \quad t_2 = t+\tau$$

$$g^{(2)}(r,r,t,t+\tau) \leq \sqrt{g^{(2)}(r,r,t,t)\; g^{(2)}(r,r,t+\tau,t+\tau)}$$

Consider stationary case (time invariant)

$$g^{(2)}(r,r,t,t+\tau) = g^{(2)}(r,r,0,\tau) =: g^{(2)}(r,\tau)$$

$$g^{(2)}(r,\tau) \leq g^{(2)}(r,0)$$

Bunching - photons tend to arrive together (in bunches)

a) $g^{(2)}(r\,\tau) < g^{(2)}(r,0)$ bunching

b) $g^{(2)}(r,\tau) > g^{(2)}(r,0)$ anti-bunching

c) $g^{(2)}(r,\tau) = g^{(2)}(r,0)$ independent „arrivals"..

Conclusion

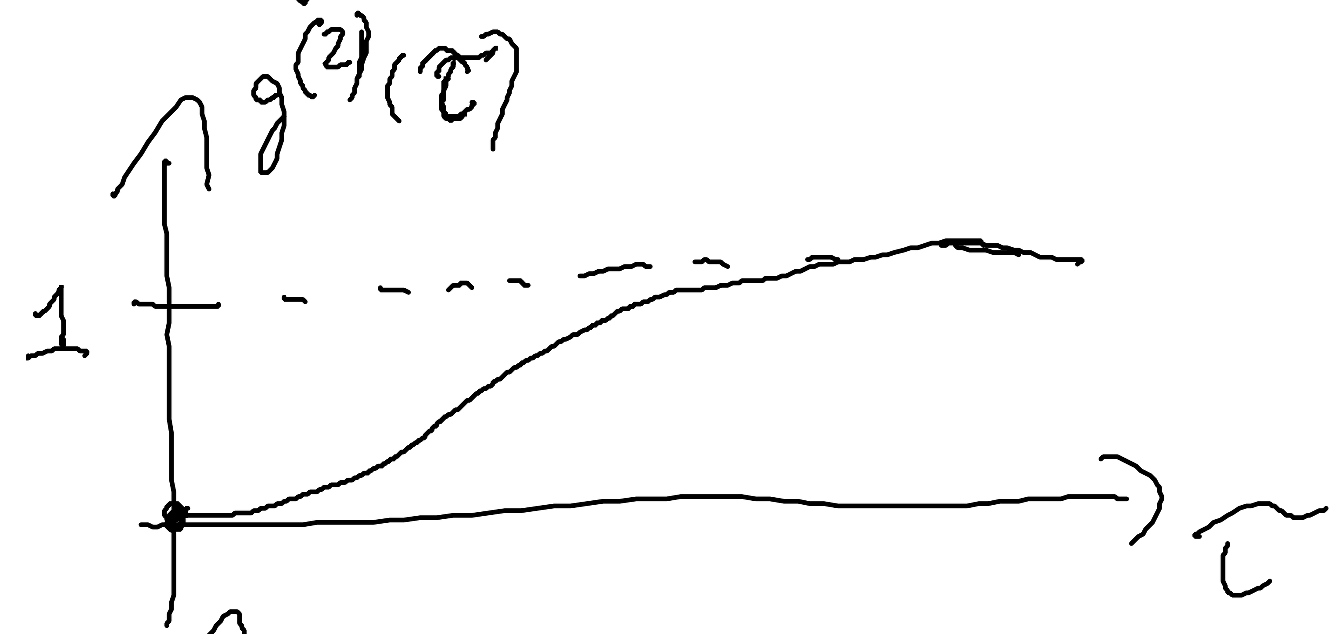
If we observe anti-bunching (in stationary case) then we deal with a non-classical state of light

sub-Poissonian statistics	$g^{(2)}(0) < 1 \Rightarrow$	non-classical state of light.
Anti-bunching	$g^{(2)}(\tau) > g^{(2)}(0) \Rightarrow$	

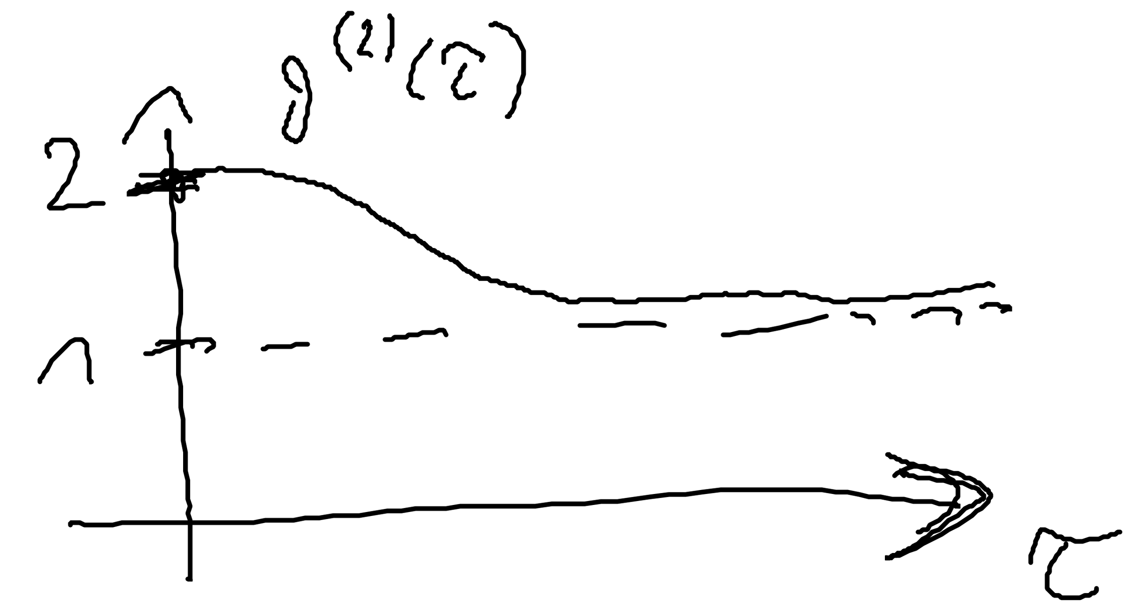
Typically $g^{(2)}(\tau) \xrightarrow{\tau \to \infty} 1$ so

– then sub-poissonian statistics ($g^{(2)}(0) < 1$)

implies anti-bunching ($g^{(2)}(\tau) \to 1 \geq g^{(2)}(0)$).

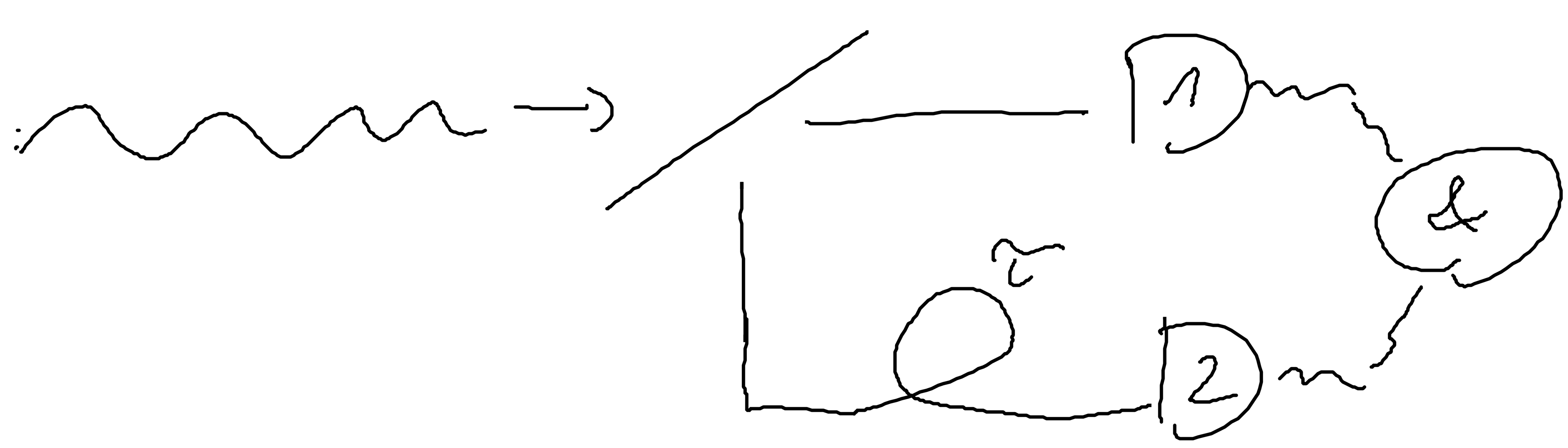


single photon emitters
$g^{(2)}(0) \approx 0$



thermal light

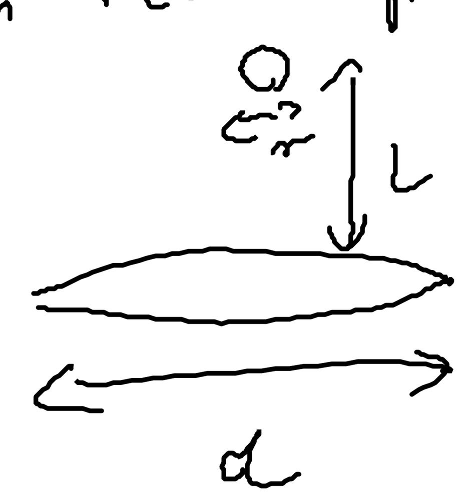
How to measure $g^{(2)}(\tau)$ experimentally.



Coincidence count at time delayed detectors.

Hanbury – Brown – Twiss interferometer (Classically)

Idea: in telescope the bigger the lens the better the resolution

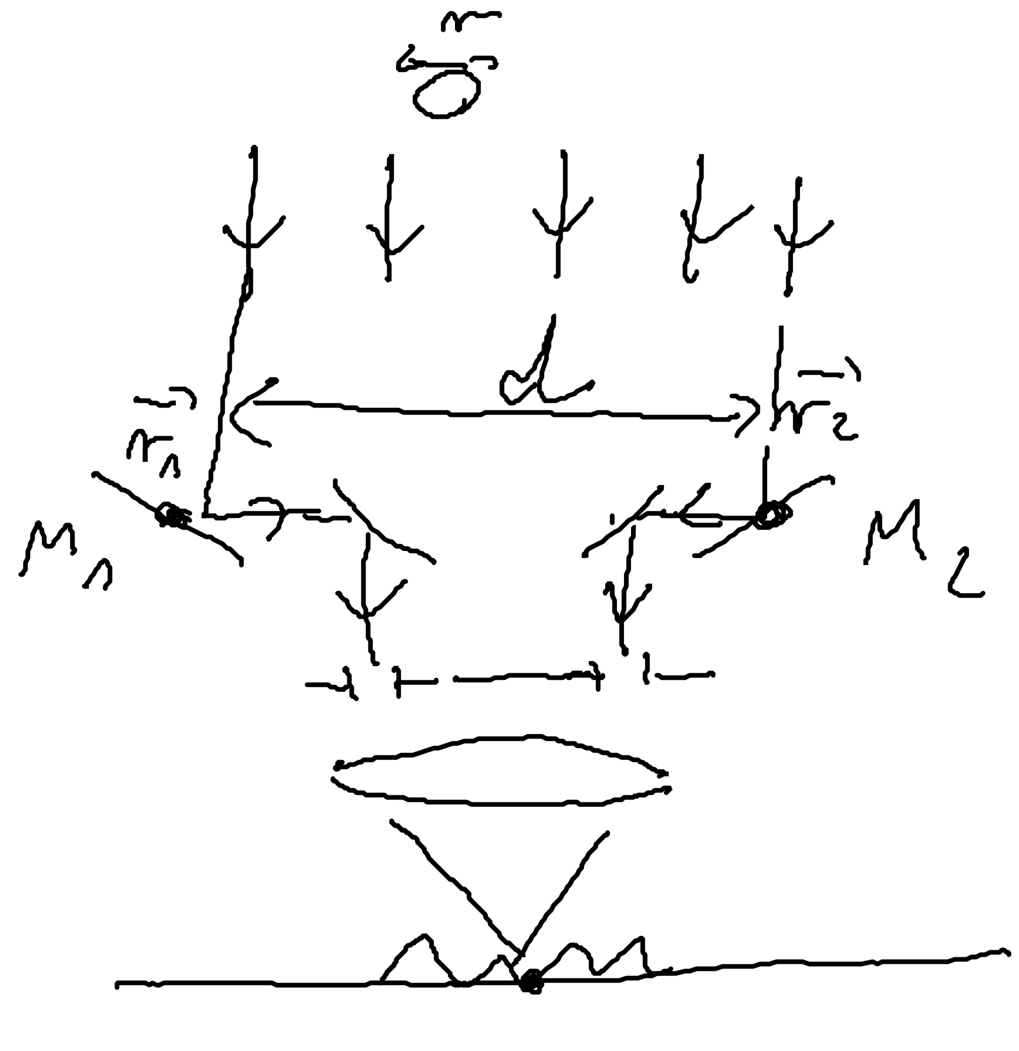


λ – wavelength

angular resolution $\delta\Theta \sim \frac{\lambda}{d}$

We can measure size of an object provided $\frac{r}{L} > \frac{\lambda}{d}$

Michelson stellar interferometer:



d – "effective" size of the lens.

$$\vec{E}(t) = K(\vec{E}(\vec{r}_1, t-\tau) + \vec{E}(\vec{r}_2, \underbrace{t-\tau}_{t'})) \quad *$$

$$I(t) = |K|^2 \left[|E(r_1, t')|^2 + |E(r_2, t')|^2 + 2\,\mathrm{Re}\left(E^*(r_1, t')\, E(r_2, t')\right) \right]$$

Sources fluctuate $\Rightarrow$ Average intensity:

$$\langle I \rangle = |K|^2 \left(\langle I_1 \rangle + \langle I_2 \rangle + 2\,\mathrm{Re}\big(\underbrace{\langle E^*(\vec{r}_1, t') E(\vec{r}_2, t') \rangle}_{\text{first order spatial coherence function}} \big) \right)$$

different parts of an object emits light incoherently

$$E(\underbrace{x, y}_{\vec{r}}, z \approx \infty) \sim$$

far field (Fraunhoffer approximation)

$$\sim \int_{\text{object}} dx' dy' \, E(\underbrace{x', y'}_{\vec{r}\,'}, 0)\, e^{-\frac{ik}{z}(xx' + yy')}$$

z

$$\langle E^*(\vec{r}_1, z) E(\vec{r}_2, z) \rangle \sim \iint_{\text{object}} d^2 r_1' \, d^2 r_2' \langle E^*(\vec{r}_1{}', 0) E(\vec{r}_2{}', 0) \rangle \cdot$$

$$\cdot\, e^{\frac{ik}{z}\left(-\vec{r}_1 \cdot \vec{r}_1{}' + \vec{r}_2 \cdot \vec{r}_2{}'\right)}$$

since the sources are not coherent $\langle E^*(r_1', 0) E(r_2', 0) \rangle =$
(independent phases at different points
and phases unstable) $= \langle I(\vec{r}_1{}') \rangle \delta(\vec{r}_1{}' - \vec{r}_2{}')$

$$\langle E^*(\vec{r}_1, z) E(\vec{r}_2, z) \rangle = \underbrace{\int_{\text{object}} d^2 r' \, e^{\frac{ik}{z}(\vec{r}_2 - \vec{r}_1)\vec{r}\,'} \langle I(\vec{r}\,') \rangle}_{\text{Fourier transform of the intensity profile of the object.}}$$

Problem: stability of „interferometer arms"

$\vec{E}$

If arms are not stable:

$$\vec{E}(t) = K\left(e^{i\varphi_1(t)} E(\vec{r}_1, t') + e^{i\varphi_2(t)} E(\vec{r}_2, t')\right)$$

φ_i - fluctuating phases due to unstable optical length of arms of the interferometer..

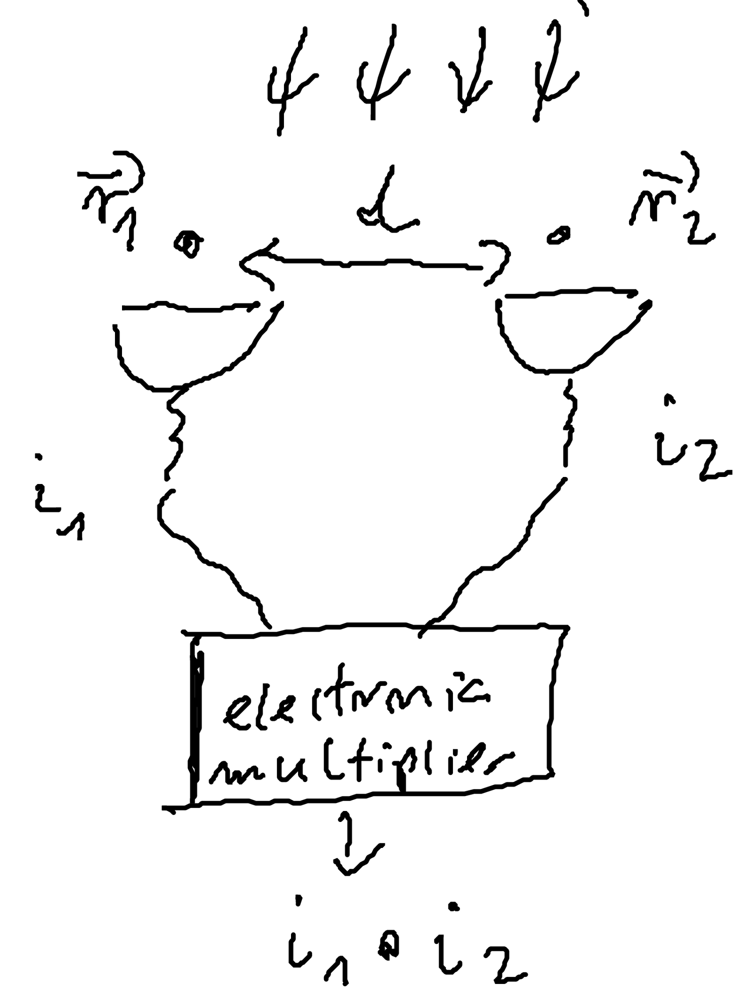
$$\langle I(t)\rangle = |K|^2\left(\langle I(\vec{r}_1)\rangle + \langle I(\vec{r}_2)\rangle + 2\,\mathrm{Re}\,\langle e^{i(\varphi_1-\varphi_2)} E^*(\vec{r}_1, t') E(\vec{r}_2, t')\rangle\right)$$

if φ_i fluctuate too much this term goes to 0.

no information on the object.

Hanbury-Brown-Twiss idea:

instead of interfering amplitudes let us "iterfere intensities"



i_2 - photocurrents (proportional to intensities)

$$i_1 \sim I(\vec{r}_1, t) = E^*(\vec{r}_1, t)\, E(\vec{r}_1, t)$$

$$i_2 \sim I(\vec{r}_2, t) = E^*(\vec{r}_2, t)\, E(\vec{r}_2, t)$$

$$\langle I(\vec{r}_1)\cdot I(\vec{r}_2)\rangle = \underline{\langle E^*(\vec{r}_1, t) E^*(\vec{r}_2, t) E(\vec{r}_1, t) E(\vec{r}_2, t)\rangle}$$

Does it tell us anything about the distant object?

Assume: $E_i = x_i + i y_i$ $\quad x_i, y_i$ - Gaussian random variables

$\{E_1 = E(\vec{r}_1, t),\ E_2 = E(\vec{r}_2, t)$

For Gaussian random variables: $\vec{x} = [x_1, \dots, x_N]^T$, $\langle x_i\rangle = 0$

$$p(\vec{x}) = \sqrt{\frac{\pi^N}{2^N \det C}}\, e^{-\frac{1}{2}\vec{x}^T C^{-1}\vec{x}} \qquad C \text{ - covariance matrix}$$

Thanks to Gaussianity, we can express fourth moments via second moments

$$\langle x_i^2 x_j^2\rangle = \langle x_i^2\rangle\langle x_j^2\rangle + 2\langle x_i x_j\rangle^2$$

we will need: $\langle E_1^* E_2^* E_1 E_2 \rangle$, $E_i = x_i + i y_i$

$$\langle (x_1 - i y_1)(x_2 - i y_2)(x_1 + i y_1)(x_2 + i y_2) \rangle =$$

$$= \langle (x_1^2 + y_1^2)(x_2^2 + y_2^2) \rangle = \langle x_1^2 x_2^2 \rangle + \langle y_1^2 y_2^2 \rangle + \langle x_1^2 y_2^2 \rangle + \langle x_2^2 y_1^2 \rangle$$

$$\overset{\Delta}{=} (\langle x_1^2 \rangle + \langle y_1^2 \rangle)(\langle x_2^2 \rangle + \langle y_2^2 \rangle) + 2(\langle x_1 x_2 \rangle^2 + \langle x_1 y_2 \rangle^2 + \langle y_1 x_2 \rangle^2 + \langle y_1 y_2 \rangle^2)$$

$$= \langle |E_1|^2 \rangle \langle |E_2|^2 \rangle + \underline{|\langle E_1^* E_2 \rangle|^2} + |\underline{\langle E_1 E_2 \rangle}|^2$$

Absolute phase of E_i is not stable:

$E_i \rightarrow E_i e^{i\varphi}$ $\quad$ $\langle .. \rangle$ includes averaging over this absolute phase φ.

$$\Downarrow$$

$$\langle E_1 E_2 \rangle \rightarrow 0$$

this implies:

$$\langle I(r_1,t)\, I(r_2,t) \rangle = \langle I(r_1,t) \rangle \cdot \langle I(r_2,t) \rangle + \underbrace{|\langle E^*(r_1,t)\, E(r_2,t) \rangle|^2}_{|g^{(1)}(r_1,r_2)|^2}$$

this contains information about the imaged object