Quantum Estimation and Measurement Theory

Problem set 11

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Problem 1 We have found that the optimal state in the optimal Bayesian phase estimation strategy using N qubits reads:

$$|\Psi\rangle = \sum_{n=0}^{N} c_n |n, N-n\rangle, \quad \text{where } c_n = \sqrt{\frac{2}{N+2}} \sin\left(\frac{(n+1)\pi}{N+2}\right). \tag{1}$$

We have realized that within the Bayesian framework , had we considered the N00N state, $|\Phi_N\rangle = \frac{1}{\sqrt{2}}(|N,0\rangle + |0,N\rangle)$, we would not obtain any advantage due to the fact that the N00N state is unable to discriminate phases that differ by a multiple of $2\pi/N$. Instead consider the following state, which is a tensor product of N00N states with photon numbers which correspond to subsequent powers of 2:

$$|\Psi_{\text{Kitaev}}\rangle = \bigotimes_{k=0}^{K=\log_2(N+1)-1} |\Phi_{2^k}\rangle, \tag{2}$$

where we have assumed that $K = \log_2(N+1)$ is a natural number. Note that in total we have utilized N particiles. This kind of state appears in the so called Kitaev phase estimation algorithm which is an important element of Shor's quantum factoring algorithm. For this state there is no ambiguity in estimating phase and while increasing N we will be able to estimate the phase with increasing precision. Check how will the Bayesian cost behave for this state. Compare with the strategy that does not utilize entanglement between the particles at all. *Hint.* It is convenient to us the notation where $|i_0\rangle\otimes\cdots\otimes|i_K\rangle$, where $i_k \in \{0,1\}$ and the state $|i_k = 0\rangle$ denotes the state $|0, 2^k\rangle$ and the state $|i_k = 1\rangle$ represents the state $|2^k, 0\rangle$ where the k-th N00N state lives. Note that, in this case $U_\varphi^{\otimes N}|i_0\rangle \otimes \cdots \otimes |i_K\rangle = e^{\mathrm{i}\varphi 2^0 i_0}|i_0\rangle \otimes \cdots \otimes e^{\mathrm{i}\varphi 2^K i_K}|i_K\rangle$. Recall that the optimal covariant measurement takes the form $\Pi_0 = |f\rangle\langle f|$, where $|f\rangle = \sum_{i_0,\dots,i_K=0}^{1} |i_0\rangle\otimes\cdots\otimes|i_K\rangle$.