Quantum Estimation and Measurement Theory

Problem set 3

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Problem 1 Consider a generalization of the Cramer-Rao bound, where instead of estimating the parameter θ itslef we want to estimate a value of a function $g(\theta)$. Prove that that if $p_\theta(x)$ is a family of probability distribution then for arbitrary locally unbiased estimator $\tilde{g}(x)$, we have

$$\Delta^2 \tilde{g} \geq \frac{g'(\theta)^2}{F} \tag{1}$$

where $g'(\theta) = \frac{\mathrm{d}g(\theta)}{\mathrm{d}\theta}$ and F is the Fisher inforomation for $p_\theta(x)$.

Problem 2 We say hthat $p_\theta(x)$ belongs to the exponential family of proability distributions if and only if

$$p_\theta(x) = e^{a(\theta)+b(x)+c(\theta)d(x)} \tag{2}$$

Prove, that in this case, there is always a function $g(\theta)$ for which there exist an efficient estimator—the estimator that saturates the CR bound. A lot of probability distributions belong to this family (see `http://en.wikipedia.org/wiki/Exponential_family`).

Problem 3 Consider a probabilistic model where we register values of N independent random variables x_n, $(n = 0, \dots, N-1)$, where $x_n \sim \mathcal{N}(an + b, \sigma^2)$ (linear dependence + gaussian noise).

a) Write down the Fisher matrix corresponding to the two-parameter estimation problem of estimating a and b parameters.

b) Using the CR bound derive a lower bound on the minimal achievable estimation variance of the two parameters: $\Delta\tilde{a}$, $\Delta\tilde{b}$. Which one is „easier" to estimate.

c) Try to provide estimators saturating the bound—check if by chance these are the same estimators that one uses in the heuristic minimum squared distance method ...