Quantum Estimation and Measurement Theory

Problem set 8,9

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Problem 1 Consider a qubit as a model of a two-level atom, where $|0\rangle$, $|1\rangle$ are respectively ground and excited state. Let us assume that we want to estimate transition frequency ω between the levels. We prepare the atom in state $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and evolve it subsequently for a known time t. As a result we obtain the state:

$$|\psi_\omega\rangle = e^{\mathrm{i}\omega t\sigma_z/2}|\psi\rangle. \tag{1}$$

We assume that the prior distribution representing our prior knowledge about the frequency is gaussian:

$$p(\omega) = \frac{1}{\sqrt{2\pi\Delta^2\omega}} e^{-(\omega-\omega_0)^2/2\Delta^2\omega}, \tag{2}$$

where ω_0 and $\Delta^2\omega$ are the mean and the variance of the distribution respectively.

a) Find the formula for the minimal Bayesian cost in this problem as a function of time of evolution t. Plot $\overline{\Delta^2\tilde{\omega}}/\Delta^2\omega$ as a function of t, which will show relative reduction of uncertainty as a result of the estimation procedure. Hint: In order to reduce you effort try to make use of the relation between the Bayesian cost and the Fisher information so you can use some of the results from the previous problem set.

b) Determine the optimal evolution time for which the Bayesian cost is the lowest possible.

c) For the optimal time provide the measurement and the values of estimated frequencies that result in the optimal estimation strategy.

d) What would happen of somebody focused just on the quantum Fisher information of the state $|\psi_\omega\rangle$ in equation (1). At what conclusions he/she would arrive regarding the optimal evolution time using just the concept of Fisher information. Would these conclusions be sensible.

Problem 2 During the lecture , it was proven that optimal Bayesian strategy of estimating φ using N copies of state $|\psi_\varphi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \exp(\mathrm{i}\varphi)|1\rangle)$ and assuming the flat prior leads to the following formula for the minimal cost:

$$\bar{C}^{(N)}_{\mathrm{opt}} = 2 - \frac{1}{2^{N-1}} \sum_{n=1}^{N} \sqrt{\binom{N}{n}\binom{N}{n-1}}, \tag{3}$$

where the cost function was chosen to be $C(\varphi, \tilde{\varphi}) = 4\sin^2[(\varphi - \tilde{\varphi})/2]$.

Since the optimal strategy in general requires application of collective measurements on many particles, we would like to compare it with the performance of a simply single particle measurement and simple estimation based on the maximum likelihood estimator. Consider the following estimation strategy:

a) We perofm a measurement on a single particle described using 4 measurement operators: $\Pi_0 = \frac{1}{2}|+\rangle\langle+|$, $\Pi_1 = \frac{1}{2}|-\rangle\langle-|$, $\Pi_2 = \frac{1}{2}|+\mathrm{i}\rangle\langle+\mathrm{i}|$, $\Pi_3 = \frac{1}{2}|-\mathrm{i}\rangle\langle-\mathrm{i}|$, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, $|\pm\mathrm{i}\rangle = (|0\rangle \pm \mathrm{i}|1\rangle)/\sqrt{2}$,. One can equivalently think about this measurement as a performing a projective measurement in basis $|+\rangle$, $|-\rangle$ with probability $1/2$ and with probability $1/2$ measuring in the $|+\mathrm{i}\rangle$, $|-\mathrm{i}\rangle$ basis. The measurement is performed subsequently on N particles. This way we obtain a sequence of measurement results $\vec{x} = (x_1, \ldots, x_N)$, where $x_i \in \{0, 1, 2, 3\}$.

b) Based on results $\vec{x}$ we estimate phase $\tilde{\varphi}$ using the maximum likelihood estimator.

c) We want to compare efficiency of this strategy with the optimal strategy. In order to do so, we fix some true value φ, and repeat the above steps approx 1000 times. For each realization we calculate the cost $C(\varphi, \tilde{\varphi})$. Since we want to compare the performance with the one optimal for the flat prior we repeat this procedure for different values of φ (e.g. 30 different values) uniformly placed within the interval $[0, 2\pi]$. We calculate the average cost $C^{(N)}$.

d) We repeat the above strategy for different N and observe when we observe convergence to the value of the optimal Bayesian cost. We expect this to happen, since asymptotically the optimal cost behaves like $1/N$ and this we know we should be able to saturate using simple measurements. It will be instructive to identify N for which the advantage of the optimal strategy is the largest compared with the simple strategy, i.e. when $C^{(N)}/C^{(N)}_{\mathrm{opt}}$ will be the largest