ON THE ENERGY-MOMENTUM SPECTRUM OF THE HOMOGENEOUS FERMI GAS

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Based on joint work with

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A quasiparticle system is a $1+d$-tuple of commuting self-adjoint operators $(H,P)=(H_{\rm fr},P_{{\rm fr},1},\dots,P_{{\rm fr},d})$:

$$\begin{aligned} H_{\rm fr} &= \sum_{i\in\mathcal{Q}_{\rm s}\cup\mathcal{Q}_{\rm a}} \int_{I_i} \omega_i({\rm k}) b_i^*({\rm k}) b_i({\rm k}) {\rm dk}, \\ P_{{\rm fr},j} &= \sum_{i\in\mathcal{Q}_{\rm s}\cup\mathcal{Q}_{\rm a}} \int_{I_i} {\rm k}_j b_i^*({\rm k}) b_i({\rm k}) {\rm dk}, \end{aligned}$$

for some dispersion relations $I_i \ni {\rm k} \mapsto \omega_i({\rm k})$, and creation/annihilation operators $b_i^*({\rm k})$, $b_i({\rm k})$. $\mathcal{Q}_{\rm s}/\mathcal{Q}_{\rm a}$ labels bosons/fermions

Define the fermionic number operator:

$$N_{\mathrm{a}} = \sum_{i \in \mathcal{Q}_{\mathrm{a}}} b_i^*(\mathrm{k}) b_i(\mathrm{k}).$$

The fermionic parity $(-1)^{N_{\mathrm{a}}}$ provides a natural superselection rule $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ so that the Hamiltonian and momentum decompose as

$$(H_{\mathrm{fr}}, P_{\mathrm{fr}}) = (H_{\mathrm{fr}}^+, P_{\mathrm{fr}}^+) \oplus (H_{\mathrm{fr}}^-, P_{\mathrm{fr}}^-). \tag{1}$$

(1) will be called a two-sector quasiparticle system.

Consider for example the non-interacting Fermi gas

$$\begin{aligned} H_{\mathrm{fr}} &= \int_{\mathbb{R}^d} (\mathrm{k}^2 - \mu) a^*(\mathrm{k}) a(\mathrm{k}) \mathrm{dk}, \\ P_{\mathrm{fr}} &= \int_{\mathbb{R}^d} \mathrm{k} a^*(\mathrm{k}) a(\mathrm{k}) \mathrm{dk}. \end{aligned}$$

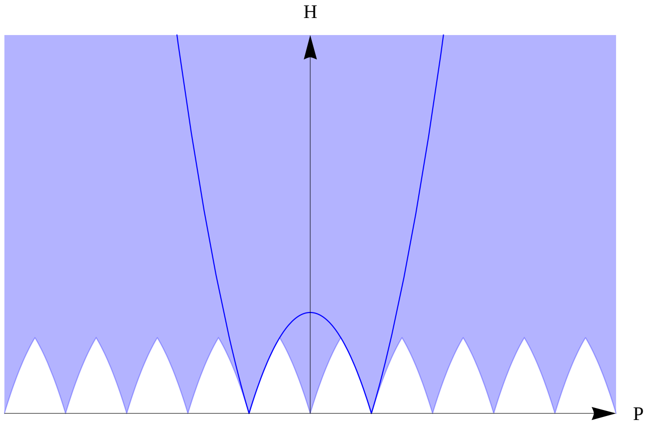
We introduce new creation/annihilation operators

$$\begin{aligned} b_{\mathrm{k}}^* : &= a_{\mathrm{k}}^*, \ b_{\mathrm{k}} := a_{\mathrm{k}}, \ \mathrm{k}^2 > \mu, \\ b_{\mathrm{k}}^* : &= a_{-\mathrm{k}}, \ b_{\mathrm{k}} := a_{-\mathrm{k}}^*, \ \mathrm{k}^2 \leq \mu. \end{aligned}$$

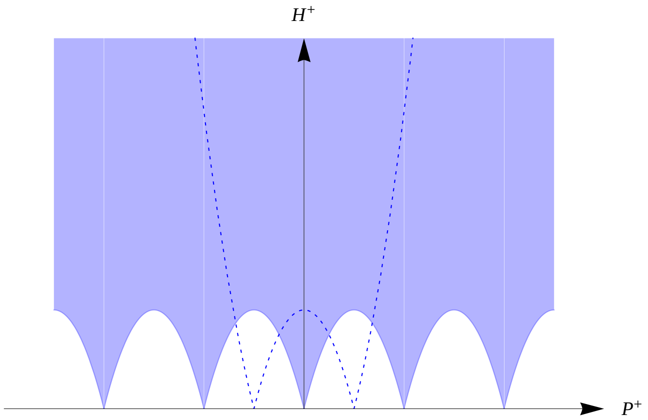
Dropping a constant from the Hamiltonian and setting $\omega(\mathrm{k}) = |\mathrm{k}^2 - \mu|$, we obtain

$$\begin{aligned} H_{\mathrm{fr}} &= \int_{\mathbb{R}^d} \omega(\mathrm{k}) b^*(\mathrm{k}) b(\mathrm{k}) \mathrm{dk}, \\ P_{\mathrm{fr}} &= \int_{\mathbb{R}^d} \mathrm{k} b^*(\mathrm{k}) b(\mathrm{k}) \mathrm{dk}. \end{aligned}$$

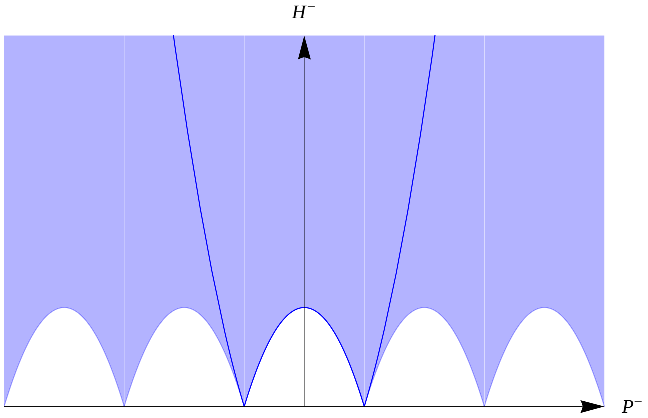
In dimension 1 its energy-momentum spectrum looks quite interesting:



$\mathrm{sp}(H, P)$ in the non-interacting case, $d = 1$.

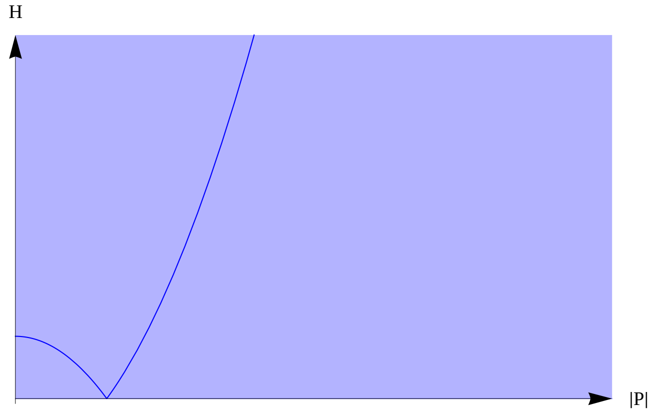


$\mathrm{sp}(H^+, P^+)$ in the non-interacting case, $d = 1$.



$\mathrm{sp}(H^-, P^-)$ in the non-interacting case, $d = 1$.

Clearly, for $d \geq 2$ the energy-momentum spectrum is rather boring:

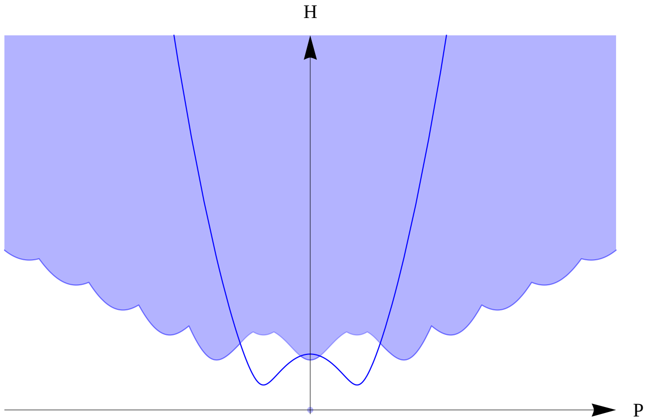


$\mathrm{sp}(H, P)$, $\mathrm{sp}(H^+, P^+)$, $\mathrm{sp}(H^-, P^-)$ in the non-interacting case, $d \geq 2$.

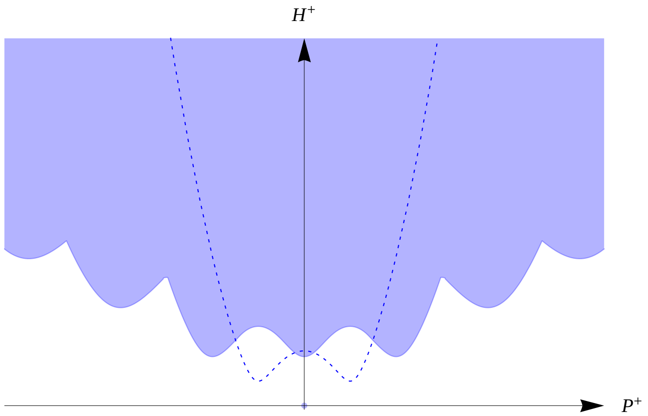
For interacting Fermi gas, the Hartree-Fock-Bogoliubov method with the Bardeen-Cooper-Schrieffer ansatz suggests that the dispesion relation may be similar to

$$\omega(\mathrm{k}) = \sqrt{(\mathrm{k}^2 - \mu)^2 + \gamma^2}. \tag{2}$$

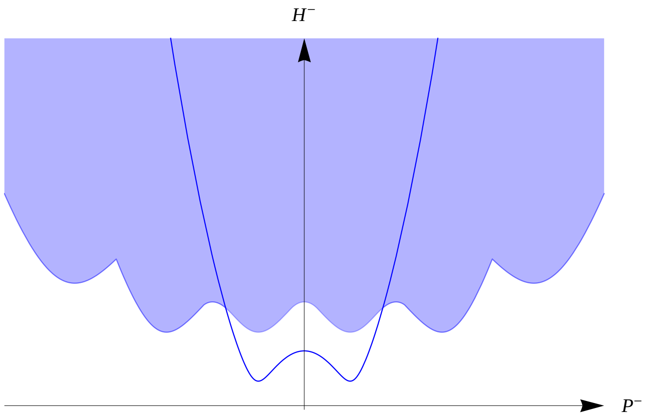
In all dimensions the energy-momentum spectrum the energy gap and the critical velocity are strictly positive.



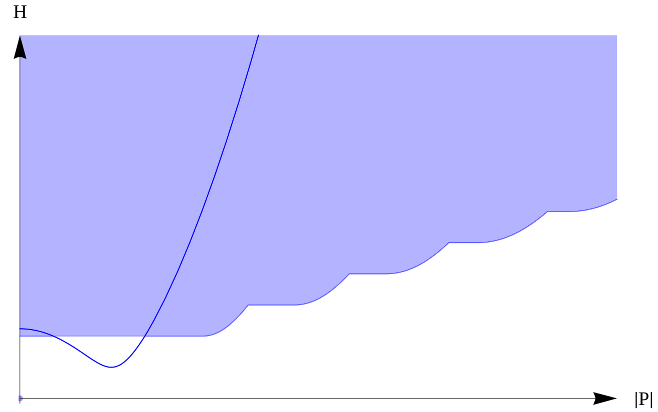
$\mathrm{sp}(H, P)$ in the interacting case, $d = 1$.



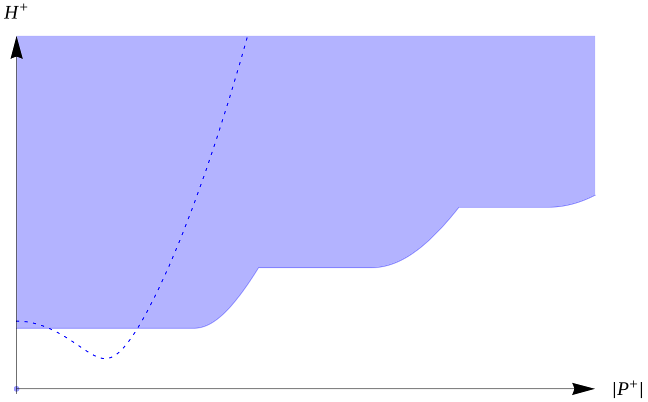
$\mathrm{sp}(H^+, P^+)$ in the interacting case, $d = 1$.



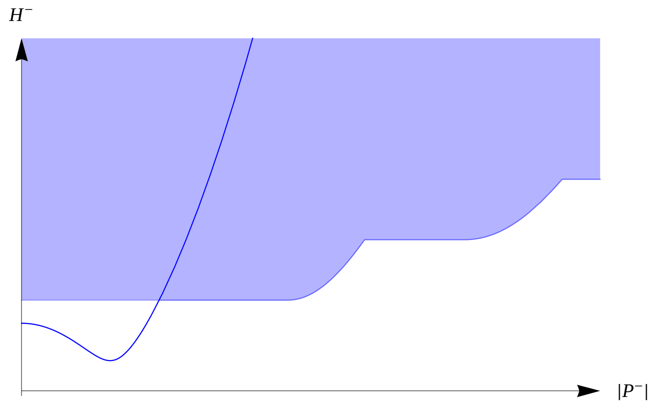
$\mathrm{sp}(H^-, P^-)$ in the interacting case, $d = 1$.



$\mathrm{sp}(H, P)$ in the interacting case, $d \geq 2$.



$\mathrm{sp}(H^+, P^+)$ in the interacting case, $d \geq 2$.



$\mathrm{sp}(H^-, P^-)$ in the interacting case, $d \geq 2$.

Suppose we have a translation invariant quantum system in infinite volume described by the Hamiltonian H and the momentum $P = (P_1, \ldots, P_d)$. Instead of the energy-momentum spectrum of (H, P).

$$\mathrm{sp}(H, P)$$

we will often use the excitation spectrum of (H, P)

$$\mathrm{sp}(H - E, P)$$

where $E := \inf \mathrm{sp} H$ is the ground state energy.

Let (H, P) be a quasiparticle system. The energy-momentum spectrum of such systems has special properties. First

$$(0, 0) \in \mathrm{sp}(H, P),$$

because of the Fock vacuum state. Moreover,

$$\mathrm{sp}(H, P) = \mathrm{sp}(H, P) + \mathrm{sp}(H, P).$$

Assume now that all the dispersion relations are non-negative. Then the ground state energy is zero and the excitation spectrum coincides with the energy-momentum spectrum. Thus

$$\begin{aligned} (0,0) &\in \mathrm{sp}(H-E,P), \\ \mathrm{sp}(H-E,P) &= \mathrm{sp}(H-E,P)+\mathrm{sp}(H-E,P). \end{aligned}$$

Let (H, P) be a translation invariant system on a Hilbert space $\mathcal{H}$. We will say that it is a quasiparticle-like system if it is unitarily equivalent to a quasiparticle system.

In practice we expect that the unitary equivalence mentioned in this definition is in some sense natural and constructed in the framework of scattering theory. For at least two important classes of many body systems the basic idea of scattering theory can be described as follows.

Using the evolution $\mathrm{e}^{\mathrm{i}tH}$ for $t \to \pm\infty$, we define two isometric wave operators

$$S^{\pm} : \bigotimes_{i \in \mathcal{Q}_{\mathrm{s}}} \Gamma_{\mathrm{s}} \left(L^2(I_i) \right) \otimes \bigotimes_{j \in \mathcal{Q}_{\mathrm{a}}} \Gamma_{\mathrm{a}} \left(L^2(I_j) \right) \to \mathcal{H},$$

For an appropriate quasiparticle system $(H_{\mathrm{fr}}, P_{\mathrm{fr}})$:

$$HS^{\pm} = S^{\pm} H_{\mathrm{fr}}, \quad PS^{\pm} = S^{\pm} P_{\mathrm{fr}},$$

Asymptotic completeness means that $S^{\pm}$ are unitary. Asymptotically complete systems are quasiparticle-like.

The first class where this kind of scattering theory holds is the 2nd quantization of Schrödinger many body operators with 2-body short range interactions. One can show that they are asymptotically complete.

Note that they are invariant wrt. the Galileian group and the dispersion relations have the form $\mathbb{R}^d \ni \mathrm{k} \mapsto E + \frac{\mathrm{k}^2}{2m}$. Quasiparticles obtained in this context can be "elementary" – typically electrons and nuclei – as well as "composite" – atoms, ions, molecules, etc.

Another important class of systems where the concept of asymptotic quasiparticles has a rigorous foundation belongs to (relativistic) quantum field teory, as axiomatized by the Haag-Kastler or Wightman axioms. If we assume the existence of discrete mass shells, the so-called Haag-Ruelle theory allows us to construct the wave operators. The system is covariant wrt. the Poincaré group and the dispersion relation has the form $\mathbb{R}^d \ni \mathrm{k} \mapsto \sqrt{m^2 + \mathrm{k}^2}$. Here, quasiparticles are the usual stable particles.

Let us stress that both classes of systems can be interacting in spite of the fact that they are equivalent to free quasiparticle systems. In particular, their scattering operator can be nontrivial.

The concept of a quasiparticle is useful also in other contexts, without the Galilei or Poincaré covariance. This allows us to consider more general dispersion relations.

An interesting system which admits a quasiparticle interpretation is the free Fermi gas with a positive chemical potential. In this case the scattering theory is trivial, however the spectrum can be quite interesting, as we saw above.

It seems that condensed matter physicists apply successfully the concept of a quasiparticle also to various interacting Bose and Fermi translation invariant gas without the Galilei or Poincaré invariance.

Bose gas with repulsive interactions at zero temperature and positive density, apparently, is well described by a free Bose gas of quasiparticles of (at least) two kinds: at low momenta we have phonons with an approximately linear dispersion relation, and at higher momenta we have rotons. This follows from the famous Bogoliubov approximation. Superfluidity can be to a large extent explained within this picture. The model of free asymptotic phonons seems to work well in real experiments.

By using the HFB approach and the BCS ansatz, Fermi gas with attractive interactions at zero temperature and positive chemical potential can be approximated by a simple model explaining its superconductivity at very low temperatures. The corresponding quasiparticles are sometimes called partiholes.

The concept of a quasiparticle-like system is probably too strong for many applications. Let us propose a weaker property. We will say that the excitation spectrum of (H, P) is quasiparticle-like if it coincides with the excitation spectrum of a quasiparticle system.

There exists a heuristic, but, we believe, a relatively convincing general argument why realistic translation invariant quantum systems in thermodynamic limit at zero temperature should satisfy

$$\begin{aligned}(0,0) &\in \mathrm{sp}(H-E,P),\\ \mathrm{sp}(H-E,P) &= \mathrm{sp}(H-E,P)+\mathrm{sp}(H-E,P).\end{aligned}$$

Consider a quantum gas in a box of a very large side length L, described by (H^L, P^L). For shortness, let us drop the superscript L. Assume that the system possesses a translation invariant ground state, which we will denote by Φ, so that $H\Phi = E\Phi$, $P\Phi = 0$.

Let $(E + e_i, \mathrm{k}_i) \in \mathrm{sp}(H, P)$, $i = 1, 2$. We can find eigenvectors with these eigenvalues, that is, vectors Φ_i satisfying $H\Phi_i = (E + e_i)\Phi_i$, $P\Phi_i = \mathrm{k}_i\Phi_i$.

Assume that it is possible to find operators A_i well localized in configuration space such that $PA_i \approx A_i(P + \mathrm{k}_i)$, and which approximately create the vectors Φ_i from the ground state, that is $\Phi_i \approx A_i\Phi$. By replacing Φ_2 with $\mathrm{e}^{\mathrm{iy}P}\Phi_2$ for some y and A_2 with $\mathrm{e}^{\mathrm{iy}P}A_2\mathrm{e}^{-\mathrm{iy}P}$, we can make sure that the regions of localization of A_1 and A_2 are separated by a large distance. (Note that a large size of L plays a role in the argument).

Now consider the vector $\Phi_{12} := A_1 A_2 \Phi$. Clearly,

$$P\Phi_{12} \approx (\mathrm{k}_1 + \mathrm{k}_2)\Phi_{12}.$$

Φ_{12} looks like the vector Φ_i in the region of localization of A_i, elsewhere it looks like Φ. We expect that

$$H\Phi_{12} \approx (E + e_1 + e_2)\Phi_{12}.$$

If this is the case, it implies that $(E + e_1 + e_2, \mathrm{k}_1 + \mathrm{k}_2) \in \mathrm{sp}(H, P)$.

Consider now an arbitrary translation invariant system with two superselection sectors $(H, P) = (H^+, P^+) \oplus (H^-, P^-)$ with a translation invariant ground state in the sector $\mathcal{H}^+$.

We will say that its excitation spectrum is two-sector quasiparticle-like if it coincides with the excitation spectrum of a two-sector quasiparticle system.

Such systems satisfy

$$\begin{aligned}(0,0) &\in \mathrm{sp}(H^+ - E, P^+),\\ \mathrm{sp}(H^+ - E, P^+) &= \mathrm{sp}(H^+ - E, P^+) + \mathrm{sp}(H^+ - E, P^+)\\ &\supset \mathrm{sp}(H^- - E, P^-) + \mathrm{sp}(H^- - E, P^-),\\ \mathrm{sp}(H^- - E, P^-) &= \mathrm{sp}(H^- - E, P^-) + \mathrm{sp}(H^+ - E, P^+).\end{aligned}$$

A heuristic general argument, similar to the one described above, shows that realistic quantum systems in thermodynamic limit should satisfy these conditions.

Consider a fermionic many body system with spin on $\mathbb{R}^d$. Its one-particle space of the system is $L^2(\mathbb{R}^d, \mathbb{C}^2)$. Its kinetic energy has the form

$$(T\Phi)_i(\mathrm{x}_1) = \int t(\mathrm{x}_1, \mathrm{x}_2)\Phi_i(\mathrm{x}_2)\mathrm{d}\mathrm{x}_2.$$

We assume that t is Hermitian and translation invariant:

$$\begin{aligned} t(\mathrm{x}_1, \mathrm{x}_2) &= \overline{t(\mathrm{x}_2, \mathrm{x}_1)} = t(\mathrm{x}_1 + \mathrm{y}, \mathrm{x}_2 + \mathrm{y}) \\ &= (2\pi)^{-d} \int \tau(\mathrm{k}) \mathrm{e}^{\mathrm{i}\mathrm{k}(\mathrm{x}_1 - \mathrm{x}_2)} \mathrm{d}\mathrm{k}. \end{aligned}$$

Its interaction will be given by a 2-body operator

$$\begin{aligned}(V\Phi)_{i_1,i_2}(\mathrm{x}_1,\mathrm{x}_2) = \frac{1}{2}\int\int\Big(&v(\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4)\Phi_{i_2,i_1}(\mathrm{x}_4,\mathrm{x}_1)\\ &-v(\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_4,\mathrm{x}_3)\Phi_{i_1,i_2}(\mathrm{x}_3,\mathrm{x}_4)\Big)\mathrm{dx}_3\mathrm{dx}_4,\end{aligned}$$

where $\Phi \in \Gamma_{\mathrm{a}}^2\left(L^2(\mathbb{R}^d,\mathbb{C}^2)\right)$. We will assume that it is Hermitian and translation invariant:

$$\begin{aligned} & v(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4) = v(\mathrm{x}_2, \mathrm{x}_1, \mathrm{x}_4, \mathrm{x}_3) \\ = \ & \overline{v(\mathrm{x}_4, \mathrm{x}_3, \mathrm{x}_2, \mathrm{x}_1)} = v(\mathrm{x}_1 + \mathrm{y}, \mathrm{x}_2 + \mathrm{y}, \mathrm{x}_3 + \mathrm{y}, \mathrm{x}_4 + \mathrm{y}) \\ = \ & (2\pi)^{-4d} \int \mathrm{e}^{\mathrm{i}\mathrm{k}_1\mathrm{x}_1 + \mathrm{i}\mathrm{k}_2\mathrm{x}_2 - \mathrm{i}\mathrm{k}_3\mathrm{x}_3 - \mathrm{i}\mathrm{k}_4\mathrm{x}_4} q(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4) \\ & \times \delta(\mathrm{k}_1 + \mathrm{k}_2 - \mathrm{k}_3 - \mathrm{k}_4) \mathrm{d}\mathrm{k}_1 \mathrm{d}\mathrm{k}_2 \mathrm{d}\mathrm{k}_3 \mathrm{d}\mathrm{k}_4, \end{aligned}$$

where q is a function defined on the subspace $\mathrm{k}_1 + \mathrm{k}_2 = \mathrm{k}_3 + \mathrm{k}_4$.

An example of a 1-particle energy is

$$\begin{aligned} T &= -\frac{1}{2m}\Delta - \mu, \\ \tau(\mathrm{k}) &= \mathrm{k}^2 - \mu. \end{aligned}$$

An example of interaction is a 2-body potential $V(\mathrm{x})$ such that $V(\mathrm{x}) = V(-\mathrm{x})$, which corresponds to

$$\begin{aligned} v(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4) &= V(\mathrm{x}_1 - \mathrm{x}_2)\delta(\mathrm{x}_1 - \mathrm{x}_4)\delta(\mathrm{x}_2 - \mathrm{x}_3), \\ q(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4) &= \int \mathrm{dq}\hat{V}(\mathrm{q})\delta(\mathrm{k}_1 - \mathrm{k}_4 - \mathrm{q})\delta(\mathrm{k}_2 - \mathrm{k}_3 + \mathrm{q}). \end{aligned}$$

Let $T_{(i)}$ denote the operator T acting on the ith variable and $V_{(ij)}$ denote the operator V acting on the (ij)th pair of variables. The n-body Hamiltonian and momentum of homogeneous Fermi gas acts on the Hilbert space $\Gamma_{\mathrm{a}}^{n}\left(L^2(\mathbb{R}^d,\mathbb{C}^2)\right)$:

$$\begin{aligned} H^n &= \sum_{1\leq i\leq n} T_{(i)} + \sum_{1\leq i<j\leq n} V_{(ij)}, \\ P^n &:= \sum_{i=1}^{n} -\mathrm{i}\nabla_{\mathrm{x}_i}. \end{aligned}$$

To investigate homogeneous Fermi gas at positive density in thermodynamic limit it is convenient to put the system on a box $\Lambda = [-L/2, L/2]^d$ with periodic boundary conditions. This means in particular that the kinetic energy and interaction are replaced by

$$t^L(x_1, x_2) = \frac{1}{L^d} \sum_{\mathrm{k} \in \frac{2\pi}{L}\mathbb{Z}^d} \mathrm{e}^{\mathrm{ik}\cdot(\mathrm{x}_1 - x_2)} \tau(\mathrm{k}),$$

$$
\begin{aligned}
& v^L(\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4) = \\
= \ & \frac{1}{L^{3d}} \sum_{\mathrm{k}_1,\dots,\mathrm{k}_4\in\frac{2\pi}{L}\mathbb{Z}^d, \quad \mathrm{k}_1+\mathrm{k}_2=\mathrm{k}_3+\mathrm{k}_4} \\
& \times \mathrm{e}^{\mathrm{i}\mathrm{k}_1\cdot\mathrm{x}_1+\mathrm{i}\mathrm{k}_2\mathrm{x}_2-\mathrm{i}\mathrm{k}_3\mathrm{x}_3-\mathrm{i}\mathrm{k}_4\mathrm{x}_4} q(\mathrm{k}_1,\mathrm{k}_2,\mathrm{k}_3,\mathrm{k}_4).
\end{aligned}
$$

The Hamiltonian

$$
H^{L,n} = \sum_{1\leq i\leq n} T^L_{(i)} + \sum_{1\leq i<j\leq n} V^L_{(ij)}
$$

acts on $\Gamma^n_{\mathrm{a}}\big(L^2(\Lambda,\mathbb{C}^2)\big)$.

It is convenient to put all the n-particle spaces into a single Fock space

$$\Gamma_{\mathrm{a}}(L^2(\Lambda,\mathbb{C}^2)) := \bigoplus_{n=0}^{\infty} \Gamma_{\mathrm{a}}^{n}\left(L^2(\Lambda,\mathbb{C}^2)\right)$$

and rewrite the Hamiltonian and momentum in the language of 2nd quantization:

$$\begin{aligned}
H^L &:= \bigoplus_{n=0}^{\infty} H^{L,n} \\
&= \sum_i \int a^*_{\mathrm{x},i} t^L(\mathrm{x}_{i_1} - \mathrm{x}_{i_2}) a_{\mathrm{x},i_2} \mathrm{dx}_1 \mathrm{dx}_2 \\
&\quad + \frac{1}{2} \sum_{i_1,i_2} \int \int a^*_{\mathrm{x}_1,i_1} a^*_{\mathrm{x}_2,i_2} v^L(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4) a_{\mathrm{x}_3,i_2} a_{\mathrm{x}_4,i_1} \\
&\qquad \mathrm{dx}_1 \mathrm{dx}_2 \mathrm{dx}_3 \mathrm{dx}_4, \\
P^L &:= \bigoplus_{n=0}^{\infty} P^{n,L} = -\mathrm{i} \int a^*_{\mathrm{x},i} \nabla^L_{\mathrm{x}} a_{\mathrm{x},i} \mathrm{dx}.
\end{aligned}$$

In the momentum representation,

$$
\begin{aligned}
H^L &= \sum_i \sum_{\mathrm{k}} \tau(\mathrm{k}) a^*_{\mathrm{k},i} a_{\mathrm{k},i} \\
&\quad + \frac{1}{2L^d} \sum_{i_1,i_2} \sum_{\mathrm{k}_1+\mathrm{k}_2=\mathrm{k}_3+\mathrm{k}_4} q(\mathrm{k}_1,\mathrm{k}_2,\mathrm{k}_3,\mathrm{k}_4) a^*_{\mathrm{k}_1,i_1} a^*_{\mathrm{k}_2,,i_2} a_{\mathrm{k}_3,i_2} a_{\mathrm{k}_4,i_1}, \\
P^L &= \sum_i \sum_{\mathrm{k}} \mathrm{k} a^*_{\mathrm{k},i} a_{\mathrm{k},i}.
\end{aligned}
$$

$H^{L,\pm}(\mathrm{k})$ will denote the operator H^L restricted to the subspace $P^L = \mathrm{k}$, $(-1)^{N^L} = \pm 1$.

For a large class of potentials the finite volume Hamiltonians H^L are bounded from below and have a discrete spectrum.

The ground state energy is defined as

$$E^L = \inf \operatorname{sp} H^L.$$

For $\mathrm{k} \in \frac{2\pi}{L}\mathbb{Z}^d$ we define the infimum of the excitation spectrum in the even/odd sector in finite volume:

$$\begin{aligned}
\epsilon^{L,+}(\mathrm{k}) &:= \inf \mathrm{sp} H^{L,+}(\mathrm{k}) - E^L, \ \ \mathrm{k} \neq 0, \\
\epsilon^{L,+}(0) &:= \inf \big(\mathrm{sp}(H^{L,+}(0) - E^L) \backslash \{0\}\big), \\
\epsilon^{L,-}(\mathrm{k}) &:= \inf \mathrm{sp} H^{L,-}(\mathrm{k}) - E^L.
\end{aligned}$$

For $\mathrm{k} \in \mathbb{R}^d$, we would like to define the infimum of the excitation spectrum in thermodynamic limit. To this end, first we define its finite volume version in a “window” given by $\delta > 0$:

$$\epsilon^{L,\delta,\pm}(\mathrm{k}) := \inf\left\{\epsilon^{L,\pm}(\mathrm{k}'_L) \ : \ \mathrm{k}'_L \in \frac{2\pi}{L}\mathbb{Z}^d, \ |\mathrm{k} - \mathrm{k}'_L| < \delta\right\}.$$

Then we set

$$\epsilon^{\pm}(\mathrm{k}) = \sup_{\delta>0}\left(\liminf_{L\to\infty}\left(\epsilon^{L,\delta,\pm}(\mathrm{k})\right)\right).$$

Conjecture 1 *We expect that for a large class of potentials with attractive interactions the following statements hold true:*

1. *The functions* $\mathbb{R}^d \ni \mathrm{k} \mapsto \epsilon^{\pm}(\mathrm{k}) \in \mathbb{R}$ *are continuous.*
2. *Let* $\mathrm{k} \in \mathbb{R}^d$. *Let* $(\mathrm{k}_s, L_s) \in \frac{2\pi}{L_s}\mathbb{Z}^d \times [0,\infty)$ *obey* $\mathrm{k}_s \to \mathrm{k}$, $L_s \to \infty$. *Then* $\epsilon^{L_s,\pm}(\mathrm{k}_s) \to \epsilon^{\pm}(\mathrm{k})$.
3. *If* $d \geq 2$, *then* $\inf_{\mathrm{k}} \min\big(\epsilon^{-}(\mathrm{k}), \epsilon^{+}(\mathrm{k})\big) =: \varepsilon > 0$.
4. *If* $d \geq 2$, *then* $\inf_{\mathrm{k}\neq 0} \frac{\min\left(\epsilon^{-}(\mathrm{k}),\epsilon^{+}(\mathrm{k})\right)}{|\mathrm{k}|} =: c_{\mathrm{cr}} > 0$.

5. We have the following subadditivity properties:

$$\epsilon^-(\mathrm{k}_1+\mathrm{k}_2) \le \epsilon^-(\mathrm{k}_1) + \epsilon^+(\mathrm{k}_2),$$

$$\epsilon^+(\mathrm{k}_1+\mathrm{k}_2) \le \epsilon^-(\mathrm{k}_1) + \epsilon^-(\mathrm{k}_2),$$

$$\epsilon^+(\mathrm{k}_1+\mathrm{k}_2) \le \epsilon^+(\mathrm{k}_1) + \epsilon^+(\mathrm{k}_2).$$

For $\mathrm{k} \in \frac{2\pi}{L}\mathbb{Z}^d$ and $j, n \in \mathbb{N}$, we define the jth shell in finite volume in the n-body case

$$\begin{aligned}
\nu_j^{L,n,+}(\mathrm{k}) &:= \text{the } j\text{th lowest eigenvalue of } H^{L,n,+}(\mathrm{k}) - E^L, \quad \mathrm{k} \neq 0,\\
\nu_j^{L,n,+}(0) &:= \text{the } j+1\text{st lowest eigenvalue of } H^{L,n,+}(0) - E^L,\\
\nu_j^{L,n,-}(\mathrm{k}) &:= \text{the } j\text{th lowest eigenvalue of } H^{L,n,-}(\mathrm{k}) - E^L.
\end{aligned}$$

(Of course, when counting eigenvalues we take into account their multiplicity).

Let $\mathbb{N}^+ := \{0, 2, 4, \dots\}$ and $\mathbb{N}^- := \{1, 3, 5, \dots\}$.

For $\mathrm{k} \in \mathbb{R}^d$, we would like to define the the jth shell in thermodynamic limit. Given by $\delta > 0$, we set

$$\nu_j^{L,\delta,+}(\mathrm{k}) \\ := \inf_{\mathrm{k}'_L \in \frac{2\pi}{L}\mathbb{Z}^d,\ |\mathrm{k}-\mathrm{k}'_L|<\delta,\ n\in\mathbb{N}^\pm} \left\{ \nu_j^{L,n}(\mathrm{k}'_L) \ : \ \inf \mathrm{sp} H^{L,n} - E^L < \delta \right\}.$$

Then

$$\nu_j^{\pm}(\mathrm{k}) = \sup_{\delta>0} \left(\liminf_{L\to\infty} \left(\nu_j^{L,\delta,\pm}(\mathrm{k}) \right) \right).$$

Clearly,

$$\begin{aligned} \nu_1^{\pm}(\mathrm{k}) &= \epsilon^{\pm}(\mathrm{k}), \\ \nu_j^{\pm}(\mathrm{k}) &\leq \nu_{j+1}^{\pm}(\mathrm{k}). \end{aligned}$$

Set

$$\epsilon_{\mathrm{ess}}^{\pm}(\mathrm{k}) := \sup\{\nu_j^{\pm}(\mathrm{k}) \ : \ j = 1, 2, \dots\}.$$

Conjecture 2 *We expect that for a large class of attractive potentials the following statements hold true:*

1. *The functions* $\mathbb{R}^d \ni \mathrm{k} \mapsto \epsilon_{\mathrm{ess}}^{\pm}(\mathrm{k}) \in \mathbb{R}_+$ *are continuous.*

2. *Let* $(\mathrm{k}_s, L_s, n_s) \in \frac{2\pi}{L_s}\mathbb{Z}^d \times [0, \infty[\times \mathbb{N}^{\pm}$ *obey* $\mathrm{k}_s \to \mathrm{k}$, $\inf \mathrm{sp} H^{L_s, n_s} - E^{L_s} \to 0$, $L_s \to \infty$. *Then*

$$\epsilon_{\mathrm{ess}}^{L_s, n_s, \pm}(\mathrm{k}_s) \to \epsilon_{\mathrm{ess}}^{\pm}(\mathrm{k}).$$

3. $\epsilon_{\rm ess}^{\pm}$ are related to $\epsilon^{\pm}$ as follows:

$$\begin{aligned} \epsilon_{\rm ess}^{-}(\mathrm{k}) &= \inf\{\epsilon^{-}(\mathrm{k}_1) + \epsilon^{+}(\mathrm{k}_2) \ : \ \mathrm{k} = \mathrm{k}_1 + \mathrm{k}_2\}, \\ \epsilon_{\rm ess}^{+}(\mathrm{k}) &= \inf\{\epsilon^{+}(\mathrm{k}_1) + \epsilon^{+}(\mathrm{k}_2), \\ &\qquad\quad \epsilon^{-}(\mathrm{k}_1) + \epsilon^{-}(\mathrm{k}_2) \ : \ \mathrm{k} = \mathrm{k}_1 + \mathrm{k}_2\}. \end{aligned}$$

One can try to compute the excitation spectrum of the Fermi gas by approximate methods. We will use the Hartree-Fock-Bogoliubov method. One can start the HFB method with a Bogoliubov rotation. For any k this corresponds to a substitution

$$a_{\mathrm{k}}^{*} = c_{\mathrm{k}} b_{\mathrm{k}}^{*} + s_{\mathrm{k}} b_{-\mathrm{k}}, \; a_{\mathrm{k}} = \bar{c}_{\mathrm{k}} b_{\mathrm{k}} + \bar{s}_{\mathrm{k}} b_{-\mathrm{k}}^{*},$$

where $c_{\rm k}$ and $s_{\rm k}$ are matrices on $\mathbb{C}^2$ satisfying

$$\begin{aligned} c_{\rm k}c_{\rm k}^* + s_{\rm k}s_{\rm k}^* &= 1, \\ c_{\rm k}s_{-\rm k}^\# + s_{\rm k}c_{-\rm k}^\# &= 0. \end{aligned}$$

($\cdot^*$ denotes the hermitian conjugation, $\cdot^\#$ denotes the transposition and $\bar{\cdot}$ denotes the complex conjugation).

For a sequence $\frac{2\pi}{L}\mathbb{Z}^d \ni \mathrm{k} \mapsto \theta_{\mathrm{k}}$ with values in matrices on $\mathbb{C}^2$ such that $\theta_{\mathrm{k}} = \theta_{-\mathrm{k}}$, set

$$U_\theta := \prod_{\mathrm{k}} \mathrm{e}^{-\frac{1}{2}\theta_{\mathrm{k}} a_{\mathrm{k}}^* a_{-\mathrm{k}}^* + \frac{1}{2}\theta_{\mathrm{k}}^* a_{\mathrm{k}} a_{-\mathrm{k}}}.$$

Such U_θ implement Bogoliubov rotations:

$$U_\theta^* a_{\mathrm{k}} U_\theta = b_{\mathrm{k}}, \qquad U_\theta^* a_{\mathrm{k}}^* U_\theta = b_{\mathrm{k}}^*.$$

Every even Bogoliubov transformation commuting with P^L is of the form U_θ.

Our Hamiltonian after the Bogoliubov rotation and the Wick ordering becomes

$$
\begin{aligned}
H = {} & B \\
& + \frac{1}{2}\sum_{\mathrm{k}} O(\mathrm{k}) b^*_{\mathrm{k}} b^*_{-\mathrm{k}} + \frac{1}{2}\sum_{\mathrm{k}} \overline{O}(\mathrm{k}) b_{-\mathrm{k}} b_{\mathrm{k}} + \sum_{\mathrm{k}} D(\mathrm{k}) b^*_{\mathrm{k}} b_{\mathrm{k}} \\
& + \text{terms higher order in } b\text{'s.}
\end{aligned}
$$

Let Ω denote the vacuum vector. $\Omega_\theta := U_\theta^* \Omega$ is the general form of an even fermionic Gaussian vector of zero momentum. We look for Ω_θ minimizing the expectation of H, which coincides with B. Minimizing is equivalent to $O(\mathrm{k}) = 0$.

(This is a special case of a general fact from the folklore of many body quantum physics. Under the name of the Beliaev Theorem, it was recently formulated and proven in a paper by M.Napiórkowski, J.P.Solovej and J.D.)

Thus, if we choose the Bogoliubov transformation according to the minimization procedure, the Hamiltonian equals

$$H = B + \sum_{\mathrm{k}} D(\mathrm{k}) b_{\mathrm{k}}^* b_{\mathrm{k}} + \text{terms higher order in } b\text{'s.} \tag{3}$$

We assume that the interaction is real, that is, $t(\mathrm{x}_1, \mathrm{x}_2)$ and $v(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4)$ are real. We make the BCS ansatz:

$$
\begin{aligned}
c_{\mathrm{k}} &= \cos\theta_{\mathrm{k}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
s_{\mathrm{k}} &= \sin\theta_{\mathrm{k}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\end{aligned}
$$

Then

$$\begin{aligned} B \;=\; & \sum_{\mathrm{k}} \tau(\mathrm{k})(1-\cos 2\theta_{\mathrm{k}}) \\ & + \frac{1}{4L^d} \sum_{\mathrm{k},\mathrm{k}'} \alpha(\mathrm{k},\mathrm{k}') \sin 2\theta_{\mathrm{k}} \sin 2\theta_{\mathrm{k}'} \\ & + \frac{1}{4L^d} \sum_{\mathrm{k},\mathrm{k}'} \beta(\mathrm{k},\mathrm{k}')(1-\cos 2\theta_{\mathrm{k}})(1-\cos 2\theta_{\mathrm{k}'}). \end{aligned}$$

Here,

$$\begin{aligned}\alpha(\mathrm{k},\mathrm{k}') &:= \frac{1}{2}\big(q(\mathrm{k},-\mathrm{k},-\mathrm{k}',\mathrm{k}')+q(-\mathrm{k},\mathrm{k},-\mathrm{k}',\mathrm{k}')\big),\\ \beta(\mathrm{k},\mathrm{k}') &= 2q(\mathrm{k},\mathrm{k}',\mathrm{k}',\mathrm{k})-q(\mathrm{k}',\mathrm{k},\mathrm{k}',\mathrm{k}).\end{aligned}$$

In particular, in the case of local potentials we have

$$\begin{aligned}\alpha(\mathrm{k},\mathrm{k}') &:= \frac{1}{2}\big(\hat{V}(\mathrm{k}-\mathrm{k}')+\hat{V}(\mathrm{k}+\mathrm{k}')\big),\\ \beta(\mathrm{k},\mathrm{k}') &= 2\hat{V}(0)-\hat{V}(\mathrm{k}-\mathrm{k}').\end{aligned}$$

We are looking for a minimum of B. To this end, we first analyze critical points of B. The condition $\partial_{2\theta_{\mathrm{k}}} B = 0$, or equivalently $O(\mathrm{k}) = 0$, has many solutions. We can have

$$\sin 2\theta_{\mathrm{k}} = 0, \quad \cos 2\theta_{\mathrm{k}} = \pm 1, \tag{4}$$

They correspond to Slater determinants and have a fixed number of particles. The solution of this kind minimizing B, is called the normal or Hartree-Fock solution.

Set

$$\begin{aligned}\delta(\mathrm{k}) &= \frac{1}{2L^d}\sum_{\mathrm{k}'}\alpha(\mathrm{k},\mathrm{k}')\sin 2\theta_{\mathrm{k}'},\\ \xi(\mathrm{k}) &= \tau(\mathrm{k}) + \frac{1}{2L^d}\sum_{\mathrm{k}'}\beta(\mathrm{k},\mathrm{k}')(1-\cos 2\theta_{\mathrm{k}'}).\end{aligned}$$

One expects that under some conditions the normal solution is not the global minimum of B.

More precisely, one expects that a global minimum is reached by a configuration satisfying

$$\sin 2\theta_{\mathrm{k}} = -\frac{\delta(\mathrm{k})}{\sqrt{\delta^2(\mathrm{k}) + \xi^2(\mathrm{k})}}, \quad \cos 2\theta_{\mathrm{k}} = \frac{\xi(\mathrm{k})}{\sqrt{\delta^2(\mathrm{k}) + \xi^2(\mathrm{k})}},$$

where at least some of $\sin 2\theta_{\mathrm{k}}$ are different from 0. It is sometimes called a superconducting solution.

For a superconducting solution we get

$$D(\mathrm{k}) = \sqrt{\xi^2(\mathrm{k}) + \delta^2(\mathrm{k})}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus we obtain a positive dispersion relation. One can expect that it is strictly positive, since otherwise the two functions δ and ξ would have a coinciding zero, which seems unlikely. Thus we expect that the dispersion relation $D(\mathrm{k})$ has a positive energy gap.

If the interaction is small, then $\xi(\mathrm{k})$ is close to $\tau(\mathrm{k})$ and $\delta(\mathrm{k})$ is small. This implies that $D(\mathrm{k})$ is close to $|\tau(\mathrm{k})|$. If $\tau(\mathrm{k})$ has a critical velocity for large k and $D(\mathrm{k})$ has a positive energy gap, then this implies that $D(\mathrm{k})$ also has a positive critical velocity.

Conditions guaranteeing that a superconducting solution minimizes the energy should involve some kind of negative definiteness of the quadratic form α – this is what we vaguely indicated by saying that the interaction is attractive. Indeed, multiply the definition of $\delta(\mathrm{k})$ with $\sin 2\theta_{\mathrm{k}}$ and sum it up over k. We then obtain

$$\sum_{\mathrm{k}} \sin^2 2\theta_{\mathrm{k}} \sqrt{\delta^2(\mathrm{k}) + \xi^2(\mathrm{k})}$$
$$= -\frac{1}{2L^d} \sum_{\mathrm{k},\mathrm{k}'} \sin 2\theta_{\mathrm{k}} \alpha(\mathrm{k},\mathrm{k}') \sin 2\theta_{\mathrm{k}'}.$$

The left hand side is positive. This means that the quadratic form given by the kernel $\alpha(\mathrm{k},\mathrm{k}')$ has to be negative at least at the vector given by $\sin 2\theta_{\mathrm{k}}$.

To sum up, for a large class of real, spin-independent and attractive interactions the HFB method yields a dispersion relations that has a positive energy gap and a positive critical velocity uniformly as $L \to \infty$, that is,

$$\inf_{\mathrm{k}} D(\mathrm{k}) > 0, \quad \inf_{\mathrm{k} \neq 0} \frac{D(\mathrm{k})}{|\mathrm{k}|} > 0.$$