

Homogeneous rank one perturbations and inverse square potentials

Jan Dereziński

Abstract. Following [2, 4, 6], I describe several exactly solvable families of closed operators on $L^2[0, \infty[$. Some of these families are defined by the theory of singular rank one perturbations. The remaining families are Schrödinger operators with inverse square potentials and various boundary conditions. I describe a close relationship between these families. In all of them one can observe interesting “renormalization group flows” (action of the group of dilations).

Mathematics Subject Classification (2010). 34L40, 33C10.

Keywords. Closed operators, rank one perturbations, one-dimensional Schrödinger operators, Bessel functions, renormalization group.

1. Introduction

My contribution consists of an introduction and 3 sections, each describing an interesting family of exactly solvable closed operators on $L^2[0, \infty[$.

The first two sections seem at first unrelated. Only in the third section the reader will see a relationship.

Section 2 is based on [2]. It is devoted to two families of operators, $H_{m,\lambda}$ and H_0^p , obtained by a rank one perturbation of a certain generic self-adjoint operator. The operators can be viewed as an elementary toy model illustrating some properties of the renormalization group. Note that in this section we do not use special functions. However we use a relatively sophisticated technique to define an operator, called sometimes *singular perturbation theory* or the *Aronszajn–Donoghue method*, see e.g., [1, 9, 5].

Section 3 is based on my joint work with Bruneau and Georgescu [4], and also with Richard [6]. It is devoted to Schrödinger operators with potentials proportional to $\frac{1}{x^2}$. Both $-\partial_x^2$ and $\frac{1}{x^2}$ are homogeneous of degree

The financial support of the National Science Center, Poland, under the grant UMO-2014/15/B/ST1/00126, is gratefully acknowledged. The author is grateful to Serge Richard, Vladimir Georgescu and Laurent Bruneau for discussions and collaboration.

–2. With appropriate homogeneous boundary conditions, we obtain a family of operators H_m , which we call *homogeneous Schrödinger operators*. They are also homogeneous of degree -2. One can compute all basic quantities for these operators using special functions—more precisely, *Bessel-type functions* and the *Gamma function*.

The operators H_m are defined only for $\operatorname{Re} m > -1$. We conjecture that they cannot be extended to the left of the line $\operatorname{Re} m = -1$ in the sense described in our paper. This conjecture was stated in [4]. It has not been proven or disproved so far.

Finally, Section 4 is based on my joint work with Richard [6], and also on [2]. It describes more general Schrödinger operators with the inverse square potentials. They are obtained by mixing the boundary conditions. These operators in general are no longer homogeneous, because their homogeneity is (weakly) broken by their boundary condition—hence the name *almost homogeneous Schrödinger operators*. They can be organized in two families $H_{m,\kappa}$ and H'_0 .

It turns out that there exists a close relationship between the operators from Section 4 and from Section 2: they are similar to one another. In particular, they have the same point spectrum.

Almost homogeneous Schrödinger operators in the self-adjoint case have been described in the literature before, see e.g., [7]. However, the non-self-adjoint case seems to have been first described in [6]. A number of new exact formulas about these operators is contained in [4, 10, 6] and [2].

Let us also mention one amusing observation, which seems to be original, about self-adjoint extensions of

$$-\partial_x^2 + \left(-\frac{1}{4} + \alpha\right)\frac{1}{x^2}.$$

The “renormalization group” acts on the set of these extensions, as described in a table after Prop. 15. Depending on $\alpha \in \mathbb{R}$, we obtain 4 “phases” of the problem. Some analogies to the condensed matter physics are suggested.

2. Toy model of renormalization group

Consider the Hilbert space $\mathcal{H} = L^2[0, \infty[$ and the operator X

$$Xf(x) := xf(x).$$

Let $m \in \mathbb{C}$, $\lambda \in \mathbb{C} \cup \{\infty\}$. Following [2], we consider a family of operators formally given by

$$H_{m,\lambda} := X + \lambda|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|. \quad (1)$$

In the perturbation $|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ we use the Dirac ket-bra notation, hopefully self-explanatory. Unfortunately, the function $x \mapsto x^{\frac{m}{2}}$ is never square integrable. Therefore, this perturbation is never an operator. It can be however understood as a quadratic form. We will see below how to interpret (1) as an operator.

If $-1 < \operatorname{Re} m < 0$, the perturbation $|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ is form bounded relatively to X , and then $H_{m,\lambda}$ can be defined by the form boundedness technique. The perturbation is formally rank one. Therefore,

$$\begin{aligned} (z - H_{m,\lambda})^{-1} &= (z - X)^{-1} \\ &+ \sum_{n=0}^{\infty} (z - X)^{-1} |x^{\frac{m}{2}}\rangle (-\lambda)^{n+1} \langle x^{\frac{m}{2}}| (z - X)^{-1} |x^{\frac{m}{2}}\rangle^n \langle x^{\frac{m}{2}}| (z - X)^{-1} \\ &= (z - X)^{-1} \\ &+ \left(\lambda^{-1} - \langle x^{\frac{m}{2}}| (z - X)^{-1} |x^{\frac{m}{2}}\rangle \right)^{-1} (z - X)^{-1} |x^{\frac{m}{2}}\rangle \langle x^{\frac{m}{2}}| (z - X)^{-1}. \end{aligned}$$

It is an easy exercise in complex analysis to compute

$$\langle x^{\frac{m}{2}}| (z - X)^{-1} |x^{\frac{m}{2}}\rangle = \int_0^{\infty} x^m (z - x)^{-1} \underline{X} = (-z)^m \frac{\pi}{\sin \pi m}.$$

Therefore, the resolvent of $H_{m,\lambda}$ can be given in a closed form:

$$\begin{aligned} (z - H_{m,\lambda})^{-1} &= (z - X)^{-1} \\ &+ \left(\lambda^{-1} - (-z)^m \frac{\pi}{\sin \pi m} \right)^{-1} (z - X)^{-1} |x^{\frac{m}{2}}\rangle \langle x^{\frac{m}{2}}| (z - X)^{-1}. \end{aligned}$$

The rhs of the above formula defines a function with values in bounded operators satisfying the resolvent equation for all $-1 < \operatorname{Re} m < 1$ and $\lambda \in \mathbb{C} \cup \{\infty\}$. Therefore, the method of pseudoresolvent [8] allows us to define a holomorphic family of closed operators $H_{m,\lambda}$. Note that $H_{m,0} = X$.

The case $m = 0$ is special: $H_{0,\lambda} = X$ for all λ . One can however introduce another holomorphic family of operators H_0^ρ for any $\rho \in \mathbb{C} \cup \{\infty\}$ by

$$(z - H_0^\rho)^{-1} = (z - X)^{-1} - (\rho + \ln(-z))^{-1} (z - X)^{-1} |x^0\rangle \langle x^0| (z - X)^{-1}.$$

In particular, $H_0^\infty = X$.

Let $\mathbb{R} \ni \tau \mapsto U_\tau$ be the group of dilations on $L^2[0, \infty[$, that is

$$(U_\tau f)(x) = e^{\tau/2} f(e^\tau x).$$

We say that B is homogeneous of degree ν if

$$U_\tau B U_\tau^{-1} = e^{\nu\tau} B.$$

E.g., X is homogeneous of degree 1 and $|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ is homogeneous of degree $1 + m$.

The group of dilations (“the renormalization group”) acts on our operators in a simple way:

$$\begin{aligned} U_\tau H_{m,\lambda} U_\tau^{-1} &= e^\tau H_{m,e^{\tau m} \lambda}, \\ U_\tau H_0^\rho U_\tau^{-1} &= e^\tau H_0^{\rho + \tau}. \end{aligned}$$

The essential spectrum of $H_{m,\lambda}$ and H_0^ρ is $[0, \infty[$. The point spectrum is more intricate, and is described by the following theorem:

Theorem 1.

1. $z \in \mathbb{C} \setminus [0, \infty[$ belongs to the point spectrum of $H_{m,\lambda}$ iff

$$(-z)^{-m} = \lambda \frac{\pi}{\sin \pi m}.$$

2. H_0^ρ possesses an eigenvalue iff $-\pi < \operatorname{Im} \rho < \pi$, and then it is $z = -e^\rho$.

For a given pair (m, λ) all eigenvalues form a geometric sequence that lies on a logarithmic spiral, which should be viewed as a curve on the Riemann surface of the logarithm. Only its “physical sheet” gives rise to eigenvalues. For m which are not purely imaginary, only a finite piece of the spiral is on the “physical sheet” and therefore the number of eigenvalues is finite.

If m is purely imaginary, this spiral degenerates to a half-line starting at the origin.

If m is real, the spiral degenerates to a circle. But then the operator has at most one eigenvalue.

The following theorem about the number of eigenvalues of $H_{m,\lambda}$ is proven in [6]. It describes an interesting pattern of “phase transitions” when we vary the parameter m . In this theorem, we denote by $\operatorname{spec}_p(A)$ denotes the set of eigenvalues of an operator A and $\#X$ denotes the number of elements of the set X .

Theorem 2. Let $m = m_r + im_i \in \mathbb{C} \setminus \{0\}$ with $|m_r| < 1$.

(i) Let $m_r = 0$.

(a) If $\frac{\ln(|s|)}{m_i} \in]-\pi, \pi[$, then $\#\operatorname{spec}_p(H_{m,\lambda}) = \infty$,

(a) if $\frac{\ln(|\lambda \frac{\pi}{\sin \pi m}|)}{m_i} \notin]-\pi, \pi[$ then $\#\operatorname{spec}_p(H_{m,\lambda}) = 0$.

(ii) If $m_r \neq 0$ and if $N \in \mathbb{N}$ satisfies $N < \frac{m_r^2 + m_i^2}{|m_r|} \leq N + 1$, then

$$\#\operatorname{spec}_p(H_{m,\lambda}) \in \{N, N + 1\}.$$

3. Homogeneous Schrödinger operators

Let $\alpha \in \mathbb{C}$. Consider the differential expression

$$L_\alpha = -\partial_x^2 + \left(-\frac{1}{4} + \alpha\right) \frac{1}{x^2}.$$

L_α is homogeneous of degree -2 . Following [4], we would like to interpret L_α as a closed operator on $L^2[0, \infty[$ homogeneous of degree -2 .

L_α , and closely related operators H_m that we introduce shortly, are interesting for many reasons.

- They appear as the radial part of the Laplacian in all dimensions, in the decomposition of Aharonov-Bohm Hamiltonian, in the membranes with conical singularities, in the theory of many body systems with contact interactions and in the Efimov effect.
- They have rather subtle and rich properties illustrating various concepts of the operator theory in Hilbert spaces (eg. the Friedrichs and Krein extensions, holomorphic families of closed operators).

- Essentially all basic objects related to H_m , such as their resolvents, spectral projections, Møller and scattering operators, can be explicitly computed.
- A number of nontrivial identities involving special functions, especially from the Bessel family, find an appealing operator-theoretical interpretation in terms of the operators H_m . E.g. the Barnes identity leads to the formula for Møller operators.

We start the Hilbert space theory of the operator L_α by defining its two naive interpretations on $L^2[0, \infty[$:

1. The minimal operator L_α^{\min} : We start from L_α on $C_c^\infty]0, \infty[$, and then we take its closure.
2. The maximal operator L_α^{\max} : We consider the domain consisting of all $f \in L^2[0, \infty[$ such that $L_\alpha f \in L^2[0, \infty[$.

We will see that it is often natural to write $\alpha = m^2$. Let us describe basic properties of $L_{m^2}^{\max}$ and $L_{m^2}^{\min}$:

Theorem 3.

1. For $1 \leq \operatorname{Re} m$, $L_{m^2}^{\min} = L_{m^2}^{\max}$.
2. For $-1 < \operatorname{Re} m < 1$, $L_{m^2}^{\min} \subsetneq L_{m^2}^{\max}$, and the codimension of their domains is 2.
3. $(L_\alpha^{\min})^* = L_\alpha^{\max}$. Hence, for $\alpha \in \mathbb{R}$, L_α^{\min} is Hermitian.
4. L_α^{\min} and L_α^{\max} are homogeneous of degree -2 .

Let ξ be a compactly supported cutoff equal 1 around 0.

Let $-1 \leq \operatorname{Re} m$. It is easy to check that $x^{\frac{1}{2}+m}\xi$ belongs to $\operatorname{Dom} L_{m^2}^{\max}$. We define the operator H_m to be the restriction of $L_{m^2}^{\max}$ to

$$\operatorname{Dom} L_{m^2}^{\min} + \mathbb{C}x^{\frac{1}{2}+m}\xi.$$

The operators H_m are in a sense more interesting than $L_{m^2}^{\max}$ and $L_{m^2}^{\min}$:

Theorem 4.

1. For $1 \leq \operatorname{Re} m$, $L_{m^2}^{\min} = H_m = L_{m^2}^{\max}$.
2. For $-1 < \operatorname{Re} m < 1$, $L_{m^2}^{\min} \subsetneq H_m \subsetneq L_{m^2}^{\max}$ and the codimension of the domains is 1.
3. $H_m^* = \bar{H}_m$. Hence, for $m \in]-1, \infty[$, H_m is self-adjoint.
4. H_m is homogeneous of degree -2 .
5. $\operatorname{spec} H_m = [0, \infty[$.
6. $\{\operatorname{Re} m > -1\} \ni m \mapsto H_m$ is a holomorphic family of closed operators.

The theorem below is devoted to self-adjoint operators within the family H_m .

Theorem 5.

1. For $\alpha \geq 1$, $L_\alpha^{\min} = H_{\sqrt{\alpha}}$ is essentially self-adjoint on $C_c^\infty]0, \infty[$.
2. For $\alpha < 1$, L_α^{\min} is Hermitian but not essentially self-adjoint on $C_c^\infty]0, \infty[$. It has deficiency indices 1, 1.

3. For $0 \leq \alpha < 1$, the operator $H_{\sqrt{\alpha}}$ is the Friedrichs extension and $H_{-\sqrt{\alpha}}$ is the Krein extension of L_{α}^{\min} .
4. $H_{\frac{1}{2}}$ is the Dirichlet Laplacian and $H_{-\frac{1}{2}}$ is the Neumann Laplacian on halfline.
5. For $\alpha < 0$, L_{α}^{\min} has no homogeneous selfadjoint extensions.

Various objects related to H_m can be computed with help of functions from the Bessel family. Indeed, we have the following identity

$$x^{-\frac{1}{2}} \left(-\partial_x^2 + \left(-\frac{1}{4} + \alpha \right) \frac{1}{x^2} \pm 1 \right) x^{\frac{1}{2}} = -\partial_x^2 - \frac{1}{x} \partial_x + \left(-\frac{1}{4} + \alpha \right) \frac{1}{x^2} \pm 1,$$

where the rhs defines the well-known (modified) Bessel equation.

One can compute explicitly the resolvent of H_m :

Theorem 6. Denote by $R_m(-k^2; x, y)$ the integral kernel of the operator $(k^2 + H_m)^{-1}$. Then for $\operatorname{Re} k > 0$ we have

$$R_m(-k^2; x, y) = \begin{cases} \sqrt{xy} I_m(kx) K_m(ky) & \text{if } x < y, \\ \sqrt{xy} I_m(ky) K_m(kx) & \text{if } x > y, \end{cases}$$

where I_m is the modified Bessel function and K_m is the MacDonalld function.

The operators H_m are similar to self-adjoint operators. Therefore, they possess the spectral projection onto any Borel subset of their spectrum $[0, \infty[$. In particular, below we give a formula for the spectral projection of H_m onto the interval $[a, b]$:

Proposition 7. For $0 < a < b < \infty$, the integral kernel of $\mathbb{1}_{[a,b]}(H_m)$ is

$$\mathbb{1}_{[a,b]}(H_m)(x, y) = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_m(kx) J_m(ky) k \, dk,$$

where J_m is the Bessel function.

One can diagonalize the operators H_m in a natural way, using the so-called Hankel transformation \mathcal{F}_m , which is the operator on $L^2[0, \infty[$ given by

$$(\mathcal{F}_m f)(x) := \int_0^{\infty} J_m(kx) \sqrt{kx} f(x) \, dx \quad (2)$$

Theorem 8. \mathcal{F}_m is a bounded invertible involution on $L^2[0, \infty[$ diagonalizing H_m , more precisely

$$\mathcal{F}_m H_m \mathcal{F}_m^{-1} = X^2.$$

It satisfies $\mathcal{F}_m A = -A \mathcal{F}_m$, where

$$A = \frac{1}{2i} (x \partial_x + \partial_x x)$$

is the self-adjoint generator of dilations.

It turns out that the Hankel transformation can be expressed in terms of the generator of dilations. This expression, together with the Stirling formula for the asymptotics of the Gamma function, proves the boundedness of \mathcal{F}_m .

Theorem 9. *Set*

$$\mathcal{I}f(x) = x^{-1}f(x^{-1}), \quad \Xi_m(t) = e^{i \ln(2)t} \frac{\Gamma(\frac{m+1+it}{2})}{\Gamma(\frac{m+1-it}{2})}.$$

Then

$$\mathcal{F}_m = \Xi_m(A)\mathcal{I}.$$

Therefore, we have the identity

$$H_m := \Xi_m^{-1}(A)X^{-2}\Xi_m(A). \quad (3)$$

(Result obtained independently by Bruneau, Georgescu, and myself in [4], and by Richard and Pankrashkin in [10]).

The operators H_m generate 1-parameter groups of bounded operators. They possess scattering theory and one can explicitly compute their Møller (wave) operators and the scattering operator.

Theorem 10. *The Møller operators associated to the pair H_m, H_k exist and*

$$\Omega_{m,k}^{\pm} := \lim_{t \rightarrow \pm\infty} e^{itH_m} e^{-itH_k} = e^{\pm i(m-k)\pi/2} \mathcal{F}_m \mathcal{F}_k = e^{\pm i(m-k)\pi/2} \frac{\Xi_k(A)}{\Xi_m(A)}.$$

The formula (3) valid for $\operatorname{Re} m > -1$ can be used as an alternative definition of the family H_m also beyond this domain. It defines a family of closed operators for the parameter m that belongs to

$$\{m \mid \operatorname{Re} m \neq -1, -2, \dots\} \cup \mathbb{R}. \quad (4)$$

Their spectrum is always equal to $[0, \infty[$ and they are analytic in the interior of (4).

In fact, $\Xi_m(A)$ is a unitary operator for all real values of m . Therefore, for $m \in \mathbb{R}$, (3) is well-defined and self-adjoint.

$\Xi_m(A)$ is bounded and invertible also for all m such that $\operatorname{Re} m \neq -1, -2, \dots$. Therefore, formula (3) defines an operator for all such m .

One can then pose various questions:

- What happens with these operators along the lines $\operatorname{Re} m = -1, -2, \dots$?
- What is the meaning of these operators to the left of $\operatorname{Re} = -1$? (They are not differential operators!)

Let us describe a certain precise conjecture about the family H_m . In order to state it we need to define the concept of a holomorphic family of closed operators.

The definition (or actually a number of equivalent definitions) of a holomorphic family of bounded operators is quite obvious and does not need to be recalled. In the case of unbounded operators the situation is more subtle, and is described e.g., in [8], see also [3].

Suppose that Θ is an open subset of \mathbb{C} , \mathcal{H} is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on \mathcal{H} . We say that this is a holomorphic family of closed operators if for each $z_0 \in \Theta$ there exists a neighborhood Θ_0 of z_0 , a Banach space \mathcal{K} and a holomorphic

family of injective bounded operators $\Theta_0 \ni z \mapsto B(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\text{Ran } B(z) = \mathcal{D}(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)B(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

We have the following practical criterion:

Theorem 11. *Suppose that $\{H(z)\}_{z \in \Theta}$ is a function whose values are closed operators on \mathcal{H} . Suppose in addition that for any $z \in \Theta$ the resolvent set of $H(z)$ is nonempty. Then $z \mapsto H(z)$ is a holomorphic family of closed operators if and only if for any $z_0 \in \Theta$ there exists $\lambda \in \mathbb{C}$ and a neighborhood Θ_0 of z_0 such that λ belongs to the resolvent set of $H(z)$ for $z \in \Theta_0$ and $z \mapsto (H(z) - \lambda)^{-1} \in B(\mathcal{H})$ is holomorphic on Θ_0 .*

The above theorem indicates that it is more difficult to study holomorphic families of closed operators that for some values of the complex parameter have an empty resolvent set. We have the following conjecture (formulated as an open question in [4]), so far unproven:

Conjecture 12. *It is impossible to extend*

$$\{\text{Re } m > -1\} \ni m \mapsto H_m$$

to a holomorphic family of closed operators on a larger connected open subset of \mathbb{C} .

4. Almost homogeneous Schrödinger operators

For $-1 < \text{Re } m < 1$ the codimension of $\text{Dom}(L_{m^2}^{\min})$ in $\text{Dom}(L_{m^2}^{\max})$ is two. Therefore, following [6], one can fit a 1-parameter family of closed operators in between $L_{m^2}^{\min}$ in $L_{m^2}^{\max}$, mixing the boundary condition $x^{\frac{1}{2}+m}$ and $x^{\frac{1}{2}-m}$. These operators in general are no longer homogeneous—their homogeneity is broken by the boundary condition. We will say that they are *almost homogeneous*.

More precisely, for any $\kappa \in \mathbb{C} \cup \{\infty\}$ let $H_{m,\kappa}$ be the restriction of $L_{m^2}^{\max}$ to the domain

$$\text{Dom}(H_{m,\kappa}) = \left\{ f \in \text{Dom}(L_{m^2}^{\max}) \mid \text{for some } c \in \mathbb{C}, \right. \\ \left. f(x) - c(x^{1/2-m} + \kappa x^{1/2+m}) \in \text{Dom}(L_{m^2}^{\min}) \text{ around } 0 \right\}, \quad \kappa \neq \infty;$$

$$\text{Dom}(H_{m,\infty}) = \left\{ f \in \text{Dom}(L_{m^2}^{\max}) \mid \text{for some } c \in \mathbb{C}, \right. \\ \left. f(x) - cx^{1/2+m} \in \text{Dom}(L_{m^2}^{\min}) \text{ around } 0 \right\}.$$

The case $m = 0$ needs a special treatment. For $\nu \in \mathbb{C} \cup \{\infty\}$, let H'_0 be the restriction of L_0^{\max} to

$$\text{Dom}(H'_0) = \left\{ f \in \text{Dom}(L_0^{\max}) \mid \text{for some } c \in \mathbb{C}, \right. \\ \left. f(x) - c(x^{1/2} \ln x + \nu x^{1/2}) \in \text{Dom}(L_0^{\min}) \text{ around } 0 \right\}, \quad \nu \neq \infty;$$

$$\text{Dom}(H_0^\infty) = \{f \in \text{Dom}(L_0^{\max}) \mid \text{for some } c \in \mathbb{C}, \\ f(x) - cx^{1/2} \in \text{Dom}(L_0^{\min}) \text{ around } 0\}.$$

Here are the basic properties of almost homogeneous Schrödinger operators.

Proposition 13.

1. For any $|\text{Re}(m)| < 1$, $\kappa \in \mathbb{C} \cup \{\infty\}$

$$H_{m,\kappa} = H_{-m,\kappa^{-1}}.$$

2. $H_{0,\kappa}$ does not depend on κ , and these operators coincide with H_0^∞ .
3. We have

$$U_\tau H_{m,\kappa} U_{-\tau} = e^{-2\tau} H_{m,e^{-2\tau}\kappa}, \\ U_\tau H_0^\nu U_{-\tau} = e^{-2\tau} H_0^{\nu+2\tau}.$$

In particular, only

$$H_{m,0} = H_{-m}, \quad H_{m,\infty} = H_m, \quad H_0^\infty = H_0$$

are homogeneous.

The following proposition describes self-adjoint cases among these operators.

Proposition 14.

$$H_{m,\kappa}^* = H_{\bar{m},\bar{\kappa}} \quad \text{and} \quad H_0^{\nu*} = H_0^{\bar{\nu}}.$$

In particular,

- (i) $H_{m,\kappa}$ is self-adjoint for $m \in]-1, 1[$ and $\kappa \in \mathbb{R} \cup \{\infty\}$, and for $m \in i\mathbb{R}$ and $|\kappa| = 1$.
- (ii) H_0^ν is self-adjoint for $\nu \in \mathbb{R} \cup \{\infty\}$.

The essential spectrum of $H_{m,\kappa}$ and H_0^ν is always $[0, \infty[$. The following proposition describes the point spectrum in the self-adjoint case.

Proposition 15.

1. If $m \in]-1, 1[$ and $\kappa \geq 0$ or $\kappa = \infty$, then $H_{m,\kappa}$ has no eigenvalues.
2. If $m \in]-1, 1[$ and $\kappa < 0$, then $H_{m,\kappa}$ has a single eigenvalue at $-4\left(\frac{\Gamma(m)}{\kappa\Gamma(-m)}\right)^{\frac{1}{m}}$.
3. If $m \in i\mathbb{R}$ and $|\kappa| = 1$, then $H_{m,\kappa}$ has an infinite sequence of eigenvalues accumulating at $-\infty$ and 0. If $m = im_1$ and $e^{i\alpha} = \frac{\kappa\Gamma(-im_1)}{\Gamma(im_1)}$, then these eigenvalues are $-4\exp\left(-\frac{\alpha+2\pi n}{m_1}\right)$, $n \in \mathbb{Z}$.

It is interesting to analyze how the set of self-adjoint extensions of the Hermitian operator

$$L_\alpha^{\min} = -\partial_x^2 + \left(-\frac{1}{4} + \alpha\right)\frac{1}{x^2}$$

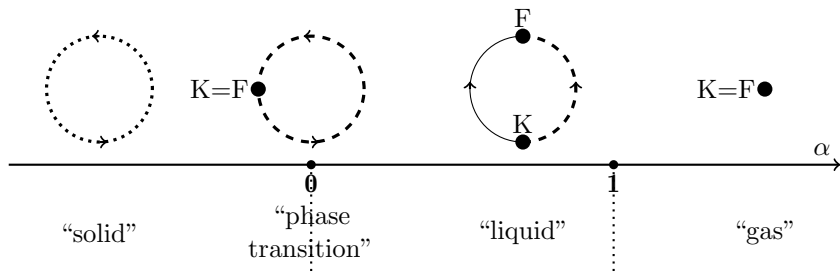
depends on the real parameter α . Self-adjoint extensions form a set isomorphic either to a point or to a circle. The “renormalization group” acts on this

set by a continuous flow, as described by Proposition 13. This flow may have fixed points.

The following table describes the various “phases” of the theory of self-adjoint extensions of L_α^{\min} . To each phase I give a name inspired by condensed matter physics. The reader does not have to take these names very seriously, however I suspect that they have some deeper meaning.

$1 \leq \alpha$	“gas”	point	Unique fixed point: Friedrichs extension=Krein extension.
$0 < \alpha < 1$	“liquid”	circle	Two fixed points: Friedrichs and Krein extension. Ren. group flows from Krein to Friedrichs. On one semicircle of non-fixed points all have one bound state; on the other all have no bound states.
$\alpha = 0$	“liquid–solid phase transition”	circle	Unique fixed point: Friedrichs extension=Krein extension. Ren. group flows from Krein to Friedrichs. Non-fixed points have one bound state.
$\alpha < 0$	“solid”	circle	No fixed points. Ren. group rotates the circle. All have infinitely many bound states.

The above table can be represented by the following picture, hopefully self-explanatory:



There exists a close link between almost homogeneous Schrödinger operators described in this section and the “toy model of renormalization group” described in Section 2. It turns out that the corresponding operators are similar to one another.

Define the unitary operator

$$(If)(x) := x^{-\frac{1}{4}} f(2\sqrt{x}).$$

Its inverse is

$$(I^{-1}f)(x) := \left(\frac{y}{2}\right)^{\frac{1}{2}} f\left(\frac{y^2}{4}\right).$$

Note that

$$I^{-1}XI = \frac{X^2}{4}, \quad I^{-1}AI = \frac{A}{2}.$$

We change slightly notation: the operators H_m , $H_{m,\kappa}$ and H_0^ν of this section will be denoted \tilde{H}_m , $\tilde{H}_{m,\kappa}$ and \tilde{H}_0^ν . Recall that in (2) we introduced the Hankel transformation \mathcal{F}_m , which is a bounded invertible involution satisfying

$$\begin{aligned} \mathcal{F}_m \tilde{H}_m \mathcal{F}_m^{-1} &= X^2, \\ \mathcal{F}_m A \mathcal{F}_m^{-1} &= -A. \end{aligned}$$

Recall also that in Section 2 we introduced the operators $H_{m,\lambda}$ and H_0^ρ . The following theorem is proven in [2]:

Theorem 16. .

1. If $\lambda \frac{\pi}{\sin(\pi m)} = \kappa \frac{\Gamma(m)}{\Gamma(-m)}$, then the operators $H_{m,\lambda}$ are similar to $\tilde{H}_{m,\kappa}$:

$$\mathcal{F}_m^{-1} I^{-1} H_{m,\lambda} I \mathcal{F}_m = \frac{1}{4} \tilde{H}_{m,\kappa},$$

2. If $\rho = -2\nu$, then the operators H_0^ρ are similar to \tilde{H}_0^ν :

$$\mathcal{F}_m^{-1} I^{-1} H_0^\rho I \mathcal{F}_m = \frac{1}{4} \tilde{H}_0^\nu,$$

References

1. Albeverio, S., Kurasov, P.: Singular Perturbations of Differential Operators, Cambridge Univ. Press, Cambridge 2000.
2. Dereziński, J.: Homogeneous rank one perturbations, Ann. Henri Poincaré, DOI 10.1007/s00023-017-0585-y
3. Dereziński, J., Wrochna, M.: Continuous and holomorphic functions with values in closed operators, Journ. Math. Phys. 55 (2014) 083512
4. Bruneau, L., Dereziński, J., Georgescu, V.: Homogeneous Schrödinger operators on half-line. Ann. Henri Poincaré 12 no. 3, 547–590 (2011).
5. Dereziński, J., Früboes, R.: Renormalization of Friedrichs Hamiltonians, Rep. Math. Phys. 50 (2002) 433-438
6. Dereziński, J., Richard, S.: On Schrödinger operators with inverse square potentials on the half-line, Annales Henri Poincaré 18 (2017), 869–928, DOI 10.1007/s00023-016-0520-7

7. Gitman, D.M., Tyutin, I.V., Voronov, B.L.: Self-adjoint Extensions in Quantum Mechanics. General Theory and Applications to Schrödinger and Dirac Equations with Singular Potentials, Birkhäuser 2012
8. Kato, T.: *Perturbation theory for linear operators*. Classics in mathematics, Springer, 1995.
9. Kiselev, A., Simon, B.: Rank one perturbations with infinitesimal coupling, J. Funct. Anal. 130(2) (1995), 345–356.
10. Pankrashkin, K., Richard, S.: Spectral and scattering theory for the Aharonov-Bohm operators. Rev. Math. Phys. **23** no. 1, 53–81 (2011).

Jan Dereziński

Department of Mathematical Methods in Physics

Faculty of Physics

University of Warsaw

Pasteura 5

02-093 Warszawa, Poland

e-mail: jan.derezinski@fuw.edu.pl