CANONICAL COMMUTATION RELATIONS, BOGOLIUBOV TRANSFORMATIONS QUADRATIC HAMILTONIANS

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CANONICAL COMMUTATION RELATIONS

Let $(\mathcal{Y}, \omega)$ be a real vector space equipped with an antisymmetric form. We will usually assume that ω is symplectic, which means that if it is nondegenerate. We will denote by $Sp(\mathcal{Y})$ the group of linear transformations preserving ω.

Heuristically, we are interested in a linear map

$$\mathcal{Y} \ni y \mapsto \hat{\phi}(y)$$

with values in self-adjoint operators such that the Heisenberg commutation relations hold:

$$[\hat{\phi}(y), \hat{\phi}(y')] \;=\; \mathrm{i}y{\cdot}\omega y'$$

This is unfortunately a non-rigorous statement, since typically such $\hat{\phi}(y)$ are unbounded. It is however possible to give a rigorous formulation of the above idea.

A regular representations of the canonical commutation relations or a regular CCR representation over $(\mathcal{Y}, \omega)$ on a Hilbert space $\mathcal{H}$ is a map

$$\mathcal{Y} \ni y \mapsto \hat{\phi}(y)$$

with values in self-adjoint operators on $\mathcal{H}$ such that

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\hat{\phi}(y)}\mathrm{e}^{\mathrm{i}\hat{\phi}(y')} &= \mathrm{e}^{-\frac{\mathrm{i}}{2}y\cdot\omega y'}\mathrm{e}^{\mathrm{i}\hat{\phi}(y+y')}, \\ \hat{\phi}(ty) &= t\phi(y), \quad t \in \mathbb{R} \end{aligned}$$

$\hat{\phi}(y)$ are called field operators. It is easy to show that they depend linearly on y and satisfy the Heisenberg commutation relations on appropriate domains.

Consider a regular CCR representation

$$\mathcal{Y} \ni y \mapsto \hat{\phi}(y). \tag{1}$$

Let $R \in Sp(\mathcal{Y})$. Then

$$\mathcal{Y} \ni y \mapsto \hat{\phi}(Ry) \tag{2}$$

is also a regular CCR representation. We say that (2) has been obtained from (1) by a Bogoliubov transformation.

One can ask whether there exists a unitary U such that

$$U\hat{\phi}(y)U^* = \hat{\phi}(Ry), \quad y \in \mathcal{Y}.$$

Such a U is called a Bogoliubov implementer.

If $\mathcal{Y} = \mathbb{R}^{2d}$ is finite dimensional, then it is possible to characterize all Bogoliubov implementers. They are products of operators of the form $\mathrm{e}^{\mathrm{i}\hat{H}}$, where $\hat{H}$ is a Bogoliubov Hamiltonian

$$\hat{H} = \sum b_{ij}\hat{\phi}_i\hat{\phi}_j + c.$$

Let us describe two basic constructions of CCR representations in the symplectic case:

1. the Schrödinger representation,
2. the Fock representation

Strictly speaking, the former works only for a finite number of degrees of freedom. The latter works for any dimension of $\mathcal{Y}$.

Consider the Hilbert space $L^2(\mathbb{R}^d)$. Let ϕ_i denote the ith coordinate of $\mathbb{R}^d$. Let $\hat{\phi}_i$ denote the operator of multiplication by the variable ϕ_i on and $\hat{\pi}_i$ the momentum operator $\frac{1}{\mathrm{i}}\partial_{\phi_i}$. Then,

$$\mathbb{R}^d \oplus \mathbb{R}^d \ni (\eta, q) \mapsto \eta \cdot \hat{\phi} + q \cdot \hat{\pi} \tag{3}$$

is an irreducible regular CCR representation on $L^2(\mathbb{R}^d)$. (3) is called the Schrödinger representation over the symplectic space $\mathbb{R}^d \oplus \mathbb{R}^d$.

Let $(\mathcal{Y}, \omega)$ be a finite dimensional symplectic space. Clearly, $\mathcal{Y}$ is always equivalent to $\mathbb{R}^d \oplus \mathbb{R}^d$ with the natural symplectic form.

The Stone–von Neumann Theorem says that all irreducible regular CCR representations over $\mathcal{Y}$ are unitarily equivalent to the Schrödinger representation.

Let $\mathcal{Z}$ be a complex Hilbert space. Consider the bosonic Fock space $\Gamma_{\rm s}(\mathcal{Z})$. We use the standard notation for creation/annihilation operators $\hat{a}^*(z)$, $\hat{a}(z)$, $z \in \mathcal{Z}$.

We equip $\mathcal{Z}$ with the symplectic form

$$z{\cdot}\omega z' := \mathrm{Im}(z|z').$$

The following regular CCR representation is called the Fock representation.

$$\mathcal{Z} \ni z \mapsto \hat{\phi}(z) := \frac{1}{\sqrt{2}}\big(\hat{a}^*(z) + \hat{a}(z)\big).$$

BOGOLIUBOV TRANSFORMATIONS AND QUADRATIC HAMILTONIANS IN FOCK REPRESENTATION

For simplicity, we will assume that the one-particle space is finite dimensional: $\mathcal{Z} = \mathbb{C}^m$. Operators on $\mathbb{C}^m$ are identified with $m \times m$ matrices. If $h = [h_{ij}]$ is a matrix, then $\overline{h}$, h^* and $h^\#$ will denote its complex conjugate, hermitian conjugate and transpose.

We are interested in operators on the bosonic Fock space $\Gamma_{\rm s}(\mathbb{C}^m)$. $\hat{a}_i, \hat{a}_j^*$ will denote the standard annihilation and creation operators on $\Gamma_{\rm s}(\mathbb{C}^m)$, where $\hat{a}_i^*$ is the Hermitian conjugate of $\hat{a}_i$,

$$
\begin{aligned}
[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^*, \hat{a}_j^*] \; &= \; 0, \\
[\hat{a}_i, \hat{a}_j^*] \; &= \; \delta_{ij}.
\end{aligned}
$$

It is convenient to consider the doubled Hilbert space $\mathbb{C}^m \oplus \mathbb{C}^m$ equipped with the complex conjugation

$$J(z_1, z_2) = (\overline{z}_2, \overline{z}_1). \tag{4}$$

Operators that commute with J have the form

$$R = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}, \tag{5}$$

and will be called J-real.

We also introduce the charge form

$$S = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}. \tag{6}$$

We say that a J-real operator

$$R = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}. \tag{7}$$

is symplectic if

$$R^* S R = S.$$

Here are the equivalent conditions

$$p^* p - q^\# \overline{q} = \mathbb{1}, \quad p^* q - q^\# \overline{p} = 0,$$

$$p p^* - q q^* = \mathbb{1}, \quad p q^\# - q p^\# = 0.$$

We denote by $Sp(\mathbb{R}^{2m})$ the group of all symplectic transformations.

Note that

$$pp^* \geq \mathbb{1}, \qquad p^*p \geq \mathbb{1}.$$

Hence p^{-1} and p^{*-1} are well defined, and we can set

$$\begin{aligned} d_1 &:= q^{\#}(p^{\#})^{-1}, \\ d_2 &:= q\overline{p}^{-1}. \end{aligned}$$

Note that $d_1^{\#} = d_1$, $d_2 = d_2^{\#}$. One has the following factorization:

$$R = \begin{bmatrix} \mathbb{1} & d_2 \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} (p^*)^{-1} & 0 \\ 0 & \overline{p} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ \overline{d}_1 & \mathbb{1} \end{bmatrix}. \tag{8}$$

In the present context, U is a (Bogoliubov) implementer of a symplectic transformation R if

$$\begin{aligned} U\hat{a}_i^* U^* &= p_{ij}\hat{a}_j^* + q_{ij}\hat{a}_j, \\ U\hat{a}_i U^* &= \overline{q}_{ij}\hat{a}_j^* + \overline{p}_{ij}\hat{a}_j. \end{aligned}$$

Every symplectic transformation has an implementer, unique up to a choice of a phase factor.

We will need a compact notation for double annihilators/creators: if $d = [d_{ij}]$ is a symmetric matrix, then

$$\begin{aligned} \hat{a}^*(d) &= \sum_{ij} d_{ij} \hat{a}_i^* \hat{a}_j^*, \\ \hat{a}(d) &= \sum_{ij} \overline{d}_{ij} \hat{a}_i \hat{a}_j, \end{aligned}$$

We have the following canonical choices: the natural implementer $U_R^{\rm nat}$, and a pair of metaplectic implementers $\pm U_R^{\rm met}$:

$$\begin{aligned} U_R^{\rm nat} &:= |\det pp^*|^{-\frac14} {\rm e}^{-\frac12 \hat{a}^*(d_2)} \Gamma\big((p^*)^{-1}\big) {\rm e}^{\frac12 \hat{a}(d_1)}, \\ \pm U_R^{\rm met} &:= \pm(\det p^*)^{-\frac12} {\rm e}^{-\frac12 \hat{a}^*(d_2)} \Gamma\big((p^*)^{-1}\big) {\rm e}^{\frac12 \hat{a}(d_1)}. \end{aligned}$$

Bogoliubov implementers fom a group called sometimes the c-metaplectic group $Mp^c(\mathbb{R}^{2m})$. Metaplectic Bogoliubov implementers form a subgroup of $Mp^c(\mathbb{R}^{2m})$ called the metaplectic group $Mp(\mathbb{R}^{2m})$.

We have an obvious homomorphism $Mp^c(\mathbb{R}^{2m}) \ni U \mapsto R \in Sp(\mathbb{R}^{2m})$.

Various homomorphisms related to the metaplectic group can be described by the following diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathbb{Z}_2 & \to & U(1) & \to & U(1) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Mp(\mathbb{R}^{2m}) & \to & Mp^{\mathrm{c}}(\mathbb{R}^{2m}) & \to & U(1) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Sp(\mathbb{R}^{2m}) & \to & Sp(\mathbb{R}^{2m}) & \to & 1 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Of special importance are positive symplectic transformations. They satisfy

$$p = p^*, \quad p > 0, \quad q = q^{\#}. \tag{9}$$

For such transformations $d_1 = d_2$ will be simply denoted by

$$d \ := \ q(p^{\#})^{-1}$$

For positive symplectic transformations the natural implementer coincides with one of the metaplectic implementers:

$$U_R^{\mathrm{nat}} \ := \ \det p^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\hat{a}^*(d)} \Gamma\big(p^{-1}\big) \mathrm{e}^{\frac{1}{2}\hat{a}(d)}.$$

By a quadratic classical Hamiltonians, we will mean an expression of the form

$$H = 2\sum h_{ij}a_i^* a_j + \sum g_{ij}a_i^* a_j^* + \sum \overline{g}_{ij}a_i a_j,$$

where a_i, a_j^* are classical (commuting) variables such that a_i^* is the complex conjugate of a_i and the following Poisson bracket relations hold:

$$\begin{aligned} \{a_i, a_j\} = \{a_i^*, a_j^*\} &= 0, \\ \{a_i, a_j^*\} &= -\mathrm{i}\delta_{ij}. \end{aligned}$$

We will assume that $h = h^*$, $g = g^\#$.

Classical Hamiltonians can be identified with self-adjoint J-real operators on the doubled space:

$$H = \begin{bmatrix} h & g \\ \overline{g} & \overline{h} \end{bmatrix},$$

We also introduce

$$B := SH = \begin{bmatrix} h & g \\ -\overline{g} & -\overline{h} \end{bmatrix}.$$

By a quantization of H we will mean an operator on the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathbb{C}^m)$ of the form

$$\sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \sum \overline{g}_{ij}\hat{a}_i\hat{a}_j + 2\sum h_{ij}\hat{a}_i^*\hat{a}_j + c,$$

where c is an arbitrary real constant.

Two quantizations of H are especially useful: the Weyl quantization $\hat{H}^{\mathrm{w}}$ and the normally ordered (or Wick) quantization $\hat{H}^{\mathrm{n}}$:

$$\begin{aligned}
\hat{H}^{\mathrm{w}} &:= \sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \sum \overline{g}_{ij}\hat{a}_i\hat{a}_j + \sum h_{ij}\hat{a}_i^*\hat{a}_j + \sum h_{ij}\hat{a}_j\hat{a}_i^*, \\
\hat{H}^{\mathrm{n}} &:= \sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \sum \overline{g}_{ij}\hat{a}_i\hat{a}_j + 2\sum h_{ij}\hat{a}_i^*\hat{a}_j.
\end{aligned}$$

The two quantizations obviously differ by a constant:

$$\hat{H}^{\mathrm{w}} = \hat{H}^{\mathrm{n}} + \mathrm{Tr}h.$$

For any quadratic Hamiltonian H, we have $\mathrm{e}^{\mathrm{i}t\hat{H}^{\mathrm{w}}} \in Mp(\mathbb{R}^{2m})$.

Theorem Suppose that $H > 0$.

1. B has real nonzero eigenvalues.
2. $\mathrm{sgn}(B)$ is symplectic.
3. $K := S\mathrm{sgn}B$ is symplectic and has positive eigenvalues.
4. Using the positive square root, define $R := K^{\frac{1}{2}}$. Then R is symplectic and diagonalizes H. That means, for some h_1,

$$R^{*-1}HR^{-1} = \begin{bmatrix} h_1 & 0 \\ 0 & h_1^{\#} \end{bmatrix}.$$

Here is an alternative exppression for K:

$$K = H^{\frac{1}{2}}\big(H^{\frac{1}{2}}SHSH^{\frac{1}{2}}\big)^{-\frac{1}{2}}H^{\frac{1}{2}}.$$

On the quantum level, if R diagonalizes H, then the corresponding unitary Bogoliubov implementers U remove double annihilators/creators from $\hat{H}$:

$$\begin{aligned} U\hat{H}^{\mathrm{w}}U^* &= 2h_{1,ij}\hat{a}_i^*\hat{a}_j + E^{\mathrm{w}}, \\ U\hat{H}^{\mathrm{n}}U^* &= 2h_{1,ij}\hat{a}_i^*\hat{a}_j + E^{\mathrm{n}}, \end{aligned}$$

where E^{w}, resp. E^{n} is the infimum of $\hat{H}^{\mathrm{w}}$, resp. of $\hat{H}^{\mathrm{n}}$.

We can compute the infimum of the Bogoliubov Hamiltonians The simplest expression is valid for the Weyl quantization, which we present in various equivalent forms:

$$\begin{aligned} E^{\mathrm{w}} := \inf \hat{H}^{\mathrm{w}} &= \frac{1}{2}\mathrm{Tr}\sqrt{B^2} \\ &= \frac{1}{2}\mathrm{Tr}\sqrt{H^{\frac12}SHSH^{\frac12}} \\ &= \mathrm{Tr}\int \frac{B^2}{B^2+\tau^2}\frac{\mathrm{d}\tau}{2\pi} \\ &= \frac{1}{2}\mathrm{Tr}\begin{bmatrix} h^2 - gg^* & -hg + gh^{\#} \\ g^*h - h^{\#}g^* & h^{\#2} - g^*g \end{bmatrix}^{\frac12} \end{aligned}$$

Suppose now that

$$H_0 = \begin{bmatrix} h_0 & 0 \\ 0 & \overline{h}_0 \end{bmatrix} \tag{10}$$

is a "free" Hamiltonian. We set

$$B_0 := SH_0 = \begin{bmatrix} h_0 & 0 \\ 0 & -\overline{h}_0 \end{bmatrix}, \qquad V = B^2 - B_0^2, \tag{11}$$

and we assume that

$$\mathrm{Tr}(h - h_0) = 0, \quad h_0 > 0. \tag{12}$$

The infimum of the Weyl quantization of H can be rewritten as

$$E^{\mathrm{w}} = \sum_{j=0}^{\infty} L_j,$$

where

$$\begin{aligned} L_0 &= \mathrm{Tr} \int \frac{B_0^2}{B_0^2+\tau^2}\frac{\mathrm{d}\tau}{2\pi} = \frac{1}{2}\mathrm{Tr}|B_0| = \mathrm{Tr} h, \\ L_j &= \mathrm{Tr} \int \frac{(-1)^{j+1}}{B_0^2+\tau^2}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^j \tau^2 \frac{\mathrm{d}\tau}{2\pi} \\ &= \mathrm{Tr} \int \frac{(-1)^j}{2j}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^j \frac{\mathrm{d}\tau}{2\pi}, \qquad j = 1, 2, \ldots . \end{aligned}$$

One can view $\hat{H}^{\mathrm{n}}$ as a Hamiltonian renormalized by subtracting L_0:

$$\hat{H}^{\mathrm{n}} = \hat{H}^{\mathrm{w}} - L_0.$$

Note the formula for the infimum:

$$E^{\mathrm{n}} = \mathrm{Tr} \int \frac{1}{B^2 + \tau^2} V \frac{1}{B_0^2 + \tau^2} \tau^2 \frac{\mathrm{d}\tau}{2\pi}$$

Formally,

$$E^{\mathrm{n}} = \sum_{j=1}^{\infty} L_j.$$

Sometimes one needs to renormalize the Hamiltonian further by subtracting L_1 as well:

$$\begin{aligned}\hat{H}^{\mathrm{ren}} &:= \hat{H}^{\mathrm{w}} - L_0 - L_1 \\ &= \hat{H}^{\mathrm{n}} - L_1.\end{aligned}$$

Here is the formula for the infimum:

$$E^{\mathrm{ren}} := \inf \hat{H}^{\mathrm{ren}} = -\mathrm{Tr} \int \frac{1}{B_0^2 + \tau^2} V \frac{1}{B^2 + \tau^2} V \frac{1}{B_0^2 + \tau^2} \tau^2 \frac{\mathrm{d}\tau}{2\pi}$$

Formally,

$$E^{\mathrm{ren}} = \sum_{j=2}^{\infty} L_j.$$

The constant L_j arises in the diagramatic expasions as the evaluation of the loop with 2j vertices. To see this, introduce the "propagator"

$$G(t) := \frac{\mathrm{e}^{-|B_0|t}}{2|B_0|}.$$

Clearly

$$\frac{1}{B_0^2 + \tau^2} = \int G(s)\mathrm{e}^{\mathrm{i}s\tau}\mathrm{d}s.$$

Therefore,

$$\begin{aligned}
L_j &= \int \mathrm{d}t_{j-1} \cdots \int \mathrm{d}t_1 \mathrm{Tr} VG(t_j - t_1) VG(t_1 - t_2) \cdots VG(t_{j-1} - t_j) \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}t_j \int_{-T}^{T} \mathrm{d}t_{j-1} \cdots \int_{-T}^{T} \mathrm{d}t_1 \\
&\quad \mathrm{Tr} VG(t_j - t_1) VG(t_1 - t_2) \cdots VG(t_{j-1} - t_j).
\end{aligned}$$

EXAMPLE:
SCALAR FIELD WITH
POSITION DEPENDENT MASS

Consider classical variables parametrized by $\vec{x} \in \mathbb{R}^3$ satisfying the Poisson bracket relations

$$\begin{aligned} \{\phi(\vec{x}), \phi(\vec{y})\} = \{\pi(\vec{x}), \pi(\vec{y})\} &= 0, \\ \{\phi(\vec{x}), \pi(\vec{y})\} &= \delta(\vec{x} - \vec{y}). \end{aligned}$$

Consider quadratic classical Hamiltonians of the free scalar field:

$$H_0 = \int \Big(\frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\phi(\vec{x})\big)^2 + \frac{1}{2}m^2\phi^2(\vec{x})\Big)\mathrm{d}\vec{x},$$

We can assume that the mass squared depends on a position, obtaining a perturbed Hamiltonian

$$H = \int \Big(\frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\phi(\vec{x})\big)^2 + \frac{1}{2}(m^2 + \kappa(\vec{x}))\phi^2(\vec{x})\Big)\mathrm{d}\vec{x},$$

Let us replace classical variables ϕ, π with quantum operators $\hat{\phi}, \hat{\pi}$ satisfying the commutation relations

$$[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0,$$
$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = \mathrm{i}\delta(\vec{x} - \vec{y}).$$

It is well-known how to quantize H_0. The one-particle space consists of positive-frequency modes. The normally ordered Hamiltonian

$$\hat{H}_0^{\mathrm{n}} = \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\hat{\phi}(\vec{x})\big)^2 + \frac{1}{2}m^2\hat{\phi}^2(\vec{x})\Big):\mathrm{d}\vec{x},$$

acts on the corresponding Fock space. The infimum of $\hat{H}_0$ is zero.

(The Weyl prescription $\hat{H}_0^{\mathrm{w}}$ is ill-defined).

In the case of H, the normally-ordered prescription does not work. One has to renormalize by subtracting the (infinite) contribution of the loop with 2 vertices L_1, which can be formally written as

$$\hat{H}^{\mathrm{ren}} = \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\hat{\phi}(\vec{x})\big)^2 + \frac{1}{2}(m^2 + \kappa(\vec{x}))\hat{\phi}^2(\vec{x})\Big):\mathrm{d}\vec{x} - L_1,$$

Let us stress that $\hat{H}^{\mathrm{ren}}$ is a well-defined self-adjoint operator acting on the same space as $\hat{H}_0^{\mathrm{n}}$

The infimum of $\hat{H}^{\mathrm{ren}}$ is the sum of loops

$$\sum_{j=2}^{\infty} L_j$$

with at least 4 vertices. It is called the vacuum energy and is closely related to the so-called effective action.