Notes on Quantization and Canonical Commutation Relations

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0.1 Introduction

Canonical commutation relations (abbreviated CCR) most often appear either in the Schrödinger representation or in the Fock representation.

The Schrödinger representation acts in the Hilbert space $L^2(\mathbf{R}^d)$. It involves self-adjoint operators x_i, called the position operators, and $D_i := -\mathrm{i}\nabla_i$, called the momentum operators. They satisfy the identities,

$$\begin{aligned} &[x_i, x_j] = 0, \quad [D_i, D_j] = 0, \\ &[x_i, D_j] = \mathrm{i}\delta_{ij}, \qquad i, j = 1, \dots, d, \end{aligned} \tag{0.1.1}$$

called the Heisenberg commutation relations.

The Fock representation acts on the symmetric (=bosonic) Fock space, which we denote $\Gamma_{\mathrm{s}}(\mathbf{C}^d)$. It involves the so-called creation operators a_i^* and annihillation operators a_i satisfying

$$\begin{aligned} &[a_i, a_j] = 0, \quad [a_i^*, a_j^*] = 0, \\ &[a_i, a_j^*] = \delta_{ij}, \qquad i, j = 1, \dots, d. \end{aligned} \tag{0.1.2}$$

If Ω denotes the vacuum vector, then the annihillation operators satisfy additionally

$$a_i\Omega = 0, \quad i = 1, \dots, d.$$

If we identify $\pi^{-\frac{d}{4}}\mathrm{e}^{-\frac{x^2}{2}} \in L^2(\mathbf{R}^d)$ with the vacuum vector Ω, a_i with $\frac{1}{\sqrt{2}}(x_i + \nabla_i)$ a_i^* with $\frac{1}{\sqrt{2}}(x_i - \nabla_i)$, then we obtain an identification of the Fock space $\Gamma_{\mathrm{s}}(\mathbf{C}^d)$ with the space $L^2(\mathbf{R}^d)$. Thus, for the same number of degrees of freedom, the Schrödinger and Fock representations are unitarily equivalent.

Representations of CCR appear in many disguises in quantum mechanics, bosonic quantum field theory and partial differential equations. These notes are devoted to a unified and rigorous description of various mathematical structures that arise in the context of CCR.

It is convenient to introduce the following abstract version of the CCR. Let $\phi_1, \dots \phi_m$ be a family of self-adjoint operators on a Hilbert space $\mathcal{H}$. Let ω be a symplectic form in $\mathbf{R}^m$. Then we say that these operators satisfy CCR over $(\mathbf{R}^m, \omega)$, if

$$[\phi_i, \phi_j] = \mathrm{i}\omega_{ij}. \tag{0.1.3}$$

The relations (0.1.1) are CCR over the symplectic space $\mathbf{R}^d \oplus \mathbf{R}^d$ endowed with the canonical symplectic form. In fact, if we set $m = 2d$, $\phi_i = x_i$, $\phi_{d+i} = D_i$, $i = 1, \dots, d$, then (0.1.1) are an example of (0.1.3).

Likewise, the relations (0.1.2) are CCR over the space $\mathbf{C}^d$ endowed with the symplectic structure given by the imaginary part of the scalar product. To see this we need to set $m = 2d$, $\phi_i = \frac{1}{\sqrt{2}}(a_i^* + a_i)$, $\phi_{d+i} := \frac{1}{\mathrm{i}\sqrt{2}}(a_i^* - a_i)$.

The definition of CCR (0.1.3) is unfortunately not quite satisfactory, because of the problems due to the unboundedness of ϕ_i. One can easily find pathological examples of operators satisfying a version of the formula (0.1.3), but not equivalent to the standard representations of CCR. In order to avoid these examples, it is convenient to use a different definition of CCR. This definition involves the unitary operators $W(y) := \mathrm{e}^{\mathrm{i}\phi(y)}$, sometimes called the Weyl operators. One assumes that $W(ty)$ depends strongly continuously on t and the so-called Weyl commutation relations are satisfied:

$$W(y_1)W(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\omega y_2}W(y_1 + y_2). \tag{0.1.4}$$

It is easy to see that the Weyl commutation relations (0.1.4) imply the Heisenberg commutation relations (0.1.3).

The theory of CCR is closely related to the theory of symplectic spaces. Recall that a symplectic space $\mathcal{Y}$ is a real vector space with a nondegenerate antisymmetric form ω.

One can equip $\mathcal{Y}$ with an additional structure:

1) A conjugation, that is a linear map κ satisfying

$$\kappa^2 = 1, \quad \omega\kappa = -\kappa'\omega. \tag{0.1.5}$$

This fixes an isomorphism of $\mathcal{Y}$ with the so-called canonical symplectic space $\mathcal{X}' \oplus \mathcal{X}$, where $\mathcal{X}$ is often called the configuration space and $\mathcal{X}'$ the momentum space.
2) A Kähler anti-involution (or complex structure), that is a map $\mathbf{i}$ such that

$$\mathbf{i}^2 = -1, \quad \mathbf{i}'\omega + \omega\mathbf{i} = 0, \quad \omega\mathbf{i} \geq 0. \tag{0.1.6}$$

A Kähler anti-involution fixes a natural scalar product in $\mathcal{Y}$.
3) A Kähler anti-involution $\mathbf{i}$ together with a conjugation κ, satisfying, besides (0.1.5) and (0.1.6) the condition

$$\kappa\mathbf{i} + \mathbf{i}\kappa = 0. \tag{0.1.7}$$

In this case one can identify $\mathcal{Y}$ with a space $\mathcal{X} \oplus \mathcal{X}$, where $\mathcal{X}$ is a real Euclidean space.

If we fix the structure 1) and the dimension of $\mathcal{Y}$ is finite, then we have a natural Schrödinger representation of CCR in the Hilbert space $L^2(\mathcal{X})$.

If we fix the structure 2), we have a distinguished Fock representation in the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$, where $\mathcal{Z}$ is the so-called holomorphic subspace of $\mathbf{C}\mathcal{Y}$. One can also use the closely related complex wave representation (also called the Bargmann-Segal representation), which reperesents the Fock space in a certain Hilbert space of holomorphic functions.

In the case 3) we can define the so-called real wave representation, also called the Q-space representation. In the finite-dimensional case this is nothing else but the Schrödinger representation with the Gaussian weight. In the infinite dimensional case this is a representation acting on the Hilbert space of the Gaussian stochastic process indexed by the vector space $\mathcal{X}$.

In the real wave representation, just as in the Schrödinger representation, position operators have a meaning of multiplication operators. The drawback of the Schrödinger representation is that it does not make sense in an infinite dimension. On the the other hand, the real wave representation can be defined in any dimension, and is used in quantum field theory as a substitute for the Schrödinger representation.

There exists a close relationship between functions on a symplectic space $\mathcal{Y}'$ and operators on a Hilbert space $\mathcal{H}$. This relationship is one of the main manifestations of the so-called correspondence principle between the classical and quantum mechanics. To formulate this relationship, one needs to introduce a "quantization" – a map that associates an operator to a function on the symplectic space. The inverse of this map will be called a "symbol". In our notes we will consider a number of different quantizations.

There is one distinguished quantization, which is fixed by the symplectic structure of $\mathcal{Y}'$. It is called the Weyl-Wigner quantization (or the Weyl quantization). If b is a function on the symplectic space $\mathcal{Y}$, then the Weyl quantization of b will be denoted by $\mathrm{Op}(b)$. (Another notation, commonly used in the context of the Schrödinger representation, is $b^{\mathrm{w}}(x, D)$).

In the context of the Schrödinger representation, two quantizations are especially simple to compute. We call them the x, D- and the D, x-quantizations. The x, D quantization (also called the Kohn-Nirenberg quantization) amounts to putting all the position opetators to the left and the momentum operators to the right. We will denote it by $\mathrm{Op}^{x,D}(b)$. (In the literature, the notation $b(x, D)$ is commonly used).

In the case of a Fock representation it is natural to introduce two other quantizations: the Wick (or normally ordered) quantization, which we denote $\mathrm{Op}^{a^*,a}(b)$, and the anti-Wick quantization, which we denote $\mathrm{Op}^{a,a^*}(b)$. In the Wick quantization we put all the creation operators to the left of the annihillation operators. In the anti-Wick quantization – the other way around. (The notation often used for the Wick quantization in the literature is $: b(\phi) :$).

If we fix a vector $\psi_0 \in \mathcal{H}$, then for every $v \in \mathcal{Y}'$ we can define the so-called coherent states associated with the vector ψ_0, by setting $\psi_v := W(-\omega^{-1}v)\psi_0$. The coherent states satisfy the following identity

$$(2\pi)^{-d} \int |\psi_v)(\psi_v| \mathrm{d}v = 1. \tag{0.1.8}$$

Note that if $(\rho_i)_{i\in I}$ is an orthonormal basis, then

$$\sum_{i\in I} |\rho_i)(\rho_i| = 1. \tag{0.1.9}$$

The identity (0.1.8) resembles (0.1.9), therefore the family of coherent states has many properties similar to an orthonormal basis.

One can use the coherent states to define two kinds of quantizations: the contravariant quantization

$$\mathrm{Op}^{\mathrm{ct}}(b) = (2\pi)^{-d} \int b(v)|\psi_v)(\psi_v| \mathrm{d}v,$$

and the covariant quantization, most conveniently described by defining the covariant symbol of an operator B to be

$$\mathrm{s}_B^{\mathrm{cv}}(v) := (\psi_v|B\psi_v).$$

If we consider a Fock representation and if we choose ψ_0 to be the vacuum, then the contravariant quantization coincides with the Wick quantization and the covariant quantization coincides with the anti-Wick quantization.

Quadratic polynomials on a symplectic space form a Lie algebra wrt the Poisson bracket. This Lie algebra is isomorphic to the central extension of the inhomogeneous symplectic Lie algebra. Quadratic polynomials in the field operators $\phi(y)$ also form a Lie algebra wrt $-\mathrm{i}[\cdot,\cdot]$. The Weyl quantization gives an isomorphism between these two Lie algebras. This isomorphism plays a key role in the correspondence principle. In fact, it implies that, on the level of quadratic Hamiltonians, the quantum and classical mechanics are essentially identical.

One can ask whether the above described isomorphism of Lie algebras can be lifted to a homomorphism of Lie groups. The answer is somewhat subtle. It can be lifted to a homomorphism of the inhomogeneous symplectic group into the group of $*$-automorphisms of the algebra $B(\mathcal{H})$. However, this homomorphism cannot be lifted to a homomorphism of this group into unitary operators. This is related to the non-trivial homotopy group of the symplectic group. Instead, there exists a double-valued homomorphism (with an ambiguity involving two possible signs), called the metaplectic representation. We present several approaches to the metaplectic representation.

In the Schrödinger representation, elements of the metaplectic group can be described using the so-called generating function of a symplectic transformation – an object well known from the classical mechanics. Note that this observation is at the foundations of the Feynmann path integral method. Unfortunately, not all symplectic transformations have a well defined generating function. Therefore, in the Schrödinger representation not all elements of the metaplectic group have regular integral kernels, which complicates the analysis of the metaplectic group in the Schrödinger representation.

The situation is better in the case of the Fock representation, which is actually the most convenient set-up to prove that the metaplectic representation is double-valued.

The simplest algebraic formulas for the metaplectic group one obtains in the fully symplectic approach – that is using the Weyl quantization. Unfortunately, here again one has exceptional elements of the metaplectic group that do not possess a regular Weyl symbol.

By the Stone-vonNeumann theorem, in the case of a fixed finite number of degrees of freedom, all representations of CCR are isomorphic up to multiplicity. This is no longer true for an infinite number degrees of freedom, where there are many non-equivalent representations. In our notes we present a number of them.

The so-called "coherent state representations" are very close to Fock representations, but in general they do not have a vacuum. A simple physically motivated construction that leads to such representations is the so-called van-Hove model with the infra-red divergence. It is a simple exactly solvable model, which can be viewed as a toy model of quantum field theory. The van Hove model with an ultraviolet divergence is also of interest – it illustrates certain aspects of the ultraviolet renormalization.

If we have two Kähler structures, one can ask when they generate unitarily equivalent Fock representations. The answer to this question is contained in a theorem due to Shale, which we present in our notes.

0.2 Literature

In the mathematical literature the x, D-quantization is attributed to Kohn-Nirenberg [KN], who used it to study elliptic partial differential equations. Following their papers, applications of pseudodifferential operators to PDE's were developed to a very efficient and sophisticated tool, especially by Beals-Feffermann [BF] and Hörmander [Hö2]. Basic identities for the Kohn-Nirenberg quantization (such as for the superposition of pseudodifferential operators) can be found in [Hö2].

The Weyl commutation relations appeared in [Weyl]. The Stone-von Neumann theorem 2.1.13 was announced by Stone in 1930, but the first published proof was given by von Neumann a year later [vN]. Different proofs of this theorem exist in the literature ([BR] vol. 2, Corollary 5.2.15, [Emch], [LV]). The proof given in [BR], for example, is based on the properties of the number operator in a Fock representation.

The CCR were first introduced by Dirac. A textbook presentation of CCR can be found in [Emch] and [BR] vol II, where the reader will find thorough bibliographical notes. The symplectic pont of view applied to CCR seems have been first stressed by I. Segal [Seg1], [Seg2] see also [BSZ] and references therein. The operators $\phi(y)$ (where y is an element of a symplectic space, without a splitting into "fields" or "positions" and "conjugate fields" or "momenta") are therefore sometimes called Segal field operators.

The Weyl quantization first appeared (in the context of the Schrödinger representation) in [Weyl]. It seems, however, that it took a long time before its importance was fully realized. It was applied to physics problems by Wigner [Wig] and further studied by Moyall [Moy]. Therefore, sometimes the name the Weyl-Wigner (or Weyl-Wigner-Moyall) quantization is also used. If ψ is a vector, then what we call the Weyl symbol of the projection onto a vector ψ in the literature is often called the Wigner function of ψ.

In the applications to PDE's, the Weyl quantization seems to have appeared quite late. A paper by Voros [Vor] seems to have drawn the attention of mathematicians to this concept. Then, Hörmander gave a very thorough and rigorous treatment of this concept in [Hö1, Hö2], introducing a very general class of algebras of pseudodifferential operators using the Weyl quantization. The relationship between the Weyl quantization and the parity operator was noted in [Gr] (see also [Der1]). The "prequantization" is a term that belongs to the so-called geometric quantization [GS, Wood, Sn]. There exist many monographs on pseudodifferential operators, which describe basic formulas about the $x, D-$quantization, sometimes also about the Weyl quantization. Among them let us note [Hö2, Tr, Ta1, Ta2, Ku, Shu, Ro].

A presentation of various quantizations close in the spirit to ours is contained in the monographs by Berezin-Shubin [BS] and Folland [Fol].

Coherent states first appeared in the paper of Schrödinger. A systematic presentation of the covariant and contravariant quantization is due to Berezin [Ber2, BS].

The Fock space was introduced on a formal level by Fock [Fock]. Its rigorous study, including the self-adjointness of field operators, was first given by Cook [Cook]. An attempt to understand quantum field theory in rigorous terms was undertaken by Freiedrichs [Frie1, Frie2] and Segal [Seg2]. Segal stressed the role of what we call a Kähler anti-involution in defining creation/annihilation operators. The term Wick and anti-Wick quantization seems to be due to Berezin, who studied these quantizations in [Ber3, BS]. The Wick (or normal) ordering is of course much older, it is attributed to Wick [Wick]. The term complex wave representation is due to Segal, see [Seg2, BSZ] and references therein. The first published account of an essentially identical construction was given by Bargmann [Ba1]. This representation (or some minor modification thereof) is also known under the name the Bargmann or Bargmann-Segal representation. Its usefulness for practical computation was stressed and exploited by Berezin in [Ber1] – what we denote ψ^{cw} and B^{cw}, he called the generating functional of ψ and B.

The idea of using a weighted Schrödinger representation to generalize the Schrödinger representation to infinite number of degrees of freedom was studied by Segal, see [Seg2, BSZ] and references therein. The name "real wave representation" is also due to Segal. This representation found its application in constructive quantum field theory, under the name Q-representation, where Q is meant to be a measure space on which the Q-representation is defined – see Simon's monograph [Sim].

The nonequivalence of coherent state representations seem to have been first realized by van Hove [vH] and studied by [Frie1].

The action of the symplectic group on operators on the CCR Hilbert space has been first noted by Segal [Seg1]. It was successfully used by Bogolubov to solve concrete problems of many body quantum physics. The construction of the corresponding unitary representation is attributed to Shale [Sha] and Weil [Weil]. This representation has to use the double covering of the symplectic group, called the metaplectic group. It is called the metaplectic, or sometimes the Shale-Weil representation.

The metaplectic group is closely related to the so-called Maslov index, introduced in [Mas]. A particularly complete treatment of the metaplectic group in the Schrödinger representation and of the Malsov index the reader will find in [LV], see also [Ar1, GS, Hö2, Sou, Ler]. The relationship of the metaplectic group to generating functions of classical mechanics can be traced back to the method of Feynman path integrals.

Our presentation of the metaplectic group in the Fock space follows that of Fowland [Fol] and of Berezin [Ber1]. In particular, Theorem 2.7.27 can be found in [Fol] and [Itz].

The formula for the Weyl-Wigner symbols of elements of the metaplectic group we learned from lectures at the Ecole Polytechnique by M. Bony [Bony].

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Chapter 1

Elements of linear algebra

Many concepts of linear algebra are used in the theory of CCR. Almost all of them are well known and elementary. Nevertheless, they can be sometimes confusing and it is good to have a consistent notation. This chapter presents a systematic review of these concepts and a description of the notation that will be used in our notes.

1.1 Basic notation

1.1.1 Linear operators

We will consider vector spaces over $\mathbf{R}$ and over $\mathbf{C}$. The symbol $\mathbf{K}$ will denote either $\mathbf{R}$ or $\mathbf{C}$.

If $\mathcal{V}, \mathcal{W}$ are vector spaces, both over $\mathbf{K}$, then $L(\mathcal{V}, \mathcal{W})$ denotes the space of $\mathbf{K}$-linear operators from $\mathcal{V}$ to $\mathcal{W}$.

We will write $\overset{\circ}{L}(\mathcal{V}, \mathcal{W})$ for the space of finite rank linear operators from $\mathcal{V}$ to $\mathcal{W}$.

We will write $L(\mathcal{V})$ for $L(\mathcal{V}, \mathcal{V})$.

Let $IL(\mathcal{V}, \mathcal{W})$ denote $\mathcal{W} \times L(\mathcal{V}, \mathcal{W})$ acting on $\mathcal{V}$ as follows: if $(w, a) \in IL(\mathcal{V}, \mathcal{W})$ and $v \in \mathcal{V}$, then $(w, a)v := w + av$. $IL(\mathcal{V}, \mathcal{W})$ is called the space of affine (inhomogeneous) maps from $\mathcal{V}$ to $\mathcal{W}$. We set $IL(\mathcal{V}) := IL(\mathcal{V}, \mathcal{V})$.

1.1.2 Normed spaces

Let $\mathcal{V}$ be a normed space. We define $\mathcal{V}^{\mathrm{cpl}}$ to be the completion of $\mathcal{V}$.

If $\mathcal{V}, \mathcal{W}$ are normed spaces, then $B(\mathcal{V}, \mathcal{W})$ denotes the space of bounded operators from $\mathcal{V}$ to $\mathcal{W}$. We set $B(\mathcal{V}) := B(\mathcal{V}, \mathcal{V})$.

1.1.3 Direct sums

If $(\mathcal{V}_i)_{i\in I}$ is a family of vector spaces, then

$$\overset{\circ}{\oplus}_{i\in I}\mathcal{V}_i$$

denotes the algebraic direct sum of the spaces $\mathcal{V}_i$. If $\mathcal{V}_1,\dots,\mathcal{V}_n$ is a finite family of vector spaces, then we will also use the notation

$$\mathop{\oplus}_{i=1}^{n}\mathcal{V}_i$$

for the direct sum of the spaces $\mathcal{V}_i$. Thus for a finite family of spaces the symbols $\oplus$ and $\overset{\circ}{\oplus}$ can be used interchangeably.

If $\mathcal{V}_i$, $\mathcal{W}_i$, $i\in I$ are families of vector spaces, and $a_i\in L(\mathcal{V}_i,)$, then we define $\mathop{\oplus}\limits_{i\in I} a_i\in L(\overset{\circ}{\oplus}_{i\in I}\mathcal{V}_i,\overset{\circ}{\oplus}_{i\in I}\mathcal{W}_i)$.

1.1.4 Linear relations

Let $\mathcal{V}$, $\mathcal{W}$ be linear spaces, both over $\mathbf{K}$. Let $R(\mathcal{V},\mathcal{W})$ denote the space of linear relations from $\mathcal{V}$ to $\mathcal{W}$, that is linear sebspaces of $\mathcal{V}\oplus\mathcal{W}$.

If $a_1\in R(\mathcal{V}_1,\mathcal{V})$ and $a_2\in R(\mathcal{V},\mathcal{V}_2)$ one defines the superposition of a_2 and a_1 denoted $a_2a_1\in R(\mathcal{V}_1,\mathcal{V}_2)$.

If $a\in R(\mathcal{V},\mathcal{W})$, one defines

$$\mathcal{D}(a):=\{v\in\mathcal{V}\ :\ \text{there exists}\ \ w\in\mathcal{W}\ \ \text{such that}\ \ (v,w)\in a\},$$

$$\mathrm{Ran}(a):=\{w\in\mathcal{V}\ :\ \text{there exists}\ \ v\in\mathcal{V}\ \ \text{such that}\ \ (v,w)\in a\}.$$

One defines also $a^{-1}\in R(\mathcal{W},\mathcal{V})$

If $(v_1,w),(v_2,w)\in a$ implies $v_1=v_2$, then, by a commonly accepted abuse of terminology, we will say that a is an operator from $\mathcal{V}$ to $\mathcal{W}$. (But we will write $a\in L(\mathcal{V},\mathcal{W})$ only if $\mathcal{D}(a)=\mathcal{V}$). In such a case, instead of $(v,w)\in a$ we will write $av=w$.

1.1.5 General linear group

The general linear group is

$$Gl(\mathcal{Y}):=\{r\in L(\mathcal{Y})\ :\ r^{-1}\ \text{exists}\}.$$

$$\overset{\circ}{Gl}(\mathcal{Y}):=\{r\in Gl(\mathcal{Y})\ :\ r-1\in\overset{\circ}{L}(\mathcal{Y})\}$$

is its subgroup.

If G is a subgroup of $Gl(\mathcal{Y})$, then we will use the notation $\overset{\circ}{G}:=\overset{\circ}{Gl}(\mathcal{Y})\cap G$.

For $r \in \overset{\circ}{Gl}(\mathcal{Y})$ we define $\det r$.

$$\overset{\circ}{Sl}(\mathcal{Y}) := \{r \in \overset{\circ}{Gl}(\mathcal{Y}) \ : \ \det r = 1\}$$

is a subgroup of $\overset{\circ}{Gl}(\mathcal{Y})$.

The general linear Lie algebra is

$$gl(\mathcal{Y}) := L(\mathcal{Y}),$$

equipped with the bracket $[a_1, a_2] := a_1 a_2 - a_2 a_1$. $\overset{\circ}{gl}(\mathcal{Y}) := \overset{\circ}{L}(\mathcal{Y})$ is its subalgebra.

If g is a subalgebra of $gl(\mathcal{Y})$, then we will write $\overset{\circ}{g} := \overset{\circ}{gl}(\mathcal{Y}) \cap g$.

For $a \in \overset{\circ}{gl}(\mathcal{Y})$ we define $\mathrm{Tr} a$.

$$\overset{\circ}{sl}(\mathcal{Y}) := \{a \in \overset{\circ}{gl}(\mathcal{Y}) \ : \ \mathrm{Tr} a = 0\}$$

is a Lie subalgebra of $\overset{\circ}{gl}(\mathcal{Y})$.

Theorem 1.1.1 *$\overset{\circ}{Gl}(\mathcal{Y})$ is generated by e^a for $a \in \overset{\circ}{gl}(\mathcal{Y})$ and $\overset{\circ}{Sl}(\mathcal{Y})$ is generated by e^a for $a \in \overset{\circ}{sl}(\mathcal{Y})$.*

1.1.6 Inhomogeneous general linear group

The inhomogeneous general linear group is

$$IGl(\mathcal{Y}) := \{(y, r) \in IL(\mathcal{Y}) \ : \ r \in Gl(\mathcal{Y})\}.$$

equipped with the multiplication

$$(y_2, r_2)(y_1, r_1) = (y_2 + r_2 y_1, r_2 r_1).$$

Note that $IGl(\mathcal{Y})$ is isomorphic to a subgroup of $Gl(\mathcal{Y} \oplus \mathbf{K})$:

$$IGl(\mathcal{Y}) \ni (y, r) \mapsto \begin{bmatrix} r & y \\ 0 & 1 \end{bmatrix} \in Gl(\mathcal{Y} \oplus \mathbf{K}).$$

We have

$$0 \to \mathcal{Y} \to IGl(\mathcal{Y}) \to Gl(\mathcal{Y}) \to 0.$$

If G is a subgroup of $Gl(\mathcal{Y})$, then we set

$$IG := \{(y, r) \in IGl(\mathcal{Y}) \ : \ r \in G\}.$$

Clearly, IG is a subgroup of $IGl(\mathcal{Y})$ and

$$0 \to \mathcal{Y} \to IG \to G \to 0.$$

The inhomogeneous general linear Lie algebra is

$$igl(\mathcal{Y}) := IL(\mathcal{Y})$$

equipped with the bracket

$$[(y_2, a_2), (y_1, a_1)] = (a_2 y_1 - a_1 y_2, a_2 a_1 - a_1 a_2).$$

Note that $igl(\mathcal{Y})$ is isomorphic to a subalgebra of $gl(\mathcal{Y} \oplus \mathbf{K})$:

$$igl(\mathcal{Y}) \ni (y, a) \mapsto \begin{bmatrix} a & y \\ 0 & 0 \end{bmatrix} \in gl(\mathcal{Y} \oplus \mathbf{K}).$$

We have

$$0 \to \mathcal{Y} \to igl(\mathcal{Y}) \to gl(\mathcal{Y}) \to 0.$$

If g is a subalgebra of $gl(\mathcal{Y})$, then we set

$$ig := \{(y, a) \in igl(\mathcal{Y}) \ : \ a \in g\}.$$

Clearly,

$$0 \to \mathcal{Y} \to ig \to g \to 0.$$

1.1.7 Vector fields

Let $\mathcal{Y}$ be a vector space. Let y denote the generic variable in $\mathcal{Y}$. For any $f \in C^\infty(\mathcal{Y}, \mathcal{Y})$, let $f(y)\nabla_y$ denote the corresponding vector field on $\mathcal{Y}$. Let $T(\mathcal{Y})$ denote the space of vector fields over $\mathcal{Y}$. The vector fields on $\mathcal{Y}$ form a Lie algebra with the bracket

$$[f_1(y)\nabla_y, f_2(y)\nabla_y] = \Big(f_1(y)\nabla_y f_2(y) - f_2(y)\nabla_y f_1(y)\Big)\nabla_y, \qquad f_i(y)\nabla_y \in T(\mathcal{Y}).$$

We have the homomorphism

$$igl(\mathcal{Y}) \ni (w, a) \mapsto -(w + ay)\nabla_y \in T(\mathcal{Y}).$$

If $f(y)\nabla_y \in T(\mathcal{Y})$, then under some conditions, there exists a 1-parameter group $\mathbf{R} \ni t \in r_t \in C^\infty(\mathcal{Y}, \mathcal{Y})$ such that

$$\partial_t r_t(y) = f(y).$$

We will then write

$$\left(\mathrm{e}^{tf(y)\nabla_y} g\right)(y) := g(r_t(y)).$$

For example, if $(w, a) \in igl(\mathcal{Y})$ and $w \in \mathrm{Ran}a$, then

$$\mathrm{e}^{-t(w+ay)\nabla_y} g(y) = g(\mathrm{e}^{-ta}(y + a^{-1}w) - w).$$

1.2 Elementary structures in vector spaces

1.2.1 Complex conjugate spaces

Let $\mathcal{V}$ be a complex space. Then $\overline{\mathcal{V}}$ will denote another complex space equipped with an antilinear isomorphism

$$\mathcal{V} \ni v \mapsto \overline{v} \in \overline{\mathcal{V}}.$$

We will use the convention that $\overline{\overline{\mathcal{V}}} = \mathcal{V}$ and $\overline{\overline{v}} = v$, $v \in \mathcal{V}$.

Let $\mathcal{V}$, $\mathcal{W}$ be complex spaces, If $a \in L(\mathcal{V}, \mathcal{W})$, then $\overline{a} \in \overline{L(\mathcal{V}, \mathcal{W})}$ can be identified with a map in $L(\overline{\mathcal{V}}, \overline{\mathcal{W}})$ as follows:

$$\overline{a}\,\overline{v} := \overline{av}.$$

Thus $\overline{L(\mathcal{V}, \mathcal{W})}$ can be identified with the space of antilinear maps from $\mathcal{V}$ to $\mathcal{W}$, and also with $L(\overline{\mathcal{V}}, \overline{\mathcal{W}})$.

1.2.2 Involutions

Let $\mathcal{Y}$ be a vector space. $\kappa \in L(\mathcal{Y})$ is called an involution if $\kappa^2 = 1$. Every involution determines a decomposition $\mathcal{Y} = \mathcal{Y}_\kappa \oplus \mathcal{Y}_{-\kappa}$, where $\mathcal{Y}_{\pm\kappa} := \{y \in \mathcal{Y} \ : \ y = \pm\kappa y\}$. The operators $\frac{1}{2}(1 \pm \kappa)$ are the projections onto $\mathcal{Y}_{\pm\kappa}$.

Conversely, a decomposition $\mathcal{Y} = \mathcal{Y}_+ \oplus \mathcal{Y}_-$ determines an involution given by the matrix $\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Operators commuting with κ are of the form $r = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

1.2.3 Anti-involutions

We say that $\mathbf{i} \in L(\mathcal{Y})$ is a anti-involution, if

$$\mathbf{i}^2 = -1.$$

Remark 1.2.1 *Antiinvolutions are often called complex structures.*

If $\mathcal{Y}$ is a real vector space with an involution $\mathbf{i}$, then for $\lambda + \mathrm{i}\mu \in \mathbf{C}$, we have two natural complex structures in $\mathcal{Y}$:

$$(\lambda + \mathrm{i}\mu)y := \lambda y + \mu \mathbf{i} y, \quad y \in \mathcal{Y}, \tag{1.2.1}$$

$$(\lambda + \mathrm{i}\mu)y := \lambda y - \mu \mathbf{i} y, \quad y \in \mathcal{Y}, \tag{1.2.2}$$

$\mathcal{Y}$ converted into a space over $\mathbf{C}$ with the multiplication (1.2.1) , which will be denoted $\mathcal{Y}^{\mathbf{C}}$. It will be called the complex form of $\mathcal{Y}$. $\mathcal{Y}$ converted into a space over $\mathbf{C}$ with the multiplication (1.2.2) , which will be denoted $\overline{\mathcal{Y}^{\mathbf{C}}}$.

Conversely, any complex space $\mathcal{V}$ can be considered a real vector space, called the real form of $\mathcal{V}$ and denoted $\mathcal{V}_{\mathbf{R}}$. It is equipped with a map $\mathbf{i} \in L(\mathcal{V}_{\mathbf{R}})$ (the multiplication by the imaginary unit i).

Let $\mathcal{Y}$, $\mathcal{W}$ be real spaces with anti-involutions $\mathbf{i}_{\mathcal{Y}}$, $\mathbf{i}_{\mathcal{W}}$. Then

$$L(\mathcal{Y}^{\mathbf{C}}, \mathcal{W}^{\mathbf{C}}) = \{a \in L(\mathcal{Y}, \mathcal{W}) \ : \ a\mathbf{i}_{\mathcal{Y}} = \mathbf{i}_{\mathcal{W}} a\}.$$

$$\overline{L(\mathcal{Y}^{\mathbf{C}}, \mathcal{W}^{\mathbf{C}})} = \{a \in L(\mathcal{Y}, \mathcal{W}) \ : \ a\mathbf{i}_{\mathcal{Y}} = -\mathbf{i}_{\mathcal{W}} a\}.$$

1.2.4 Conjugations

Let a space $\mathcal{Y}$ be equipped with an anti-involution $\mathbf{i} \in L(\mathcal{Y})$. We say that $\kappa \in L(\mathcal{Y})$ is a conjugation if it is an involution and $\mathbf{i}\kappa = -\kappa\mathbf{i}$.

Let us write $\mathcal{X} := \mathcal{Y}_{\kappa}$. Then $\mathbf{i}\mathcal{X} = \mathcal{Y}_{-\kappa}$. Thus $\mathcal{Y}$ can be identified with $\mathcal{X} \oplus \mathcal{X}$, so that

$$\mathbf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $r \in L(\mathcal{X} \oplus \mathcal{X})$ commutes with $\mathbf{i}$ iff it is of the form

$$r = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \tag{1.2.3}$$

where $a, b \in L(\mathcal{X})$.

Let $\mathcal{X}$ be a real space and let $\mathbf{C}\mathcal{X}$ denote the $(\mathcal{X} \oplus \mathcal{X})^{\mathbf{C}}$, with the anti-involution given by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which will be denoted simply by i, and the conjugation given by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and denoted as $\mathbf{C}\mathcal{X} \ni v \mapsto \overline{v} \in \mathbf{C}\mathcal{X}$. Note that $L(\mathbf{C}\mathcal{X})$, in the representation $\mathcal{X} \oplus \mathcal{X}$, consists of matrices of the form (1.2.3)

Every operator $r \in L(\mathcal{X})$ has a unique extension to a complex linear operator on $\mathbf{C}\mathcal{Y}$. Usually, this extension will be denoted by the same letter r. Only in the next subsection, it will be denoted $\mathbf{C}r \in L(\mathbf{C}\mathcal{X})$.

1.2.5 Complexification of an anti-involution

Assume that a real space $\mathcal{Y}$ is equipped with an anti-involution $\mathbf{i} \in L(\mathcal{Y})$. Thus $(\mathbf{C}\mathcal{Y})_{\mathbf{R}}$ has two distiguished anti-involutions: the usual i, and also $\mathbf{C}\mathbf{i}$.

Define the subspace $\mathcal{Z} := \{y - \mathrm{i}\mathbf{i}y \ : \ y \in \mathcal{Y}\}$. $\mathcal{Z}$ will be called the holomorphic subspace of $\mathbf{C}\mathcal{Y}$. $\overline{\mathcal{Z}} := \{y + \mathrm{i}\mathbf{i}y \ : \ y \in \mathcal{Y}\}$ will be called the antiholomorphic subspace of $\mathbf{C}\mathcal{Y}$. We have $\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$ and the corresponding projections equal $1_{\mathcal{Z}} := \frac{1}{2}(1 - \mathrm{i}\mathbf{C}\mathbf{i})$ and $1_{\overline{\mathcal{Z}}} := \frac{1}{2}(1 + \mathrm{i}\mathbf{C}\mathbf{i})$. On $\mathcal{Z}$ we have $\mathbf{C}\mathbf{i} = \mathrm{i}$ and on $\overline{\mathcal{Z}}$ we have $\mathbf{C}\mathbf{i} = -\mathrm{i}$.

If r is a real linear operator on $\mathcal{Y}$, then its extension to $\mathbf{C}\mathcal{Y}$, denoted $\mathbf{C}r$, preserves $\mathcal{Z}$ iff $r\mathbf{i} = \mathbf{i}r$. Then we can define a complex linear operator p on $\mathcal{Z}$ by setting

$$p := \mathbf{C}r\Big|_{\mathcal{Z}}. \tag{1.2.4}$$

Note that $\overline{p} = \mathbf{C}r\Big|_{\overline{\mathcal{Z}}}$ and $\mathbf{C}r = p \oplus \overline{p}$, where we use the decomposition $\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$.

The converse construction is as follows. Let $\mathcal{Z}$ be a complex vector space. Then we set $\mathcal{Y}$ to be the (real) subspace of $(\mathcal{Z} \oplus \overline{\mathcal{Z}})_{\mathbf{R}}$ given by

$$\mathcal{Y} := \{(z, \overline{z}) \in \mathcal{Z} \oplus \overline{\mathcal{Z}} \ : \ z \in \mathcal{Z}\}.$$

If $p \in L(\mathcal{Z})$, then we can define a complex linear operator on $\mathcal{Y}$ by setting

$$r(z + \overline{z}) := pz + \overline{pz}.$$

In other words, $r := (p + \overline{p})\Big|_{\mathcal{Y}}$. Clearly, this is the converse construction to (1.2.4).

Let $r \in L(\mathcal{Y})$. Using the decomposition $\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$ we can write $\mathbf{C}r$ as

$$\mathbf{C}r = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix},$$

where $p := 1_{\mathcal{Z}}\mathbf{C}r1_{\mathcal{Z}} \in L(\mathcal{Z}, \mathcal{Z})$, $q := 1_{\mathcal{Z}}\mathbf{C}r1_{\overline{\mathcal{Z}}} \in L(\overline{\mathcal{Z}}, \mathcal{Z})$. We have $r \in L(\mathcal{Y}^{\mathbf{C}})$ iff

$$\mathbf{C}r = \begin{bmatrix} p & 0 \\ 0 & \overline{p} \end{bmatrix}.$$

Suppose that $\mathcal{Y}$ is finite dimensional and is equipped with the Lebesgue measure $\mathrm{d}y$. On $(\mathbf{C}\mathcal{Y})_{\mathbf{R}}$ (treated as $\mathcal{Y} \oplus \mathcal{Y}$) we have a natural Lebesgue measure, which can be denoted, say, $\mathrm{d}y\mathrm{d}\mathrm{i}y$. We claim that this fixes in a natural way a Lebesgue measure on $\mathcal{Z}_{\mathbf{R}}$.

In fact, we have the map $\mathcal{Z} \ni z \mapsto \overline{z} \in \overline{\mathcal{Z}}$. Therefore, given a Lebesgue measure on $\mathcal{Z}_{\mathbf{R}}$, which we temporarily denote by $|\mathrm{d}z|$, we have a Lebesgue measure on $\overline{\mathcal{Z}}_{\mathbf{R}}$, temporarily denoted $|\mathrm{d}\overline{z}|$. Now the condition

$$\mathrm{d}y\mathrm{d}\mathrm{i}y = |\mathrm{d}z||\mathrm{d}\overline{z}|$$

fixes a Lebesgue measure $\mathrm{d}z|$ on $\mathcal{Z}_{\mathbf{R}}$.

In what follows, instead of writing $|\mathrm{d}z|$, this Lebesgue measure on $\mathcal{Z}_{\mathbf{R}}$ will be denoted $(2\mathrm{i})^{-d}\mathrm{d}z\mathrm{d}\overline{z}$, where d is the (complex) dimension of $\mathcal{Z}$. (To justify this notation, consider the one dimensional case, identify $\mathcal{Z}$ with $\mathbf{C}$, then $\mathrm{d}z = \mathrm{d}x + \mathrm{i}\mathrm{d}y$, $\mathrm{d}\overline{z} = \mathrm{d}x - \mathrm{i}\mathrm{d}y$, and then $\mathrm{d}z \wedge \mathrm{d}\overline{z} = \mathrm{i}2\mathrm{d}x \wedge \mathrm{d}y$.)

The map $\mathcal{Z} \ni z \mapsto z + \overline{z} =: y \in \mathcal{Y}^{\mathbf{C}}$ is a linear isomorphism with the inverse $\mathcal{Y}^{\mathbf{C}} \ni y \mapsto \frac{1 - \mathrm{i}\mathbf{C}\mathrm{i}}{2}y =: z \in \mathcal{Z}$. Note that under this transformation the Lebesgue measure $\mathrm{d}y$ is transformed into 2^d times the Lebesgue measure on $\mathcal{Z}_{\mathbf{R}}$, that is into $\mathrm{i}^{-d}\mathrm{d}z\mathrm{d}\overline{z}$.

1.3 Duality

1.3.1 Dual pairs

Let $(\mathcal{X}, \mathcal{X}')$ be a pair of vector spaces over $\mathbf{K}$. We say that $(\mathcal{X}, \mathcal{X}')$ is a (bilinear) dual pair if it is equipped with a bilinear map

$$\mathcal{X}' \times \mathcal{X} \ni (\xi, x) \mapsto \langle \xi | x \rangle \in \mathbf{K}$$

such that

$$\forall x \in \mathcal{X},\ \exists \xi \in \mathcal{X}' \quad \langle \xi | x \rangle \neq 0, \qquad \forall \xi \in \mathcal{X}',\ \exists x \in \mathcal{X} \quad \langle \xi | x \rangle \neq 0.$$

We will usually simply write $x\xi$ or ξx instead of $\langle \xi | x \rangle$.

If $(\mathcal{X}, \mathcal{X}')$ is a dual pair, then $(\mathcal{X}', \mathcal{X})$ is also a dual pair. We will say that $\mathcal{X}'$ is a (linear) dual of $\mathcal{X}$.

If $(\mathcal{X}_i, \mathcal{X}_i')$, $i \in I$ is a family of dual pairs, then $\left(\overset{\circ}{\oplus}_{i\in I}\mathcal{X}_i, \overset{\circ}{\oplus}_{i\in I}\mathcal{X}_i'\right)$ is a dual pair.

The space $\mathcal{X}^{/\mathrm{al}} := L(\mathcal{X}, \mathbf{K})$ is called the algebraic dual of $\mathcal{X}$. $(\mathcal{X}, \mathcal{X}^{/\mathrm{al}})$ is an example of a dual pair. Note that if $(\mathcal{X}, \mathcal{X}')$ is a dual pair, then $\mathcal{X}'$ is naturally embedded in $\mathcal{X}^{/\mathrm{al}}$.

If $\mathcal{X}$ is finite dimensional, then up to a natural isomorphism all spaces dual to $\mathcal{X}$ coincide with the algebraic dual. In this case, we will denote it simply by $\mathcal{X}'$.

If $\mathcal{X}$ is finite dimensional and $\{q_1, \dots, q_d\}$ is a basis in $\mathcal{X}$ then there exists a unique basis $\{\eta_1, \dots, \eta_d\}$ in $\mathcal{X}'$ satisfying $\langle \eta_i, q_j \rangle = \delta_{ij}$. We say that the basis $\{\eta_1, \dots, \eta_d\}$ is dual to $\{q_1, \dots, q_d\}$. Note that a Lebesgue measure on $\mathcal{X}$ determines uniquely a Lebesgue measure on $\mathcal{X}'$.

If $\mathcal{X}$ is a Banach space, then $(\mathcal{X}, B(\mathcal{X}, \mathbf{K}))$ is a dual pair. In this case, one usually reserves the notation $\mathcal{X}'$ for $B(\mathcal{X}, \mathbf{K})$ (the topological dual).

1.3.2 Conjugate operators

Suppose that $(\mathcal{X}, \mathcal{X}')$ and $(\mathcal{Y}, \mathcal{Y}')$ are dual pairs. Let $a \in R(\mathcal{X}, \mathcal{Y})$. Then $a' in R(\mathcal{Y}', \mathcal{X}')$ is defined as follows

$$(\eta, \xi) \in a' \Leftrightarrow \Big((x, y) \in a \ \Rightarrow\ \eta y = \xi x\Big).$$

We have

$$(a')' = a.$$

If $\mathcal{X}$, $\mathcal{Y}$ are finite dimensional and $a \in L(\mathcal{X}, \mathcal{Y})$, then $a' \in L(\mathcal{Y}', \mathcal{X})$. The same is true if $\mathcal{X}'$ and $\mathcal{Y}'$ are algebraic duals. We have in this case

$$Gl(\mathcal{Y}) \ni r \mapsto (r')^{-1} \in Gl(\mathcal{Y}^{/\mathrm{al}}),$$

$$gl(\mathcal{Y}) \ni a \mapsto -a' \in gl(\mathcal{Y}^{/\mathrm{al}})$$

are homomorphisms (isomorphisms if $\mathcal{Y}$ is finite dimensional).

If $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces and $\mathcal{X}'$ and $\mathcal{Y}'$ are their topological duals, then $a \in B(\mathcal{X}, \mathcal{Y})$ implies $a' \in B(\mathcal{Y}', \mathcal{X}')$.

1.3.3 Sesquilinear dual pairs

Let $(\mathcal{Z}, \mathcal{Z}^*)$ be a pair of vector spaces over $\mathbf{C}$. We say that $(\mathcal{Z}, \mathcal{Z}^*)$ is a hermitian dual pair if it is equipped with a map

$$\mathcal{Z}^* \times \mathcal{Z} \ni (\xi, x) \mapsto (\xi|x) \in \mathbf{C}$$

which is antilinear wrt the first component and linear wrt the second component, and such that

$$\forall z \in \mathcal{Z},\ \exists w \in \mathcal{Z}^* \quad (w|z) \neq 0, \qquad \forall w \in \mathcal{Z}^*,\ \exists z \in \mathcal{Z} \quad (w|z) \neq 0.$$

Given a complex space $\mathcal{Z}$, linear dual spaces $\mathcal{Z}'$ and antilinear dual spaces $\mathcal{Z}^*$ are in a 1-1 correspondence given by $\mathcal{Z}' = \overline{\mathcal{Z}^*}$, or equivalently, $\mathcal{Z}^* = \overline{\mathcal{Z}'}$.

The space $\mathcal{Z}^{*\mathrm{al}} := \overline{L(\mathcal{Z}, \mathbf{C})}$, that is the space of all antilinear functionals on $\mathcal{Z}$, is called the algebraic antilinear dual of $\mathcal{Z}$. $(\mathcal{Z}, \mathcal{Z}^{*\mathrm{al}})$ is an example of a sesquilinear dual pair.

If $\mathcal{Z}$ is finite dimensional, then up to a natural isomorphism all the spaces hermitian dual to $\mathcal{Z}$ coincide with the algebraic antilinear dual. In this case, we will denote it simply by $\mathcal{Z}^*$.

If $\mathcal{Z}$ is a Banach space, then $(\mathcal{Z}, \overline{B(\mathcal{Z}, \mathbf{C})})$ is a sesquilinear dual pair. In this case, one usually reserves the notation $\mathcal{Z}^*$ for $\overline{B(\mathcal{Z}, \mathbf{C})}$ (the topological antilinear dual).

1.3.4 Hermitian conjugate operators

Suppose that $(\mathcal{V}, \mathcal{V}^*)$ and $(\mathcal{W}, \mathcal{W}^*)$ are sesquilinear dual pairs. Let $a \in R(\mathcal{V}, \mathcal{W})$. Then $a^* in R(\mathcal{W}^*, \mathcal{V}'^*)$ is defined as follows

$$(\eta, \xi) \in a^* \Leftrightarrow \Big((v, w) \in a \ \Rightarrow \ (\eta|w) = (\xi v)\Big).$$

Clearly, if $\mathcal{Z}' = \overline{\mathcal{Z}^*}$ and $\mathcal{V}' = \overline{\mathcal{V}^*}$, then a' exists in the sense of the dual pairs $(\mathcal{Z}, \mathcal{Z}')$, $\mathcal{V}, \mathcal{V}')$ iff a^* exists in the sense of the hermitian dual pairs $(\mathcal{Z}, \mathcal{Z}^*)$, $(\mathcal{V}, \mathcal{V}^*)$ and $a^* = \overline{a'}$ or, equivalently, $a' = \overline{a^*}$.

1.3.5 The dual of a space with antiinvolution

Suppose that $(\mathcal{Y}, \mathbf{i})$ is a real vector space with an antiinvolution. Let $(\mathcal{Y}, \mathcal{Y}')$ be a dual pair and assume that $\mathbf{i}' \in L(\mathcal{Y}')$. Clearly, $(\mathcal{Y}', \mathbf{i}')$ is a space with an antiinvolution.

Theorem 1.3.1 *$(\mathcal{Z}, (\mathcal{Y}')^{\mathbf{C}})$ and $(\overline{\mathcal{Z}}, \overline{(\mathcal{Y}')^{\mathbf{C}}})$ are dual pairs.*

Proof. Let $y \in \mathbf{C}\mathcal{Y}$, $v \in \mathcal{Y}'$. Suppose that y is a linear functional on $(\mathcal{Y}')^{\mathbf{C}}$. Then

$$\mathrm{i}\langle v|y\rangle = \langle \mathbf{i}'v|y\rangle = \langle v|\mathbf{i}y\rangle.$$

Thus $\mathrm{i}y = \mathbf{i}y$. This means that $y \in \mathcal{Z}$. $\square$

Let us remark though that treating $\mathcal{Z}$ as the dual of $(\mathcal{Y}')^{\mathbf{C}}$ leads to a certain apparent paradox, which we describe below.

Suppose that $\mathcal{Y}$ is finite dimensional of dimension d, with the Lebesgue measure $\mathrm{d}y$. Recall from Subsection **??** that it leads to a natural Lebesgue measure on $\mathcal{Z}$ and on $\overline{\mathcal{Z}}$. Treating $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ as real spaces, we obtain natural Lebesgue measures on $\mathcal{Z}'$ and on $\overline{\mathcal{Z}}'$.

On the other hand, the Lebesgue measure on $\mathcal{Y}$ fixes a Lebesgue measure on $\mathcal{Y}'$. As the set $\mathcal{Y}'$ equals $(\mathcal{Y}')^{\mathbf{C}}$ and $\overline{(\mathcal{Y}')^{\mathbf{C}}}$, which by theorem 1.3.1 can be identified with $\mathcal{Z}'$ and $\overline{\mathcal{Z}}'$. But the Lebesgue measure on $\mathcal{Y}'$ is equal 2^d times the natural Lebesgue measures on $\mathcal{Z}'$ and on $\overline{\mathcal{Z}}'$.

1.4 (Pseudo-)Euclidean spaces

1.4.1 Symmetric forms

Let $(\mathcal{Y}, \mathcal{Y}')$ be a dual pair. Let $\alpha \in L(\mathcal{Y}, \mathcal{Y}')$. We will say that α is symmetric iff

$$y_1 \alpha y_2 = y_2 \alpha y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

The space of $\alpha \in L(\mathcal{Y}, \mathcal{Y}')$ that are symmetric will be denoted $L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}')$ In other words,

$$L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}') = \{\alpha \in L(\mathcal{Y}, \mathcal{Y}') \ : \ \alpha' = \alpha\}.$$

We say that $\alpha \in L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}')$ is nondegenerate iff $\mathrm{Ker}\alpha = \{0\}$. If α is nondegenerate, then $(\mathcal{Y}, \mathrm{Ran}\alpha)$ is a dual pair. We will usually assume that $\mathcal{Y}' = \mathrm{Ran}\alpha$. We can then define $\alpha^{-1} \in L_{\mathrm{s}}(\mathcal{Y}', \mathcal{Y})$, which is also nondegenerate.

In the finite dimensional case, where the choice of $\mathcal{Y}'$ is unique, if α is nondegenerate, then $\mathrm{Ran}\alpha$ automatically equals $\mathcal{Y}'$.

Let $\mathcal{Y}$ be a space over $\mathbf{R}$. We will write $\alpha > 0$ and we will say that α is positive definite if

$$y \alpha y > 0, \quad y \in \mathcal{Y}, \ y \neq 0.$$

Clearly, if α is positive definite, then it is non-degenerate on $\mathcal{Y}$. A symmetric nondegenerate form α is called a pseudo-Euclidean form (scalar product). It is called Euclidean if it is positive.

Let $\mathcal{Y}$ be a real space of dimension $d < \infty$. Every $\alpha \in L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}')$ can be diagonalized so that on the diagonal n_+ entries equal 1, n_- entries equal -1 and the remaining $d - n_- - n_+ = \dim \mathrm{Ker}\alpha$ entries are zero. The number $\mathrm{inert}\alpha := n_+ - n_-$ is called the inertia of α.

If $\mathcal{Y}$ is a space over $\mathbf{C}$, then a nondegenerate symmetric form is called a complex Euclidean form.

1.4.2 (Pseudo-)orthogonal group

Let $(\mathcal{Y}_1, \alpha_1)$, $(\mathcal{Y}_2, \alpha_2)$ be (pseudo-)Euclidean spaces. Their dual spaces are set to be $\mathcal{Y}_i' := \alpha_i \mathcal{Y}_i$.

Let $r \in L(\mathcal{Y}_1, \mathcal{Y}_2)$. We say that r is (pseudo-)orthogonal if r is invertible and

$$(ry)\alpha_2 ry = r\alpha_1 r, \qquad y \in \mathcal{Y}$$

Theorem 1.4.1 *$r \in L(\mathcal{Y}_1, \mathcal{Y}_2)$ is (pseudo-)orthogonal iff $r' \in L(\mathcal{Y}_2', \mathcal{Y}_1')$ and*

$$r'\alpha_2 r = \alpha_1, \qquad r\alpha_1^{-1} r' = \alpha_2^{-1}.$$

Consider now a single (pseudo-)Euclidean space $(\mathcal{Y}, \alpha)$, with $\mathcal{Y}' := \alpha\mathcal{Y}$. The set of (pseudo-)orthogonal elements of $Gl(\mathcal{Y})$ forms a group called the (pseudo-)orthogonal group of $\mathcal{Y}$ and denoted $O(\mathcal{Y})$.

Theorem 1.4.2 *If $r \in \mathring{O}(\mathcal{Y})$, then $\det r = \pm 1$.*

$$\mathring{SO}(\mathcal{Y}) := \{r \in \mathring{O}(\mathcal{Y}) \ : \ \det r = 1\}$$

is a subgroup of $\mathring{O}(\mathcal{Y})$.

Let $a \in L(\mathcal{Y})$. We say that a is (pseudo-)self-adjoint if

$$(ay_1)\alpha y_2 = y_1 \alpha a y_2, \qquad y_1, y_2 \in \mathcal{Y}.$$

We say that a is (pseudo-)anti-self-adjoint if

$$(ay_1)\alpha y_2 = -y_1 \alpha a y_2 \qquad y_1, y_2 \in \mathcal{Y}.$$

Theorem 1.4.3 *1) $a \in L(\mathcal{Y})$ is (pseudo-)self-adjoint iff $a' \in L(\mathcal{Y}')$ and*

$$a'\alpha = \alpha a.$$

2) $a \in L(\mathcal{Y})$ is (pseudo-)anti-self-adjoint iff $a' \in L(\mathcal{Y}')$ and

$$a'\alpha = -\alpha a.$$

The set of (pseudo)-anti-self-adjoint elements of $gl(\mathcal{Y})$ is a Lie algebra called the (pseudo-)orthogonal Lie algebra of $\mathcal{Y}$ and denoted $o(\mathcal{Y})$.

Theorem 1.4.4 *1) $\mathring{o}(\mathcal{Y}) \subset \mathring{sl}(\mathcal{Y})$.*
2) $\mathring{O}(\mathcal{Y})$ is generated by e^a for $a \in \mathring{o}(\mathcal{Y})$.
3) $so(\mathcal{Y})$ equals $\{b\alpha \ : \ b \in L_{\mathrm{s}}(\mathcal{Y}', \mathcal{Y})\}$

1.4.3 Real Hilbert spaces

If $\mathcal{Y}, \alpha$ is a real Euclidean space, then $\|y\| := \sqrt{y\alpha y}$ defines a norm. A complete real Euclidean space is called a real Hilbert space.

If $\mathcal{Y}, \alpha$ is a real Hilbert space, then α is an isomorphism of the real Hilbert spaces $(\mathcal{Y}, \alpha)$ and $(\mathcal{Y}', \alpha^{-1})$. We will thus identify $\mathcal{Y}$ and $\mathcal{Y}'$. We will use the notation $y_1 \cdot y_2$ or $y_1 y_2$ instead of $y_1 \alpha y_2$ and a' instead of $\alpha a' \alpha^{-1}$.

The most convenient way to study a real Hilbert space is often to complexify it and then to study the complex Hilbert space with a conjugation (see Subsection **??**).

The properties and terminology for real Hilbert spaces is essentially the same as for complex Hilbert spaces. (The exception is that the analogs of orthogonal operators ar unitary operators). Therefore, we refer the reader to the Subsection **??**, which treats complex Hilbert spaces for more material on real Hilbert spaces.

1.5 Symplectic spaces

1.5.1 Antisymmetric bilinear forms

Let $(\mathcal{Y}, \mathcal{Y}')$ be a dual pair. Let $\omega \in L(\mathcal{Y}, \mathcal{Y}')$. We will say that ω is antisymmetric iff

$$y_1 \omega y_2 = -y_2 \omega y_2, \quad y_1, y_2 \in \mathcal{Y},$$

or equivalently

$$y \omega y = 0, \quad y \in \mathcal{Y}.$$

The space of $\omega \in L(\mathcal{Y}, \mathcal{Y}')$ such that ω is antisymmetric will be denoted $L_{\mathrm{a}}(\mathcal{Y}, \mathcal{Y}')$. In other words,

$$L_{\mathrm{a}}(\mathcal{Y}, \mathcal{Y}') = \{\omega \in L(\mathcal{Y}, \mathcal{Y}') \ : \ \omega' = -\omega\}.$$

A subspace $\mathcal{X} \subset \mathcal{Y}$ is called isotropic iff

$$y_1 \omega y_2 = 0, \quad y_1, y_2 \in \mathcal{Y}.$$

A maximal isotropic subspace is called Lagrangian.

We say that $\omega \in L_{\mathrm{a}}(\mathcal{Y}, \mathcal{Y}')$ is nondegenerate iff $\mathrm{Ker}\omega = \{0\}$. If ω is nondegenerate, then $(\mathcal{Y}, \mathrm{Ran}\omega)$ is a dual pair. Moreover, we can define $\omega^{-1} \in L_{\mathrm{a}}(\mathcal{Y}', \mathcal{Y})$, which is also nondegenerate.

In the finite dimensional case, where the choice of $\mathcal{Y}'$ is unique, if ω is nondegenerate, then $\mathrm{Ran}\omega$ automatically equals $\mathcal{Y}'$.

A nondegenerate antisymmetric bilinear form is also called a symplectic form.

1.5.2 Symplectic group

Let $(\mathcal{Y}_i, \omega_i)$ be symplectic spaces. Their dual spaces are set to be $\mathcal{Y}_i' := \omega_i \mathcal{Y}_i$.

A map $r \in L(\mathcal{Y}_1, \mathcal{Y}_2)$ is called a linear symplectomorphism if it is invertible and

$$(ry_1)\omega_2 r y_2 = y_1 \omega_1 y_2, \qquad y_1, y_2 \in \mathcal{Y}_1.$$

Note that $r^{-1} = \omega_1^{-1} r' \omega_2$. Therefore the definition of a symplectomorphism can be restated as follows:

Theorem 1.5.1 *$r \in L(\mathcal{Y}_1, \mathcal{Y}_2)$ is a symplectomorphism iff $r' \in L(\mathcal{Y}_2', \mathcal{Y}_1'$ and*

$$1)\ r'\omega_2 r = \omega_1, \qquad 2)\ r\omega_1^{-1} r' = \omega_2^{-1}.$$

Proof. $\Rightarrow$ We check that $r' = \omega_1 r^{-1} \omega_2^{-1}$ is the conjugate of r and satisfies 1) and 2).

$\Leftarrow$ We check that $r^{-1} = \omega_1^{-1} r' \omega_2$ is the inverse of r. □

We denote by $Sp(\mathcal{Y})$ the group of all linear symplectic maps on $\mathcal{Y}$.

Theorem 1.5.2 *$\overset{\circ}{S}p(\mathcal{Y}) \subset \overset{\circ}{S}l(\mathcal{Y})$.*

The Lie algebra $sp(\mathcal{Y})$ is defined as

$$sp(\mathcal{Y}) := \{a \in gl(\mathcal{Y})\ :\ (ay_1)\omega y_2 + y_1 \omega a y_2 = 0,\ y_1, y_2 \in \mathcal{Y}\}.$$

Theorem 1.5.3 *$a \in sp(\mathcal{Y})$ iff $a' \in L(\mathcal{Y}')$ and*

$$a'\omega + \omega a = 0.$$

Theorem 1.5.4 *1) $\overset{\circ}{s}p(\mathcal{Y}) \subset \overset{\circ}{s}l(\mathcal{Y})$.*
2) $\overset{\circ}{S}p(\mathcal{Y})$ is generated by e^a for $a \in \overset{\circ}{s}p(\mathcal{Y})$.
3) $sp(\mathcal{Y})$ equals $\{b\omega\ :\ b \in L_{\mathrm{s}}(\mathcal{Y}', \mathcal{Y})\}$

1.5.3 Extensions of the symplectic group

By $HSp(\mathcal{Y})$ we denote the group $\mathbf{K} \times ISp(\mathcal{Y}) = \mathbf{K} \times \mathcal{Y} \times Sp(\mathcal{Y})$ equipped with the product

$$[(t_1, y_1, r_1), (t_2, y_2, r_2)] = (\tfrac{1}{2} y_1 \omega y_2, r_1 y_2 + y_1, r_1 r_2).$$

We have

$$0 \to \mathbf{K} \to HSp(\mathcal{Y}) \to ISp(\mathcal{Y}) \to 0.$$

$$0 \to \mathbf{K} \to \mathbf{K} \times \mathcal{Y} \to HSp(\mathcal{Y}) \to Sp(\mathcal{Y}) \to 0.$$

$\mathbf{K} \times \mathcal{Y}$ is called the Heisenberg group.

By $hsp(\mathcal{Y})$ we denote the Lie algebra $\mathbf{R}\times\mathcal{Y}\times sp(\mathcal{Y})$ equipped with the bracket

$$[(t_1,y_1,a_1),(t_2,y_2,a_2)] = (y_1\omega y_2, a_1y_2-a_2y_1, a_1a_2-a_2a_1).$$

Note that

$$0\to\mathbf{K}\to hsp(\mathcal{Y})\to isp(\mathcal{Y})\to 0,$$
$$0\to\mathbf{K}\to\mathbf{K}\times\mathcal{Y}\to hsp(\mathcal{Y})\to sp(\mathcal{Y})\to 0,$$

and

$$hsp(\mathcal{Y})\ni(t,y,a)\mapsto(-t,\omega y,\omega a\omega^{-1})\in hsp(\mathcal{Y}').$$

is an isomorphism of Lie algebras. $\mathbf{K}\times\mathcal{Y}$ is called the Heisenberg Lie algebra.

1.5.4 Poisson bracket

For $f,g\in C^1(\mathcal{Y}')$ we can define the Poisson bracket

$$\{f,g\}(v) := \nabla f(v)\omega\nabla g(v).$$

$C^\infty(\mathcal{Y}')$ equipped with $\{\cdot,\cdot\}$ is a Lie algebra. The map

$$C^\infty(\mathcal{Y}')\ni f\mapsto \mathrm{H}_f := -(\omega\nabla_v f(v))\nabla_v\in T(\mathcal{Y}')$$

is a homomorphism of Lie algebras, called the Hamiltonian vector field of f. The function f is called a Hamiltonian of a vector field $a(v)\nabla_v$ iff $a(v)\nabla_v = \mathrm{H}_f$.

Note that $\{f,g\} = \mathrm{H}_f\nabla g$.

Let the space of quadratic polynomials on $\mathcal{Y}'$ be denoted $\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\le 2}(\mathcal{Y})$ (see Subsection **??**). It is a Lie subalgebra of $C^\infty(\mathcal{Y}')$ wrt the Poisson bracket. More precisely, if $t_1,t_2\in\mathbf{K}$, $y_1,y_2\in\mathcal{Y}$, $b_1,b_2\in\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{2}(\mathcal{Y})\simeq\overset{\circ}{L}_{\mathrm{s}}(\mathcal{Y}',\mathcal{Y})$, then

$$\{(t_1+y_1+\tfrac{1}{2}b_1),(t_2+y_2+\tfrac{1}{2}b_2)\} = t+y+b,$$

$$\text{where}\quad t=y_1\omega y_2\in\mathbf{K},\quad y=b_1\omega y_1-b_2\omega y_2\in\mathcal{Y},\quad b=b_1\omega b_2-b_2\omega b_1\in\overset{\circ}{L}_{\mathrm{s}}(\mathcal{Y}',\mathcal{Y}).$$

We have the isomorphisms:

$$\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\le 2}(\mathcal{Y})\ni(t+y+\tfrac{1}{2}b)\mapsto(t,y,b\omega)\in hsp(\mathcal{Y}),$$
$$\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\le 2}(\mathcal{Y})\ni(t+y+\tfrac{1}{2}b)\mapsto(-t,\omega y,\omega b)\in hsp(\mathcal{Y}').$$

The Hamiltonian vector field of a quadratic polynomial $g(v)=t+yv+\frac{1}{2}vbv$ has the form

$$\mathrm{H}_g = -(\omega y+\omega bv)\nabla_v\in T(\mathcal{Y}'),$$

(here, v denotes the variable in $\mathcal{Y}'$).

1.6 Canonical symplectic spaces

1.6.1 Conjugations in a symplectic space

Let $(\mathcal{Y}, \omega)$ be a symplectic space. $\kappa \in L(\mathcal{Y})$ is called a conjugation if it is an involution and

$$y_1 \omega y_2 = -(\kappa y_1)\omega \kappa y_2, \qquad y_1, y_2 \in \mathcal{Y}.$$

Theorem 1.6.1 *κ is a conjugation iff κ is an involution and possesses a conjugate $\kappa' \in L(\mathcal{Y}')$ such that*

$$\kappa' \omega = -\omega \kappa.$$

The subspaces $\mathcal{Y}_\kappa$, $\mathcal{Y}_{-\kappa}$ are Lagrangian. They form a dual pair, by $\mathcal{Y}_\kappa \times \mathcal{Y}_{-\kappa} \ni (\xi, x) \mapsto \xi \omega x$. They will be denoted $\mathcal{X}' := \mathcal{Y}_\kappa$, $\mathcal{X} := \mathcal{Y}_{-\kappa}$. Thus we can identify $\mathcal{Y}$ with $\mathcal{X}' \oplus \mathcal{X}$ so that the symplectic form and the conjugation are given by

$$\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.6.5}$$

Note that the dual space $\mathcal{Y}$, defined as $\omega\mathcal{Y}$, is equal to $\mathcal{Y}' = \mathcal{X} \oplus \mathcal{X}'$. It is also a symplectic space with a conjugation. Its symplectic form and its conjugation are given by

$$-\omega^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \kappa' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Conversely, if $\mathcal{X}$ is a vector space, then $\mathcal{X}' \oplus \mathcal{X}$ equipped with (1.6.5) is a symplectic space with conjugation. It is often called the canonical symplectic space.

Theorem 1.6.2 *Let $\mathcal{Y}$ be a finite dimensional symplectic space.*
There exists a conjugation in $L(\mathcal{Y})$. In other words, there exists a vector space $\mathcal{X}$ such that $\mathcal{Y}$ is isomorphic to $\mathcal{X}' \oplus \mathcal{X}$.
2) If $\kappa_1, \kappa_2 \in L(\mathcal{Y})$ are two conjugations, then there exists $r \in Sp(\mathcal{Y})$ such that $r\kappa_1 r^{-1} = \kappa_2$.

1.6.2 Symplectic group in a canonical symplectic space

Let us assume until the end of this section that $\mathcal{Y} = \mathcal{X}' \oplus \mathcal{X}$ is a canonical symplectic space.

$a \in^(\mathcal{X}' \oplus \mathcal{X})$ iff

$$a = \begin{bmatrix} -c' & \alpha \\ -\beta & c \end{bmatrix},$$

where $\alpha \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}')$, $c \in L(\mathcal{X}, \mathcal{X})$, $\beta \in L_{\mathrm{s}}(\mathcal{X}', \mathcal{X})$.

$$\mathcal{X} \oplus \mathcal{X}' \ni (x, \xi) \mapsto b(x, \xi) = \frac{1}{2} x\alpha x + \xi c x + \frac{1}{2} \xi \beta \xi$$

is a Hamiltonian of a.

The Poisson bracket on $\mathcal{X} \oplus \mathcal{X}'$ equals

$$\{f, g\} = \nabla_x f \cdot \nabla_\xi g - \nabla_\xi f \cdot \nabla_x g, \quad f, g \in C^\infty(\mathcal{X} \oplus \mathcal{X}').$$

Let $r \in L(\mathcal{X}' \oplus \mathcal{X})$. We can write $r = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Recall that $r \in Sp(\mathcal{X}' \oplus \mathcal{X})$ iff $r' \in L(\mathcal{X} \oplus \mathcal{X}')$ and

$$1)\ r'\omega r = \omega, \qquad 2)\ r\omega^{-1}r' = \omega^{-1}.$$

Now 1) is equivalent to

$$a'd - c'b = 1,$$

$$c'a = a'c, \qquad d'b = b'd.$$

2) is equivalent to

$$ad' - bc' = 1,$$

$$ab' = ba', \qquad cd' = dc'.$$

If b is invertible, then we have the factorization

$$r = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e & 1 \end{bmatrix} \begin{bmatrix} 0 & b \\ b^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix},$$

where

$$e = db^{-1} = b'^{-1}d' \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}'), \qquad f = b^{-1}a = a'b'^{-1} \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}').$$

We can then define the so-called generating funtion

$$\mathcal{X} \times \mathcal{X} \ni (x_1, x_2) \mapsto S(x_1, x_2) := \frac{1}{2}x_1 e x_1 - (b^{-1}x_2)x_1 + \frac{1}{2}x_2 f x_2.$$

Note that if $r(\xi_1, x_1) = (\xi_2, x_2)$, then

$$\nabla_{x_1} S(x_1, x_2) = \xi_1, \quad \nabla_{x_2} S(x_1, x_2) = \xi_2.$$

1.6.3 Point transformations

The set of elements of $sp(\mathcal{X}' \oplus \mathcal{X})$ that commute with the conjugation κ is the image of the following injective homomorphism:

$$gl(\mathcal{X}) \ni c \mapsto \begin{bmatrix} -c' & 0 \\ 0 & c \end{bmatrix} \in sp(\mathcal{X}' \oplus \mathcal{X}). \tag{1.6.6}$$

$\mathcal{X} \oplus \mathcal{X}' \ni (x, \xi) \mapsto \xi c x = x c' \xi$ is a Hamiltonian of (1.6.6).

The set of elements of $Sp(\mathcal{X}' \oplus \mathcal{X})$ that commute with the conjugation κ is the image of the following injective homomorphism:

$$Gl(\mathcal{X}) \ni m \mapsto \begin{bmatrix} (m')^{-1} & 0 \\ 0 & m \end{bmatrix} \in Sp(\mathcal{X}' \oplus \mathcal{X}). \tag{1.6.7}$$

The exponential of (1.6.6) equals (1.6.7) with $m = \mathrm{e}^c$.

1.6.4 Transformations fixing $\mathcal{X}'$

The set of the elements of $sp(\mathcal{X}' \oplus \mathcal{X})$ that send $\mathcal{X}'$ to zero is the image of the following injective homomorphism of Lie algebra:

$$L_{\rm s}(\mathcal{X}, \mathcal{X}') \ni \alpha \mapsto \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \in sp(\mathcal{X}' \oplus \mathcal{X}) \tag{1.6.8}$$

A Hamiltonian of (1.6.8) is $\frac{1}{2}x\alpha x$.

The set of elements of $Sp(\mathcal{X}' \oplus \mathcal{X})$ that fix elements of $\mathcal{X}'$ is the image of the following injective homomorphism of groups

$$L_{\rm s}(\mathcal{X}, \mathcal{X}') \ni \alpha \mapsto \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \in Sp(\mathcal{X}' \oplus \mathcal{X}). \tag{1.6.9}$$

The exponential of (1.6.8) equals (1.6.9).

1.6.5 Lagrangian subspaces in a canonical symplectic space

Let $\mathcal{X}_1$ be a Lagrangian subspace in $\mathcal{X}' \oplus \mathcal{X}$ such that $\mathcal{X}_1 \cap \mathcal{X}' = \{0\}$. Then there exists $\alpha \in L(\mathcal{X}, \mathcal{X}')$ such that $(x, \xi) \in \mathcal{X}_1$, iff $\xi = \alpha x$.

Lemma 1.6.3 *$\mathcal{X}_1$ is Lagrangian iff α is symmetric.*

Proof.

$$0 = (\alpha x_1, x_1)\omega(\alpha x_2, x_2) = (\alpha x_2)x_1 - (\alpha x_1)x_2.$$

□

The function $\mathcal{X} \ni x \mapsto S(x) := \frac{1}{2}x\alpha x$ is called the generating function of $\mathcal{X}_1$. Note that

$$\mathcal{X}_1 = \{(\nabla S(x), x) \ : \ x \in \mathcal{X}\}.$$

1.7 (Pseudo-)unitary and (pseudo-)Kähler spaces

1.7.1 Hermitian forms

Let $(\mathcal{Z}, \mathcal{Z}^*)$ be a sesquilinear pair of complex vector space. Let $\chi \in L(\mathcal{Z}, \mathcal{Z}^*)$. We we will call χ a hermitian form, iff

$$y_1\chi y_2 = \overline{y_2\chi y_1}, \quad y_1, y_2 \in \mathcal{Z},$$

or, equivalently,

$$y\chi y \in \mathbf{R}, \quad y \in \mathcal{Z}.$$

The space of $\chi \in L(\mathcal{Z}, \mathcal{Z}^*)$ such that χ is hermitian will be denoted $L_{\rm h}(\mathcal{Z}, \mathcal{Z}^*)$. In other words

$$L_{\rm h}(\mathcal{Z}, \mathcal{Z}^{*\rm al}) = \{\chi \in L(\mathcal{Z}, \mathcal{Z}^*) \ : \ \chi = \chi^*\}.$$

We say that $\chi \in L_{\rm h}(\mathcal{Z}, \mathcal{Z}^{*\rm al})$ is nondegenerate iff ${\rm Ker}\chi = \{0\}$. If χ is nondegenerate, then $(\mathcal{Z}, {\rm Ran}\chi)$ is a sesquilinear dual pair. We will usually assume that that $\mathcal{Y}^* = {\rm Ran}\mathcal{Y}$. We can then define $\chi^{-1} \in L_{\rm h}(\mathcal{Z}^*, \mathcal{Z})$, which is also nondegenerate.

In the finite dimensional case, where the choice of $\mathcal{Z}^*$ is unique, if χ is nondegenerate, then ${\rm Ran}\chi$ automatically equals $\mathcal{Z}^*$.

We write $\chi > 0$ and we say that χ is positive definite if

$$y\chi y > 0 \ \Leftrightarrow \ y \in \mathcal{Z}, \ y \neq 0.$$

Clearly, if χ is positive definite, then it is non-degenerate on $\mathcal{Z}$.

A hermitian nondegenerate form χ is called a hermitian scalar product. A space equiped with such a form is called a pseudo-unitary space. It is called a unitary space if the scalar product is positive definite.

1.7.2 (Pseudo-)unitary groups

Let $(\mathcal{Z}_1, \chi_1)$, $(\mathcal{Z}_2, \chi_2)$ be (pseudo-)unitary spaces. Set $\mathcal{Z}_i^* := \chi_i \mathcal{Z}_i$.

$r \in L(\mathcal{Z}_1, \mathcal{Z}_2)$ is called a (pseudo-)unitary transformation if it is invertible and

$$(ry_1)\chi_2 r y_2 = y_1 \chi_1 y_2, \quad y_1, y_2 \in \mathcal{Z}.$$

Theorem 1.7.1 *r is (pseudo-)unitary iff $r^* \in L(\mathcal{Z}_2^*, \mathcal{Z}_1^*)$ and*

$$r^*\chi_2 r = \chi_1, \qquad r\chi_1^{-1} r^* = \chi_2^{-1}.$$

Suppose now that $(\mathcal{Z}, \chi)$ is a single (pseudo-)unitary space. The set of (pseudo-)unitary elements of $Gl(\mathcal{Z})$ forms a group denoted $U(\mathcal{Z})$ and called the (pseudo-)unitary group of $\mathcal{Z}$.

We say that $a \in L(\mathcal{Z})$ is (pseudo-)self-adjoint if

$$(ay_1)\chi y_2 = y_1 \chi a y_2, \qquad y_1, y_2 \in \mathcal{Z}.$$

We say that $a \in L(\mathcal{Z})$ is (pseudo-)anti-self-adjoint if

$$(ay_1)\chi y_2 = -y_1 \chi a y_2, \qquad y_1, y_2 \in \mathcal{Z}.$$

Theorem 1.7.2 *1) $a \in L(\mathcal{Z})$ is (pseudo-)self-adjoint iff $a^* \in L(\mathcal{Z}^*)$ and*

$$a^*\chi = \chi a.$$

2) a is (pseudo-)anti-self-adjoint iff $a^* \in L(\mathcal{Z}^*)$ and

$$a^*\chi = -\chi a.$$

3) a is anti-(pseudo-)self-adjoint iff ${\rm i}a$ is (pseudo-)self-adjoint.

The set of (pseudo)-anti-self-adjoint elements of $gl(\mathcal{Z})$ forms a Lie algebra, denoted $u(\mathcal{Z})$ and called the (pseudo)-unitary Lie algebra of $\mathcal{Z}$.

1.7.3 Hilbert spaces

If $\mathcal{Z}, \chi$ is a unitary space, then $\|y\| := \sqrt{y\chi y}$ defines a norm. A complete unitary space is called a (complex) Hilbert space.

If $\mathcal{Z}, \chi$ is a Hilbert space, then χ is an isomorphism of the Hilbert spaces $(\mathcal{Z}, \chi)$ and $(\mathcal{Z}^*, \chi^{-1})$. We will thus identify $\mathcal{Z}$ and $\mathcal{Z}^*$. We will write $(y_1|y_2)$ instead of $y_1\chi y_2$ and a^* instead of $\chi a^* \chi^{-1}$.

If $(\mathcal{V}_i)_{i\in I}$ is a family of Hilbert spaces, then $\overset{\circ}{\oplus}_{i\in I}\mathcal{V}_i$ has the structure of a unitary space. We write

$$\underset{i\in I}{\oplus} \mathcal{V}_i := \left(\overset{\circ}{\oplus}_{i\in I}\mathcal{V}_i\right)^{\mathrm{cpl}}.$$

Note that Hilbert spaces possess their own terminology concerning unbounded operators (closed, self-adjoint, essentially self-adjoint, symmetric operators, etc.) which at some points differs from the purely algebraic terminology which we use outside of the context of Hilbert spaces.

Theorem 1.7.3 (Spectral Theorem) *Let a be a bounded operator such that $a^* = a$. Then there exists a unique *-homomorphism*

$$\mathcal{L}^\infty(\sigma(a)) \ni f \mapsto f(a) \in B(\mathcal{Z})$$

such that $f(a) = a$ if $f(x) = x$.

Theorem 1.7.4 (Polar decomposition) *Suppose that b is a closed operator on $\mathcal{Z}$ such that* $\mathrm{Ker}b = \{0\}$ *and* $\mathrm{Ran}b$ *is dense in $\mathcal{Z}$. Then there exists a unique unitary operator u and a unique positive operator k such that $b = wk$.*

An operator $a \in B(\mathcal{Z}, \mathcal{W})$ belongs to $l^p(\mathcal{Z}, \mathcal{W})$, if $\mathrm{Tr}(a^*a)^{\frac{p}{2}} < \infty$. We will write $l^p(\mathcal{Z}) := l^p(\mathcal{Z}, \mathcal{Z})$.

The set of self-adjoint operators in $B(\mathcal{Z})$ is denoted $B_{\mathrm{h}}(\mathcal{Z})$ and in $l^p(\mathcal{Z})$ by $l^p_{\mathrm{h}}(\mathcal{Z})$.

1.7.4 Complexification of a real Hilbert space

Let $(\mathcal{X}, \cdot)$ be a real Hilbert space. Then $(\mathbf{C}\mathcal{X}, (\cdot|\cdot))$ is a complex Hilbert space with the scalar product

$$(x_1 + \mathrm{i}y_1|x_2 + \mathrm{i}y_2) := x_1x_2 + y_1y_2 + \mathrm{i}x_1y_2 - \mathrm{i}x_2y_1.$$

For $r \in L(\mathcal{X})$, let $\mathbf{C}r$ denote its extension to $L(\mathbf{C}\mathcal{X})$. Then

$$\mathbf{C}r' = (\mathbf{C}r)^*. \tag{1.7.10}$$

In particular, if $r' = r$, then $(\mathbf{C}r)^* = \mathbf{C}r$. If $r \in O(\mathcal{X})$, then $\mathbf{C}r \in U(\mathbf{C}\mathcal{X})$.

Note that in the sequel we will use the same letter to denote the operator r on $\mathcal{X}$ and its extension $\mathbf{C}r$ to $\mathbf{C}\mathcal{X}$.

1.7.5 (Pseudo-)Kähler spaces

We say that $(\mathcal{Y},\alpha,\omega,\mathbf{i})$ is a pseudo-Kähler space if $\mathcal{Y}$ is a real vector space, $\alpha\in L_{\mathrm{s}}(\mathcal{Y},\mathcal{Y}^{/\mathrm{al}})$, $\omega\in L_{\mathrm{a}}(\mathcal{Y},\mathcal{Y}^{/\mathrm{al}})$ are nondegenerate, $\mathbf{i}$ is an anti-involution on $\mathcal{Y}$ and $\omega\mathbf{i}=\alpha$. If $\alpha>0$, then we say that it is a Kähler space.

We will use the dual space $\mathcal{Y}'=\omega\mathcal{Y}=\alpha\mathcal{Y}$.

We say that a pair $(\omega,\mathbf{i})$ is (pseudo-)Kähler if there exists a (pseudo-)Euclidean form α such that $(\alpha,\omega,\mathbf{i})$ is (pseudo-)Kähler. If we are fix ω, then we say that an anti-involution $\mathbf{i}$ is (pseudo-)Kähler iff $(\omega,\mathbf{i})$ is (pseudo-)Kähler. If we fix an anti-involution $\mathbf{i}$, then we say that a symplectic form ω is (pseudo-)Kähler if $(\omega,\mathbf{i})$ is (pseudo-)Kähler.

We introduce similar definitions for the pairs $(\alpha,\mathbf{i})$ and $(\omega,\mathbf{i})$.

Theorem 1.7.5 *Let ω be a symplectic form.*
1) $\mathbf{i}\in Sp(\mathcal{Y})\cap sp(\mathcal{Y})$, iff $\mathbf{i}$ is a anti-involution such that $(\omega,\mathbf{i})$ is pseudo-Kähler.
2) Let $\mathbf{i}$ be an anti-involution. The pair $(\omega,\mathbf{i})$ is pseudo-Kähler iff one of the following equivalent conditions holds:
(a) $\omega\mathbf{i}$ is symmetric, or equivalently, $\mathbf{i}\in sp(\mathcal{Y})$.
(b) $\mathbf{i}\in Sp(\mathcal{Y})$;

Theorem 1.7.6 *Let α be a (pseudo-)Euclidean form.*
1) $\mathbf{i}\in O(\mathcal{Y})\cap o(\mathcal{Y})$, iff $\mathbf{i}$ is an antiinvolution such that $(\alpha,\mathbf{i})$ is (pseudo-)Kähler.
2) Let $\mathbf{i}$ be an anti-involution. Then the pair $(\alpha,\mathbf{i})$ is (pseudo-)Kähler iff one of the following equivalent conditions holds:
(a) $\alpha\mathbf{i}$ is antisymmetric, or equivalently, $\mathbf{i}\in o(\mathcal{Y})$.
(b) $\mathbf{i}\in O(\mathcal{Y})$;

Theorem 1.7.7 *Let α be a (pseudo-)Euclidean form and ω a symplectic form. Then the pair (α,ω) is (pseudo-)Kähler iff*

$$(\omega^{-1}\alpha)^2=-1.$$

Theorem 1.7.8 *1) Let the triple $(\mathcal{Y},\alpha,\omega,\mathbf{i})$ be a (pseudo-)Kähler space. Define $\chi\in L(\mathcal{Y},\mathbf{C}\mathcal{Y}')$ by setting*

$$\begin{aligned}\chi:=\alpha+\mathrm{i}\omega\ &=\omega(\mathbf{i}+\mathrm{i})=(-\mathbf{i}'+\mathrm{i})\omega\\ &=\alpha(1-\mathrm{i}\mathbf{i})=(1+\mathrm{i}\mathbf{i}')\alpha.\end{aligned}\tag{1.7.11}$$

Then, $\mathrm{Ran}\chi=\overline{(\mathcal{Y}^{\mathbf{C}})'}$ and $(\mathcal{Y}^{\mathbf{C}},\chi)$ is a (pseudo-)unitary space.
2) Let $(\mathcal{V},\chi)$ be a (pseudo-)unitary space. Then $(\mathcal{V}_{\mathbf{R}},\mathrm{Re}\chi,\mathrm{Im}\chi,\mathrm{i})$ is a (pseudo-)Kähler space.

1.7.6 Complexification of a Kähler space

Consider a Kähler space $(\mathcal{Y}, \cdot, \omega, \mathbf{i})$. The space $\mathbf{C}\mathcal{Y}$ has a natural unitary structure determined by the Euclidean structure of $\mathcal{Y}$ (see Subsection **??**)

$$((x_1 + \mathrm{i}y_1)|(x_2 + \mathrm{i}y_2)) = x_1 \cdot x_2 + y_1 \cdot y_2 - \mathrm{i}y_1 \cdot x_2 + \mathrm{i}y_1 \cdot x_2, \qquad x_i, y_i \in \mathcal{Y}. \tag{1.7.12}$$

$$\mathrm{i}(x + \mathrm{i}y) = -y + \mathrm{i}x.$$

Recall from Subsection 1.2.3 that we can introduce the subspaces (determined by the anti-involution $\mathbf{i}$ in $\mathcal{Y}$)

$$\mathcal{Z} := \{y - \mathrm{i}\mathbf{i}y, \quad y \in \mathcal{Y}\}$$

such that

$$\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$$

Note that the subspaces $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ are complex and orthogonal to one another in terms of the scalar product (1.7.12). The operator $\mathbf{i}$ is unitary on $\mathbf{C}\mathcal{Y}$. If we use the decomposition $\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$, then

$$\mathbf{Ci} = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}.$$

The elements of the Kähler structure on $\mathcal{Y}$ can be expressed using the scalar product $(\cdot|\cdot)$ and the map $\mathbf{i}$ on $\mathbf{C}\mathcal{Y}$:

$$y_1 \cdot y_2 = (y_1|y_2), \quad y_1, y_2 \in \mathcal{Y},$$

$$y_1 \omega y_2 = -(y_1|\mathbf{i}y_2), \quad y_1, y_2 \in \mathcal{Y}.$$

The scalar product $\cdot$ and the symplectic form ω have the unique bilinear extensions to $\mathbf{C}\mathcal{Y}$. In terms of the scalar product $(\cdot|\cdot)$ they are given by

$$v_1 \cdot v_2 = (\overline{v_1}|v_2), \quad v_1, v_2 \in \mathbf{C}\mathcal{Y},$$

$$v_1 \omega v_2 = -(\overline{v_1}|\mathbf{i}v_2), \quad v_1, v_2 \in \mathbf{C}\mathcal{Y}.$$

Bilinear extensions of $(\cdot, \omega, \mathbf{i})$ to $\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$ are also given by

$$\begin{aligned}
&(z_1 + \overline{z_2})(w_1 + \overline{w_2}) := ((z_2|w_1) + (w_2|z_1)), \\
&(z + \overline{z})\omega(w + \overline{w}) := \tfrac{1}{\mathrm{i}}((z_2|w_1) - (w_2|z_1)), \\
&\mathbf{i}(z_1 + \overline{z_2}) := \mathrm{i}z_1 - \mathrm{i}\overline{z_2}, \qquad z_1, z_2, w_1, w_2 \in \mathcal{Z}.
\end{aligned}$$

Note that both $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ are isotropic subspaces for $\cdot$ and ω. Moreover,

$$\overline{z_1} \cdot z_2 = (z_1|z_2), \quad z_1, z_2 \in \mathcal{Z},$$

$$\overline{z_1}\omega z_2 = -\mathrm{i}(z_1|z_2), \qquad z_1, z_2 \in \mathcal{Z}.$$

Let $\mathcal{Z}$ be a Hilbert space. Set

$$\mathcal{Y} = \{(z, \overline{z}) \ : \ z \in \mathcal{Z}\}.$$

Then we have a natural identification $\mathcal{Z} \oplus \overline{\mathcal{Z}} = \mathbf{C}\mathcal{Y}$. $\mathcal{Y}$ has a natural Kähler structure:

$$(z + \overline{z})(w + \overline{w}) := 2\mathrm{Re}(z|w),$$

$$(z + \overline{z})\omega(w + \overline{w}) := 2\mathrm{Im}(z|w),$$

$$\mathbf{i}(z + \overline{z}) := \mathrm{i}z - \mathrm{i}\overline{z}, \qquad z, w \in \mathcal{Z}.$$

The above construction is converse to the construction of $\mathcal{Z}$ starting from $\mathcal{Y}$.

Remark 1.7.9 *We have a natural unitary identification of $\mathcal{Y}$ with $\mathcal{Z}$: if $y \in \mathcal{Y}$, then we identify it with $(z, \overline{z}) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}$ by*

$$z = \frac{1}{\sqrt{2}}(y - \mathrm{i}\mathbf{i}y) \quad \overline{z} = \frac{1}{\sqrt{2}}(y + \mathrm{i}\mathbf{i}y), \qquad y = \frac{1}{\sqrt{2}}(z + \overline{z}) \tag{1.7.13}$$

Note, however, that we will avoid using this identification.

1.7.7 Unitary group in a Kähler space

Note that

$$\begin{aligned} U(\mathcal{Y}^{\mathbf{C}}) &= O(\mathcal{Y}) \cap Sp(\mathcal{Y}) \\ &= O(\mathcal{Y}) \cap GL(\mathcal{Y}^{\mathbf{C}}) \\ &= Sp(\mathcal{Y}) \cap GL(\mathcal{Y}^{\mathbf{C}}). \end{aligned}$$

$$\begin{aligned} u(\mathcal{Y}^{\mathbf{C}}) &= o(\mathcal{Y}) \cap sp(\mathcal{Y}) \\ &= o(\mathcal{Y}) \cap gl(\mathcal{Y}^{\mathbf{C}}) \\ &= sp(\mathcal{Y}) \cap gl(\mathcal{Y}^{\mathbf{C}}). \end{aligned}$$

In what follows we extend transformations from $\mathcal{Y}$ to $\mathbf{C}\mathcal{Y}$ and use the decomposition $\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$.

$a \in u(\mathcal{Y}^{\mathbf{C}})$ iff

$$a = \begin{bmatrix} \mathrm{i}h & 0 \\ 0 & -\mathrm{i}\overline{h} \end{bmatrix},$$

with $h \in B_{\mathrm{h}}(\mathcal{Z})$.

$r \in U(\mathcal{Y}^{\mathbf{C}})$ iff

$$r = \begin{bmatrix} p & 0 \\ 0 & \overline{p} \end{bmatrix},$$

with $p^*p = pp^* = 1$ (in other words, $p \in U(\mathcal{Z})$).

1.7.8 Symplectic group in a Kähler space

We assume that $\mathcal{Y}$ is a complete Kähler space.

Recall that $a \in sp(\mathcal{Y})$ iff $a'\omega + \omega a = 0$. If we identify $\mathcal{Y}$ and $\mathcal{Y}'$, and retain the notation a' for the adjoint of a in the sense of real Hilbert spaces, then this is equivalent to $a'\mathbf{i} + \mathbf{i}a = 0$. If we extend a to $\mathbf{C}\mathcal{Y}$ and use the notation a^* for the adjoint of a in the sense of complex Hilbert spaces, then this is equivalent to $a^*\mathbf{i} + \mathbf{i}a = 0$.

In the following matrix notation we will use the decomposition $\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$.

Theorem 1.7.10 *$a \in sp(\mathcal{Y}) \cap B(\mathcal{Y})$ iff its extension to $\mathbf{C}\mathcal{Y}$ equals*

$$a = \begin{bmatrix} \mathrm{i}h & d \\ \overline{d} & \mathrm{i}\overline{h} \end{bmatrix},$$

such that $h \in B_h(\mathcal{Y})$, $d \in B_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$.

Recall that $r \in Sp(\mathcal{Y}) \cap B(\mathcal{Y})$ iff

$$1)\ r'\omega r = \omega, \qquad 2)\ r\omega^{-1}r' = \omega^{-1}.$$

In terms of the extension of r to $\mathbf{C}\mathcal{Y}$, this is equivalent to

$$1)\ r^*\mathbf{i}r = \mathbf{i}, \qquad 2)\ r\mathbf{i}r^* = \mathbf{i}.$$

Theorem 1.7.11 *$r \in Sp(\mathcal{Y}) \cap B(\mathcal{Y})$ iff its extension to $\mathbf{C}\mathcal{Y}$ equals*

$$\mathbf{C}r = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix},$$

and the following conditions hold:

$$\textit{conditions implied by 1):} \qquad \begin{array}{ll} p^*p - q'\overline{q} = 1, & p^*q - q'\overline{p} = 0, \\ -p'\overline{q} + q^*p = 0, & -p'\overline{p} + q^*q = -1. \end{array}$$

(Note that eg the first column implies the second).

$$\textit{conditions implied by 2):} \qquad \begin{array}{ll} pp^* - qq^* = 1, & \overline{p}q' - qp' = 0, \\ \overline{q}p^* - \overline{p}q^* = 0, & \overline{q}q' - \overline{p}p' = -1. \end{array}$$

(Again, the first column implies the second).

Note that we have

$$pp^* \geq 1, \quad p^*p \geq 1,$$

hence p^{-1} exists.

Set

$$d = q\overline{p}^{-1} = (p^*)^{-1}q' \in B_{\rm s}(\overline{\mathcal{Z}}, \mathcal{Z}), \quad c = p^{-1}q = q'(p^*)^{-1} \in B_{\rm s}(\overline{\mathcal{Z}}, \mathcal{Z}).$$

Then

$$c^*c < 1, \quad d^*d < 1,$$

and

$$r = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (p^*)^{-1} & 0 \\ 0 & \overline{p} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \overline{c} & 1 \end{bmatrix}.$$

We have the identities:

$$p = 1_{\mathcal{Z}} r 1_{\mathcal{Z}} = \tfrac{r + r^{*(-1)}}{2} 1_{\mathcal{Z}} = 1_{\mathcal{Z}} \tfrac{r + r^{*(-1)}}{2},$$

$$q = 1_{\mathcal{Z}} r 1_{\overline{\mathcal{Z}}} = \tfrac{r - r^{*(-1)}}{2} 1_{\overline{\mathcal{Z}}} = 1_{\mathcal{Z}} \tfrac{r - r^{*(-1)}}{2},$$

$$d = \tfrac{r^*r-1}{r^*r+1} 1_{\mathcal{Z}} = 1_{\overline{\mathcal{Z}}} \tfrac{r^*r-1}{r^*r+1}$$

$$c = \tfrac{r^*r-1}{r^*r+1} 1_{\mathcal{Z}} = 1_{\overline{\mathcal{Z}}} \tfrac{r^*r-1}{r^*r+1}$$

Theorem 1.7.12 *Let $r \in Sp(\mathcal{Y}) \cap B(\mathcal{Y})$. Then there exists a unique $k \in Sp(\mathcal{Y}) \cap B(\mathcal{Y})$ and $u \in U(\mathcal{Y}^{\mathbf{C}})$ such that $k' = k$, $k > 0$ and $r = vk$. v and k are given by the polar decomposition of r.*

Proof. Set $t := rr'$. Then $t' = t$, $t > 0$ and $t \in Sp(\mathcal{Y})$, which means $\mathbf{i}^{-1}t\mathbf{i} = -t$. Take $k := \sqrt{t}$. Then k also satisfies $k' = k$, $k > 0$ and $\mathbf{i}^{-1}k\mathbf{i} = -k$.

Set $v := rt^{-1}$. t is symplectic, hence so is v. v is orthogonal (by the properties of the polar decomposition). Therefore v is unitary. □

1.7.9 Positive symplectic transformations in a Kähler space

$a \in sp(\mathcal{Y}) \cap B(\mathcal{Y})$ satisfying $a = a'$ is always of the form

$$a = \begin{bmatrix} 0 & d \\ \overline{d} & 0 \end{bmatrix}, \tag{1.7.14}$$

where $d \in B_{\rm s}(\overline{\mathcal{Z}}, \mathcal{Z})$.

If $r \in Sp(\mathcal{Y}) \cap B(\mathcal{Y})$ satisfies $r = r'$ and $r > 0$, then for some $p \in B_{\rm h}(\mathcal{Z})$, $q \in B_{\rm s}(\mathcal{Z}, \overline{\mathcal{Z}})$ satisfying $p > 0$ and $p^2 - qq^* = 1$ we have

$$r = \begin{bmatrix} p & q \\ q^* & \overline{p} \end{bmatrix} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^{-1} & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c^* & 1 \end{bmatrix}.$$

with
$$c = p^{-1}q = qp^{-1} \in B_{\rm s}(\overline{\mathcal{Z}}, \mathcal{Z}),$$
satisfying
$$c^*c < 1. \tag{1.7.15}$$
$$p = \tfrac{r+r^{-1}}{2}1_{\mathcal{Z}} = 1_{\mathcal{Z}}\tfrac{r+r^{-1}}{2},$$
$$q = \tfrac{r-r^{-1}}{2}1_{\overline{\mathcal{Z}}} = 1_{\mathcal{Z}}\tfrac{r-r^{-1}}{2},$$
$$c = \tfrac{r^2-1}{r^2+1}1_{\mathcal{Z}} = 1_{\overline{\mathcal{Z}}}\tfrac{r^2-1}{r^2+1}.$$

Conversely, if c satisfies (1.7.15), then it corresponds to a unique $r \in Sp(\mathcal{Y}) \cap B(\mathcal{Y})$ satisfying $r' = r$, $r > 0$, which can be obtained from the formula

$$r = (1 - c - \overline{c})^{\frac{1}{2}}(1 + c + \overline{c})^{-\frac{1}{2}} = \begin{bmatrix} (1+c^*c)^{-1} & (1+cc^*)^{-1}c \\ c^*(1+cc^*)^{-1} & (1+cc^*)^{-1} \end{bmatrix}^{\frac{1}{2}}.$$

If $a \in sp(\mathcal{Y})$, $a' = a$, then $r = \mathrm{e}^a$ belongs to $Sp(\mathcal{Y}) \cap B(\mathcal{Y})$ and satisfies $r' = r$, $r > 0$. If a is equal (1.7.14), then
$$r = \begin{bmatrix} \mathrm{ch}\sqrt{d^*d} & \frac{\mathrm{sh}\sqrt{dd^*}}{\sqrt{dd^*}}d \\ d^*\frac{\mathrm{sh}\sqrt{dd^*}}{\sqrt{dd^*}} & \mathrm{ch}\sqrt{dd^*} \end{bmatrix}.$$

1.7.10 Kähler structures in a symplectic space

Suppose that $(\mathcal{Y}, \cdot, \omega, \mathbf{i})$ is a complete Kähler space. Thus $\mathbf{i}^2 = -1$ and $\mathbf{i}' = -\mathbf{i}$.

Let $\mathbf{j}$ be another anti-involution such that $(\mathbf{j}, \omega)$ is Kähler. This means

$$\mathbf{j}^2 = -1, \quad (\mathbf{ij})' = \mathbf{ij}, \quad \mathbf{ij} < 0.$$

Set $k := -\mathbf{ij}$. Then
$$k = k', \quad \mathbf{i}^{-1}k\mathbf{i} = k^{-1}, \quad k > 0. \tag{1.7.16}$$
Conversely, if k satisfies (1.7.16), and $\mathbf{j} := \mathbf{i}k$, then $(\mathbf{j}, \omega)$ is Kähler.

Note also, that for any $t > 0$ we can define the power k^t, which will satisfy all the conditions of (1.7.16). In particular, set $r := k^{\frac{1}{2}}$. Then r satisfies

$$\mathbf{i}^{-1}r\mathbf{i} = r^{-1}, \quad r' = r, \quad r > 0, \quad r^{-1}\mathbf{i}r = \mathbf{j}.$$

Thus r is a positive element of the symplectic group intertwining $\mathbf{i}$ and $\mathbf{j}$.

Theorem 1.7.13 *Suppose that* $\mathbf{j}$ *and* k *are as above. Then there exists an involution* κ *such that*
$$\kappa\mathbf{i}\kappa = -\mathbf{i}, \quad \kappa\mathbf{j}\kappa = -\mathbf{j}, \quad \kappa k\kappa = k.$$
(Clearly, two of the above conditions imply the third).

If $\mathrm{Ker}(\mathbf{i}-\mathbf{j})=\{0\}$ *or, equivalently,* $\mathrm{Ker}(1-k)=\{0\}$*, then there is a canonical choice of* κ*.*

The operator k *maps* $\mathcal{Y}_\kappa$ *and* $\mathcal{Y}_{-\kappa}$ *into themselves. Set* $m := k\big|_{\mathcal{Y}_\kappa}$ *and identify* $\mathcal{Y}_\kappa$ *with* $\mathcal{Y}_{-\kappa}$ *using* $\mathbf{i}$*. Then* $m' = m$*,* $m > 0$ *and in the direct sum* $\mathcal{Y} = \mathcal{Y}_\kappa \oplus \mathcal{Y}_{-\kappa}$ *we have*

$$\mathbf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 & -m \\ m^{-1} & 0 \end{bmatrix}, \quad k = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}.$$

Proof. Set $\mathcal{Y}_1 = \mathrm{Ker}(k-1)$, $\mathcal{Y}^+ := \mathrm{Ran}1_{]1,\infty[}(k)$, $\mathcal{Y}^- = \mathrm{Ran}1_{]0,1[}(k)$. $\mathcal{Y}_1$ is invariant wrt $\mathbf{i}$, so it can be split as $\mathcal{Y}_1^\pm$ with $\mathbf{i}\mathcal{Y}_1^+ = \mathcal{Y}_1^-$. We set $\mathcal{Y}_{\pm\kappa} := \mathcal{Y}_1^\pm \oplus \mathcal{Y}^\pm$. □

1.7.11 Unitarization

Theorem 1.7.14 *Let* $(\mathcal{Y},\cdot)$ *be a real Hilbert space.*
1) Let a *be an essentially anti-self-adjoint operator on* $\mathcal{Y}$ *such that* $\mathrm{Ker}a = \{0\}$*. Then there exists a unique anti-involution* $\mathbf{i}$ *on* $\mathcal{Y}$ *such that* $(\cdot,\mathbf{i})$ *is Kähler and*

$$-\mathbf{i}a = -a\mathbf{i} > 0.$$

2) Let $r \in O(\mathcal{Y})$ *such that* $\mathrm{Ker}(r \pm 1) = \{0\}$*. Then there exists a unique anti-involution* $\mathbf{i}$ *and a unique self-adjoint operator* c *such that* $(\cdot,\mathbf{i})$ *is Kähler,* $-1 < c < 1$*,* $c\mathbf{i} = \mathbf{i}c$ *and* $r = c + \mathbf{i}\sqrt{1-c^2}$*.*

Proof. 1) By the polar decomposition, $a = b\mathbf{i}$ with $b > 0$ self-adjoint and $\mathbf{i} \in O(\mathcal{Y})$. The relation $a' = -a$ implies $b\mathbf{i} = \mathbf{i}b$.

2) $c := \frac{1}{2}(r + r')$ and $d := \frac{1}{2}(r - r')$ are self-adjoint and anti-self-adjoint operators with trivial kernels. Clearly, $r = c + d$ and $r' = c - d$. From $rr' = 1$ and $r'r = 1$ we obtain $c^2 + d^2 = 1$. By 1), $d = e\mathbf{i} = \mathbf{i}e$ for a uniquely determined anti-involution $\mathbf{i}$ and a positive self-adjoint e. Hence $e = \sqrt{1-c^2}$. □

If a is essentially anti-self-adjoint on $\mathcal{Y}$, and $\mathcal{Y}_1 := \mathrm{Ker}a^\perp$, then a maps $\mathcal{D}(a) \cap \mathcal{Y}_1$ into $\mathcal{Y}_1$, is essentially anti-self-adjoint on $\mathcal{D}(a) \cap \mathcal{Y}_1$ and $a\big|_{\mathcal{Y}_1}$ has a trivial kernel.

Let $r \in O(\mathcal{Y})$. Set $\mathcal{Y}_\pm := \mathrm{Ker}(r \mp 1)$ and $\mathcal{Y}_1 := (\mathcal{Y}_- + \mathcal{Y}_+)^\perp$. Then $\mathcal{Y}_1$ is a subspace invariant wrt r and $r\big|_{\mathcal{Y}_1}$ has a trivial kernel.

Thus the assumptions on a and r in Theorem 1.7.14 are not really restrictive.

Theorem 1.7.15 *Let* $(\mathcal{Y},\omega)$ *be a symplectic space.*
1) Let $d \in L(\mathcal{Y})$ *such that* $\omega d \in L_s(\mathcal{Y})$ *and* $\omega d > 0$*. Let* $\mathcal{Y}_1$ *be the completion of* $\mathcal{Y}$ *wrt* ωd*. Let us define the operator* d^{-1} *with the domain* $d\mathcal{Y}$*. Clearly,* d^{-1} *is anti-symmetric.*

Assume that it is essentially anti-self-adjoint. Then there exists a unique anti-involution $\mathbf{i} \in B(\mathcal{Y}_1)$ *such that* $(\omega, \mathbf{i})$ *is Kähler and*

$$d\mathbf{i} = \mathbf{i}d.$$

2) Suppose in addition that

$$y_1 \omega d y_1 + y_2 \omega d y_2 > |y_1 \omega y_2|, \quad y_1, y_2 \in \mathcal{Y}. \tag{1.7.17}$$

Then $-\mathbf{i}d \geq \frac{1}{2}$.

Proof. 1) d^{-1} satisfies the assumptions of Theorem 1.7.14. Hence there exists a unique anti-involution $\mathbf{i}$ such that $(\omega d, \mathbf{i})$ is Kähler and $d^{-1}\mathbf{i} = \mathbf{i}d^{-1} > 0$. This implies in particular that $d\mathbf{i} = \mathbf{i}d$.

We have

$$\begin{aligned} y_1 \omega \mathbf{i} y_2 &= y_1 \omega d d^{-1} y_2 = (d^{-1}\mathbf{i} y_2) \omega d y_1 \\ &= y_2 \omega d d^{-1} \mathbf{i} y_1 = y_2 \omega \mathbf{i} y_1, \quad y_1, y_2 \in \mathcal{D}(d^{-1}), \\ y \omega \mathbf{i} y &= y \omega d \mathbf{i} d^{-1} y > 0, \quad y \in \mathcal{D}(d^{-1}). \end{aligned}$$

Therefore, $\omega \mathbf{i}$ is a Euclidean form. Thus, $(\omega, \mathbf{i})$ is Kähler.

2) We set $y_1 = y$ and $y_2 = \mathbf{i}y$ in (1.7.17). Then (1.7.17) yields

$$2 y \omega \mathbf{i}(-\mathbf{i}d) y > y \omega \mathbf{i} y.$$

□

1.7.12 Pseudo-Kähler spaces with conjugation

Suppose that $(\mathcal{X}, \gamma)$ is a (pseudo-)Euclidean vector space. Then $\mathcal{X} \oplus \mathcal{X}$ is naturally a (pseudo-)Kähler space with a conjugation:

$$\mathbf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in L(\mathcal{X} \oplus \mathcal{X}),$$

$$\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in L_{\mathrm{a}}(\mathcal{X} \oplus \mathcal{X}), \quad \alpha = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} \in L_{\mathrm{s}}(\mathcal{X} \oplus \mathcal{X}).$$

Conversely, suppose that $\mathcal{Y}$ is a (pseudo-)Kähler space with a conjugation κ. Let $\mathcal{Y} = \mathcal{Y}_\kappa \oplus \mathcal{Y}_{-\kappa}$ be the corresponding decomposition of $\mathcal{Y}$. Clearly, we have a (pseudo-)Euclidean structure on $\mathcal{Y}_\kappa$. If we rename $\mathcal{Y}_\kappa$ to be $\mathcal{X}$, then $\mathcal{Y}$ is naturally isomorphic to $\mathcal{X} \oplus \mathcal{X}$.

$u(\mathcal{X}\oplus\mathcal{X})$ consists of matrices

$$a = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

satisfying

$$c' = c, \qquad d' = -d.$$

$U(\mathcal{X}\oplus\mathcal{X})$ consists of matrices

$$r = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

satisfying

$$c'c + d'd = 1, \qquad c'd - d'c = 0.$$

1.7.13 Complexification of a Kähler space with conjugation

Suppose that $(\mathcal{X},\cdot)$ is a Euclidean space and $\mathcal{Y} := \mathcal{X}\oplus\mathcal{X}$ is a Kähler space with a conjugation, as described in the previous subsection. We can consider $\mathbf{C}\mathcal{Y}$ and define subspaces $\mathcal{Z},\overline{\mathcal{Z}}\subset\mathcal{Y}$, as in Subsection **??**.

We have two distinct direct sum decompositions of $\mathbf{C}\mathcal{Y}$:

$$\mathbf{C}\mathcal{Y} = \mathcal{Z}\oplus\overline{\mathcal{Z}} = \mathbf{C}\mathcal{X}\oplus\mathbf{C}\mathcal{X}.$$

As usual, let $1_{\mathcal{Z}}$, $1_{\overline{\mathcal{Z}}}$ denote the orthogonal projections from $\mathbf{C}\mathcal{Y}$ onto $\mathcal{Z}$ and $\overline{\mathcal{Z}}$. Recall that $\frac{1+\kappa}{2}$ and $\frac{1-\kappa}{2}$ are orthogonal projections from $\mathbf{C}\mathcal{Y}$ onto $\mathbf{C}\mathcal{X}\oplus\{0\}$ and $\{0\}\oplus\mathbf{C}\mathcal{X}$.

Theorem 1.7.16 *$1_{\mathcal{Z}}\frac{1\pm\kappa}{\sqrt{2}}$ and $1_{\overline{\mathcal{Z}}}\frac{1\pm\kappa}{\sqrt{2}}$ are partial isometries.*

1.8 Tensor products

1.8.1 Tensor products

Let $\mathcal{V},\mathcal{W}$ be vector spaces over $\mathbf{K}$. Let $\mathcal{Z}$ be the vector space over $\mathbf{K}$ of finite linear combinations of $(v,w)\in\mathcal{V}\times\mathcal{W}$. Let $\mathcal{Z}_0$ be its subspace of $\mathcal{Z}$ spanned by elements of the form $(v,w_1+w_2)-(v,w_1)-(v,w_2)$, $(v_1+v_2,w)-(v_1,w)-(v_2,w)$, $(\lambda v,w)-\lambda(v,w)$, $(v,\lambda w)-\lambda(v,w)$, where $\lambda\in\mathbf{K}$. Then we define

$$\mathcal{V}\overset{\circ}{\otimes}\mathcal{W} := \mathcal{Z}/\mathcal{Z}_0$$

– the algebraic tensor product of $\mathcal{V}$ and $\mathcal{W}$. If one of the spaces is finite dimensional then we can write $\otimes$ instead of $\overset{\circ}{\otimes}$.

$v\otimes w:=[(v,w)]$ defines the bilinear map

$$\mathcal{V}\times\mathcal{W}\ni(v,w)\mapsto v\otimes w\in\mathcal{V}\overset{\circ}{\otimes}\mathcal{W}.$$

If $\{v_i\ :\ i\in I\}$ is a Hamel basis of $\mathcal{V}$ and $\{w_j\ :\ j\in J\}$ is a Hamel basis of $\mathcal{W}$, then $\{v_i\otimes w_j\ :\ (i,j)\in I\times J\}$ is a Hamel basis of $\mathcal{V}\times\mathcal{W}$.

Note that we have natural isomorphisms

$$\mathcal{V}\simeq\mathbf{K}\overset{\circ}{\otimes}\mathcal{V}\simeq\mathcal{V}\overset{\circ}{\otimes}\mathbf{K}$$

and

$$\mathcal{V}_1\overset{\circ}{\otimes}(\mathcal{V}_2\overset{\circ}{\otimes}\mathcal{V}_3)\simeq(\mathcal{V}_1\overset{\circ}{\otimes}\mathcal{V}_2)\overset{\circ}{\otimes}\mathcal{V}_3. \tag{1.8.18}$$

(1.8.18) will be denoted simply $\mathcal{V}_1\overset{\circ}{\otimes}\mathcal{V}_2\overset{\circ}{\otimes}\mathcal{V}_3$. The tensor multiplication $\otimes$ is associative.

If $\mathcal{V}$, $\mathcal{W}$ are real vector spaces, then we have the identification

$$\mathbf{C}(\mathcal{V}\otimes\mathcal{W})\simeq\mathbf{C}\mathcal{V}\otimes\mathbf{C}\mathcal{W}.$$

If $a_1\in L(\mathcal{V}_1,\mathcal{W}_1)$ and $a_2\in L(\mathcal{V}_2,\mathcal{W}_2)$, then there exists a unique $a_1\otimes a_2\in L(\mathcal{V}_1\overset{\circ}{\otimes}\mathcal{V}_2,\mathcal{W}_1\overset{\circ}{\otimes}\mathcal{W}_2)$ such that

$$(a_1\otimes a_2)(v_1\otimes v_2):=a_1v_1\otimes a_2v_2.$$

Let S^n denote the permutation group of n elements. If $\sigma\in S_n$, then there exists a unique

$$\Theta(\sigma)\in L(\mathcal{V}_1\overset{\circ}{\otimes}\cdots\overset{\circ}{\otimes}\mathcal{V}_n,\mathcal{V}_{\sigma 1}\overset{\circ}{\otimes}\cdots\overset{\circ}{\otimes}\mathcal{V}_{\sigma n})$$

such that

$$\Theta(\sigma)v_1\otimes\cdots\otimes v_n:=v_{\sigma 1}\otimes\cdots\otimes v_{\sigma n}.$$

We have the natural identification

$$L(\mathcal{W},\mathcal{V}^{/\mathrm{al}})\simeq(\mathcal{V}\overset{\circ}{\otimes}\mathcal{W})^{/\mathrm{al}}.$$

Thus we have 2 possible notations for a bilinear form a: if $v\in\mathcal{V}$, $w\in cW$, then we can write

$$vaw=\langle a|v\otimes w\rangle.$$

The first expression treats a as an element of $L(\mathcal{W},\mathcal{V}^{/\mathrm{al}})$, whereas the second as an element of $(\mathcal{V}\overset{\circ}{\otimes}\mathcal{W})^{/\mathrm{al}}$. Note that in the literature a third notation is often used: $a(v,w)$ – we will not use this notation.

If $(\mathcal{V}_i,\mathcal{V}_i')$ are dual pairs, then

$$\left(\mathcal{V}_1\overset{\circ}{\otimes}\cdots\overset{\circ}{\otimes}\mathcal{V}_n,\mathcal{V}_1'\overset{\circ}{\otimes}\cdots\overset{\circ}{\otimes}\mathcal{V}_n'\right)$$

is a dual pair. Thus if $\xi \in \mathcal{V}_1'\mathring{\otimes}\cdots\mathring{\otimes}\mathcal{V}_n'$ and $b \in \mathcal{V}_1\mathring{\otimes}\cdots\mathring{\otimes}\mathcal{V}_n$, then we have a natural pairing

$$\langle\xi|b\rangle \in \mathbf{K}.$$

If $\xi \in \mathcal{V}_1'\mathring{\otimes}\cdots\mathring{\otimes}\mathcal{V}_m'$ and $b \in \mathcal{V}_1\mathring{\otimes}\cdots\mathring{\otimes}\mathcal{V}_n$, then we define the interior product $i(\xi)b$ as follows. We set

$$i(\xi)b = 0, \qquad m > n.$$

If $m \leq n$, then

$$i(\xi) \in L(\mathcal{V}_1\mathring{\otimes}\cdots\mathring{\otimes}\mathcal{V}_n, \mathcal{V}_{m+1}\mathring{\otimes}\cdots\mathring{\otimes}\mathcal{V}_n)$$

is uniquely determined by the following condition: If b is of the form $b = c \otimes d$ with $c \in \mathcal{V}_1\mathring{\otimes}\cdots\mathring{\otimes}\mathcal{V}_m$, then

$$i(\xi)b := \langle\xi|c\rangle d \in \mathcal{V}_{m+1}\mathring{\otimes}\cdots\mathring{\otimes}\mathcal{V}_n.$$

Theorem 1.8.1 *If $\xi \in \mathcal{V}_1' \otimes \cdots \otimes \mathcal{V}_k'$, $\eta \in \mathcal{V}_{k+1}' \otimes \cdots \otimes \mathcal{V}_m'$ and $a \in \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$, then*

$$i(\xi \otimes \eta)a = i(\eta)i(\xi)a.$$

1.8.2 Tensor product of Hilbert spaces

If $\mathcal{V}$, $\mathcal{W}$ are Hilbert spaces, then $\mathcal{V}\mathring{\otimes}\mathcal{W}$ has a unique scalar product such that

$$(v_1 \otimes w_1|v_2 \otimes w_2) := (v_1|v_2)(w_1|w_2), \quad v_1, v_2 \in \mathcal{V}, \quad w_1, w_2 \in \mathcal{W}.$$

We set

$$\mathcal{V} \otimes \mathcal{W} := (\mathcal{V}\mathring{\otimes}\mathcal{W})^{\mathrm{cpl}}$$

If $a_1 \in B(\mathcal{V}_1, \mathcal{W}_1)$, $a_2 \in B(\mathcal{V}_2, \mathcal{W}_2)$, then $a_1 \otimes a_2$ extends to an element of $B(\mathcal{V}_1 \otimes \mathcal{V}_2, \mathcal{W}_1 \otimes \mathcal{W}_2)$, also denoted $a_1 \otimes a_2$.

If $\sigma \in S_n$ and $\mathcal{V}, \ldots \mathcal{V}_n$ are Hilbert spaces, then $\Theta(\sigma)$ extends to unitary operator from $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$ to $\mathcal{V}_{\sigma 1} \otimes \cdots \otimes \mathcal{V}_{\sigma n}$, also denoted $\Theta(\sigma)$.

If $\{v_i \ : \ i \in I\}$ is an orthonormal basis of $\mathcal{V}$ and $\{w_j \ : \ j \in J\}$ is an orthonormal basis of $\mathcal{W}$, then $\{v_i \otimes w_j \ : \ (i,j) \in I \times J\}$ is an orthonormal basis of $\mathcal{V} \otimes \mathcal{W}$.

We have the natural identifications

$$B(\mathcal{W}, \mathcal{V}) \subset (\mathcal{V}\mathring{\otimes}\overline{\mathcal{W}})^{*\mathrm{al}}, \quad l^2(\mathcal{W}, \mathcal{V}) \simeq \mathcal{V} \otimes \overline{\mathcal{W}}.$$

Thus if $a \in B(\mathcal{W}, \mathcal{V})$, $v \in \mathcal{V}$, $w \in \mathcal{W}$, we have two notations

$$(v|aw) = (a|v \otimes w).$$

In the first we treat a as an element of $B(\mathcal{W}, \mathcal{V})$, and in the second as an element of $(\mathcal{V}\mathring{\otimes}\overline{\mathcal{W}})^{*\mathrm{al}}$.

1.8.3 Tensor algebra

Fix a vector space $\mathcal{V}$. We will write

$$\overset{\circ}{\Gamma}{}^{n}(\mathcal{V}) = \mathcal{V}^{\overset{\circ}{\otimes} n}, \qquad \overset{\circ}{\Gamma}(\mathcal{V}) := \overset{\circ}{\oplus}{}_{n=0}^{\infty} \overset{\circ}{\Gamma}{}^{n}(\mathcal{V}),$$

where $\overset{\circ}{\Gamma}{}^{0}(\mathcal{V})$ denotes $\mathbf{K}$. The element $1 \in \overset{\circ}{\Gamma}{}^{0}(\mathcal{V})$ is often called the vacuum and denoted Ω. $\overset{\circ}{\Gamma}(\mathcal{V})$ is an associative algebra with the operation $\otimes$ and the identity Ω.

If $p \in L(\mathcal{V}_1, \mathcal{V}_2)$, then we will use the notation

$$\Gamma^n(p) := p^{\otimes n} \in L\Big(\overset{\circ}{\Gamma}{}^{n}(\mathcal{V}_1), \overset{\circ}{\Gamma}{}^{n}(\mathcal{V}_2)\Big),$$

$$\Gamma(p) := \mathop{\oplus}_{n=0}^{\infty} \Gamma^n(p) \in L\Big(\overset{\circ}{\Gamma}(\mathcal{V}_1), \overset{\circ}{\Gamma}(\mathcal{V}_2)\Big).$$

Likewise, we define, for $h \in L(\mathcal{V})$,

$$\mathrm{d}\Gamma^n(h) = \sum_{j=0}^{n} 1_{\mathcal{V}}^{\otimes j} \otimes h \otimes 1_{\mathcal{V}}^{\otimes(n-j-1)} \in L\Big(\overset{\circ}{\Gamma}{}^{n}(\mathcal{V})\Big),$$

$$\mathrm{d}\Gamma(h) := \mathop{\oplus}_{n=0}^{\infty} \mathrm{d}\Gamma^n(h) \in L\Big(\overset{\circ}{\Gamma}(\mathcal{V})\Big).$$

The number operator is defined as $N = \mathrm{d}\Gamma(1)$. The parity operator is

$$I := (-1)^N. \tag{1.8.19}$$

Theorem 1.8.2 *1) Let $h, h_1, h_2 \in L(\mathcal{V})$, $p \in L(\mathcal{V}_1, \mathcal{V}_2)$, $p_1 \in L(\mathcal{V}_1, \mathcal{V})$, $p_2 \in L(\mathcal{V}, \mathcal{V}_1)$. We then have*

$$\Gamma(\mathrm{e}^h) = \exp(\mathrm{d}\Gamma(h)),$$

$$\Gamma(p_2)\Gamma(p_1) = \Gamma(p_2 p_1),$$

$$[\mathrm{d}\Gamma(h_1), \mathrm{d}\Gamma(h_2)] = \mathrm{d}\Gamma([h_1, h_2]).$$

2) Let $a, b \in \overset{\circ}{\Gamma}(\mathcal{V})$, $h \in L(\mathcal{V})$, $p \in L(\mathcal{V}, \mathcal{V}_1)$. Then

$$\Gamma(p)(a \otimes b) = (\Gamma(p)a) \otimes (\Gamma(p)b),$$

$$[\mathrm{d}\Gamma(h), (a \otimes b)] = [\mathrm{d}\Gamma(h), a] \otimes b + a \otimes [\mathrm{d}\Gamma(h), b],$$

3) Let $b \in \overset{\circ}{\Gamma}(\mathcal{V})$, $\xi \in \overset{\circ}{\Gamma}(\mathcal{V}')$, $p \in L(\mathcal{V}, \mathcal{V}_1)$. Then

$$\Gamma(p) i(\xi) b = i(\Gamma(p'^{-1}\xi))(\Gamma(p)b),$$

$$[\mathrm{d}\Gamma(h), i(\xi)b] = -i([\mathrm{d}\Gamma(h')\xi])b + i(\xi)[\mathrm{d}\Gamma(h), b],$$

1.8.4 Tensor algebra over a Hilbert space – full Fock space

Suppose that $\mathcal{V}$ is a Hilbert space. We define the Hilbert spaces

$$\Gamma^n(\mathcal{V}) = \mathcal{V}^{\otimes n} = \left(\overset{\circ}{\Gamma}{}^n(\mathcal{V})\right)^{\mathrm{cpl}},$$

$$\Gamma(\mathcal{V}) = \bigoplus_{n=0}^{\infty} \Gamma^n(\mathcal{V}) = \left(\overset{\circ}{\Gamma}(\mathcal{V})\right)^{\mathrm{cpl}}.$$

If $\|p\| \leq 1$, then $\Gamma(p)$ extends to a bounded operator of norm 1. If p is unitary, then so is $\Gamma(p)$. If h is self-adjoint, then $\Gamma(h)$ and $\mathrm{d}\Gamma(h)$ extend to self-adjoint operators.

1.9 Symetric tensor product

1.9.1 Symmetric tensor product

Let $\mathcal{V}$ be a vector space over $\mathbf{K}$.

$S^n \ni \sigma \mapsto \Theta(\sigma) \in L(\overset{\circ}{\Gamma}{}^n(\mathcal{V}))$ is a representation of the permutation group S^n. We define

$$\Theta_{\mathrm{s}}^n := \frac{1}{n!} \sum_{\sigma \in S^n} \Theta(\sigma) \in L(\overset{\circ}{\Gamma}{}^n(\mathcal{V})),$$

$$\Theta_{\mathrm{s}} := \overset{\infty}{\underset{n=0}{\oplus}} \Theta_{\mathrm{s}}^n \in L(\overset{\circ}{\Gamma}(\mathcal{V})).$$

Θ_{s}^n is a projection in $\overset{\circ}{\Gamma}{}^n(\mathcal{V})$ and Θ_{s} is a projection in $\overset{\circ}{\Gamma}(\mathcal{V})$. We set

$$\overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V}) := \Theta_{\mathrm{s}}^n \overset{\circ}{\Gamma}{}^n(\mathcal{V}),$$

$$\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V}) := \overset{\circ}{\oplus}{}_{n=0}^{\infty} \overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V}) = \Theta_{\mathrm{s}} \overset{\circ}{\Gamma}(\mathcal{V}).$$

If $a, b \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V})$, then we define

$$a \otimes_{\mathrm{s}} b := \Theta_{\mathrm{s}} a \otimes b \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V}).$$

The product $\otimes_{\mathrm{s}}$ is associative and commutative.

If $(\mathcal{V}, \mathcal{V}')$ is a dual pair, then $\left(\overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V}), \overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V}')\right)$ and $\left(\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V}), \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V}')\right)$ are dual pairs.

For $\xi \in \overset{\circ}{\Gamma}{}_{\mathrm{s}}^m(\mathcal{V}')$ and $b \in \overset{\circ}{\Gamma}{}_{\mathrm{s}}^{m+n}(\mathcal{V})$ we define

$$\xi \nabla b := \frac{(m+n)!}{n!} i(\xi) b \in \overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V}).$$

Elements of the form $v^{\otimes n}$, $v \in \mathcal{V}$ span $\overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V})$.

Let $\{v_i \ : \ i \in I\}$ be a Hamel basis of $\mathcal{V}$. Let $C_0(I, \mathbf{N})$ denote the set of sequences indexed by I with values in nonnegative integers with all by a finite number of entries zero. For $k = (k_i)_{i \in I} \in C_0(I, \mathbf{N})$ set $|k| := \sum_{i \in I} k_i$, $k! := \prod\limits_{i \in I} k_i!$ and $v_k := \underset{i \in I}{\otimes_{\mathrm{s}}} v_i^{\otimes k_i}$. Then $\{v_k \ : \ k \in C_0(I, \mathbf{N}), \ |k| = n\}$ is a Hamel basis of $\overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V})$.

Suppose that $\mathcal{V}$ is finite dimensional with a basis $\{v_1, \ldots, v_d\}$. Let $\{\xi_i \ : \ i = 1, \ldots, d\}$ be the dual basis in $\mathcal{V}'$. Then $\{k!\xi_k \ : \ k \in C_0(I, \mathbf{N}), \ |k| = n\}$ is the dual basis in $\overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V}')$.

If $p \in L(\mathcal{V}, \mathcal{W})$, then $\Gamma^n(p)$ maps $\overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V})$ into $\overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{W})$. Hence $\Gamma(p)$ maps $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V})$ into $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{W})$. We will use the same symbols $\Gamma^n(p)$ and $\Gamma(p)$ to denote the corresponding restricted operators.

If $h \in L(\mathcal{V})$, then $\mathrm{d}\Gamma^n(h)$ maps $\overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V})$ into itself. Hence, $\mathrm{d}\Gamma(h)$ maps $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V})$ into itself. We will use the same symbols $\mathrm{d}\Gamma^n(h)$ and $\mathrm{d}\Gamma(h)$ to denote the corresponding restricted operators.

We have the identification of $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V}_1) \overset{\circ}{\otimes} \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V}_2)$ with $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V}_1 \oplus \mathcal{V}_2)$ defined as follows: Let $a_i \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V}_i)$ and let j_i be the imbedding of $\mathcal{V}_i$ in $\mathcal{V}_1 \oplus \mathcal{V}_2$. Then $a_1 \otimes a_2$ is identified with $(\Gamma(j_1)a_1) \otimes_{\mathrm{s}} (\Gamma(j_2)a_2)$.

Theorem 1.9.1 *1) Let $a, b \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V})$, $\xi, \eta \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V}')$, $\nu \in \mathcal{V}'$. Then*

$$(\xi\nabla)(\eta\nabla)b = (\xi \otimes_{\mathrm{s}} \eta)\nabla b,$$

$$\nu\nabla(a \otimes_{\mathrm{s}} b) = (\nu\nabla a) \otimes_{\mathrm{s}} b + a \otimes_{\mathrm{s}} (\nu\nabla b)$$

2) Let $p \in L(\mathcal{V}, \mathcal{V}_1)$, $h \in L(\mathcal{V})$, $a, b \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V})$. Then

$$\Gamma(p)(a \otimes_{\mathrm{s}} b) = (\Gamma(p)a) \otimes_{\mathrm{s}} (\Gamma(p)b),$$

$$[\mathrm{d}\Gamma(h), (a \otimes_{\mathrm{s}} b)] = [\mathrm{d}\Gamma(h), a] \otimes_{\mathrm{s}} b + a \otimes_{\mathrm{s}} [\mathrm{d}\Gamma(h), b].$$

3) Let $p \in L(\mathcal{V}, \mathcal{V}_1)$, $h \in L(\mathcal{V})$, $b \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V})$, $\xi \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{V}')$. Then

$$\Gamma(p)c\nabla b = (\Gamma(p'^{-1}c)\nabla(\Gamma(p)b),$$

$$[\mathrm{d}\Gamma(h), \xi\nabla b] = -([\mathrm{d}\Gamma(h')\xi])\nabla b + \xi\nabla[\mathrm{d}\Gamma(h), b].$$

1.9.2 Symmetric tensor algebras over a Hilbert space – bosonic Fock space

If $\mathcal{V}$ is a Hilbert space, then Θ_{s}^n extends to an orthogonal projection in $\Gamma^n(\mathcal{V})$ and Θ_{s} extends to an orthogonal projection in $\Gamma(\mathcal{V})$. We define Hilbert spaces

$$\Gamma_{\mathrm{s}}^n(\mathcal{V}) = \Theta_{\mathrm{s}}^n \Gamma_{\mathrm{s}}^n(\mathcal{V}) = (\overset{\circ}{\Gamma}{}_{\mathrm{s}}^n(\mathcal{V}))^{\mathrm{cpl}},$$

$$\Gamma_{\rm s}(\mathcal{V}) = \mathop{\oplus}_{n=0}^{\infty} \Gamma_{\rm s}^n(\mathcal{V}) = (\mathop{\Gamma_{\rm s}}\limits^{\circ}(\mathcal{V}))^{\rm cpl}.$$

Let $\{v_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{V}$. Then $\{\sqrt{k!}v_k \ : \ k \in C_0(I,\mathbf{N}), \ |k| = n\}$ is an orthonormal basis of $\Gamma_{\rm s}^n(\mathcal{V})$.

We set

$$\Gamma_{\rm s}^{\rm fin}(\mathcal{V}) := \mathop{\oplus}\limits^{\circ}{}_{n=0}^{\infty}\Gamma_s^n(\mathcal{V}),$$

(the space of finite particle vectors).

Note the identifications

$$B_{\rm s}(\overline{\mathcal{V}},\mathcal{V}) \subset (\mathop{\Gamma}\limits^{\circ}{}_{\rm s}^{2}(\mathcal{V}))^{*{\rm al}},$$

$$l_{\rm s}^2(\overline{\mathcal{V}},\mathcal{V}) \simeq \Gamma_{\rm s}^2(\mathcal{V}).$$

It is convenient to introduce also the space

$$\Gamma_{\rm s}^{\rm cw}(\mathcal{V}) := \{\psi \in \Gamma_{\rm s}(\mathcal{V}) \ : \ (\psi|N!\psi) < \infty\},$$

equipped with the scalar product

$$(\psi_1|\psi_2)_{\rm cw} := (\psi_1|N!\psi_2).$$

Clearly, the map

$$\Gamma_{\rm s} \ni \psi \mapsto \psi^{\rm cw} := \frac{1}{\sqrt{N!}}\psi \in \Gamma_{\rm s}^{\rm cw}(\mathcal{V})$$

is unitary.

Remark 1.9.2 *Symmetric tensor algebra over a Hilbert space in the context of quantum physics is usually called a bosonic Fock space. Segal uses the name bosonic Fock-Cook space.*

In the notation that we adopted we distinguish between the norm $(\cdot|\cdot)$ and the norm $(\cdot|\cdot)_{\rm cw}$. The subscript cw stands for "complex wave" and is related to the "complex wave representation" which will be described in Subsection **??** *– see remark in that subsection.*

1.9.3 Polynomials

Let $\mathcal{Y}$ be vector space over $\mathbf{K}$ and $(\mathcal{Y},\mathcal{Y}')$ a dual pair. If $a \in \mathop{\Gamma_{\rm s}}\limits^{\circ}(\mathcal{Y})$ and $a = \sum\limits_{p=0}^{\infty} a_p$ with $a_p \in \mathop{\Gamma}\limits^{\circ}{}_{\rm s}^{p}(\mathcal{V})$, then we set

$$\mathcal{Y}' \ni v \mapsto P_a(v) := \sum_{p=0}^{\infty}\langle v^{\otimes p}, a_p\rangle \in \mathbf{K}.$$

Note that $P_a \in C(\mathcal{Y}',\mathbf{K})$.

The operation $\otimes_{\mathrm{s}}$ corresponds to the multiplication of such functions:

$$P_{a\otimes_{\mathrm{s}}b}(v) = P_a(v)P_b(v), \quad a,b \in \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y}).$$

Functions P_a for $a \in \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y})$ are called polynomials on $\mathcal{Y}'$. In what follows we will write $a(v)$ instead of $P_a(v)$, thus treating $\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y})$ as a subspace of $C(\mathcal{Y}', \mathbf{K})$.

Note that we have the identification

$$L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}^{/\mathrm{al}}) \simeq \mathring{\Gamma}_{\mathrm{s}}^{2}(\mathcal{Y})^{/\mathrm{al}}.$$

Thus we have three different ways of writing the same thing:

$$vav = \langle a|v \otimes v\rangle = a(v).$$

The first expression treats a as an element of $L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}^{/\mathrm{al}})$, the second as an element of $\mathring{\Gamma}_{\mathrm{s}}^{2}(\mathcal{Y})^{/\mathrm{al}}$, and the third as a polynomial.

1.9.4 Complex polynomials

$\mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y})$ can be treated as a subspace of $C(\mathcal{Y}', \mathbf{C}) = C(\mathcal{Y}')$ and will be called the space of complex polynomials on $\mathcal{Y}'$, as opposed to $\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y})$, which will be called real polynomials on $\mathcal{Y}'$.

If $\mathcal{Z}$ is a complex space, then we will call $\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}) \subset \mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}'_{\mathbf{R}}) \subset C(\mathcal{Z}'_{\mathbf{R}})$ the holomorphic polynomials on $\mathcal{Z}'$.

If $\mathcal{Y}$ is a real vector space, then we have a natural identification

$$\mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y}) \simeq \mathring{\Gamma}_{\mathrm{s}}(\mathbf{C}\mathcal{Y}).$$

This identification corresponds to the analytic continuation of complex polynomials on $\mathcal{Y}'$ to holomorphic polynomials on $(\mathbf{C}\mathcal{Y})'$.

Now assume that a real vector space $\mathcal{Y}$ is equipped with an anti-involution $\mathbf{i}$. $\mathcal{Y}$ can be viewed as a complex space in two natural ways: as $\mathcal{Y}^{\mathbf{C}}$ and $\overline{\mathcal{Y}^{\mathbf{C}}}$. Correspondingly, we have the space of holomorphic polynomials on $\mathcal{Y}'$, $\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y}^{\mathbf{C}}) \subset \mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y})$, and antiholomorphic polynomials on $\mathcal{Y}'$ that is $\mathring{\Gamma}_{\mathrm{s}}(\overline{\mathcal{Y}^{\mathbf{C}}}) \subset \mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y})$.

In the usual way we define the subspaces $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ of $\mathbf{C}\mathcal{Y}$, so that $\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$. Note that we can identify $\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y}^{\mathbf{C}})$ with $\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z})$ and $\mathring{\Gamma}_{\mathrm{s}}(\overline{\mathcal{Y}^{\mathbf{C}}})$ with $\mathring{\Gamma}(\overline{\mathcal{Z}})$.

Recall that we have $\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$. Therefore, the holomorphic and antiholomorphic polynomials generate all complex polynomials on $\mathcal{Y}'$:

$$\mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y}) \simeq \mathring{\Gamma}_{\mathrm{s}}(\mathbf{C}\mathcal{Y}) = \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z})\mathring{\otimes}_{\mathrm{s}}\mathring{\Gamma}_{\mathrm{s}}(\overline{\mathcal{Z}}).$$

1.10 Integration

1.10.1 Gaussian integrals

Let $\mathcal{X}$ be a finite dimensional real vector space of dimension d with a Lebesgue measure $\mathrm{d}x$.

Let α be a positive symmetric form on $\mathcal{Y}$. Then

$$\int \mathrm{e}^{-x\alpha x}\mathrm{d}x = \pi^{\frac{d}{2}}(\det \alpha)^{-\frac{d}{2}}.$$

If α is any symmetric form on $\mathcal{Y}$, then

$$\int \mathrm{e}^{\mathrm{i}x\alpha x}\mathrm{d}x = \pi^{\frac{d}{2}}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathrm{inert}\alpha}(\det |\alpha|)^{-\frac{d}{2}}.$$

1.10.2 $\mathcal{S}(\mathcal{X})$ and $\mathcal{S}'(\mathcal{X})$ spaces

The space of Schwartz functions on $\mathcal{X}$ is defined as

$$\mathcal{S}(\mathcal{X}) := \{\psi \in C^\infty(\mathcal{X}) \ : \int |x^\alpha \nabla_x^\beta \psi(x)|^2 \mathrm{d}x < \infty, \quad \text{for any multiindices } \alpha, \beta\} \ . \tag{1.10.20}$$

(In the definition we use an identification of $\mathcal{X}$ with $\mathbf{R}^d$, it is clear that $\mathcal{S}(\mathcal{X})$ does not depend on this identification).

Remark 1.10.1 *The definition (1.10.20) is equivalwent to*

$$\mathcal{S}(\mathcal{X}) = \{\psi \in C^\infty(\mathcal{X}) \ : \ |x^\alpha \nabla_x^\beta \psi(x)| \leq c_{\alpha,\beta}, \quad \textit{for any multiindeces } \alpha, \beta\}. \tag{1.10.21}$$

The definition (1.10.21) is more common in the literature.

$\mathcal{S}(\mathcal{X})$ is a Frechet space with seminorms $\|\psi\|_{\alpha,\beta} := (\int |x^\alpha \nabla_x^\beta \psi(x)|^2 \mathrm{d}x)^{\frac{1}{2}}$. Let $\mathcal{S}'(\mathcal{X})$ denote its (topological) dual. Note that we have the inclusions

$$\mathcal{S}(\mathcal{X}) \subset L^2(\mathcal{X}) \simeq L^2(\mathcal{X})^* \subset \mathcal{S}'(\mathcal{X}).$$

If $\psi \in \mathcal{S}(\mathcal{X})$ and $f \in \mathcal{S}(\mathcal{X})$, then the action of f on ψ will be denoted as if f were a function, that means

$$\int f(x)\psi(x)\mathrm{d}x.$$

1.10.3 Fourier transformation

For $f \in L^1(\mathcal{X})$, the Fourier transform of f is denoted either $\mathcal{F}f$ or $\hat{f}$, belongs to $C_\infty(\mathcal{X}')$, and is given by the formula

$$\hat{f}(\xi) = \int f(x) \mathrm{e}^{-\mathrm{i}x\xi} \mathrm{d}\xi.$$

It is well known that $\mathcal{F}$ extends to a unique bounded operator from $L^2(\mathcal{X})$ to $L^2(\mathcal{X}')$. $(2\pi)^{-\frac{d}{2}}\mathcal{F}$ is unitary, and we have the Fourier inversion formula

$$f(x) = (2\pi)^{-d} \int \hat{f}(x) \mathrm{e}^{\mathrm{i}x\xi} \mathrm{d}x.$$

The space $\mathcal{S}(\mathcal{X})$ is mapped by $\mathcal{F}$ onto $\mathcal{S}(\mathcal{X}')$.

$\mathcal{F}$ can be extended to a unique continuous linear map from $\mathcal{S}'(\mathcal{X})$ onto $\mathcal{S}'(\mathcal{X}')$.

In particular, if $f(x) = \mathrm{e}^{-\frac{1}{2}x\alpha x}$, then

$$\hat{f}(\xi) = (2\pi)^{\frac{d}{2}} (\det \alpha)^{-\frac{d}{2}} \mathrm{e}^{-\frac{1}{2}\xi\alpha^{-1}\xi}.$$

If $g(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}x\alpha x}$, then

$$\hat{g}(\xi) = (2\pi)^{\frac{d}{2}} \mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathrm{inert}\alpha} (\det \alpha)^{-\frac{d}{2}} \mathrm{e}^{-\frac{\mathrm{i}}{2}\xi\alpha^{-1}\xi}.$$

1.10.4 Convolution operators

Let $D := -\mathrm{i}\nabla_x$. Note the identities:

$$\begin{aligned} f(D)\psi(x) &= (2\pi)^{-d} \int \mathrm{e}^{\mathrm{i}(x-y)\xi} f(\xi)\psi(y) \mathrm{d}\xi \mathrm{d}y \\ &= (2\pi)^{-d} \int \hat{f}(y-x)\psi(y)\mathrm{d}y. \end{aligned}$$

For example,

$$\mathrm{e}^{-\mathrm{i}t\nabla_x\nabla_\xi}\psi(x,\xi) = (2\pi t)^{-d} \int \mathrm{e}^{\frac{\mathrm{i}}{t}(x-x_1)(\xi-\xi_1)} \psi(x_1,\xi_1)\mathrm{d}x_1\mathrm{d}\xi_1.$$

If $\mathcal{Y}$ is a Euclidean space, we can write

$$\mathrm{e}^{-\frac{t}{2}D^2}\psi(y) = \mathrm{e}^{\frac{t}{2}\nabla_y^2}\psi(y) = (2\pi t)^{-\frac{d}{2}} \int \mathrm{e}^{-\frac{1}{2t}(y-y_1)^2}\psi(y_1)\mathrm{d}y_1. \tag{1.10.22}$$

Sometimes it is convenient in this context to use the decomposition $\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$. Instead of y, ∇_y, one then uses $z = 1_{\mathcal{Z}}y$, $\overline{z} = 1_{\overline{\mathcal{Z}}}y$, $\nabla_z = 1_{\mathcal{Z}}\nabla_y$, $\nabla_{\overline{z}} = 1_{\overline{\mathcal{Z}}}\nabla_y$. Note the identities

$$\frac{1}{2}y^2 = z\overline{z}, \quad \frac{1}{2}\nabla_y^2 = \nabla_z\nabla_{\overline{z}}, \quad \mathrm{d}y = \frac{1}{\mathrm{i}}\mathrm{d}z\mathrm{d}\overline{z}.$$

Then (1.10.22) can be rewritten as

$$\mathrm{e}^{t\nabla_z\nabla_{\overline{z}}}\psi(z,\overline{z}) = (2\pi\mathrm{i}t)^{-d} \int \mathrm{e}^{-\frac{1}{t}(z-z_1)(\overline{z}-\overline{z_1})}\mathrm{d}z_1\mathrm{d}\overline{z_1}.$$

Chapter 2

Canonical commutation relations

2.1 Schrödinger representation of CCR

2.1.1 Heisenberg commutation relations

Let $\mathcal{X}$ be a finite dimensional real vector space with a Lebesgue measure $\mathrm{d}x$. Consider the Hilbert space $L^2(\mathcal{X}, \mathrm{d}x)$ (denoted usually for simplicity $L^2(\mathcal{X})$).

Define the position operator and the momentum operator:

$$\begin{array}{ll}(\eta x\psi)(x) := \eta x\psi(x), & \eta \in \mathcal{X}', \\ & \psi \in \mathcal{D}(\eta x) := \{\psi \in L^2(\mathcal{X}) \ : \ \int |\eta x|^2 |\psi(x)|^2 \mathrm{d}x < \infty\}, \\ (qD\psi)(x) := -\mathrm{i}q\nabla\psi(x), & q \in \mathcal{X}, \\ & \psi \in \mathcal{D}(qD) := \{\psi \in L^2(\mathcal{X}) \ : \ \int |q\nabla_x\psi(x)|^2 \mathrm{d}x < \infty\}.\end{array}$$

ηx and qD are self-adjoint operators.

Theorem 2.1.1 *The Schwartz space $\mathcal{S}(\mathcal{X})$ is the intersection of*

$$\{\psi \in L^2(\mathcal{X}) \ : \psi \in \mathcal{D}((\eta_1 x)\cdots(\eta_n x)(q_1 D)\cdots(q_m D))\}$$

for $\eta_1 \dots \eta_n \in \mathcal{X}'$, $q_1 \dots q_m \in \mathcal{X}$. Therefore, $\mathcal{S}(\mathcal{X})$ is the largest subspace of $L^2(\mathcal{X})$ preserved by the operators ηx and qD.

The operator ηx and qD, viewed as operators on $\mathcal{S}(\mathcal{X})$, satisfy the so-called Heisenberg commutation relation:

$$[\eta_1 x, \eta_2 x] = [q_1 D, q_2 D] = 0, \qquad [\eta x, qD] = \mathrm{i}\eta q.$$

The algebra of polynomials in x and D, treated as operators on $\mathcal{S}(\mathcal{X})$, will be denoted $\mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X}) \subset L(\mathcal{S}(\mathcal{X}))$.

Remark 2.1.2 $\mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X})$ *is sometimes called the Weyl algebra. (There are, however, other objects which are sometimes called by the same name).*

2.1.2 The x, D- and the D, x-quantizations

Clearly, if $h_1 \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}')$, then $h_1(x)$ is well defined as an operator on $\mathcal{S}(\mathcal{X})$. Such operators form a commutative subalgebra in $\mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X})$.

Likewise, if $h_2 \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$, then $h_2(D)$ is well defined as an operator on $\mathcal{S}(\mathcal{X})$. Such operators form another commutative algebra in $\mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X})$.

We define the x, D- and the D, x-quantizations as the maps

$$\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}' \oplus \mathcal{X}) \ni h \mapsto \mathrm{Op}^{x,D}(h) \in \mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X}),$$

$$\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}' \oplus \mathcal{X}) \ni h \mapsto \mathrm{Op}^{D,x}(h) \in \mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X}),$$

as follows: if $h = h_1 \otimes_{\mathrm{s}} h_2$ where $h_1 \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}')$, $h_2 \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$, then

$$\mathrm{Op}^{x,D}(h) = h_1(x)h_2(D), \tag{2.1.1}$$

$$\mathrm{Op}^{D,x}(h) = h_2(D)h_1(x). \tag{2.1.2}$$

We extend the definition to $\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}' \oplus \mathcal{X})$ by linearity.

Remark 2.1.3 *An alternative notation, commonly used in the literature on partial differential equations:* $\mathrm{Op}^{x,D}(h) = h(x, D)$. *The x, D-quantization is sometimes called the Kohn-Nirenberg quantization.*

The maps inverse to (2.1.1) and (2.1.2) are denoted

$$\mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X}) \ni H \mapsto \mathrm{s}_H^{x,D} \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}' \oplus \mathcal{X}), \tag{2.1.3}$$

$$\mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X}) \ni H \mapsto \mathrm{s}_H^{D,x} \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}' \oplus \mathcal{X}). \tag{2.1.4}$$

and are called the x, D- and D, x-symbols of an operator H.

We will treat the ordering x, D as the standard one.

Theorem 2.1.4 *0)* $\mathrm{s}_1^{x,D} = 1$.
1) If $H \in \mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X})$, then

$$\mathrm{s}_{H^*}^{x,D} = \overline{\mathrm{s}_H^{D,x}} \tag{2.1.5}$$

2)

$$\mathrm{s}_{\eta x}^{x,D} = \eta, \quad \eta \in \mathcal{X}', \qquad \mathrm{s}_{qD}^{x,D} = q, \quad q \in X, \tag{2.1.6}$$

More generally, if $f \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}')$ and $g \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$, then

$$\mathrm{Op}^{x,D}(f) = f(x), \quad \mathrm{Op}^{x,D}(g) = g(D). \tag{2.1.7}$$

3) If $H \in \mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X})$, then

$$\begin{aligned} \mathrm{s}_H^{x,D}(x,\xi) &= \mathrm{e}^{-\mathrm{i}\nabla_x\nabla_\xi}\mathrm{s}_H^{D,x}(x,\xi) \\ &= (2\pi)^{-d}\int_{\mathcal{X}\oplus\mathcal{X}'}\mathrm{e}^{-\mathrm{i}(x-x_1)(\xi-\xi_1)}\mathrm{s}_H^{D,x}(x_1,\xi_1)\mathrm{d}x_1\mathrm{d}\xi_1. \end{aligned} \tag{2.1.8}$$

4) If $G, H \in \mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X})$, then

$$\begin{aligned} \mathrm{s}_{GH}^{x,D}(x,\xi) &= \mathrm{e}^{-\mathrm{i}\nabla_{\xi_1}\nabla_{x_1}}s_G^{x,D}(x,\xi_1)s_H^{x,D}(x_1,\xi)\Big|_{\substack{x=x_1,\\ \xi=\xi_1}} \\ &= (2\pi)^{-d}\int_{\mathcal{X}\oplus\mathcal{X}'}\mathrm{e}^{-\mathrm{i}(x-x_1)(\xi-\xi_1)}\mathrm{s}_G^{x,D}(x,\xi_1)\mathrm{s}_H^{x,D}(x_1,\xi)\mathrm{d}x_1\mathrm{d}\xi_1. \end{aligned} \tag{2.1.9}$$

The operator $\mathrm{e}^{-\mathrm{i}\nabla_x\nabla_\xi}$ in (2.1.8) and the similar operator in (2.1.9) can be understood as a sum of differential operators. In the case of this theorem, the sum is finite, because we deal with polynomial symbols.

The integral formula in (2.1.8) and (2.1.9)should be understood in the sense of oscillatory integrals.

Proof. (2.1.8) can be proven by a simple combinatorial argument. (2.1.9) follows easily from (2.1.9).

(2.1.8) and (2.1.9) follow also (in a much larger generality) from integral formulas considered later on (see Subsection **??**). □

The following formula is a version of Wick's theorem. It follows from (2.1.9).

Theorem 2.1.5 *Let $H_1, \dots, H_n \in \mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X})$. Then*

$$\mathrm{s}_{H_1\cdots H_n}^{x,D}(x,\xi) = \exp\Big(-\mathrm{i}\sum_{i<j}\nabla_{x_i}\nabla_{\xi_j}\Big)\mathrm{s}_{H_1}^{x,D}(x_1,\xi_1)\cdots\mathrm{s}_{H_n}^{x,D}(x_n,\xi_n)\Big|_{\substack{x=x_1=\cdots=x_n,\\ \xi=\xi_1=\cdots=\xi_n.}}$$

2.1.3 Weyl operators

Let us consider the 1-parameter unitary groups on $L^2(X)$ generated by the position and the momentum operators.

Theorem 2.1.6 *Let $\eta \in \mathcal{X}'$, $q \in \mathcal{X}$. We have the so-called Weyl commutation relations*

$$\mathrm{e}^{\mathrm{i}\eta x}\mathrm{e}^{\mathrm{i}qD} = \mathrm{e}^{-\mathrm{i}\eta q}\mathrm{e}^{\mathrm{i}qD}\mathrm{e}^{\mathrm{i}\eta x}. \tag{2.1.10}$$

The operator $\eta x + qD$ is essentially self-adjoint on $\mathcal{S}(\mathcal{X})$. For $\psi \in L^2(\mathcal{X})$ we have

$$\mathrm{e}^{\mathrm{i}(\eta x+qD)}\psi(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\eta q+\mathrm{i}\eta x}\psi(x+q). \tag{2.1.11}$$

Moreover, the following identities are true:

$$\begin{aligned} \mathrm{e}^{\mathrm{i}(\eta x+qD)} &= \mathrm{e}^{\frac{\mathrm{i}}{2}\eta q}\mathrm{e}^{\mathrm{i}\eta x}\mathrm{e}^{\mathrm{i}qD} \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2}\eta q}\mathrm{e}^{\mathrm{i}qD}\mathrm{e}^{\mathrm{i}\eta x} \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}\eta x}\mathrm{e}^{\mathrm{i}qD}\mathrm{e}^{\frac{\mathrm{i}}{2}\eta x} \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}qD}\mathrm{e}^{\mathrm{i}\eta x}\mathrm{e}^{\frac{\mathrm{i}}{2}qD}. \end{aligned} \tag{2.1.12}$$

Proof. Clearly, we have

$$\mathrm{e}^{\mathrm{i}qD}\psi(x) = \psi(x+q).$$

This easily implies (2.1.10).

Define

$$U(t) := \mathrm{e}^{\frac{\mathrm{i}}{2}t^2\eta q}\mathrm{e}^{\mathrm{i}t\eta x}\mathrm{e}^{\mathrm{i}tqD},$$

or

$$U(t)\psi(x) := \mathrm{e}^{\frac{\mathrm{i}}{2}t^2\eta q+\mathrm{i}t\eta x}\psi(x+tq).$$

We compute

$$\partial_t U(t)\psi = \mathrm{i}(\eta x + qD)U(t)\psi.$$

Clearly, if $\psi \in \mathcal{S}(\mathcal{X})$, then $U(t)\psi \in \mathcal{S}(\mathcal{X})$ for all t. Therefore,

$$U(t) = \mathrm{e}^{\mathrm{i}t(\eta x+qD)}$$

and $\mathcal{S}(\mathcal{X})$ is a core of $\eta x + qD$. This implies (2.1.11)

The identities (2.1.12) follow from (2.1.11). □

Theorem 2.1.7 *If $B \in B(L^2(\mathcal{X}))$ commutes with all operators in $\{\mathrm{e}^{\mathrm{i}\eta x},\ \ \mathrm{e}^{\mathrm{i}qD}\ :\ \eta \in \mathcal{X}',\ q \in \mathcal{X}\}$, then B is proportional to identity. In other words, the set $\{\mathrm{e}^{\mathrm{i}\eta x},\ \ \mathrm{e}^{\mathrm{i}qD}\ :\ \eta \in \mathcal{X}',\ q \in \mathcal{X}\}$ is irreducible in $B(L^2(\mathcal{X}))$.*

Proof. $L^\infty(\mathcal{X})$, identified with multiplication operators in $L^2(\mathcal{X})$, is a maximal abelian algebra in $B(L^2(\mathcal{X}))$. By the Fourier transformation, linear combinations of operators of the form $\mathrm{e}^{\mathrm{i}\eta x}$ are weakly dense in $L^\infty(\mathcal{X})$. Hence if B commutes with all operators $\mathrm{e}^{\mathrm{i}\eta x}$, it has to be of the form $f(x)$ with $f \in L^\infty(\mathcal{X})$.

We have $\mathrm{e}^{\mathrm{i}qD}f(x)\mathrm{e}^{-\mathrm{i}qD} = f(x+q)$. Hence if $f(x)$ commutes with $\mathrm{e}^{\mathrm{i}qD}$, then $f(x+q) = f(x)$. If this is the case for all $q \in \mathcal{X}$, f has to be constant. □

Theorem 2.1.8 *1) Let $\psi \in L^2(\mathcal{X})$. Then $\psi \in \mathcal{S}(\mathcal{X})$ iff*

$$\mathcal{X}' \oplus \mathcal{X} \ni (\eta, q) \mapsto (\psi|\mathrm{e}^{\mathrm{i}\eta x}\mathrm{e}^{\mathrm{i}qD}\psi) \tag{2.1.13}$$

belongs to $\mathcal{S}(\mathcal{X}' \oplus \mathcal{X})$.
2) If $\psi_1, \psi_2 \in \mathcal{S}(\mathcal{X})$, then

$$\mathcal{X}' \oplus \mathcal{X} \ni (\eta, q) \mapsto (\psi_1|\mathrm{e}^{\mathrm{i}\eta x}\mathrm{e}^{\mathrm{i}qD}\psi_2)$$

belongs to $\mathcal{S}(\mathcal{X}' \oplus \mathcal{X})$.

Proof. Let us prove 1). (2.1.13) is a partial Fourier transform of the function $\mathcal{X} \oplus \mathcal{X} \ni (x, q) \mapsto \overline{\psi}(x)\psi(x+q)$. Thus (2.1.13) belongs to $\mathcal{S}(\mathcal{X}' \oplus \mathcal{X})$ iff $\overline{\psi}(x)\psi(x+q)$ belongs to $\mathcal{S}(\mathcal{X} \oplus \mathcal{X})$, which is equivalent to $\psi \in \mathcal{S}(\mathcal{X})$. □

2.1.4 Quadratic forms on $\mathcal{S}(\mathcal{X})$

We say that $H \in \mathcal{F}'(\mathcal{X}' \oplus \mathcal{X})$, if H is a continuous map

$$\mathcal{S}(\mathcal{X}) \times \mathcal{S}(\mathcal{X}) \ni (\psi_1, \psi_2) \mapsto (\psi_1|H\psi_2) \in \mathbf{C},$$

antilinear wrt the first component and linear wrt the second. $\mathcal{F}'(\mathcal{X}' \oplus \mathcal{X}))$ is called the space of (sesquilinear) forms on $\mathcal{S}(\mathcal{X})$ and can be regarded as a natural extension of $B(L^2(\mathcal{X}))$. It can be viewed as the "space of noncommutative tempered distributions on $\mathcal{X} \oplus \mathcal{X}'$".

By the Schwartz theorem, if $H \in \mathcal{F}'(\mathcal{X}' \oplus \mathcal{X})$, there exists a tempered distribution $H(x_1, x_2) \in \mathcal{S}'(\mathcal{X} \oplus \mathcal{X})$ such that

$$(\psi_1|H\psi_2) = \int \overline{\psi_1}(x_1)H(x_1, x_2)\psi_2(x_2)\mathrm{d}x_1\mathrm{d}x_2.$$

$H(x_1, x_2)$ will be called the distributional kernel of H. We have a bijection between $\mathcal{F}'(\mathcal{X}' \oplus \mathcal{X})$ and $\mathcal{S}'(\mathcal{X} \oplus \mathcal{X})$ given by the distributional kernel.

Note that $\mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X}) \subset \mathcal{F}'(\mathcal{X}' \oplus \mathcal{X})$. In addition, $\mathcal{F}'(\mathcal{X}' \oplus \mathcal{X})$ is a module over $\mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X})$.

Note that $H \in l^2(L^2(\mathcal{X}))$ iff $H(x_1, x_2)$ belongs to $L^2(\mathcal{X} \oplus \mathcal{X})$. Moreover, if $H, G \in l^2(L^2(\mathcal{X}))$, then

$$\mathrm{Tr}H^*G = \int_{\mathcal{X}}\int_{\mathcal{X}} \overline{H}(x_2, x_1)G(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2.$$

2.1.5 The x, D quantization of distributional symbols

If $h \in \mathcal{S}'(\mathcal{X} \oplus \mathcal{X}')$, then we define the elements of $\mathcal{F}'(\mathcal{X}' \oplus \mathcal{X})$, $\mathrm{Op}^{x,D}(h)$ and $\mathrm{Op}^{D,x}(h)$, whose distributional kernels are equal

$$\begin{aligned} \mathrm{Op}^{x,D}(h)(x_1, x_2) &= (2\pi)^{-d} \textstyle\int_{\mathcal{X}'} h(x_1, \xi)\mathrm{e}^{\mathrm{i}(x_1 - x_2)\xi}\mathrm{d}\xi, \\ \mathrm{Op}^{D,x}(h)(x_1, x_2) &= (2\pi)^{-d} \textstyle\int_{\mathcal{X}'} h(x_2, \xi)\mathrm{e}^{\mathrm{i}(x_1 - x_2)\xi}\mathrm{d}\xi. \end{aligned} \tag{2.1.14}$$

Theorem 2.1.9 *1) For $h \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}' \oplus \mathcal{X}) \subset \mathcal{S}'(\mathcal{X} \oplus \mathcal{X}')$, the above definition coincides with (2.1.1) and (2.1.2).*
2) The maps

$$\mathcal{S}'(\mathcal{X} \oplus \mathcal{X}') \ni h \mapsto \mathrm{Op}^{x,D}(h) \in \mathcal{F}'(\mathcal{X}' \oplus \mathcal{X}),$$
$$\mathcal{S}'(\mathcal{X} \oplus \mathcal{X}') \ni h \mapsto \mathrm{Op}^{D,x}(h) \in \mathcal{F}'(\mathcal{X}' \oplus \mathcal{X})$$

are bijective, and if their inverses (symbols) are denoted as in (2.1.3) and (2.1.4), then for $H \in \mathcal{F}'(\mathcal{X}' \oplus \mathcal{X})$ we have

$$\begin{aligned} \mathrm{s}_H^{x,D}(x,\xi) &= \textstyle\int_{\mathcal{X}} H(x, x-y)\mathrm{e}^{-\mathrm{i}\xi y}\mathrm{d}y, \\ \mathrm{s}_H^{D,x}(x,\xi) &= \textstyle\int_{\mathcal{X}} H(x+y, x)\mathrm{e}^{-\mathrm{i}\xi y}\mathrm{d}y. \end{aligned} \tag{2.1.15}$$

3) The formulas (2.1.5) and (2.1.8) are true.
4) The formula (2.1.9) is true for instance if either $G \in \mathcal{F}'(\mathcal{X}' \oplus \mathcal{X})$ and $H \in \mathrm{CCR}(\mathcal{X}' \oplus \mathcal{X})$, or the other way around.
5) (2.1.9) is also true if the Fourier transforms of $\mathrm{s}_G^{x,D}$ and $\mathrm{s}_H^{x,D}$ belong to $L^1(\mathcal{X} \oplus \mathcal{X}')$. (In this case, G and H are bounded).
6) If $f \in \mathcal{S}'(\mathcal{X})$, $g \in \mathcal{S}'(\mathcal{X}')$, then (2.1.7) is true.
7) If $h \in \mathcal{S}'(\mathcal{X} \oplus \mathcal{X}')$, then

$$\begin{aligned} \mathrm{e}^{\mathrm{i}qD}\mathrm{Op}^{x,D}(h)\mathrm{e}^{-\mathrm{i}qD} &= \mathrm{Op}^{x,D}(h(\cdot + q, \cdot)), \\ \mathrm{e}^{\mathrm{i}\eta x}\mathrm{Op}^{x,D}(h)\mathrm{e}^{-\mathrm{i}\eta x} &= \mathrm{Op}^{x,D}(h(\cdot, \cdot - \eta)). \end{aligned}$$

8) We have $H \in HS(L^2(\mathcal{X}))$ iff $\mathrm{s}_H^{x,D} \in L^2(\mathcal{X} \oplus \mathcal{X}')$ and iff $\mathrm{s}_H^{D,x} \in L^2(\mathcal{X} \oplus \mathcal{X}')$. Mreover,

$$\mathrm{Tr}H^*G = (2\pi)^{-d}\int_{\mathcal{X}\oplus\mathcal{X}'} \overline{\mathrm{s}_H^{x,D}}(x,\xi)\mathrm{s}_G^{x,D}(x,\xi)\mathrm{d}x\mathrm{d}\xi = (2\pi)^{-d}\int_{\mathcal{X}\oplus\mathcal{X}'} \overline{\mathrm{s}_H^{D,x}}(x,\xi)\mathrm{s}_G^{D,x}(x,\xi)\mathrm{d}x\mathrm{d}\xi.$$

Proof. 2) follows from (2.1.14) by the inversion of the Fourier transform. (2.1.8) follows by combining the first formula of (2.1.14) with the second formula of (2.1.15).

5) follows from the formula

$$\mathrm{Op}^{x,D}(h) = (2\pi)^{-2d}\int_{\mathcal{X}'\oplus\mathcal{X}} \hat{h}(\eta, q)\mathrm{e}^{\mathrm{i}\eta x}\mathrm{e}^{\mathrm{i}qD}\mathrm{d}\eta\mathrm{d}q. \tag{2.1.16}$$

□

Example 2.1.10 *Fix a Euclidean structure in $\mathcal{X}$. Consider the projection P_0 onto the normalized vector*

$$\psi_0 = \pi^{-\frac{d}{4}}\mathrm{e}^{-\frac{1}{2}x^2}.$$

The integral kernel of P_0 equals

$$P_0(x,y) = \pi^{-\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}x^2 - \frac{1}{2}y^2}.$$

Its x, D- and D, x-symbols equal

$$\begin{aligned} \mathrm{s}_{P_0}^{x,D}(x,\xi) &= 2^{\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}x^2 - \frac{1}{2}\xi^2 - \mathrm{i}x\xi}, \\ \mathrm{s}_{P_0}^{D,x}(x,\xi) &= 2^{\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}x^2 - \frac{1}{2}\xi^2 + \mathrm{i}x\xi}. \end{aligned}$$

2.1.6 Point transformations

Recall that if $h, g \in C^\infty(\mathcal{X} \oplus \mathcal{X}')$ then the Poisson bracket of h and g is defined as

$$\{h, g\} = \nabla_x h \nabla_\xi g - \nabla_\xi h \nabla_x g \in C^\infty(\mathcal{X} \oplus \mathcal{X}').$$

Let $h \in \mathcal{S}'(\mathcal{X} \oplus \mathcal{X}')$. If $\eta \in \mathcal{X}'$ and $q \in \mathcal{X}$, then clearly,

$$\begin{aligned}
&[\eta x, \mathrm{Op}^{x,D}(h)] = \mathrm{i}\mathrm{Op}^{x,D}(\eta \nabla_\xi h) = \mathrm{i}\mathrm{Op}^{x,D}(\{\eta x, h\}), \\
&[qD, \mathrm{Op}^{x,D}(h)] = -\mathrm{i}\mathrm{Op}^{x,D}(q \nabla_x h) = \mathrm{i}\mathrm{Op}^{x,D}(\{q\xi, h\}).
\end{aligned} \tag{2.1.17}$$

Let $b \in gl(\mathcal{X}) \simeq \mathcal{X} \otimes \mathcal{X}' \simeq \mathcal{X} \otimes_{\mathrm{s}} \mathcal{X}'$. Clearly, b can be identified with the function $\mathcal{X} \times \mathcal{X}' \ni (x, \xi) \mapsto xc'\xi = \xi cx$. We have

$$\mathrm{Op}^{x,D}(c) = xc'D.$$

Note that

$$[xc'D, \mathrm{Op}^{x,D}(h)] = -\mathrm{i}\mathrm{Op}^{x,D}(xc'\nabla_x h - \xi c \nabla_\xi h) = \mathrm{i}\mathrm{Op}^{x,D}(\{c, h\}). \tag{2.1.18}$$

We can summarize (2.1.17) and (2.1.18) as follows:

Theorem 2.1.11 *If $h \in \mathcal{S}'(\mathcal{X} \oplus \mathcal{X}')$ and*

$$g \in \mathbf{R} \oplus \mathcal{X}' \oplus \mathcal{X} \oplus (\mathcal{X}' \otimes_{\mathrm{s}} \mathcal{X}) \subset \overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\leq 2}(\mathcal{X}' \oplus \mathcal{X}),$$

then

$$[\mathrm{Op}^{x,D}(g), \mathrm{Op}^{x,D}(h)] = \mathrm{i}\mathrm{Op}^{x,D}(\{g, h\}).$$

Let $m \in Gl(\mathcal{X}_1, \mathcal{X}_2)$. We define $m_* \in B(L^2(\mathcal{X}_1, \mathcal{X}_2))$ by setting

$$m_* \psi(x) = \psi(m^{-1}x), \quad x \in \mathcal{X}_1.$$

The following (straightforward) theorem expresses the covariance of the x, D muantization wrt the point transformations.

Theorem 2.1.12 *1) Let $m_1 \in GL(\mathcal{X}_1, \mathcal{X})$ and $m_2 \in Gl(\mathcal{X}, \mathcal{X}_2)$. then*

$$(m_2 m_1)_* = m_{2*} m_{1*},$$

2) Let $m \in Gl(\mathcal{X}_1, \mathcal{X}_2)$ and $h \in \mathcal{S}'(\mathcal{X}_1, \mathcal{X}_1')$. Then

$$m_* \mathrm{Op}^{x,D}(h) m_*^{-1} = \mathrm{Op}^{x,D}(h(m^{-1}\cdot, m'\cdot)),$$

3) Let $c \in gl(\mathcal{X})$. Then

$$\mathrm{e}^{\mathrm{i}xc'D} = (\mathrm{e}^{-c})_*.$$

2.1.7 Stone–von Neumann theorem

Theorem 2.1.13 (Stone–von Neumann theorem) *Suppose that $\mathcal{X}$ is a finite dimensional vector space and we are given a pair of strongly continuous unitary representations of the abelian groups $\mathcal{X}'$ and $\mathcal{X}$ in a Hilbert space $\mathcal{H}$*

$$\mathcal{X}' \ni \eta \mapsto V(\eta) \in U(\mathcal{H}),$$

$$\mathcal{X} \ni q \mapsto T(q) \in U(\mathcal{H}),$$

satisfying the Weyl commutation relations

$$V(\eta)T(q) = \mathrm{e}^{-\mathrm{i}\eta q} T(q)V(\eta).$$

Then there exists a Hilbert space $\mathcal{K}$ and a unitary operator

$$U : L^2(\mathcal{X}) \otimes \mathcal{K} \to \mathcal{H}$$

such that

$$\begin{aligned} V(\eta)U &= U\mathrm{e}^{\mathrm{i}\eta x} \otimes 1_{\mathcal{K}}, \\ T(q)U &= U\mathrm{e}^{\mathrm{i}qD} \otimes 1_{\mathcal{K}}. \end{aligned}$$

Proof. Step 1. Clearly, the groups $V(\eta)$ and $T(q)$ can be written as

$$V(\eta) = \mathrm{e}^{\mathrm{i}\eta x_\pi}, \qquad T(q) = \mathrm{e}^{\mathrm{i}qD_\pi},$$

for some vectors of self-adjoint operators x_π and D_π. We can define

$$\begin{aligned} P_0 \ &:= (2\pi)^{-d} \int \mathrm{e}^{-\frac{1}{4}\eta^2 - \frac{1}{4}q^2 + \frac{\mathrm{i}}{2}\eta q} \mathrm{e}^{\mathrm{i}\eta x_\pi} \mathrm{e}^{\mathrm{i}qD_\pi} \mathrm{d}\eta \mathrm{d}q \\ &= (2\pi)^{-d} \int \mathrm{e}^{-\frac{1}{4}\eta^2 - \frac{1}{4}q^2 - \frac{\mathrm{i}}{2}\eta q} \mathrm{e}^{\mathrm{i}qD_\pi} \mathrm{e}^{\mathrm{i}\eta x_\pi} \mathrm{d}\eta \mathrm{d}q, \end{aligned}$$

and $\mathcal{K} := \mathrm{Ran} P_0$. We check that

$$P_0^* = P_0, \quad P_0^2 = P_0.$$

Step 2. First check that if $f \in L^1(\mathcal{X})$, then

$$\pi^{\frac{d}{2}} P_0 \mathrm{e}^{x_\pi^2} f(x_\pi) P_0 = P_0 \int f(x)\mathrm{d}x. \tag{2.1.19}$$

Next we show that there exists a unique isometric operator $U : L^2(\mathcal{X}) \otimes \mathcal{K} \to \mathcal{H}$ such that

$$U\psi \otimes \rho := \pi^{\frac{d}{4}} \mathrm{e}^{\frac{1}{2}x_\pi^2} \psi(x_\pi)\rho, \qquad \psi \in L^2(\mathcal{X}), \quad \rho \in \mathcal{K}.$$

To verify that U is isometric, we compute:

$$\begin{aligned}(U\psi_1\otimes\rho_1|U\psi_2\otimes\rho_2) &= \pi^{\frac{d}{2}}(\rho_1|\mathrm{e}^{x_\pi^2}\overline{\psi_1}(x_\pi)\psi_2(x_\pi)\rho_2)\\ &= \pi^{\frac{d}{2}}(\rho_1|P_0\mathrm{e}^{x_\pi^2}\overline{\psi_1}(x_\pi)\psi_2(x_\pi)P_0\rho_2)\\ &= (\rho_1|\rho_2)(\psi_1|\psi_2),\end{aligned}$$

where in the last step we used (2.1.19).

Step 3. We prove that U intertwines the CCR relations. To this end, using the explicit integral formula for P_0, we first obtain

$$\mathrm{e}^{\mathrm{i}qD_\pi}P_0 \;= \mathrm{e}^{-qx_\pi-\frac{1}{2}q^2}P_0. \tag{2.1.20}$$

Thus, for $\rho\in\mathcal{K}$,

$$\mathrm{e}^{\mathrm{i}qD_\pi}\rho = \mathrm{e}^{-qx_\pi-\frac{1}{2}q^2}\rho.$$

Hence,

$$\begin{aligned}\mathrm{e}^{\mathrm{i}qD_\pi}U\psi\otimes\rho &= \pi^{\frac{d}{4}}\mathrm{e}^{\mathrm{i}qD_\pi}\mathrm{e}^{\frac{1}{2}x_\pi^2}\psi(x_\pi)\rho\\ &= \pi^{\frac{d}{4}}\mathrm{e}^{\frac{1}{2}(x_\pi+q)^2}\psi(x_\pi+q)\mathrm{e}^{\mathrm{i}qD_\pi}\rho\\ &= \pi^{\frac{d}{4}}\mathrm{e}^{\frac{1}{2}x_\pi^2}\psi(x_\pi+q)\rho = U(\mathrm{e}^{\mathrm{i}qD}\psi)\otimes\rho.\end{aligned}$$

It is easier to check that U intertwines the position operators:

$$\mathrm{e}^{\mathrm{i}\eta x_\pi}U\psi\otimes\rho = \pi^{\frac{d}{4}}\mathrm{e}^{\mathrm{i}\eta x_\pi}\mathrm{e}^{\frac{1}{2}x_\pi^2}\psi(x_\pi)\rho = U\mathrm{e}^{\mathrm{i}\eta x_\pi}\psi\otimes\rho.$$

Step 5. Finally, let us show that U is surjective. Clearly, if $\rho\in\mathcal{K}$, then $U\pi^{-\frac{d}{4}}\mathrm{e}^{-\frac{1}{2}x^2}\otimes\rho = \rho$. Hence $\mathcal{K}\subset \mathrm{Ran}U$. Thus, using the intertwining property of U, it is enough to show that

$$\{\mathrm{e}^{\mathrm{i}\eta x_\pi}\mathrm{e}^{\mathrm{i}qD_\pi}\rho\;:\;\eta\in\mathcal{X}',\;q\in\mathcal{X},\;\rho\in\mathcal{K}\} \tag{2.1.21}$$

spans $\mathcal{H}$.

Let $\zeta\in\mathcal{H}$. Set $f(\eta,q):=(\zeta|\mathrm{e}^{\mathrm{i}\eta x_\pi}\mathrm{e}^{\mathrm{i}qD_\pi}\zeta)$. Assume that ζ is orthogonal to (2.1.21). Then

$$0=(\zeta|\mathrm{e}^{\mathrm{i}\eta x_\pi}\mathrm{e}^{\mathrm{i}qD_\pi}P_0\mathrm{e}^{-\mathrm{i}\eta x_\pi}\mathrm{e}^{-\mathrm{i}qD_\pi}\zeta) = (2\pi)^{-d}\int\mathrm{d}q_1\mathrm{d}\eta_1 f(\eta_1 q_1)\mathrm{e}^{-\frac{1}{4}\eta^2-\frac{1}{4}q^2+\frac{\mathrm{i}}{2}\eta q}\mathrm{e}^{\mathrm{i}\eta_1 q-\mathrm{i}\eta q_1}.$$

By the properties of the Fourier transformation, $f(\eta,q)=0$ almost everywhere. But $(\eta,q)\mapsto f(\eta,q)$ is a continuous function and $f(0,0)=(2\pi)^{-d}\|\zeta\|^2$. So $\zeta=0$. □

2.2 Symplectic theory of CCR's

2.2.1 Heisenberg-type representations of CCR

Let $\mathcal{Y}$ be a real vector space with an antisymmetric form ω. Let $\mathcal{D}$ be a unitary space. We say that a linear map

$$\mathcal{Y} \ni y \mapsto \phi_\pi(y) \in L(\mathcal{D}) \tag{2.2.22}$$

is a Heisenberg-type representation of CCR over $\mathcal{Y}$ in $\mathcal{D}$ iff $\phi_\pi(y)$ are symmetric operators satisfing

$$[\phi_\pi(y_1), \phi_\pi(y_2)] = \mathrm{i} y_1 \omega y_2, \quad y_1, y_2 \in \mathcal{Y}. \tag{2.2.23}$$

Remark 2.2.1 *If we treat $\mathcal{Y} \oplus \mathbf{R}$ as the Heisenberg algebra and if $L(\mathcal{D})$ is equipped with the bracket $\frac{1}{\mathrm{i}}[\cdot, \cdot]$, then the condition (2.2.23) is equivalent to*

$$\mathcal{Y} \oplus \mathbf{R} \ni (y, t) \mapsto \phi_\pi(y) + t 1_{\mathcal{D}} \in L(\mathcal{D})$$

being a homomorphism of Lie algebras.

By complex linearity we can extend the definition of $\phi_\pi(y)$ to $\mathbf{C}\mathcal{Y}$, by setting

$$\phi_\pi(y_1 + \mathrm{i} y_2) := \phi_\pi(y_1) + \mathrm{i}\phi_\pi(y_2), \quad y_1, y_2 \in \mathcal{Y}.$$

Thus we have

$$\phi_\pi(w)^* = \phi_\pi(\overline{w}), \quad w \in \mathbf{C}\mathcal{Y}.$$

The algebra of complex polynomials in $\phi_\pi(y)$, $y \in \mathcal{Y}$, treated as a subalgebra of $L(\mathcal{D})$, will be denoted $\mathrm{CCR}_\pi(\mathcal{Y})$.

Remark 2.2.2 $\mathrm{CCR}_\pi(\mathcal{Y})$ *is sometimes called the (abstract) Weyl algebra.*

Example 2.2.3 *Consider the symplectic space $\mathcal{X}' \oplus \mathcal{X}$, where $\mathcal{X}$ is a real vector space of dimension d and the symplectic form is given by*

$$(\eta_1, q_1)\omega(\eta_2, q_2) = \eta_1 q_2 - \eta_2 q_1.$$

The map

$$\mathcal{X}' \oplus \mathcal{X} \ni (\eta, q) \mapsto \eta x + qD \in L(\mathcal{S}(\mathcal{X}))$$

is a Heisenberg-type representation of CCR, which is a version of the Schrödinger representation.

2.2.2 Weyl-Wigner quantization

Let $y_1, \dots, y_n \in \mathcal{Y}$. We define

$$\mathrm{Op}_\pi(y_1 \otimes_\mathrm{s} \cdots \otimes_\mathrm{s} y_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \phi_\pi(y_{\sigma 1}) \cdots \phi_\pi(y_{\sigma n}). \tag{2.2.24}$$

The map extends uniquely to a linear map

$$\mathbf{C}\overset{\circ}{\Gamma}_\mathrm{s}(\mathcal{Y}) \ni h \mapsto \mathrm{Op}_\pi(\mathcal{Y}) \in \mathrm{CCR}_\pi(h) \subset L(\mathcal{D}). \tag{2.2.25}$$

Let $h_1, h_2 \in \mathbf{C}\overset{\circ}{\Gamma}_\mathrm{s}(\mathcal{Y})$. We define the Weyl-Wigner-Moyall product $\#$

$$\begin{aligned} h_1 \# h_2(v) \ &:= \exp\left(\tfrac{\mathrm{i}}{2} \nabla_{v_1} \omega \nabla_{v_2}\right) h_1(v_1) h_2(v_2)\Big|_{v=v_1=v_2} \\ &= \pi^{-2d} (\det \omega)^{-1} \int_{\mathcal{Y}'} \int_{\mathcal{Y}'} \mathrm{e}^{2\mathrm{i}(v-v_1)\omega^{-1}(v-v_2)} h_1(v_1) h_2(v_2) \mathrm{d}v_1 \mathrm{d}v_2. \end{aligned} \tag{2.2.26}$$

(The first formula can be interpreted as a finite sum of differential operators. The integral formula is valid as an oscillatory integral, if $\mathcal{Y}$ is a finite dimensional symplectic space of dimension $2d$).

Theorem 2.2.4 *0)* $1 = \mathrm{Op}_\pi(1)$.
1) If $h \in \mathbf{C}\overset{\circ}{\Gamma}_\mathrm{s}(\mathcal{Y})$, then

$$\mathrm{Op}_\pi(\overline{h}) = \mathrm{Op}_\pi(h)^*.$$

2) If $y \in \mathbf{C}\mathcal{Y}$, then

$$\mathrm{Op}_\pi(y) = \phi_\pi(y).$$

More generally, if $\mathcal{X}$ is an isotropic subspace $\mathcal{Y}$, then the operators $\phi_\pi(y)$, $y \in \mathcal{X}$ commute with one another, and it makes sense, at least for $g \in \mathbf{C}\overset{\circ}{\Gamma}_\mathrm{s}(\mathcal{X})$, (polynomials in the $\mathcal{X}'$ variables) to use the notation $g(\phi_\pi)$ for functions of ϕ_π. We then have

$$\mathrm{Op}_\pi(g) = g(\phi_\pi).$$

3) If $h_1, h_2 \in \mathbf{C}\overset{\circ}{\Gamma}_\mathrm{s}(\mathcal{Y})$, then the following formula for the product is valid:

$$\mathrm{Op}_\pi(h_1 \# h_2) = \mathrm{Op}_\pi(h_1) \mathrm{Op}_\pi(h_2),$$

4) If $h \in \mathbf{C}\overset{\circ}{\Gamma}_\mathrm{s}(\mathcal{Y})$ and $y \in \mathbf{C}\mathcal{Y}$, then

$$\frac{1}{2}\left(\phi_\pi(y)\mathrm{Op}_\pi(h) + \mathrm{Op}_\pi(h)\phi_\pi(y)\right) = \mathrm{Op}_\pi(y \otimes_\mathrm{s} h).$$

Theorem 2.2.5 *If $h_1, \dots, h_n \in \mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y})$, then the following version of the Wick theorem for the Weyl-Wigner quantization is true*

$$h_1 \# \cdots \# h_n(v) = \exp(\tfrac{\mathrm{i}}{2} \sum_{i<j} \nabla_{v_i} \omega \nabla_{v_j}) h_1(v_1) \cdots h_n(v_n)\Big|_{v=v_1=\cdots=v_n}.$$

Suppose that $\mathrm{Op}_\pi(h) = 0$ implies $h = 0$ (which will be satisfied in our applications). Then the map (2.2.25) is injective. In this case one can define its inverse, which will be called the Weyl-Wigner symbol of the operator H, denoted

$$\mathrm{CCR}_\pi(\mathcal{Y}) \ni H \mapsto \mathrm{s}_H \in \mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y}).$$

Remark 2.2.6 *The Weyl-Wigner quantization is usually called the Weyl quantization, especially in the mathematical literature devoted to PDE's, in the context of the Schrödinger representation. On the other hand, if $\psi \in \mathcal{H}$, then the Weyl-Wigner symbol of the orthogonal projection onto ψ is often called the Wigner function of ψ – especially in the physics literature. Our terminology is a compromise of those two names.*

2.2.3 Abstract definition of the x, D quantization

Suppose that $(\mathcal{X}, \mathcal{X}')$ is a dual pair of real vector spaces. Suppose that $\mathcal{Y} = \mathcal{X}' \oplus \mathcal{X}$ is equipped with the canonical symplectic form ω. Let $\mathcal{X}' \oplus \mathcal{X} \ni (\eta, q) \mapsto \phi(\eta, q) \in L(\mathcal{D})$ be a Heisenberg-type reperesentation of CCR. Then operators $\phi(\eta, 0)$, $\eta \in \mathcal{X}'$, commute with one another. So do the operators $\phi(0, q)$, $q \in \mathcal{X}$. Let us rename them

$$\eta x := \phi(\eta, 0), \quad qD := \phi(0, q).$$

Then one can introduce the x, D and D, x quantizations for symbols in $\mathring{\Gamma}_{\mathrm{s}}(\mathcal{X}' \oplus \mathcal{X})$ exactly as it was done in Subsection **??**.

2.2.4 Weyl-type representations of CCR

Let $\mathcal{H}$ be a Hilbert space. We say that a map

$$\mathcal{Y} \ni y \mapsto W_\pi(y) \in U(\mathcal{H}) \tag{2.2.27}$$

is a Weyl-type representation of CCR relations over $\mathcal{Y}$ in $\mathcal{H}$ if

$$W_\pi(y_1) W_\pi(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2} y_1 \omega y_2} W_\pi(y_1 + y_2). \tag{2.2.28}$$

Remark 2.2.7 *If we treat $\mathcal{Y} \oplus \mathbf{R}$ as the Heisenberg group, then (2.2.28) is equivalent to*

$$\mathcal{Y} \oplus \mathbf{R} \ni (y, t) \mapsto W_\pi(y) \mathrm{e}^{-\mathrm{i}t} \in U(\mathcal{H})$$

being a homomorphism of groups.

Note that (2.2.27) implies

$$\begin{aligned} &W_\pi^*(y) = W_\pi(-y), \qquad W_\pi(0) = 1, \\ &W_\pi(y_1)W_\pi(y_2) = \mathrm{e}^{-\mathrm{i}y_1\omega y_2} W_\pi(y_2)W_\pi(y_1), \\ &W_\pi(t_1 y)W_\pi(t_2 y) = W((t_1+t_2)y), \quad t_1, t_2 \in \mathbf{R}. \end{aligned} \tag{2.2.29}$$

We say that a subset $K \subset \mathcal{H}$ is cyclic if $\mathrm{Span}\{W_\pi(y)\psi \ : \ \psi \in K, \ y \in \mathcal{Y}\}$ is dense in $\mathcal{H}$. We say that $\psi_0 \in \mathcal{H}$ is cyclic if $\mathrm{Span}\{W_\pi(y)\psi_0 \ : \ \ y \in \mathcal{Y}\}$ is dense in $\mathcal{H}$.

We say that the representation $\mathcal{Y} \ni y \mapsto W_\pi(y) \in U(\mathcal{H})$ is irreducible if $H \in B(\mathcal{H})$ and $[W_\pi(y), H] = 0$ implies that H is proportional to identity. Clearly, in the case of an irreducible representation, all the nonzero vectors in $\mathcal{H}$ are cyclic.

If we have representations of CCR over $\mathcal{Y}_1$ and $\mathcal{Y}_2$ in spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\mathcal{Y}_1 \oplus \mathcal{Y}_2 \ni (y_1, y_2) \mapsto W_{\pi_1}(y_1) \otimes W_{\pi_2}(y_2)$ is representation of CCR over $\mathcal{Y}_1 \oplus \mathcal{Y}_2$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

2.2.5 Regular representations of CCR

A Weyl-type representation of CCR (2.2.27) is called a regular representation of CCR, if

$$\mathbf{R} \ni t \mapsto W_\pi(ty) \quad \text{is strongly continuous for any } y \in \mathcal{Y}. \tag{2.2.30}$$

Example 2.2.8 *Consider the symplectic space $\mathcal{X}' \oplus \mathcal{X}$, as in Example 2.2.3. Then it follows from Theorem 2.1.12 that the map*

$$\mathcal{X}' \oplus \mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}(\eta x + qD)} \in U(L^2(\mathcal{X}))$$

is an irreducible regular representation of CCR over $\mathcal{X}' \oplus \mathcal{X}$ in $L^2(\mathcal{X})$, called the Schrödinger representation.

Below we restate the Stone–von Neumann Theorem (Theorem 2.1.13) adapting it to the context of regular representations of CCR. It will say that up to a multiplicity all regular representations of CCR over a finite dimensional symplectic space are isomorphic to the Schrödinger representation.

Theorem 2.2.9 (Stone–von Neumann theorem) *Suppose that $\mathcal{Y}$ is a finite dimensional symplectic space. Let $\mathcal{Y} \ni y \mapsto W_\pi(y) \in U(\mathcal{H})$ be a regular representation of CCR over $\mathcal{Y}$ in $\mathcal{H}$. Choose an isomorphism of $\mathcal{Y}$ with $\mathcal{X}' \oplus \mathcal{X}$ (which is always possible). Then there exists a Hilbert space $\mathcal{K}$ and a unitary operator $U : L^2(\mathcal{X}) \otimes \mathcal{K} \to \mathcal{H}$ such that*

$$W_\pi(\eta, q)U = U\mathrm{e}^{\mathrm{i}(\eta x + qD)} \otimes 1_{\mathcal{K}}.$$

2.2.6 Field operators

From now on we assume that we are given a regular representation $\mathcal{Y} \ni y \mapsto W_\pi(y) \in U(\mathcal{H})$. By (2.2.29) and (2.2.30), $\mathbf{R} \ni t \mapsto W_\pi(ty)$ is a strongly continuous unitary group. By the Stone theorem, for any $y \in \mathcal{Y}$, we can define its self-adjoint generator

$$\phi_\pi(y) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} W_\pi(ty)\Big|_{t=0}.$$

$\phi_\pi(y)$ will be called the field operator corresponding to $y \in \mathcal{Y}$.

Remark 2.2.10 *Sometimes, the operators $\phi_\pi(y)$ are called Segal field operators.*

Let $\mathcal{Y}$ be of any dimension. If $\mathcal{Y}_0$ is a subspace of $\mathcal{Y}$ we define $\mathcal{G}(\mathcal{Y}_0) \subset \mathcal{H}$ to be the intersection of

$$\mathcal{D}(\phi_\pi(y_1)\cdots\phi_\pi(y_n)), \quad y_1, \dots, y_n \in \mathcal{Y}_0, \quad n = 1, 2\dots \tag{2.2.31}$$

$\mathcal{G}(\mathcal{Y}_0)$ has a natural structure of a Fréchet space with the seminorms given by $\psi \mapsto \|\phi_\pi(y_1)\cdot\phi_\pi(y_n)\psi\|$.

Theorem 2.2.11 *Let $\mathcal{Y}_0$ be a subspace of $\mathcal{Y}$.*
1) $\mathcal{Y}_0 \ni y \mapsto \phi_\pi(y) \in L(\mathcal{G}(\mathcal{Y}_0))$ is a Heisenberg-type representation of CCR over $\mathcal{Y}_0$;
2) If $\mathcal{Y}_0$ is finite dimensional, then $\mathcal{G}(\mathcal{Y}_0)$ is dense in $\mathcal{H}$;
3) If $\mathcal{Y}_0$ is finite dimensional, ω is nondegenerate on $\mathcal{Y}_0$ and $\psi_1, \psi_2 \in \mathcal{G}(\mathcal{Y}_0)$, then the function $\mathcal{Y}_0 \ni y \mapsto (\psi_1|W_\pi(y)\psi_2)$ belongs to $\mathcal{S}(\mathcal{Y}_0)$.

Proof. 1) is immediate.
To show 2) it is enough to consider two cases.
Case I. $\omega = 0$ on $\mathcal{Y}_0$. 2) follows by the spectral theorem.
Case II. ω is nondegenerate on $\mathcal{Y}_0$. Then, for some vector space $\mathcal{X}$, we can identify $\mathcal{Y}_0$ with $\mathcal{X}' \oplus \mathcal{X}$ with the canonical symplectic form, and by Theorem 2.2.9 there exists a unitary map $U : L^2(\mathcal{X}) \otimes \mathcal{K} \to \mathcal{H}$ such that

$$W_\pi(y)U = W(y) \otimes 1_\mathcal{K} U, \quad y \in \mathcal{Y}_0$$

where $W(y)$ denote the Weyl operators in the Schrödinger representation. Now we know from Theorem 2.1.1 that

$$U\mathcal{G}(\mathcal{Y}_0) = \mathcal{S}(\mathcal{X}) \otimes \mathcal{K}, \tag{2.2.32}$$

which is clearly dense in $L^2(\mathcal{X}) \otimes \mathcal{K}$.

In general, $\mathcal{Y}_0$ is a direct sum of two symplecticaly orthogonal subspaces corresponding to the Case I and Case II.

3) follows from (2.2.32) and Theorem 2.1.8 2). □

Lemma 2.2.12 *For any $y, y_1 \in \mathcal{Y}$, $W_\pi(y)$ leaves invariant $\mathcal{D}(\phi_\pi(y_1))$ and*

$$[\phi_\pi(y), W_\pi(y_1)] = y_1\omega y W_\pi(y_1). \tag{2.2.33}$$

Theorem 2.2.13 *Let $y_1, y_2 \in \mathcal{Y}$.*
1) Let $\psi_1, \psi_2 \in \mathcal{D}(\phi_\pi(y_1)) \cap \mathcal{D}(\phi_\pi(y_2))$. Then

$$(\phi_\pi(y_1)\psi_1|\phi_\pi(y_2)\psi_2) - (\phi_\pi(y_2)\psi_1|\phi_\pi(y_1)\psi_2) = \mathrm{i}y_1\omega y_2(\psi_1|\psi_2).$$

2) $\phi_\pi(y_1 + \mathrm{i}y_2)$ is a closed operator on the domain

$$\mathcal{D}(\phi_\pi(y_1 + \mathrm{i}y_2)) := \mathcal{D}(\phi_\pi(y_1)) \cap \mathcal{D}(\phi_\pi(y_2)).$$

Proof. By differentiating wrt t_1 and t_2 and setting $t_1 = t_2 = 0$ the identity

$$(W_\pi(t_1y_1)\psi_1|W_\pi(t_2y_2)\psi_2) = \mathrm{e}^{-\mathrm{i}t_1t_2y_1\omega y_2}(W_\pi(t_2y_2)\psi_1|W_\pi(t_1y_1)\psi_2)$$

we obtain 1).

Next, using 1), for $\psi \in \mathcal{D}(\phi_\pi(y_1)) \cap \mathcal{D}(\phi_\pi(y_2))$ we can write

$$\|\phi_\pi(y_1 + \mathrm{i}y_2)\psi\|^2 = \|\phi_\pi(y_1)\psi\|^2 + \|\phi_\pi(y_2)\psi\|^2 - y_1\omega y_2\|\psi\|^2.$$

We know that $\phi_\pi(y_1)$ and $\phi_\pi(y_2)$ are self-adjoint, hence closed. Hence $\mathcal{D}(\phi_\pi(y_1))$ and $\mathcal{D}(\phi_\pi(y_2))$ are complete in the graph norms. Hence so is $\mathcal{D}(\phi_\pi(y_1)) \cap \mathcal{D}(\phi_\pi(y_2))$ in the intersection norm. □

2.2.7 Quadratic forms on $\mathcal{G}(\mathcal{Y})$

We say that $H \in \mathcal{F}_\pi'(\mathcal{Y})$ if H is a continuous map

$$\mathcal{G}_\pi(\mathcal{Y}) \times \mathcal{G}_\pi(\mathcal{Y}) \ni (\psi_1, \psi_2) \mapsto (\psi_1|H\psi_2) \in \mathbf{C},$$

antilinear wrt the first component and linear wrt the second. $\mathcal{F}_\pi'(\mathcal{Y})$ is called the space of (sesquilinear) forms on $\mathcal{G}_\pi(\mathcal{Y})$ and can be regarded as a natural extension of $B(\mathcal{H})$. It can be viewed as the "space of noncommutative tempered distributions on $\mathcal{Y}'$".

Note that $\mathrm{CCR}_\pi(\mathcal{Y}) \subset \mathcal{F}'(\mathcal{Y})$. In addition, $\mathcal{F}_\pi'(\mathcal{Y})$ is a left and right module over $\mathrm{CCR}_\pi(\mathcal{Y})$.

2.2.8 Weyl-Wigner quantization of distributional symbols

Now assume that $\mathcal{Y}$ is a finite dimensional symplectic space of dimension $2d$. Suppose that $\mathcal{Y} \ni y \mapsto W_\pi(y) \in U(\mathcal{H})$ is a regular irreducible representation of CCR.

Recall that for a function $h \in \mathcal{S}'(\mathcal{Y}')$ the Fourier transform of h, denoted $\hat{h} \in \mathcal{S}'(\mathcal{Y})$, satisfies

$$h(v) = (2\pi)^{-2d} \int_{\mathcal{Y}} \hat{h}(y) \mathrm{e}^{\mathrm{i}yv} \mathrm{d}y, \quad v \in \mathcal{Y}'.$$

We can extend the definition of the Weyl-Wigner quantization to $\mathcal{S}'(\mathcal{Y}')$ as follows. We know from Theorem 2.2.11 3) that if $\psi_1, \psi_2 \in \mathcal{G}(\mathcal{Y})$, then $\mathcal{Y} \ni y \mapsto (\psi_1 | W_\pi(y)\psi_2)$ is a Schwartz function. Therefore, if $h \in \mathcal{S}'(\mathcal{Y}')$, then $\mathrm{Op}_\pi(h) \in \mathcal{F}'_\pi(\mathcal{Y})$ can be defined by the formula

$$\begin{aligned}(\psi_1 | \mathrm{Op}_\pi(h)\psi_2) &:= (2\pi)^{-2d} \textstyle\int_{\mathcal{Y}} \hat{h}(y)(\psi_1 | W_\pi(y)\psi_2)\mathrm{d}y \\ &= (2\pi)^{-2d} \textstyle\int_{\mathcal{Y}} \int_{\mathcal{Y}'} h(v)(\psi_1 | W_\pi(y)\psi_2)\mathrm{e}^{-\mathrm{i}vy}\mathrm{d}y\mathrm{d}v, \qquad \psi_1, \psi_2 \in \mathcal{G}(\mathcal{Y}).\end{aligned} \tag{2.2.34}$$

Theorem 2.2.14 *1) If $h \in \mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y})$, then the definition (2.2.24) coincides with (2.2.34);*
2) $W_\pi(y) = \mathrm{Op}_\pi(\mathrm{e}^{\mathrm{i}y})$. More generally, if $\mathcal{X}$ is an isotropic subspace of $\mathcal{Y}$, then $\mathcal{X} \ni q \mapsto \phi_\pi(q)$ is a vector of commuting self-adjoint operators. Then if $f \in \mathcal{S}'(\mathcal{X}') \subset \mathcal{S}'(\mathcal{Y}')$,

$$f(\phi_\pi) = \mathrm{Op}_\pi(f).$$

3) If $g \in \mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y})$, $h \in \mathcal{S}'(\mathcal{Y}')$, then

$$\mathrm{Op}_\pi(h)^* = \mathrm{Op}_\pi(\overline{h}),$$

$$\mathrm{Op}_\pi(g)\mathrm{Op}_\pi(h) = \mathrm{Op}_\pi(g\#h).$$

4) $W_\pi(-\omega^{-1}v)\mathrm{Op}_\pi(h)W_\pi(\omega^{-1}v) = \mathrm{Op}_\pi(h(\cdot - v))$.
5) The map

$$\mathcal{S}'(\mathcal{Y}') \ni h \mapsto \mathrm{Op}_\pi(h) \in \mathcal{F}'_\pi(\mathcal{Y})$$

is bijective. Its inverse, called the Weyl-Wigner symbol and denoted

$$\mathcal{F}'_\pi(\mathcal{Y}) \ni H \mapsto \mathrm{s}_H \in \mathcal{S}'(\mathcal{Y}'),$$

is thus well defined.
6) We have $H \in l^2(\mathcal{H})$ iff $\mathrm{s}_H \in L^2(\mathcal{Y})$ and

$$\mathrm{Tr} H^* G = (2\pi)^{-d} \int \overline{\mathrm{s}_H(v)} s_G(v) \mathrm{d}v.$$

Proof. To prove 1), it is enough to consider $y_0 \in \mathcal{Y}$ and $h = y_0^n$, because such tensors span $\mathbf{C}\Gamma_{\mathrm{s}}^n(\mathcal{Y})$. The Fourier transform of h (viewed as a function on $\mathcal{Y}'$) equals $\hat{h} = (2\pi)^{2d}\mathrm{i}^n(y_0\nabla_y)^n\delta_0$. Hence

$$\mathrm{Op}_\pi(h) = (-\mathrm{i})^n(y_0\nabla_y)^n W_\pi(y)\Big|_{y=0} = \phi_\pi(y_0)^n.$$

To prove 5) and 6) it is convenient to use the Schrödinger representation, considered in the next subsection. □

In what follows, we will often drop the subscript π if we deal with a fixed representation.

2.2.9 Weyl-Wigner quantization in the Schrödinger representation

Let $\mathcal{X}$ be a finite dimensional vector space. Consider the Schrödinger representation $\mathcal{X}\oplus\mathcal{X}' \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}(\eta x+qD)} \in U(L^2(\mathcal{X}))$. Let $\mathrm{Op}(\cdot)$ denote the Weyl-Wigner quantization of h in the Schrödinger representation.

Theorem 2.2.15 *1) Let $h \in \mathcal{S}'(\mathcal{X}\oplus\mathcal{X}')$. The distributional kernel of* $\mathrm{Op}(h)$ *can be computed as follows:*

$$\mathrm{Op}(h)(x,y) = (2\pi)^{-d}\int_{\mathcal{X}'} h((x+y)/2,\xi)\mathrm{e}^{\mathrm{i}(x-y)\xi}\mathrm{d}\xi. \qquad (2.2.35)$$

2) Let $H \in \mathcal{F}'(\mathcal{X}'\oplus\mathcal{X})$. The symbol can be obtained from the distributional kernel by the formula

$$\mathrm{s}_H(x,\xi) = \textstyle\int_{\mathcal{X}} H(x+\frac{y}{2}, x-\frac{y}{2})\mathrm{e}^{-\mathrm{i}\xi y}\mathrm{d}y.$$

3) The relationship between the x, D-symbol and the Weyl-Wigner symbol is as follows:

$$\begin{aligned}\mathrm{s}_H^{x,D}(x,\xi) &= \mathrm{e}^{-\frac{\mathrm{i}}{2}\nabla_x\nabla_\xi}s_H(x,\xi)\\ &= \pi^{-d}\int\limits_{\mathcal{X}\oplus\mathcal{X}'}\mathrm{e}^{-\mathrm{i}2(x-x_1)(\xi-\xi_1)}\mathrm{s}_H(x_1,\xi_1)\mathrm{d}x_1\mathrm{d}\xi_1.\end{aligned}$$

Proof. Let us prove 1). It is enough to check (2.2.35) for $h(x,\xi) := \mathrm{e}^{\mathrm{i}(\eta x+q\xi)}$. We know that

$$\mathrm{Op}(h) = W(\eta,q) = \mathrm{e}^{\mathrm{i}(\eta x+qD)} = \mathrm{e}^{\frac{\mathrm{i}}{2}\eta x}\mathrm{e}^{\mathrm{i}qD}\mathrm{e}^{\frac{\mathrm{i}}{2}\eta x},$$

which has the integral kernel

$$H(x,y) = (2\pi)^{-d}\int_{\mathcal{X}'}\mathrm{e}^{\frac{\mathrm{i}}{2}\eta(x+y)}\mathrm{e}^{\mathrm{i}q\xi+\mathrm{i}\xi(x-y)}\mathrm{d}\xi.$$

□

Remark 2.2.16 *In the context of the Schrödinger representation the Weyl quantization of h is often denoted by*

$$h^{\mathrm{w}}(x, D) \quad \textit{instead of} \quad \mathrm{Op}(h).$$

Example 2.2.17 *Let P_0 be the operator considered in Example 2.1.10 (the orthogonal projection onto $\pi^{-\frac{d}{4}}\mathrm{e}^{-\frac{1}{2}x^2}$). Then*

$$\mathrm{s}_{P_0}(x, \xi) = 2^d \mathrm{e}^{-x^2-\xi^2}.$$

Exercise *Compute the Weyl-Wigner symbol of the operators*

$$\Big((D - A(x))^2 + V(x)\Big),$$

$$\Big((D - A(x))^2 + V(x)\Big)^2.$$

2.2.10 The parity operator

Let $\mathcal{Y}$ be a finite dimensional symplectic space of dimension $2d$. Suppose that $\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$ is a regular representation of CCR.

Let δ_v denote the deltafunction at $v \in \mathcal{Y}'$.

Define

$$I = \mathrm{Op}(\pi^d \delta_0). \tag{2.2.36}$$

Theorem 2.2.18 *1) I is self-adjoint and $I^2 = 1$.*
2) In the Schrödinger representation

$$I\psi(x) = \psi(-x). \tag{2.2.37}$$

Proof. Let us show 2) first. The distributional kernel of I equals

$$\begin{aligned} I(x, y) &= 2^{-d} \int \delta(\tfrac{x+y}{2}, \xi)\mathrm{e}^{\mathrm{i}(x-y)\xi}\mathrm{d}\xi \\ &= 2^{-d}\delta(\tfrac{x+y}{2}) = \delta(x + y). \end{aligned}$$

This proves 2). 1) follows immediately from 2). $\square$

Define also

$$I_v = \mathrm{Op}(\pi^d \delta_v) = W(\omega^{-1}v)IW(-\omega^{-1}v).$$

Theorem 2.2.19 *1) If $h \in L^1(\mathcal{Y}')$, then $\mathrm{Op}(h)$ is a compact operator. In terms of an absolutely norm convergent integral we can write*

$$\mathrm{Op}(h) = \pi^{-d} \int I_v h(v) \mathrm{d}v. \tag{2.2.38}$$

We have

$$\|\mathrm{Op}(h)\| \leq \pi^{-d} \|h\|_1. \tag{2.2.39}$$

2) If H is trace class then $s_H \in C_\infty(\mathcal{Y}')$ and

$$s_H(v) = 2^d \mathrm{Tr} I_v H, \tag{2.2.40}$$

$$|\mathrm{s}_H| \leq 2^d \mathrm{Tr}|H|.$$

Proof. The formula (2.2.38) follows from the definition (2.2.34) and gives the estimate (2.2.39).

Let $h \in L^1$. We can approximate h with $h_n \in L^1 \cap L^2$ in the norm L^1. By the estimate (2.2.39) we have $\mathrm{Op}(h_n) \to \mathrm{Op}(h)$ in norm. We know that operators with L^2 symbols are Hilbert-Schmidt, hence compact. Therefore, the operators $\mathrm{Op}(h_n)$ are compact. Hence, so is $\mathrm{Op}(h)$. This ends the proof of 1).

Let us prove 2). Let $g \in L^2 \cap L^1$ and let H be trace class. Then H is also Hilbert-Schmidt, and using first Theorem 2.2.14 6), then (2.2.38), and finally the trace class property of H we obtain

$$\begin{aligned} (2\pi)^{-d} \int \overline{g}(v) \mathrm{s}_H(v) \mathrm{d}v = \mathrm{Tr}\mathrm{Op}(g)^* H \quad &= \pi^{-d} \mathrm{Tr}\left(\int \overline{g}(v) I_v \mathrm{d}v H \right) \\ &= (2\pi)^{-d} \int \overline{g}(v) 2^d \mathrm{Tr} I_v H \mathrm{d}v. \end{aligned}$$

This proves the identity (2.2.40) for almost all v.

Using the fact that $v \mapsto I_v$ is strongly continuous and H is trace class we see that $2^d \mathrm{Tr} I_v H$ is continuous. Using $\mathrm{w-}\lim_{v \to \infty} I_v = 0$ and the trace class property of H we conclude that $\lim_{v \to \infty} 2^d \mathrm{Tr} I_v H = 0$. □

2.2.11 "Prequantization"

Le $\mathcal{Y}$ be a finite dimensional space with an antisymmetric form ω. There exists a method to construct a representation of CCR in a fully ω-invariant way.

We define two maps

$$\mathcal{Y} \ni y \mapsto W_+(y) = \mathrm{e}^{\mathrm{i}(yv - \frac{1}{2} y \omega D_v)} \in U(L^2(\mathcal{Y}')),$$

$$\mathcal{Y} \ni y \mapsto W_-(y) = \mathrm{e}^{\mathrm{i}(yv + \frac{1}{2} y \omega D_v)} \in U(L^2(\mathcal{Y}')).$$

Then it is easy to see that the first map is a regular representation of CCR over $(\mathcal{Y},\omega)$ and the second is a regular representation of CCR over $(\mathcal{Y},-\omega)$. They commute with one another. For $h\in\mathcal{S}'(\mathcal{Y}')$ let

$$\mathrm{Op}_\pm(h) := (2\pi)^{-2d}\int \hat{h}(y)W_\pm(y)\mathrm{d}y$$

be the corresponding Weyl-Wigner quantizations of h.

We can also define the usual Schrödinger representation over $\mathcal{Y}\oplus\mathcal{Y}'$ equpped with the canonical form:

$$\mathcal{Y}\oplus\mathcal{Y}'\ni(y,\zeta)\mapsto W(y,\zeta):=\mathrm{e}^{\mathrm{i}(yv+\zeta D_v)}\in U(L^2(\mathcal{Y}')).$$

The corresponding Weyl-Wigner quantization will be denoted $\mathrm{Op}(g)$, where $g\in\mathcal{S}'(\mathcal{Y}'\oplus\mathcal{Y})$.

The relationship between these two quantizations is as follows: If $h\in\mathcal{S}'(\mathcal{Y}')$, we define $h_\pm\in\mathcal{S}'(\mathcal{Y}'\oplus\mathcal{Y})$ by setting

$$h_\pm(v,y):=h(v\mp\tfrac{1}{2}\omega y).$$

Then

$$\mathrm{Op}_\pm(h)=\mathrm{Op}(h_\pm).$$

2.3 General coherent vectors

2.3.1 Coherent states transformation

We assume that $\mathcal{Y}$ is a finite dimensional symplectic space of dimension $2d$. Let $\mathcal{Y}\ni y\mapsto W(y)\in U(\mathcal{H})$ be an irreducible regular representation of CCR. Let $\psi_0\in\mathcal{H}$ a fixed normalized vector. We define the family of coherent vectors associated with the vector ψ_0 by setting

$$\psi_v:=W(-\omega^{-1}v)\psi_0,\quad v\in\mathcal{Y}'.$$

The orthogonal projection onto the coherent vector ψ_v will be denoted

$$P_v:=W(-\omega^{-1}v)P_0W(\omega^{-1}v).$$

Remark 2.3.1 *Let us mention an additional assumption on ψ_0, which sometimes is imposed. One can demand that, for any $y\in\mathcal{Y}$, $\psi_0\in\mathcal{D}(\phi(y))$ and*

$$(\psi_0|\phi(y)\psi_0)=0.$$

This assumption implies that $\psi_v\in\mathcal{D}(\phi(y))$ and

$$(\psi_v|\phi(y)\psi_v)=vy,\quad v\in\mathcal{Y}'.$$

Thus the state given by the coherent vector ψ_v can be viewed as localized in the phase space at $v\in\mathcal{Y}'$.

Note however, that we will not use the above assumption in this section.

We also define the coherent states transformation to be

$$\mathcal{H} \ni \rho \mapsto T\rho(v) := (2\pi)^{-\frac{d}{2}}(\psi_v|\rho), \quad v \in \mathcal{Y}'.$$

Remark 2.3.2 *The coherent state representation is sometimes also called the FBI transformation, for Fourier, Bros and Iagolnitzer.*

Example 2.3.3 *Assume for the moment that $\mathcal{Y} = \mathcal{X}' \oplus \mathcal{X}$ and $\mathcal{H} = L^2(\mathcal{X})$. Consider the Schrödinger representation $\mathcal{X} \oplus \mathcal{X}' \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}(\eta x + qD)} \in U(L^2(\mathcal{X}))$. Fix a normalized vector $\psi_{(0,0)} \in L^2(\mathcal{X})$. Let $(q, \eta) \in \mathcal{Y}' = \mathcal{X} \oplus \mathcal{X}'$. The coherent states are then given by*

$$\psi_{(q,\eta)}(x) = \mathrm{e}^{\mathrm{i}(-qD+\eta x)}\psi_{(0,0)} = \mathrm{e}^{\mathrm{i}\eta x - \frac{\mathrm{i}}{2}q\eta}\psi_{(0,0)}(x - q), \tag{2.3.41}$$

$$P_{(q,\eta)}(x_1, x_2) = \psi_{(0,0)}(x_1 - q)\overline{\psi_{0,0)}}(x_2 - q)\mathrm{e}^{\mathrm{i}(x_1 - x_2)\eta}. \tag{2.3.42}$$

Remark 2.3.4 *Some authors, working in the Schrödinger representation, define a slightly different class of coherent vectors:*

$$\tilde{\psi}_{(q,\eta)}(x) = \mathrm{e}^{\mathrm{i}\eta x}\psi_{(0,0)}(x - q), \tag{2.3.43}$$

The definitions (2.3.41) and (2.3.43) lead to the same family of projections (2.3.42), so in practice it does not matter much which definition is used. The advantage of the definition (2.3.41) is its symplectic invariance.

Proposition 2.3.5 *1)*

$$(2\pi)^{-d} \int P_v \mathrm{d}v = 1. \tag{2.3.44}$$

2) T is an isometric map from $\mathcal{H}$ into $L^2(\mathcal{Y}')$.
3) T intertwines the basic representation with a certain representation of CCR in $L^2(\mathcal{Y}')$ similar to the "prequantization":

$$\mathrm{e}^{\mathrm{i}y(\frac{1}{2}v - \omega D_v)}T = TW(y), \qquad y \in \mathcal{Y}.$$

Proof. To prove 1), we use the Schrödinger representation and the formula (2.3.42). Let $\rho \in L^2(\mathcal{X})$.

$$\begin{aligned}\int_{\mathcal{X}\oplus\mathcal{X}'}(\rho|P_{(q,\eta)}\rho)\mathrm{d}q\mathrm{d}\eta &= \int_{\mathcal{X}\oplus\mathcal{X}'}\int_{\mathcal{X}}\int_{\mathcal{X}}\overline{\rho}(x_1)\psi_0(x_1 - q)\overline{\psi_0}(x_2 - q)\mathrm{e}^{\mathrm{i}(x_1 - x_2)\eta}\rho(x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}q\mathrm{d}\eta \\ &= (2\pi)^d \int_{\mathcal{X}\oplus\mathcal{X}'}\overline{\rho}(x)\psi_0(x - q)\overline{\psi_0}(x - q)\rho(x)\mathrm{d}x\mathrm{d}q = (2\pi)^d\|\rho\|^2\|\psi_0\|^2.\end{aligned}$$

2) follows immediately from 1).

To prove 3) we compute:

$$\begin{aligned}(TW(y)\rho)(v) &= (\psi_0|W(\omega^{-1}v)W(y)\rho) \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}yv}(\psi_0|W(\omega^{-1}(v + \omega y))\rho) \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}yv}\mathrm{e}^{\mathrm{i}(\omega y)D_v}(\psi_0|W(\omega^{-1}v)\rho) = \left(\mathrm{e}^{\mathrm{i}y(\frac{1}{2}v - \omega D_v)}T\rho\right)(v).\end{aligned}$$

□

2.3.2 Contravariant quantization

Let $M^1(\mathcal{Y}')$ denote the space of complex measures of bounded variation on $\mathcal{Y}'$. Note that $M^1(\mathcal{Y}')$ can be viewed as a Banach space and contains $L^1(\mathcal{Y}')$ as its subspace. In our notation we will treat elements of $M^1(\mathcal{Y}')$ as if they were functions.

$L^\infty_\infty(\mathcal{Y}')$ denotes the set of $h \in L^\infty(\mathcal{Y}')$ such that $\lim_{|v|\to\infty} h(v) = 0$.

As we will see from the following theorem, if $h \in L^\infty(\mathcal{Y}') + M^1(Y')$, then the following formula defines a bounded operator, which we will call the contravariant quantization of h:

$$\begin{aligned}\mathrm{Op}^{\mathrm{ct}}(h) \ &:= (2\pi)^{-d} \textstyle\int h(v) P_v \mathrm{d}v \\ &= T^* h(v) T.\end{aligned}$$

(In the second line, $h(v)$ has the meaning of the multiplication operator).

Theorem 2.3.6 *0)* $\mathrm{Op}^{\mathrm{ct}}(1) = 1_{\mathcal{H}}$.
1) $\mathrm{Op}^{\mathrm{ct}}(h)^* = \mathrm{Op}^{\mathrm{ct}}(\overline{h})$.
2) If $v \in \mathcal{Y}'$, *then*

$$W(-\omega^{-1}v)\mathrm{Op}^{\mathrm{ct}}(h)W(\omega^{-1}v) = \mathrm{Op}^{\mathrm{ct}}(h(\cdot - v)).$$

3) If h *is real valued measurable, then*

$$\text{ess inf} h \leq \mathrm{Op}(h) \leq \text{ess sup} h. \tag{2.3.45}$$

Hence, if $h \in L^\infty(\mathcal{Y}')$, *then* $\mathrm{Op}^{\mathrm{ct}}(h)$ *is bounded and*

$$\|\mathrm{Op}^{\mathrm{ct}}(h)\| \leq \|h\|_\infty, \tag{2.3.46}$$

4) Let $h \geq 0$. *Then* $\mathrm{Op}^{\mathrm{ct}}(h) \geq 0$. *Moreover,* $h \in M^1(\mathcal{Y}')$ *iff* $\mathrm{Op}^{\mathrm{ct}}(h) \in l^1(\mathcal{H})$ *and*

$$\mathrm{Tr}\mathrm{Op}^{\mathrm{ct}}(h) = (2\pi)^{-d} \int h(v)\mathrm{d}v. \tag{2.3.47}$$

5) If $h \in M^1(\mathcal{Y}')$, *then* $\mathrm{Op}^{\mathrm{ct}}(h) \in l^1(\mathcal{H})$ *and (2.3.47) is true.*
6) Suppose that $h \in L^\infty_\infty(\mathcal{Y}') + M^1(\mathcal{Y}')$. *Then* $\mathrm{Op}^{\mathrm{ct}}(h)$ *is compact.*

Proof. 2) follows from $W(-\omega^{-1}v)P_w W(\omega^{-1}v) = P_{w+v}$. (2.3.45) follows immediately from (2.3.44). Applying (2.3.45) to $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\alpha}h)$ we obtain that the numerical range of $\mathrm{Op}^{\mathrm{ct}}(h)$ is contained in the ball of radius $\|h\|_\infty$. This implies (2.3.46). 4) follows from $\mathrm{Tr}P_v = 1$. To show 5) we use 4) and the decomposition $h = h_1 + \mathrm{i}h_2 - h_3 - \mathrm{i}h_3$, where $h_i \in M^1$ and $h_i \geq 0$.

If the map

$$L^\infty \cup L^1(\mathcal{Y}') \ni h \mapsto \mathrm{Op}^{\mathrm{ct}}(h) = H \in B(\mathcal{H})$$

is injective, then we can define its inverse called the contravariant symbol of the operator H, which will be denoted $\mathrm{s}^{\mathrm{ct}}_H$.

2.3.3 Covariant quantization

Let $H \in B(\mathcal{H})$. Then we define its covariant symbol by

$$\begin{aligned} \mathrm{s}_H^{\mathrm{cv}}(v) &:= \mathrm{Tr} P_v H \\ &= (\psi_0 | W(\omega^{-1}v) H W(-\omega^{-1}v)\psi_0), \qquad v \in \mathcal{Y}'. \end{aligned}$$

Theorem 2.3.7 *0)* $\mathrm{s}_{1_{\mathcal{H}}}^{\mathrm{cv}} = 1$.
1) $\mathrm{s}_{H^*}^{\mathrm{cv}} = \overline{\mathrm{s}_H^{\mathrm{cv}}}$.
2) If $H = \mathrm{Op}^{\mathrm{cv}}(h)$, $H_v := W(-\omega^{-1}v)\mathrm{Op}^{\mathrm{cv}}(h)W(\omega^{-1}v)$, *then*

$$\mathrm{s}_{H_v} = \mathrm{s}_H(\cdot - v).$$

3) If $H \in B(\mathcal{H})$, *then* $\mathrm{s}_H^{\mathrm{cv}} \in C(\mathcal{Y}') \cap L^\infty(\mathcal{Y}')$ *and*

$$\|\mathrm{s}_H^{\mathrm{cv}}\|_\infty \leq \|H\|. \tag{2.3.48}$$

4) Let $H \geq 0$. *Then* $\mathrm{s}_H^{\mathrm{cv}} \geq 0$. *Moreover,* $H \in l^1(\mathcal{H})$ *iff* $\mathrm{s}_H^{\mathrm{cv}} \in L^1(\mathcal{Y}')$ *and*

$$\mathrm{Tr} H = (2\pi)^{-d} \int \mathrm{s}_H^{\mathrm{cv}}(v)\mathrm{d}v. \tag{2.3.49}$$

5) If $H \in l^1(\mathcal{H})$, *then* $\mathrm{s}_H^{\mathrm{cv}} \in L^1(\mathcal{Y}')$ *and (2.3.49) is true.*
6) If H *is compact, then* $\mathrm{s}_H^{\mathrm{cv}} \in C_\infty(\mathcal{Y}')$.

Proof. Let us show 3). The inequality (2.3.48) is immediate. Moreover, $v \mapsto W(\omega^{-1}v)HW(-\omega^{-1}v)$ is strongly continuous. Hence $v \mapsto s_H(v)$ is continuous. To prove 6) we note that ψ_v goes weakly to zero as $|v| \to \infty$. Hence, for compact H, $\mathrm{s}_H(v) \to 0$ as $|v| \to \infty$.

To show 4) we use (2.3.44) and apply the trace to the identity

$$H = (2\pi)^{-d} \int_{\mathcal{Y}'} H^{\frac{1}{2}} P_v H^{\frac{1}{2}} \mathrm{d}v.$$

To prove 5) we note that if $H \in l^1(\mathcal{H})$, then we can decompose it as $H = H_1 + \mathrm{i}H_2 - H_3 - \mathrm{i}H_4$, with $H_i \geq 0$, $H_i \in l^1(\mathcal{H})$. □

If the map

$$B(\mathcal{H}) \ni H \mapsto \mathrm{s}_H^{\mathrm{cv}} = h \in C(\mathcal{Y}') \cap L^\infty(\mathcal{Y}')$$

is injective, then its inverse will be called the covariant quantization of the symbol h and will be denoted $\mathrm{Op}^{\mathrm{cv}}(h)$.

2.3.4 Connections between various quantizations

Let $v, w \in \mathcal{Y}'$. Clearly,

$$\begin{aligned} T\psi_w(v) &= (2\pi)^{-\frac{d}{2}}(\psi_{v-w}|\psi_0), \\ s^{\rm cv}_{P_w}(v) &= |(\psi_{w-v}|\psi_0)|^2, \\ s_{P_w}(v) &= 2^d(\psi_{w-v}|I\psi_{w-v}), \\ s^{\rm ct}_{P_w}(v) &= (2\pi)^d\delta(v-w). \end{aligned}$$

Theorem 2.3.8 *Set*

$$k_1(v) := (2\pi)^{-d}s_{P_0}(v) = (\pi)^{-d}(\psi_v|I\psi_v),$$

$$k_2(v) := (2\pi)^{-d}s^{\rm cv}_{P_0}(v) = (2\pi)^{-d}|(\psi_v|\psi_0)|^2.$$

Then

$$k_2(v) = \int k_1(v-v_1)k_1(v_1){\rm d}v_1. \tag{2.3.50}$$

Proof. Using (2.2.40) and (2.2.38) we obtain

$$P_0 = \pi^{-d}\int {\rm s}_{P_0}(w)I_w{\rm d}w,$$

$${\rm s}_{P_0}(v-w) = 2^d{\rm Tr}I_{w-v}P_0 = 2^d{\rm Tr}I_wP_v.$$

Hence

$$\begin{aligned} s^{\rm cv}_{P_0}(v) = {\rm Tr}P_0P_v &= \pi^{-d}{\rm Tr}\int {\rm s}_{P_0}(w)I_wP_v{\rm d}w \\ &= (2\pi)^{-d}\int {\rm s}_{P_0}(w){\rm s}_{P_0}(v-w){\rm d}w. \end{aligned}$$

□

Define the integral operators K

$$K\psi(v) := \int k_1(v-v_1)\psi(v_1){\rm d}v_1.$$

Then (2.3.50) means that

$$K^2\psi(v) := \int k_2(v-v_1)\psi(v_1){\rm d}v_1.$$

We have the following relations between various symbols of an operator H:

$$\begin{aligned} s^{\rm cv}_H(v) &= \int s^{\rm ct}_H(v_1)k_2(v-v_1){\rm d}v_1, \quad &&\text{or } {\rm s}^{\rm cv}_H = K^2{\rm s}^{\rm ct}_H; \\ s^{\rm cv}_H(v) &= \int s_H(v_1)k_1(v-v_1){\rm d}v_1, \quad &&\text{or } {\rm s}^{\rm cv}_H = K{\rm s}_H; \\ s_H(v) &= \int s^{\rm ct}_H(v_1)k_1(v-v_1){\rm d}v_1 \quad &&\text{or } {\rm s}_H = K{\rm s}^{\rm ct}_H. \end{aligned}$$

Example 2.3.9 *In the context of the Schrödinger representation, the most common choice for* ψ_0 *is*

$$\psi_{(0,0)}(x) = \pi^{\frac{d}{4}} \mathrm{e}^{-\frac{1}{2}x^2}.$$

The corresponding coherent states are called Gaussian or Glauber's coherent states and are equal to

$$\psi_{(q,\eta)}(x) = \pi^{-\frac{d}{4}} \mathrm{e}^{\mathrm{i}\eta x - \frac{\mathrm{i}}{2}q\eta - (x-q)^2}.$$

The corresponding integral kernels k_1*,* k_2 *and the operators* K*,* K^2 *are equal*

$$k_1(x,\xi) = \pi^{-d}\mathrm{e}^{-x^2-\xi^2}, \qquad K = \mathrm{e}^{\frac{1}{4}\nabla_x^2 + \frac{1}{4}\nabla_\xi^2},$$

$$k_2(x,\xi) = (2\pi)^{-d}\mathrm{e}^{-\frac{1}{2}x^2-\frac{1}{2}\xi^2}, \qquad K^2 = \mathrm{e}^{\frac{1}{2}\nabla_x^2 + \frac{1}{2}\nabla_\xi^2}.$$

We will say more about Gaussian coherent states in Section **??***, because they appear naturally in the context of the Fock representation.*

2.4 CCR relations in Fock spaces

2.4.1 Fock representation of CCR

Let $\mathcal{Z}$ be a Hilbert space. Recall that in Subsection 1.9.1 we defined the Fock space over $\mathcal{Z}$, denoted $\Gamma_{\mathrm{s}}(\mathcal{Z})$, and its finite particle subspace, $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$.

If $z \in \mathcal{Z}$, we define the creation operator $z(a^*) \in L(\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z}))$, by setting

$$z(a^*)\rho := \sqrt{n+1} z \otimes_{\mathrm{s}} \rho, \qquad \rho \in \Gamma_{\mathrm{s}}^n(\mathcal{Z}).$$

We also define the annihilation operator $\overline{z}(a) \in L(\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z}))$ by setting

$$\overline{z}(a) := (z(a^*))^*.$$

These operators satisfy the commutation relations

$$[\overline{z_1}(a), \overline{z_2}(a)] = 0, \qquad [z_1(a^*), z_2(a^*)] = 0,$$

$$[\overline{z_1}(a), z_2(a^*)] = (z_1|z_2), \qquad z_1, z_2 \in \mathcal{Z}.$$

Remark 2.4.1 *In the mathematical literature another notation for creation/annihillation operators is more common. For* $z \in \mathcal{Z}$*, one writes*

$$a^*(z) \quad \textit{instead of} \quad z(a^*),$$

$$a(z) \quad \textit{instead of} \quad \overline{z}(a).$$

Thus, in particular, $a(z)$ *depends antilinearly on* z*.*

In the physical literature one usually assumes that $\mathcal{Z} = L^2(\mathcal{Q})$ for some measure space $\mathcal{Q}$ and so $z \in \mathcal{Z}$ can be represented as functions $\mathcal{Q} \ni q \mapsto z(q)$. One introduces "operator valued distributions" $\mathcal{Q} \ni q \mapsto a^*(q), a(q)$. One writes

$$\begin{aligned} &\textstyle\int z(q)a^*(q)\mathrm{d}q \quad \textit{instead of} \quad z(a^*), \\ &\textstyle\int \overline{z(q)}a(q)\mathrm{d}q \quad \textit{instead of} \quad \overline{z}(a). \end{aligned}$$

Note that the notation that we adopted is close to the "physical" one (it is independent, however, of the non-canonical and cumbersome identification of $\mathcal{Z}$ and $L^2(\mathcal{Q})$).

In our notation one treats $z(a^*)$ and $\overline{z}(a)$ as (linear) functions of the vectors of commuting operators a^* and a respectively. In particular, if $f \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z})$, then we have natural extensions of this notation to nonlinear functions: $f(a^*)$, $\overline{f}(a)$.

Let

$$\mathcal{Y} = \{(z, \overline{z}) \ : \ z \in \mathcal{Z}\} \subset \mathcal{Z} \oplus \overline{\mathcal{Z}}. \tag{2.4.51}$$

$\mathcal{Y}$ will be treated as a Kähler space. For $z, w \in \mathcal{Z}$ the corresponding Euclidean form, symplecting form and anti-involution are defined as follows

$$\begin{aligned} &(z + \overline{z})(w + \overline{w}) := 2\mathrm{Re}(z|w), \\ &(z + \overline{z})\omega(w + \overline{w}) := 2\mathrm{Im}(z|w), \\ &\mathbf{i}(z + \overline{z}) := \mathrm{i}z - \mathrm{i}\overline{z}. \end{aligned}$$

Recall that $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ can be identified with $\mathbf{C}\mathcal{Y}$. The Euclidean form $\cdot$, the symplectic form ω and the antiinvolution $\mathbf{i}$ can be extended by linearity to $\mathbf{C}\mathcal{Y}$:

$$\begin{aligned} &(z_1 + \overline{z_2})(w_1 + \overline{w_2}) := (z_2|w_1) + (w_2|z_1), \\ &(z_1 + \overline{z_2})\omega(w_1 + \overline{w_2}) := \tfrac{1}{\mathrm{i}}((z_2|w_1) - (w_2|z_1)), \\ &\mathbf{i}(z_1 + \overline{z_2}) := \mathrm{i}z_1 - \mathrm{i}\overline{z}_2, \end{aligned}$$

where $z_1, z_2, w_1, w_2 \in \mathcal{Z}$.

For $y = (z_1, \overline{z}_2) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}$ we define

$$\phi(y) := z_1(a^*) + \overline{z}_2(a).$$

Lemma 2.4.2 *For $y \in \mathcal{Y}$, $\phi(y)$ is a symmetric operator. Moreover,*

$$[\phi(y_1), \phi(y_2)] = \mathrm{i}y_1\omega y_2, \quad y_1, y_2 \in \mathbf{C}\mathcal{Y}. \tag{2.4.52}$$

In other words, the map $\mathcal{Y} \ni y \mapsto \phi(y) \in L(\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z}))$ is a Heisenberg-type representation of CCR.

Proof. To show (2.4.52), we compute

$$\begin{aligned}[\phi(z_1+\overline{z_2}),\phi(w_1+\overline{w_2})] &= [z_1(a^*),\overline{w_2}(a)]+[\overline{z_2}(a),w_1(a^*)]\\ &= -(w_2|z_1)+(z_2|w_1)=\mathrm{i}(z_1+\overline{z_2})\omega(w_1+\overline{w_2}).\end{aligned}$$

□

Theorem 2.4.3 *1) For $y\in\mathbf{C}\mathcal{Y}$, $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ is an invariant subspace of entire analytic vectors for $\phi(y)$.*
2) For $y\in\mathcal{Y}$, $\phi(y)$ is essentially self-adjoint on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$. In what follows, $\phi(y)$ will denote the corresponding self-adjoint operator and $W(y):=\mathrm{e}^{\mathrm{i}\phi(y)}$.
3) If $y_1,y_2\in\mathbf{C}\mathcal{Y}$ and $\psi\in\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$, then the relationship

$$\mathrm{e}^{\mathrm{i}\phi(y_1)}\mathrm{e}^{\mathrm{i}\phi(y_2)}\psi=\mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\omega y_2}\mathrm{e}^{\mathrm{i}\phi(y_1+y_2)}\psi \tag{2.4.53}$$

holds, where $\mathrm{e}^{\mathrm{i}\phi(y)}$ is defined in terms of the power series and all the series involved in (2.4.53) are absolutely convergent.
4) The map $\mathcal{Y}\ni y\mapsto W(y)\in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ is a regular irreducible representation of CCR.
5) The parity operator, defined in (1.8.19) as $I=(-1)^N$, coincides with the parity operator defined in (2.2.36).
6) Let $\mathcal{Y}_0$ be a finite dimensional $\mathbf{i}$-invariant subspace of $\mathcal{Y}$. Let $\mathcal{Z}_0$ be the corresponding subspace of $\mathcal{Z}$. Let $p_0\in B(\mathcal{Z})$ be the orthogonal projection onto $\mathcal{Z}_0$. Then the space $\mathcal{G}(\mathcal{Y}_0)$ defined in (2.2.31) equals

$$\mathcal{G}(\mathcal{Y}_0)=\bigcap_{n=1}^{\infty}\mathcal{D}(\mathrm{d}\Gamma(p_0)^n).$$

In particular, if the space $\mathcal{Y}$ is finite dimensional, then

$$\mathcal{G}(\mathcal{Y})=\bigcap_{n=1}^{\infty}\mathcal{D}(N^n).$$

Proof. If $\psi\in\Gamma_{\mathrm{s}}^{m}(\mathcal{Z})$, then

$$\|\phi(y)^n\psi\|\leq\left(\tfrac{(n+m)!}{m!}\right)^{\frac{1}{2}}(\sqrt{2}\|y\|)^n\|\psi\|. \tag{2.4.54}$$

This proves 1).

2) follows from 1) and the Nelson theorem.

3) follows by explicit computations using (2.4.52). Moreover, we use (2.4.54) to control the absolute convergence of the series.

4) follows from 3) if we note, that for $y_1,y_2\in\mathcal{Y}$ both sides of (2.4.53) extend to unitary operators, so (2.4.53) is valid on the whole space $\Gamma_{\mathrm{s}}(\mathcal{Z})$. □

Remark 2.4.4 *Note that the formula (2.4.53) is a special case of the so-called Baker-Campbell formula, which says the following: if A, B are operators such that $[A,B]$ commutes with A and B, then*

$$\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{\frac{1}{2}[A,B]}\mathrm{e}^{A+B} \tag{2.4.55}$$

If we are given a Hilbert space $\mathcal{Z}$ and $\mathcal{Y}$ is the Kähler space defined as in (2.4.51), then the representation of CCR $\mathcal{Y} \ni y \mapsto W(y) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ described in Theorem 2.4.3 4) will be called the Fock representation.

2.4.2 Operators $\mathrm{d}\Gamma(\cdot)$

If h is a closed operator on $\mathcal{Z}$, then we can define the closed operator $\mathrm{d}\Gamma(h)$ on $\Gamma_{\mathrm{s}}(\mathcal{Z})$. Let us note its properties related to the representation of CCR:

Theorem 2.4.5 *1) Let h denote a closed operator on $\mathcal{Z}$. Let $z \in \mathcal{D}(h) \subset \mathcal{Z}$. Then*

$$[\mathrm{d}\Gamma(h), z(a^*)] = (hz)(a^*),$$

$$[\mathrm{d}\Gamma(h), \overline{z}(a)] = -\overline{(h^*z)}(a),$$

$$W(z+\overline{z})\mathrm{d}\Gamma(h)W(-z-\overline{z}) = \mathrm{d}\Gamma(h) - \mathrm{i}(hz)(a^*) + \mathrm{i}\overline{(h^*z)}(a) + (z|hz).$$

2) Assume that h is self-adjoint. Then 1) can be reformulated as follows. Set $g := h + \overline{h}$, which can be treated as a self-adjoint operator on $\mathcal{Y}$, commuting with $\mathbf{i}$. Let $y \in \mathcal{D}(g) \subset \mathcal{Y}$. Then

$$[\mathrm{d}\Gamma(h), \mathrm{i}\phi(y)] = \phi(\mathbf{i}gy),$$

$$W(y)\mathrm{d}\Gamma(h)W(-y) = \mathrm{d}\Gamma(h) - \phi(\mathbf{i}gy) + \tfrac{1}{2}(y|gy).$$

2.4.3 Operators $\Gamma(\cdot)$

Let $\mathcal{Z}_i$, $i = 1,2$ be Hilbert spaces. Let $p \in B(\mathcal{Z}_1, \mathcal{Z}_2)$ with $\|p\| \leq 1$. Recall that we can define $\Gamma(p) \in B(\Gamma_{\mathrm{s}}(\mathcal{Z}_1), \Gamma_{\mathrm{s}}(Z_2))$. The Γ functor has the following properties related to the representation of CCR:

Theorem 2.4.6 *1) Let $z_1 \in \mathcal{Z}_1$, $z_2 \in \mathcal{Z}_2$. Let $p \in B(\mathcal{Z}_1, \mathcal{Z}_2)$, $\|p\| \leq 1$. Then*

$$\Gamma(p)z_1(a^*) = pz_1(a^*)\Gamma(q),$$

$$\Gamma(p)\overline{(p^*z_2)}(a) = z_2(a)\Gamma(p).$$

2) Assume that p is isometric, that is $p^*p = 1$. Then 1) can be reformulated as follows. Set $r := p + \overline{p}$, which can be treated as an isometric operator on $\mathcal{Y}$, commuting with $\mathbf{i}$. Let $y_1 \in \mathcal{Y}_1$. Then

$$\Gamma(p)\phi(y_1) = \phi(ry_1)\Gamma(p).$$

If p is unitary, then

$$\Gamma(p)\phi(y_1)\Gamma(p^{-1}) = \phi(ry_1).$$

2.4.4 Tensor product of Fock spaces

Let $\mathcal{Z}_i$, $i = 1, 2$ be two Hilbert spaces. Let i_1, i_2 denote the injections of $\mathcal{Z}_1, \mathcal{Z}_2$ into $\mathcal{Z}_1 \oplus \mathcal{Z}_2$. We define $U : \Gamma_{\mathrm{s}}(\mathcal{Z}_1) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_2) \to \Gamma_{\mathrm{s}}(\mathcal{Z}_1 \oplus \mathcal{Z}_2)$ as follows:

$$U\psi_1 \otimes \psi_2 := \sqrt{\frac{(n+m)!}{n!m!}}\left(\Gamma(i_1)\psi_1\right) \otimes_{\mathrm{s}} \left(\Gamma(i_2)\psi_2\right), \quad \psi_1 \in \Gamma_{\mathrm{s}}^n(\mathcal{Z}_1),\ \psi_2 \in \Gamma_{\mathrm{s}}^m(\mathcal{Z}_2). \tag{2.4.1}$$

Proposition 2.4.7 *i) U is unitary*
ii) $U\Omega_1 \otimes \Omega_2 = \Omega$
iii)

$$W(y_1 \oplus y_2)U = UW(y_1) \otimes W(y_2),$$

$$\phi(y_1 \oplus y_2)U = U(\phi(y_1) \otimes 1 + 1 \otimes \phi(y_2)), \qquad y_1 \in \mathcal{Y}_1, \quad y_2 \in \mathcal{Y}_2.$$

iv)

$$\mathrm{d}\Gamma(h_1 \oplus h_2)U = U(\mathrm{d}\Gamma(h_1) \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h_2)), \qquad h_i \in B(\mathcal{Z}_i). \tag{2.4.2}$$

$$U\Gamma(p_1) \otimes \Gamma(p_2) = \Gamma(p_1 \oplus p_2)U, \qquad p_i \in B(\mathcal{Z}_i).$$

2.4.5 Weyl operators and Gaussian coherent vectors

Theorem 2.4.8 *1) If $z \in \mathcal{Z}$, then $\mathrm{e}^{\mathrm{i}z(a^*)}$ and $\mathrm{e}^{\mathrm{i}\overline{z}(a)}$ are closable operators with the domains*

$$\mathcal{D}\left(\mathrm{e}^{\mathrm{i}z(a^*)}\right) = \mathcal{D}\left(\mathrm{e}^{\frac{1}{2}\phi(\mathrm{i}z - \mathrm{i}\overline{z})}\right),$$

$$\mathcal{D}\left(\mathrm{e}^{\mathrm{i}\overline{z}(a)}\right) = \mathcal{D}\left(\mathrm{e}^{-\frac{1}{2}\phi(\mathrm{i}z - \mathrm{i}\overline{z})}\right),$$

2) In the sense of quadratic forms, we can write

$$W(z + \overline{z}) = \mathrm{e}^{-\frac{1}{2}|z|^2}\mathrm{e}^{\mathrm{i}z(a^*)}\mathrm{e}^{\mathrm{i}\overline{z}(a)}, \quad z \in \mathcal{Z}. \tag{2.4.3}$$

Proof. To show 1) we use the Baker-Campbell formula (2.4.55):

$$\mathrm{e}^{-\mathrm{i}\overline{z}(a)}\mathrm{e}^{\mathrm{i}z(a^*)} = \mathrm{e}^{\frac{1}{2}|z|^2}\mathrm{e}^{\phi(\mathrm{i}z - \mathrm{i}\overline{z})},$$

$$\mathrm{e}^{-\mathrm{i}z(a^*)}\mathrm{e}^{\mathrm{i}\overline{z}(a)} = \mathrm{e}^{-\frac{1}{2}|z|^2}\mathrm{e}^{-\phi(\mathrm{i}z - \mathrm{i}\overline{z})}.$$

2) also follows from (2.4.55). □

Recall that in **??** we defined the coherent vectors associated with a given normalized vector ψ_0 by the formula

$$\psi_v := W(-\omega^{-1}v)\psi_0, \qquad v \in \mathcal{Y}'. \tag{2.4.4}$$

If $\mathcal{Y}$ is a Kähler space, we can identify $\mathcal{Y}'$ with $\mathcal{Y}$. An arbitrary element of $\mathcal{Y}$ can be written as $y = w + \overline{w}$ for $w \in \mathcal{Z}$. Moreover, ω^{-1} can be identified with $\mathbf{i}$. We have $\mathbf{i}(w + \overline{w}) = \mathrm{i}w - \mathrm{i}\overline{w}$. Thus, in (2.4.4), instead of $W(-\omega^{-1}v)$ we can write $W(-\mathrm{i}w + \mathrm{i}\overline{w})$ with $w \in \mathcal{Z}$. Note that by (2.4.3) we have

$$W(-\mathrm{i}w + \mathrm{i}\overline{w}) = \mathrm{e}^{-\frac{1}{2}|w|^2}\mathrm{e}^{w(a^*)}\mathrm{e}^{-\overline{w}(a)}. \tag{2.4.5}$$

The coherent vectors defined using $\psi_0 = \Omega$ will be called Glauber's or the Gaussian coherent vectors. To simplify the notation, we will write Ω_w instead of $\psi_{w+\overline{w}}$, where $w \in \mathcal{Z}$. Using (2.4.5) we can write

$$\begin{aligned}\Omega_w = W(-\mathrm{i}w + \mathrm{i}\overline{w}))\Omega \ &= \mathrm{e}^{-\frac{1}{2}|w|^2}\mathrm{e}^{w(a^*)}\Omega \\ &= \mathrm{e}^{-\frac{1}{2}|w|^2}\sum_{n=0}^{\infty}\frac{w^{\otimes n}}{\sqrt{n!}}.\end{aligned}$$

We will also write P_w instead of $P_{w+\overline{w}}$, so that P_w is the orthogonal projection onto Ω_w and

$$P_w = W(-\mathrm{i}w + \mathrm{i}\overline{w})P_0W(\mathrm{i}w - \mathrm{i}\overline{w}).$$

Recall that

$$1 = (2\pi\mathrm{i})^{-d}\int P_w \mathrm{d}w\mathrm{d}\overline{w}.$$

The corresponding FBI transformation – the Gaussian FBI transformation – is thus equal to

$$T\rho(w + \overline{w}) = (2\pi)^{\frac{d}{2}}(\Omega_w|\rho) = (2\pi)^{\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}|w|^2}\sum_{n=0}^{\infty}\frac{(w^{\otimes n}|\rho)}{\sqrt{n!}}. \tag{2.4.6}$$

Theorem 2.4.9 *We have*

$$z(a)\Omega_w = (w|z)\Omega_w.$$

Therefore,

$$(\Omega_w|z(a^*)\Omega_w) = (w|z), \quad (\Omega_w|\overline{z}(a)\Omega_w) = (z|w), \quad z \in \mathcal{Z},$$

and hence

$$(\Omega_w|\phi(y)\Omega_w) = y(w + \overline{w}), \quad y \in \mathcal{Y}.$$

2.4.6 Complex wave representation of CCR

Recall that in Subsection **??** we defined the space $\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{Z})$ and the unitary map (the "complex wave transform")

$$\Gamma_{\mathrm{s}}(\mathcal{Z}) \ni \rho \mapsto \rho^{\mathrm{cw}} := \frac{1}{\sqrt{N!}}\rho \in \Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{Z}).$$

In terms of creation operators, its inverse can be written as

$$\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{Z}) \ni \rho \mapsto \rho(a^*)\Omega \in \Gamma_{\mathrm{s}}(\mathcal{Z}).$$

Many algebraic properties of CCR have a simpler expression in the complex wave space:

$$(z(a^*)\rho)^{\mathrm{cw}} = z \otimes_{\mathrm{s}} \rho^{\mathrm{cw}}, \qquad (\overline{z}(a)\rho)^{\mathrm{cw}} = \overline{z}\nabla\rho^{\mathrm{cw}}, \quad z \in \mathcal{Z}.$$

The space $\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{Z})$ has a convenient interpretation as a Hilbert space of holomorphic functions on $\overline{\mathcal{Z}}$. This interpretation will be described in the following subsection.

2.4.7 Holomorphic L^2 spaces with a Gaussian weight I

Let $\mathcal{Z}$ be a (complex) Hilbert space. Let $\mathcal{Z}_{\mathbf{R}}$ denote its real form. Note the identifications: $\mathcal{Z}' = \overline{\mathcal{Z}}$, $\overline{\mathcal{Z}}' = \mathcal{Z}$, $\mathcal{Z}_{\mathbf{R}} = (\overline{\mathcal{Z}})_{\mathbf{R}}$. The generic variable in $\mathcal{Z}$ will be w.

If $\mathcal{Z}$ is finite dimensional of dimension d, there is a natural Lebesgue measure on $\mathcal{Z}_{\mathbf{R}}$, which we denote $(2\mathrm{i})^{-d}\mathrm{d}w\mathrm{d}\overline{w}$.

As in Subsection **??**, we can define the usual L^2 space with the Gaussian weight

$$L^2(\mathcal{Z}_{\mathbf{R}}, \mathrm{e}^{-|w|^2}(2\pi\mathrm{i})^{-d}\mathrm{d}w\mathrm{d}\overline{w}).$$

Recall that the space of analytic polynomials on $\overline{\mathcal{Z}} = \mathcal{Z}'$ is a subspace of all complex polynomials on $\mathcal{Z}_{\mathbf{R}}$. Thus we have

$$\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}) \subset \mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}_{\mathbf{R}}) \subset L^2(\mathcal{Z}_{\mathbf{R}}, \mathrm{e}^{-|w|^2}(2\pi\mathrm{i})^{-d}\mathrm{d}w\mathrm{d}\overline{w}).$$

More precisely, $\rho^{\mathrm{cw}} \in \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z})$ is identified with the polynomial

$$\begin{aligned} \overline{\mathcal{Z}} \ni \overline{w} \mapsto \rho^{\mathrm{cw}}(\overline{w}) &:= \textstyle\sum_{n=0}^{\infty}(w^{\otimes n}|\rho^{\mathrm{cw}}) \\ &= \textstyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}(w^{\otimes n}|\rho) = \mathrm{e}^{\frac{1}{2}|w|^2}(\Omega_w|\rho). \end{aligned} \tag{2.4.7}$$

We define $L^2_{\mathbf{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-|w|^2}(2\pi\mathrm{i})^{-d}\mathrm{d}w\mathrm{d}\overline{w})$ to be the closed subspace of $L^2(\mathcal{Z}_{\mathbf{R}}, \mathrm{e}^{-|w|^2}(2\pi\mathrm{i})^{-d}\mathrm{d}w\mathrm{d}\overline{w})$ spanned by $\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z})$.

Theorem 2.4.10 *(2.4.7) extends to a unique unitary map*

$$\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{Z}) \ni \psi \mapsto \psi(\cdot) \in L^2_{\mathbf{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-|w|^2}(2\pi\mathrm{i})^{-d}\mathrm{d}w\mathrm{d}\overline{w}).$$

Thus if $\rho_1, \rho_2 \in \Gamma_{\mathrm{s}}(\mathcal{Z})$, *then*

$$(\rho_1|\rho_2) = (\rho_1^{\mathrm{cw}}|\rho_2^{\mathrm{cw}})_{\mathrm{cw}} = (2\pi\mathrm{i})^{-d}\int \mathrm{e}^{-|w|^2}\overline{\rho_1^{\mathrm{cw}}}(\overline{w})\rho_2^{\mathrm{cw}}(w)\mathrm{d}w\mathrm{d}\overline{w}, \tag{2.4.8}$$

Proof. Let us choose an orthonormal basis in $\mathcal{Z}$. Then one can reduce the problem to the one-dimensional case and assume that $\mathcal{Z} = \mathbf{C}$. Then it is enough to show that

$$(2\pi\mathrm{i})^{-1}\int_{\mathbf{C}} \mathrm{e}^{-|w|^2}\overline{w}^n w^m \mathrm{d}w\mathrm{d}\overline{w} = n!\delta_{n,m}.$$

□

Remark 2.4.11 *The map* $\Gamma_{\mathrm{s}}(\mathcal{Z})\rho \mapsto \rho^{\mathrm{cw}}(\cdot) \in L^2_{\mathbf{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-|w|^2}(2\pi\mathrm{i})^{-d}\mathrm{d}w\mathrm{d}\overline{w})$ *is sometimes called the Bargmann or Bargmann-Segal transform. Segal calls it the complex wave representation. Berezin calls it the generating functional.*

2.4.8 Holomorphic L^2 spaces with a Gaussian weight II

Let $\mathcal{Y}$ be the complete Kähler space associated to $\mathcal{Z}$, as in Subsection **??**. Let v denote the generic variable of $\mathcal{Y}$ and let $\mathrm{d}v$ be the Lebesguwe measure on $\mathcal{Y}$.

Recall that $\mathcal{Y}$ can be treated as a complex space in two ways: as $\mathcal{Y}^{\mathbf{C}}$ and $\overline{\mathcal{Y}^{\mathbf{C}}}$. Correspondingly, the space $L^2(\mathcal{Y},(2\pi)^{-d}\mathrm{e}^{-\frac{1}{2}v^2}\mathrm{d}v)$ has two subspaces of functions holomorphic in two distinct senses: $L^2_{\mathbf{C}}(\overline{\mathcal{Y}^{\mathbf{C}}},(2\pi)^{-d}\mathrm{e}^{-\frac{1}{2}v^2}\mathrm{d}v)$ and $L^2_{\mathbf{C}}(\mathcal{Y}^{\mathbf{C}},(2\pi)^{-d}\mathrm{e}^{-\frac{1}{2}v^2}\mathrm{d}v)$.

Theorem 2.4.12 *1) The pointwise multiplication extends to a unitary map*

$$L^2_{\mathbf{C}}(\overline{\mathcal{Y}^{\mathbf{C}}},(2\pi)^{-d}\mathrm{e}^{-\frac{1}{2}v^2}\mathrm{d}v)\otimes L^2_{\mathbf{C}}(\mathcal{Y}^{\mathbf{C}},(2\pi)^{-d}\mathrm{e}^{-\frac{1}{2}v^2}\mathrm{d}v)\to L^2(\mathcal{Y},(2\pi)^{-d}\mathrm{e}^{-\frac{1}{2}v^2}\mathrm{d}v).$$

2) The map

$$L^2_{\mathbf{C}}(\overline{\mathcal{Z}},\mathrm{e}^{-|w|^2}(2\pi\mathrm{i})^{-d}\mathrm{d}w\mathrm{d}\overline{w})\ni f\mapsto f(1_{\overline{\mathcal{Z}}}\cdot)\in L^2_{\mathbf{C}}(\overline{\mathcal{Y}^{\mathbf{C}}},(2\pi)^{-d}\mathrm{e}^{-\frac{1}{2}v^2}\mathrm{d}v)$$

is unitary.

Proof. To see 2) note that if $v\in\mathcal{Y}$ and $w=1_{\overline{\mathcal{Z}}}v$, then $\frac{1}{2}v^2=|w|^2$. Moreover, for a finite dimensional space $\mathcal{Y}_1\subset\mathcal{Y}$, the Lebesgue measure on $\mathcal{Y}_1$, denoted $\mathrm{d}v_1$, is transformed to 2^d times the Lebesgue measure on $\mathcal{Z}_1$, that is $\mathrm{i}^{-d}\mathrm{d}w_1\mathrm{d}\overline{w}_1$. $\square$

By (2.4.6) the complex wave transform is closely related to the Gaussian FBI transform:

$$T\rho(w+\overline{w})=(2\pi)^{-\frac{d}{2}}\rho^{\mathrm{cw}}(\overline{w})\mathrm{e}^{-\frac{1}{2}|w|^2},\quad \rho\in\Gamma_{\mathrm{s}}(\mathcal{Z}),\quad w\in\mathcal{Z}.$$

Note that the formula for the scalar product in terms of the space of holomorphic functions (2.4.8) follows from the properties of the FBI transformation.

In fact, recall that under the map $\mathcal{Y}\ni v\mapsto w:=\frac{1}{2}(v-\mathrm{i}\mathbf{i}v)\in\mathcal{Z}$ we have $v^2=2|w|^2$. Therefore, the Lebesgue measure on $\mathcal{Y}$, denoted $\mathrm{d}v$, transforms into 2^d times the Lebesgue measure on $\mathcal{Z}$, which, as we recall, we denote by $(2\mathrm{i})^{-d}\mathrm{d}w\mathrm{d}\overline{w}$. Therefore, the measure $\mathrm{d}v$ is transformed into $\mathrm{i}^{-d}\mathrm{d}w\mathrm{d}\overline{w}$.

We know that the FBI transform is isometric. Therefore,

$$\begin{aligned}\|\rho\|^2=\int|T\rho(v)|^2\mathrm{d}v= & \int|T\rho(w+\overline{w})|^2\frac{\mathrm{d}w\mathrm{d}\overline{w}}{\mathrm{i}^d}\\ & =(2\pi\mathrm{i})^{-d}\int|\rho^{\mathrm{cw}}(\overline{w})|^2\mathrm{e}^{-|w|^2}\mathrm{d}w\mathrm{d}\overline{w},\end{aligned}$$

which proves (2.4.8).

2.4.9 Abstract approach to creation and annihilation

Creation and annihillation operators can be defined in an abstract way, for a general representation of CCR, not necessarily acting in a Fock space. In this and the next subsection we adopt this point of view.

Clearly, to this end one needs to endow a symplectic space with some additional structure. In our approach we assume that this additional structure is the Kähler structure.

Thus, let $(\mathcal{Y}, \cdot, \omega, \mathbf{i})$ be a Kähler space. As usual, let $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ denote the corresponding holomorphic and antiholomorphic subspaces of $\mathbf{C}\mathcal{Y}$.

Suppose that $\mathcal{Y} \ni y \mapsto \phi_\pi(y) \in L(\mathcal{D})$ is a Heisenberg type representation of CCR.

For $z \in \mathcal{Z}$ we will write

$$z(a_\pi^*) := \phi_\pi(z), \quad \overline{z}(a_\pi) := \phi_\pi(\overline{z}).$$

Recall that $\mathcal{Z}, \overline{\mathcal{Z}}$ are Lagrangian subspaces of $\mathbf{C}\mathcal{Y}$. Hence, we have two commutative subalgebras in $\mathrm{Op}_\pi(\mathcal{Y})$: the creation algebra generated by $z(a_\pi^*)$ and the annihilation algebra generated by $\overline{z}(a_\pi)$, where $z \in \mathcal{Z}$. Clearly,

$$\begin{aligned} \mathrm{Op}_\pi(f) &= f(a_\pi^*), \\ \mathrm{Op}_\pi(\overline{f}) &= \overline{f}(a_\pi), \quad f \in \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}). \end{aligned}$$

Remark 2.4.13 *Various authors used different additional structures in a symplectic space $(\mathcal{Y}, \omega)$, when they want to define creation and annihillation operators. We use a Kähler structure, that is we add a Euclidean form $\cdot$ and an antiinvolution $\mathbf{i}$ so that $(\cdot, \omega, \mathbf{i})$ is Kähler. Clearly, it is sufficient to add just a Kähler antiinvolution $\mathbf{i}$, which determines the Euclidean form uniquely). This choice is made by Segal (under the name of a "complex structure compatible with ω").*

Other formalism is used by Araki. He starts with a complex symplectic space $\mathcal{W}$ endowed with an antilinear symplectic involution Γ and a "basis projection" P satisfying $\Gamma^2 = 1$, $\Gamma P \Gamma = 1 - P$, $P' \omega P = 0$, $P^2 = P$. We have the following dictionary between our and Araki's formalism: $\mathcal{W}$ is our $\mathbf{C}\mathcal{Y}$, $\Gamma(w)$ is our $\overline{w}$, P is our $1_{\mathcal{Z}}$. Araki calls it "the self-dual formalism".

2.4.10 Vacua

Suppose that we are given a regular representation of CCR over a Kähler space: $\mathcal{Y} \ni y \mapsto W_\pi(y) \in U(\mathcal{H})$. We define the *space of vacua*

$$\mathcal{K}_\pi := \{\psi \in \mathcal{H} \mid \overline{z}(a_\pi)\psi = 0, \ z \in \mathcal{Z}\}.$$

Theorem 2.4.14 *1) $\mathcal{K}_\pi$ is a closed subspace of $\mathcal{H}$.*
2) $\mathcal{K}_\pi$ is contained in the set of analytic vectors of $\phi_\pi(y)$, $y \in \mathcal{Y}$.
3) If $\psi \in \mathcal{K}_\pi$ and ψ is normalized, then $(\psi | \phi_\pi(y)^2 \psi) = \frac{1}{2} y^2$ and

$$(\psi | W_\pi(y) \psi) = \exp(-y^2/4). \tag{2.4.9}$$

Proof. Recall that by Theorem **??**, the annihillation operators are closed. Therefore, $\mathcal{K}_\pi$ is closed as the intersection of null spaces of closed operators, which proves 1).

To prove 2) we will show 3).

Clearly, $\mathcal{K}_\pi \subset \mathcal{D}(\phi_\pi(y))$, hence for $\psi \in \mathcal{K}_\pi$, $t \in \mathbf{R}$,

$$S(t) := (\psi|W_\pi(ty)\psi)$$

is continuously differentiable. Let $z := \frac{1}{2}(y - \mathrm{i}\mathbf{i}y)$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(t) &= \mathrm{i}(\psi|\phi_\pi(y)W_\pi(ty)\psi) \\ &= \mathrm{i}(\overline{z}(a_\pi)\psi|W_\pi(ty)\psi) + \mathrm{i}(\psi|W_\pi(ty)\overline{z}(a_\pi)\psi) - \tfrac{1}{2}ty^2(\psi|W_\pi(ty)\psi) \\ &= -\tfrac{1}{2}ty^2S(t).\end{aligned}$$

Using $S(0) = 1$, we get

$$S(t) = \exp(-t^2y^2/4),$$

which shows (2.4.9). By (2.4.9), $t \mapsto W_\pi(ty)\psi$ extends to an analytic function around the real axis, and hence the spectral theorem implies that ψ is an analytic vector for $\phi_\pi(y)$. □

Example 2.4.15 *Let $\mathcal{X}$ be a finite dimensional real Hilbert space of dimension d. Consider the Kähler space $\mathcal{X}\oplus\mathcal{X}$ as in Subsection* **??**. *Consider the corresponding Schrödinger representation in $L^2(\mathcal{X})$. Then $\pi^{-\frac{d}{4}}\mathrm{e}^{-\frac{1}{2}x^2}$ is a normalized vacuum.*

Theorem 2.4.16 *Suppose that $\mathcal{Y}$ is a Kähler space. Let $\mathcal{Z}$ be the corresponding holomorphic subspace of $\mathbf{C}\mathcal{Y}$ and $\mathcal{Y} \ni y \mapsto W(y) \in U(\Gamma_\mathrm{s}(\mathcal{Z}))$ the corresponding Fock representation. Let $\mathcal{Y} \ni y \mapsto W_\pi(y) \in U(\mathcal{H})$ be an arbitrary representation of CCR. There exists a unique operator $U : \mathcal{K} \otimes \Gamma_\mathrm{s}(\mathcal{Z}) \to \mathcal{H}$ satisfying*

$$\begin{aligned} &U\rho \otimes \Omega = \rho, \qquad \rho \in \mathcal{K}, \\ &U1_\mathcal{K} \otimes W(y) = W_\pi(y)U, \quad y \in \mathcal{Y}. \end{aligned} \tag{2.4.10}$$

U is isometric. If the space of vacua $\mathcal{K}$ for the representation π is cyclic, then U is unitary.

Proof. We set

$$U\rho \otimes z_1 \otimes_\mathrm{s} \cdots \otimes_\mathrm{s} z_n := \frac{1}{\sqrt{n!}} z_1(a^*) \cdots z_n(a^*)\rho, \quad z_1, \ldots z_n \in \mathcal{Z}, \quad \rho \in \mathcal{K}.$$

□

2.4.11 Wick and anti-Wick quantization as the ordering of creation and annihillation operators

Suppose that $\mathcal{Y}$ is a Kähler space and $\mathcal{Y} \ni y \mapsto \phi_\pi(y) \in L(\mathcal{D})$ is a Heisenberg type representation of CCR.

We define two quantizations:

$$\begin{aligned} &\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Y}) \ni b \mapsto \mathrm{Op}_\pi^{a^*,a}(b) \in \mathrm{CCR}(\mathcal{Y}), \\ &\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Y}) \ni b \mapsto \mathrm{Op}_\pi^{a,a^*}(b) \in \mathrm{CCR}(\mathcal{Y}). \end{aligned} \tag{2.4.11}$$

as follows. Clearly, $\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Y}) = \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathbf{C}\mathcal{Y}) = \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z})\overset{\circ}{\otimes}_{\mathrm{s}}\overset{\circ}{\Gamma}_{\mathrm{s}}(\overline{\mathcal{Z}})$. If $b_1, b_2 \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z})$, then we set

$$\begin{aligned} \mathrm{Op}^{a^*,a}(b_1 \otimes_{\mathrm{s}} \overline{b_2}) &:= \mathrm{Op}(b_1)\mathrm{Op}(\overline{b_2}) = b_1(a^*)\overline{b_2}(a), \\ \mathrm{Op}^{a,a^*}(b_1 \otimes_{\mathrm{s}} \overline{b_2}) &:= \mathrm{Op}(\overline{b_2})\mathrm{Op}(b_1) = \overline{b_2}(a)b_1(a^*). \end{aligned}$$

The inverse maps to (2.4.11) (if they can be defined), called the Wick and anti-Wick symbols of the operator, will be denoted by

$$\begin{aligned} &\mathrm{CCR}(\mathcal{Y}) \ni B \mapsto \mathrm{s}_B^{a^*,a} \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Y}), \\ &\mathrm{CCR}(\mathcal{Y}) \ni B \mapsto \mathrm{s}_B^{a,a^*} \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Y}). \end{aligned}$$

The Wick and anti-Wick symbols of operators in $\mathrm{CCR}(\mathcal{Y})$ belong to $\mathbf{C}\Gamma_{\mathrm{s}}(\mathcal{Y})$, which we usually identify with a subspace of $C(\mathcal{Y})$ (reacall that $\mathcal{Y}'$ is identified with $\mathcal{Y}$). They are polynomials, so by analyticity they can be extended to $\mathbf{C}\mathcal{Y}$. We will often use this analytic extensions. When describing the analytically continued Wick symbols, we will identify $\mathbf{C}\mathcal{Y}$ with $\overline{\mathcal{Z}}\oplus\mathcal{Z}$ and we will denote them by $\mathrm{s}_B^{a^*,a}(\overline{w_1}, w_2)$. In the case of anti-Wick symbols, we will identify $\mathbf{C}\mathcal{Y}$ with $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ and their analytic extensions will be denoted by $\mathrm{s}_B^{a,a^*}(w_1, \overline{w_2})$. The relationship between the symbols on $\mathcal{Y}$ and on $\mathbf{C}\mathcal{Y}$ is given by

$$\begin{aligned} \mathrm{s}_B^{a^*,a}(\overline{w} + w) &= \mathrm{s}_B^{a^*,a}(\overline{w}, w), \\ \mathrm{s}_B^{a,a^*}(\overline{w} + w) &= \mathrm{s}_B^{a,a^*}(w, \overline{w}), \quad w \in \mathcal{Z}. \end{aligned}$$

Remark 2.4.17 *In the literature one can find various notations for the Wick and anti-Wick quantizations. A coomon notation is*

$$: b(\phi) : \quad \textit{instead of} \ \ \mathrm{Op}^{a^*,a}(b).$$

Often the space $\mathcal{Z}$ is assumed to be of the form $\mathcal{Z} = \mathbf{C}^n$ and then the polynomial b can be written as $\sum_{\alpha,\beta} b_{\alpha,\beta}\overline{w}^\alpha w^\beta$ (where α, β are multiindices). One introduces operators a_i^*, a_i, $i = 1, \ldots, n$. Then one can write

$$\sum_{\alpha,\beta} b_{\alpha,\beta}(a^*)^\alpha a^\beta \quad \textit{instead of} \ \ \mathrm{Op}^{a^*,a}(b),$$

$$\sum_{\alpha,\beta} b_{\alpha,\beta} a^\beta (a^*)^\alpha \quad \textit{instead of} \quad \mathrm{Op}^{a,a^*}(b),$$

which is the most straightforward, but often somewhat heavy notation for the Wick and ant-Wick quantization

Gerard and the author used the notation

$$Wick(b) \quad \textit{instead of} \quad \mathrm{Op}^{a^*,a}(b).$$

Note also that the Wick ordering is often called the normal ordering.

The following formulas are analogous to the corresponding formulas of the x, D and D, x quantization:

Theorem 2.4.18 *0)*

$$\mathrm{s}_1^{a^*,a} = 1, \qquad \mathrm{s}_1^{a,a^*} = 1,$$

1) If $B \in \mathrm{CCR}(\mathcal{Y})$*, then*

$$\mathrm{s}_{B^*}^{a,a^*} = \overline{\mathrm{s}_B^{a,a^*}}, \qquad \mathrm{s}_{B^*}^{a^*,a} = \overline{\mathrm{s}_B^{a^*,a}}.$$

2) If $B \in \mathrm{CCR}(\mathcal{Y})$*, then*

$$\begin{aligned}
\mathrm{s}_B^{a^*,a}(\overline{w_0}, w_2) &= \mathrm{e}^{\nabla_{\overline{w_0}} \nabla_{w_2}} \mathrm{s}_B^{a,a^*}(w_2, \overline{w_0}) \\
&= (2\pi\mathrm{i})^{-d} \int \mathrm{e}^{-(w_2 - w_1)(\overline{w_0} - \overline{w_1})} \mathrm{s}_B^{a,a^*}(w_1, \overline{w_1}) \mathrm{d}w_1 \mathrm{d}\overline{w_1}.
\end{aligned}$$

3) If $B_1, B_2 \in \mathrm{CCR}(\mathcal{Y})$*, then*

$$\begin{aligned}
\mathrm{s}_{B_1 B_2}^{a^*,a}(\overline{w}_0, w_2) &= \mathrm{e}^{\nabla_{w_1} \nabla_{\overline{w_1}}} \mathrm{s}_{B_1}^{a^*,a}(\overline{w_0}, w_1) \mathrm{s}_{B_2}^{a^*,a}(\overline{w_1}, w_2) \Big|_{\substack{w_1 = w_2 \\ \overline{w_1} = \overline{w_0}}} \\
&= (2\pi\mathrm{i})^{-d} \int \mathrm{e}^{-(\overline{w_0} - \overline{w_1})(w_2 - w_1)} \mathrm{s}_{B_1}^{a^*,a}(\overline{w_0}, w_1) \mathrm{s}_{B_2}^{a^*,a}(\overline{w_1}, w_2) \mathrm{d}w_1 \mathrm{d}\overline{w_1};
\end{aligned}$$

$$\mathrm{s}_{B_1 B_2}^{a,a^*}(w_0, \overline{w_2}) = \mathrm{e}^{-\nabla_{\overline{w_1}} \nabla_{w_1}} \mathrm{s}_{B_1}^{a,a^*}(w_0, \overline{w_1}) \mathrm{s}_{B_2}^{a,a^*}(w_1, \overline{w_2}) \Big|_{\substack{w_1 = w_0 \\ \overline{w_1} = \overline{w_2}}}$$

(Note that the exponents of differential operators in the above formulas can be understood as finite sums of differential operators, since here we consider polynomial symbols. Note also that in the expression for the anti-Wick symbol of a product of two operators there is no integral formula).

The following theorem gives the connection between the Weyl and the Wick and the anti-Wick quantizations. We express these connections using two alternative notations – either we treat the symbols as functions of a variable $v \in \mathcal{Y}$, or we treat them as functions of the variables $w_1 \in \mathcal{Z}$ and $\overline{w_2} \in \overline{\mathcal{Z}}$.

Theorem 2.4.19 *Let $B \in \mathrm{CCR}(\mathcal{Y})$.*
1) We can express the Wick symbol of B in terms of its Weyl-Wigner symbol as follows:

$$\begin{aligned} \mathrm{s}_B^{a^*,a}(v) &= \mathrm{e}^{\frac14 \nabla_v^2} \mathrm{s}_B(v) \\ &= \pi^{-d} \int \mathrm{e}^{-(v-v_1)^2} \mathrm{s}_B(v_1) \mathrm{d}v_1, \\ \mathrm{s}_B^{a^*,a}(\overline{w_0}, w_2) &= \mathrm{e}^{\frac12 \nabla_{\overline{w_0}} \nabla_{w_2}} \mathrm{s}_B(\overline{w_0}, w_2) \\ &= (\pi \mathrm{i})^{-d} \int \mathrm{e}^{-2(\overline{w_0}-\overline{w_1})(w_2-w_1)} \mathrm{s}_B(\overline{w_1}+w_1) \mathrm{d}\overline{w_1} \mathrm{d}w_1; \end{aligned}$$

2) We can express the Weyl-Wigner symbol of B in terms of its anti-Wick symbol as follows:

$$\begin{aligned} \mathrm{s}_B(v) &= \mathrm{e}^{\frac14 \nabla_v^2} \mathrm{s}_B^{a,a^*}(v) \\ &= \pi^{-d} \int \mathrm{e}^{-(v-v_1)^2} \mathrm{s}_B^{a,a^*}(v_1) \mathrm{d}v_1, \\ \mathrm{s}_B(w_0, \overline{w_2}) &= \mathrm{e}^{\frac12 \nabla_{w_0} \nabla_{\overline{w_2}}} \mathrm{s}_B^{a,a^*}(w_0, \overline{w_2}) \\ &= (\pi \mathrm{i})^{-d} \int \mathrm{e}^{-2(w_0-w_1)(\overline{w_2}-\overline{w_1})} \mathrm{s}_B^{a,a^*}(w_1+\overline{w_1}) \mathrm{d}w_1 \mathrm{d}\overline{w_1}. \end{aligned}$$

Proof. Let $b_1, b_2 \in \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z})$, $b = b_1 \otimes_{\mathrm{s}} \overline{b}_2$,

$$B = \mathrm{Op}^{a^*,a}(b) = b_1(a^*)\overline{b_2}(a), \qquad \mathrm{s}_B^{a^*,a}(\overline{w}, w) = b_1(\overline{w})\overline{b_2}(w).$$

We first use (2.2.26) and then by $b_1(v) = b_1(w)\big|_{w=1_{\mathcal{Z}}v}$, $b_2(v) = b_1(\overline{w})\big|_{\overline{w}=1_{\overline{\mathcal{Z}}}v}$, $1_{\mathcal{Z}}\nabla_v = \nabla_w$, $1_{\overline{\mathcal{Z}}}\nabla_v = \nabla_{\overline{w}}$, we obtain

$$\begin{aligned} \mathrm{s}_B(v) &= \exp\left(\tfrac{\mathrm{i}}{2}\nabla_{v_1}\omega\nabla_{v_2}\right) b_1(v_1)\overline{b_2}(v_2)\Big|_{v=v_1=v_2} \\ &= \exp\left(\tfrac{\mathrm{i}}{2}\nabla_{\overline{w_1}}\omega\nabla_{w_2}\right) b_1(\overline{w_1})\overline{b_2}(w_2)\Big|_{1_{\overline{\mathcal{Z}}}v=w_1,\ \ 1_{\mathcal{Z}}v=w_2} \\ &= \exp\left(\tfrac{\mathrm{i}}{2}\nabla_{\overline{w}}\omega\nabla_{w}\right) \mathrm{s}_B^{a^*,a}(\overline{w}, w)\Big|_{v=\overline{w}+w}. \end{aligned}$$

Then we use $\mathrm{i}\nabla_{\overline{w}}\omega\nabla_w = -\nabla_{\overline{w}}\nabla_w = -\frac12\nabla_v^2$. This implies 1). The proof of 2) is similar. □

Theorem 2.4.20 *Let $z \in \mathcal{Z}$, $y \in \mathcal{Y}$, $b \in \mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Y})$. Then*

$$\mathrm{Op}^{a^*,a}(z) = z(a^*), \qquad \mathrm{Op}^{a^*,a}(\overline{z}) = \overline{z}(a),$$

$$\mathrm{Op}^{a^*,a}(y) = \phi(y),$$

$$\mathrm{Op}^{a^*,a}(z \otimes_{\mathrm{s}} b) = z(a^*)\mathrm{Op}^{a^*,a}(b), \qquad \mathrm{Op}^{a^*,a}(\overline{z} \otimes_{\mathrm{s}} b) = \mathrm{Op}^{a^*,a}(b)\overline{z}(a),$$

$$[\mathrm{Op}^{a^*,a}(b), z(a^*)] = \mathrm{Op}^{a^*,a}(z\nabla b), \qquad [\overline{z}(a), \mathrm{Op}^{a^*,a}(b)] = \mathrm{Op}^{a^*,a}(\overline{z}\nabla b).$$

2.4.12 Wick and anti-Wick quantization as covariant and contravariant quantization

Suppose that $\mathcal{Y}$ is a Kähler space and $\mathcal{Y} \ni y \mapsto W(y) \in U(\Gamma_{\rm s}(\mathcal{Z}))$ is the corresponding Fock representation. We consider the Gausssian coherent vectors Ω_w and the corresponding projections P_w, $w \in \mathcal{Z}$.

Theorem 2.4.21 *1) Let $B \in {\rm CCR}(\mathcal{Y})$. Then for all $w \in \mathcal{Z}$, $\Omega_w \in \mathcal{D}(B)$and*

$$ {\rm s}_B^{a^*,a}(\overline{w}+w) = (\Omega_w|B\Omega_w), \quad w \in \mathcal{Z}. \tag{2.4.12} $$

2) Let $b \in \mathbf{C}\overset{\circ}{\Gamma}_{\rm s}(\mathcal{Y})$. Let the dimension of $\mathcal{Z}$ be finite. Then

$$ {\rm Op}^{a,a^*}(b) = (2\pi{\rm i})^{-d}\int b(\overline{w}+w)P_w{\rm d}w{\rm d}\overline{w}. \tag{2.4.13} $$

(The integral should be understood in terms of a quadratic form on $\mathcal{G}(\mathcal{Y})$).

Proof. Let $b_1, b_2 \in \overset{\circ}{\Gamma}_{\rm s}(\mathcal{Z})$. Set

$$ b := b_1 \otimes_{\rm s} \overline{b_2} \in \mathbf{C}\overset{\circ}{\Gamma}_{\rm s}(\mathcal{Y}). $$

Then

$$ \begin{aligned} (\Omega_w|{\rm Op}^{a^*,a}(b)\Omega_w) &= (\Omega_w|b_1(a^*)\overline{b_2}(a)\Omega_w) \\ &= (\Omega|W({\rm i}w-{\rm i}\overline{w})b_1(a^*)\overline{b_2}(a)W(-{\rm i}w+{\rm i}\overline{w})\Omega) \\ &= (\Omega|b_1(a^*+\overline{w})\overline{b_2}(a+w)\Omega) \\ &= b_1(\overline{w})\overline{b_2}(w) \quad = \quad b(\overline{w}+w). \end{aligned} $$

This proves (2.4.12).

$$ \begin{aligned} {\rm Op}^{a,a^*}(b) &= \overline{b_2}(a)b_1(a^*) \\ &= (2\pi{\rm i})^{-d}\textstyle\int \overline{b_2}(a)P_w b_1(a^*){\rm d}w{\rm d}\overline{w} \\ &= (2\pi{\rm i})^{-d}\textstyle\int W({\rm i}w-{\rm i}\overline{w})\overline{b_2}(a+w)P_0 b_1(a^*+\overline{w})W(-{\rm i}w+{\rm i}\overline{w}){\rm d}w{\rm d}\overline{w} \\ &= (2\pi{\rm i})^{-d}\textstyle\int \overline{b_2}(w)b_1(\overline{w})P_w{\rm d}w{\rm d}\overline{w} \quad = \quad (2\pi{\rm i})^{-d}\textstyle\int b(\overline{w}+w)P_w{\rm d}w{\rm d}\overline{w}. \end{aligned} $$

This proves (2.4.13). □

Remark 2.4.22 *Theorem 2.4.21 1) says that the Wick symbol coincides with the covariant symbol defined with help of Gaussian coherent states. Thus (2.4.12) can be denoted ${\rm s}_B^{\rm cv}(\overline{w}+w)$, using the notation of Subsection* **??**. *(Strictly speaking, however, operators in*

$\mathrm{CCR}(\mathcal{Y})$ *are usually unbounded, so they do not belong to the class considered in Subsection* **??***).*

Theorem 2.4.21 2) says that the anti-Wick quantization coincides with the contravariant quantization for Glauber's coherent states. Thus (2.4.13) can be denoted $\mathrm{Op}^{\mathrm{ct}}(b)$*, using the notation of Subsection* **??***. (Strictly speaking, however, functions in* $\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Y})$ *usually do not belong to* $M^1(\mathcal{Y})+L^\infty(\mathcal{Y})$*, so they do not belong to the class considered in Subsection* **??***).*

So far we defined the Wick symbols only for operators in $\mathrm{CCR}(\mathcal{Y})$. Motivated by Theorem 2.4.21 1), we extend the definition of the Wick symbol to a much larger class of operators. We will say that a (possibly unbounded) quadratic form B has the Wick symbol if for any $w \in \mathcal{Z}$, Ω_w belongs to the domain of B, and then we set

$$s_B(\overline{w}+w) = (\Omega_w|B\Omega_w) = \mathrm{e}^{-|w|^2}\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!}}(w^{\otimes n}|Bw^{\otimes m})\frac{1}{\sqrt{m!}}. \tag{2.4.14}$$

Its analytic extension to $\overline{\mathcal{Z}}\oplus\mathcal{Z}$ is given by

$$\begin{aligned} \mathrm{s}_B(\overline{w_1},w_2) &= \mathrm{e}^{-\overline{w_1}w_2}\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!}}(w_1^{\otimes n}|Bw_2^{\otimes m})\frac{1}{\sqrt{m!}} \\ &= \mathrm{e}^{-\overline{w_1}w_2+\frac{1}{2}|w_1|^2+\frac{1}{2}|w_2|^2}(\Omega_{w_1}|B\Omega_{w_2}).. \end{aligned}$$

Likewise, the anti-Wick quantization can be defined for a large class of symbols including polynomials $\mathbf{C}\overset{\circ}{\Gamma}(\mathcal{Y})$, bounded functions and measures of bounded variation.

Example 2.4.23 *The anti-Wick, Weyl and Wick symbols of* P_0 *(the projection onto* Ω*) are given below (compare with Examples 2.1.10 and 2.2.17).*

$$\mathrm{s}_{P_0}^{a,a^*}(\overline{w},w) = (2\pi)^d\delta_0,$$

$$\begin{aligned} \mathrm{s}_{P_0}(\overline{w},w) &= 2^d\mathrm{e}^{-2|w|^2}, \\ \mathrm{s}_{P_0}^{a^*,a}(\overline{w},w) &= \mathrm{e}^{-|w|^2}, \end{aligned}$$

Theorem 2.4.24 *Let* $h \in B(\mathcal{Z})$*,* $p \in B(\mathcal{Z})$*. Then* $\mathrm{d}\Gamma(h)$ *and* $\Gamma(p)$ *possess the Wick symbol and*

$$\begin{aligned} \mathrm{s}_{\mathrm{d}\Gamma(h)}(\overline{w},w) &= (w|hw), \\ \mathrm{s}_{\Gamma(p)}(\overline{w},w) &= \mathrm{e}^{-|w|^2+(w|pw)}. \end{aligned}$$

2.4.13 Wick quantization – the operator formalism

Let Θ_{s}^n denote the orthogonal projection from $\Gamma_{\mathrm{s}}(\mathcal{Z})$ onto $\Gamma_{\mathrm{s}}^n(\mathcal{Z})$. For $b\in B(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ set $b_{n,m} := \Theta_s^n B\Theta_{\mathrm{s}}^m$. Let

$$B^{\mathrm{fin}}(\Gamma_{\mathrm{s}}(\mathcal{Z})) = \{b\in B(\Gamma_{\mathrm{s}}(\mathcal{Z}))\ :\ \text{there exists}\ \ n_0\ \text{such that}\ \ b_{n,m}=0\ \ \text{for}\ \ n,m>n_0\}.$$

Theorem 2.4.25 *Let $b\in B^{\mathrm{fin}}(\Gamma_{\mathrm{s}}(\mathcal{Z}))$. Define a a quadratic form on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z}))$, by setting*

$$(\rho|B\psi) = \sum_{n,m=0}^{\infty}\sum_{k=0}^{\min(m,n)}\frac{\sqrt{n!m!}}{k!}(\rho|b_{n-k,m-k}\otimes 1_{\mathcal{Z}}^{\otimes k}\psi),\qquad \rho,\psi\in\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z}). \tag{2.4.15}$$

Then the Wick symbol of B (as defined in the previous subsection) equals

$$\mathrm{s}_B^{a^*,a}(\overline{w}+w) = \sum_{n,m=0}^{\infty}(w^{\otimes n}|bw^{\otimes m}).$$

Proof. Using (2.4.14) we obtain

$$\begin{aligned}
\mathrm{s}_B(\overline{w}+w)\mathrm{e}^{|w|^2} &= \sum_{n,m=0}^{\infty}\tfrac{1}{\sqrt{n!}}(w^{\otimes n}|Bw^{\otimes m})\tfrac{1}{\sqrt{m!}}\\
&= \sum_{n,m=0}^{\infty}\sum_{k=0}^{\min(n,m)}\tfrac{1}{k!}(w^{\otimes n}|b_{n-k,m-k}\otimes 1_{\mathcal{Z}}^{\otimes k}w^{\otimes m})\\
&= \sum_{n,m=0}^{\infty}\sum_{k=0}^{\min(n,m)}\tfrac{1}{k!}|w|^{2k}(w^{\otimes(n-k)}|b_{n-k,m-k}w^{\otimes(m-k)})\\
&= \sum_{n,m=0}^{\infty}\sum_{k=0}^{\infty}\tfrac{1}{k!}|w|^{2k}(w^{\otimes n}|b_{n,m}w^{\otimes m}) = \sum_{n,m=0}^{\infty}(w^{\otimes n}|b_{n,m}w^{\otimes m})\mathrm{e}^{|w|^2}.
\end{aligned}$$

Motivated by the above theorem, we introduce the following definition: if $b\in B^{\mathrm{fin}}(\Gamma_{\mathrm{s}}(\mathcal{Z}))$, then its Wick quantization is the quadratic form on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ equal to

$$(\rho|\mathrm{Op}^{a^*,a}(b)\psi) = \sum_{n,m=0}^{\infty}\sum_{k=0}^{\min(m,n)}\frac{\sqrt{n!m!}}{k!}(\rho|b_{n-k,m-k}\otimes 1_{\mathcal{Z}}^{\otimes k}\psi),\qquad \rho,\psi\in\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z}). \tag{2.4.16}$$

We will often identify $b\in B^{\mathrm{fin}}(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ with a function on $\mathcal{Y}$ by the formula

$$\mathcal{Y}\ni(\overline{w}+w)\mapsto b(\overline{w}+w) := \sum_{n,m=0}^{\infty}(w^{\otimes n}|bw^{\otimes m}), \tag{2.4.17}$$

or with its analytic continuation

$$\mathbf{C}\mathcal{Y} = \overline{\mathcal{Z}}\oplus\mathcal{Z}\ni(\overline{w}_1+w_2)\mapsto b(\overline{w}_1+w_2) := \sum_{n,m=0}^{\infty}(w_1^{\otimes n}|bw_2^{\otimes m}), \tag{2.4.18}$$

Note that if we use the identification (2.4.17) and $b\in\mathbf{C}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z})$, then $\mathrm{Op}^{a^*,a}(b)$ as defined in Subsection **??** coincides with $\mathrm{Op}^{a^*,a}(b)$ as defined in (2.4.16).

In the following identities it is convenient to use the new, more general definition of the Wick quantization.

Theorem 2.4.26 *In the following identities $b \in B^{\rm fin}(\Gamma_{\rm s}(\mathcal{Z}))$, $h \in B(\mathcal{Z}) = B(\Gamma_{\rm s}^1(\mathcal{Z})) \subset B^{\rm fin}(\Gamma_{\rm s}(\mathcal{Z}))$, $p \in B(\mathcal{Z}_1, \mathcal{Z})$.*

$$\begin{aligned}
\mathrm{Op}^{a^*,a}(h) &= \mathrm{d}\Gamma(h),\\
[\mathrm{d}\Gamma(h), \mathrm{Op}^{a^*,a}(b)] &= \mathrm{Op}^{a^*,a}(hb - bh^*),\\
\Gamma(p)\mathrm{Op}^{a^*,a}(b\Gamma(p)) &= \mathrm{Op}^{a^*,a}(\Gamma(p)b)\Gamma(p),\\
\Gamma(p)\mathrm{Op}^{a^*,a}(b) &= \mathrm{Op}^{a^*,a}(\Gamma(p)b\Gamma(p^*))\Gamma(p), \qquad \textit{if } p \textit{ is isometric},\\
\Gamma(p)\mathrm{Op}^{a^*,a}(b)\Gamma(p^*) &= \mathrm{Op}^{a^*,a}(\Gamma(p)b\Gamma(p^*)) \qquad \textit{if } p \textit{ is unitary.}
\end{aligned}$$

The following theorem describes the special class of particle preserving operators.

Theorem 2.4.27 *If $b = b_{m,m} \in B(\Gamma_{\rm s}^m(\mathcal{Z})$, then*

$$(\rho|\mathrm{Op}(b)\psi) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!}(\rho|b \otimes 1_{\mathcal{Z}}^{(n-m)}\psi) = \sum_{(i_1,\dots,i_m)} b^n_{i_1,\dots,i_m},$$

where the sum runs over all m-tuples of distinct elements of $\{1,\dots,n\}$. The operators $b^n_{i_1,\dots,i_m} \in B(\Gamma_{\rm s}^n(\mathcal{Z}))$ are defined as follows:

$$b^n_{i_1,\dots,i_m} := \Theta(\sigma) b \otimes 1_{\mathcal{Z}}^{\otimes(n-m)}\Theta(\sigma)^{-1} \in B(\Gamma_{\rm s}^n(\mathcal{Z}))$$

where $\sigma \in S_n$ is any permutation that transforms $(1,\dots,m)$ onto $(i_1,\dots,i_m)$.

2.4.14 The Bargmann kernel of an operator

In the context of the complex wave representation, instead of an operator B on $\Gamma_{\rm s}(\mathcal{Z})$, it is often more convenient to use the operator

$$B^{\rm cw} := \frac{1}{\sqrt{N!}} B \frac{1}{\sqrt{N!}}.$$

Using the identification (2.4.18), it can be viewed as an analytic function on $\overline{\mathcal{Z}} \oplus \mathcal{Z}$, sometimes called the Bargmann kernel of the operator B:

$$\begin{aligned}
\overline{\mathcal{Z}} \oplus \mathcal{Z} \ni (\overline{w_1}, w_2) \mapsto B^{\rm cw}(\overline{w_1}, w_2) \ &:= \sum_{n,m=0}^{\infty} (w_1^{\otimes n} \tfrac{1}{\sqrt{n!}} | B \tfrac{1}{\sqrt{m!}} w_2^{\otimes m})\\
&= (\mathrm{e}^{w_1(a^*)}\Omega | B \mathrm{e}^{w_2(a^*)}\Omega).
\end{aligned}$$

Theorem 2.4.28 *1) The relationship between the Bargmann kernel and the Wick symbol of an operator B on $\Gamma_{\rm s}(\mathcal{Z})$ is given by the following identity:*

$$B^{\rm cw}(\overline{w_1}, w_2) = \mathrm{e}^{\overline{w_1}w_2} s_B^{a^*,a}(\overline{w_1}, w_2) = \mathrm{e}^{\frac12|w_1|^2 + \frac12|w_2|^2}(\Omega_{w_1}|B\Omega_{w_2}).$$

2) Let $\rho \in \Gamma_{\rm s}(\mathcal{Z})$, $B, B_1, B_2 \in B(\Gamma_{\rm s}(\mathcal{Z}))$. Then

$$(B\rho)^{\rm cw}(\overline{w_1}) = (2\pi{\rm i})^{-d} \int B^{\rm cw}(\overline{w_1}, w_2){\rm e}^{|w_2|^2} \rho^{\rm cw}(\overline{w_2}){\rm d}w_2{\rm d}\overline{w_2},$$

$$(B_1B_2)^{\rm cw}(\overline{w_0}, w_2) = (2\pi{\rm i})^{-d} \int B^{\rm cw}(\overline{w_0}, w_1){\rm e}^{|w_1|^2} B^{\rm cw}(\overline{w_1}, w_2){\rm d}w_1{\rm d}\overline{w_1}.$$

Proof. 1) follows immediately from (2.3.44). To prove 2) we also use (2.3.44), which implies

$$(\Omega_{w_1}|B\rho) = (2\pi{\rm i})^{-d} \int (\Omega_{w_1}|B\Omega_{w_2})(\Omega_{w_2}|\rho){\rm d}w_2{\rm d}\overline{w_2},$$

$$(\Omega_{w_0}|B_1B_2\Omega_{w_2}) = (2\pi{\rm i})^{-d} \int (\Omega_{w_0}|B_1\Omega_{w_1})(\Omega_{w_1}|B_2\Omega_{w_2}){\rm d}w_1{\rm d}\overline{w_1}.$$

2.5 Weighted Schrödinger representations

2.5.1 Weighted Schrödinger representations in finite dimension

Suppose that $\mathcal{X}$ is finite dimensional vector space. Fix $m \in L^2(\mathcal{X})$ such that $m \neq 0$ almost everywhere. Defien the measure ${\rm d}\mu(x) = |m|^2(x){\rm d}x$. Then

$$L^2(\mathcal{X}, {\rm d}\mu) \ni \psi \mapsto U\psi = m(x)\psi \in L^2(\mathcal{X}, {\rm d}x)$$

is a unitary operator such that $U1 = m$.

The following theorem is obvious:

Theorem 2.5.1

$$\begin{aligned} \mathcal{X}' \oplus \mathcal{X} \ni (\eta, q) &\mapsto U^*{\rm e}^{{\rm i}\eta x + {\rm i}qD}U \\ &= {\rm e}^{{\rm i}\eta x + {\rm i}qD + qm^{-1}(x)\nabla m(x)} \in U(L^2(\mathcal{X}, {\rm d}\mu)) \end{aligned}$$

is a regular irreducible representation of CCR.

Remark 2.5.2 *Set $V(x) := -\frac{1}{2}m^{-1}(x)\Delta m(x)$. Then $(-\frac{1}{2}\Delta + V(x))m = 0$. The Schrödinger operator $-\frac{1}{2}\Delta + V(x)$ in the $L^2(\mathcal{X}, {\rm d}\mu)$ representation looks like*

$$U^*\left(-\frac{1}{2}\Delta + V(x)\right)U = -\frac{1}{2}\Delta - m^{-1}(x)\nabla m(x)\nabla.$$

It is called the Dirichlet form. 1 is its eigenstate.

2.5.2 L^2 spaces with a Gaussian weight

Suppose that $\mathcal{X}$ is a finite dimensional Euclidean space of dimension d. We will identify $\mathcal{X}$ with $\mathcal{X}'$. Its Lebesgue measure will be denoted $\mathrm{d}x$. Then $\pi^{-\frac{d}{2}}\mathrm{e}^{-x^2}\mathrm{d}x$ is a probabilistic measure on $\mathcal{X}$. Consider the space $L^2(\mathcal{X}, \pi^{-\frac{d}{2}}\mathrm{e}^{-x^2}\mathrm{d}x)$.

Theorem 2.5.3 *The space of polynomials, that is* $\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$*, is dense in* $L^2(\mathcal{X}, \pi^{-\frac{d}{2}}\mathrm{e}^{-x^2}\mathrm{d}x)$.

Suppose now that $\mathcal{X}$ is a real Hilbert space of an arbitrary dimension. In $\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$ we introduce the scalar product as follows. If $\rho, \psi \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$, then there exists a finite dimensional subspace $\mathcal{X}_1$ of $\mathcal{X}$ such that $d_1 = \dim \mathcal{X}_1$ and $\rho, \psi \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}_1)$. We then put

$$(\rho|\psi) := \pi^{-\frac{d_1}{2}} \int_{\mathcal{X}_1} \overline{\rho(x_1)}\psi(x_1)\mathrm{e}^{-x_1^2}\mathrm{d}x_1. \tag{2.5.19}$$

Note that the definition (2.5.19) does not depend on the choice of the subspace $\mathcal{X}_1$.

Thus $\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$ is endowed with a unitary structure. The completion of $\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$ in the corresponding norm is denoted $L^2(\mathcal{X}, \pi^{-\frac{d}{2}}\mathrm{e}^{-x^2}\mathrm{d}x)$. (In this notation the number $d = \dim \mathcal{X}$ has just a symbolic meaning if d is infinite).

Remark 2.5.4 *The space* $L^2(\mathcal{X}, \pi^{-\frac{d}{2}}\mathrm{e}^{-x^2}\mathrm{d}x)$ *is used in the probability theory as the Hilbert space of the Gaussian stochastic process indexed by* $\mathcal{X}$*. In the context of constructive field theory it is sometimes called* $L^2(Q)$ *where* Q *is an appropriate measure space whose existence follows by abstract arguments.*

2.5.3 Real wave representation

Suppose now that $\mathcal{X}$ is a real Hilbert space of an arbitrary dimension. Recall that $\mathcal{X} \oplus \mathcal{X}$ is naturally a Kähler space with conjugation, where the anti-involution $\mathbf{i}$ and the conjugation κ are given by

$$\mathbf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note also that we can write $\mathcal{X}$ instead of $\mathcal{X}'$.

Theorem 2.5.5 *Let* $\mathcal{X}$ *be a real Hilbert space of an arbitrary dimension. For* $(\eta, q) \in \mathcal{X} \oplus \mathcal{X}$*,*

$$\eta x_{\mathrm{rw}} + qD_{\mathrm{rw}} := \eta x + q\tfrac{1}{\mathrm{i}}(\nabla_x - x) = \mathrm{e}^{\frac{1}{2}x^2}(\eta x + qD)\mathrm{e}^{-\frac{1}{2}x^2}$$

can be defined as a self-adjoint operator on $L^2(\mathcal{X}, \pi^{-\frac{d}{2}}\mathrm{e}^{-x^2}\mathrm{d}x)$.

$$\mathcal{X} \oplus \mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}(\eta x_{\mathrm{rw}} + qD_{\mathrm{rw}})} \in U(L^2(\mathcal{X}, \pi^{-\frac{d}{2}}\mathrm{e}^{-x^2}\mathrm{d}x))$$

is an irreducible regular representation of CCR. Moreover, the family of operators $\{\mathrm{e}^{\mathrm{i}\eta x_{\mathrm{rw}}}\ :\ \eta\in\mathcal{X}\}$ *generates a maximal von Neumann algebra in* $B(L^2(\mathcal{X},\pi^{-\frac{d}{2}}\mathrm{e}^{-x^2}\mathrm{d}x))$. *The same is true concerning the family* $\{\mathrm{e}^{\mathrm{i}qD_{\mathrm{rw}}}\ :\ q\in\mathcal{X}\}$. *The Weyl commutation relations are satisfied:*

$$\mathrm{e}^{\mathrm{i}\eta x_{\mathrm{rw}}}\mathrm{e}^{\mathrm{i}qD_{\mathrm{rw}}}=\mathrm{e}^{-\mathrm{i}\eta q}\mathrm{e}^{\mathrm{i}qD_{\mathrm{rw}}}\mathrm{e}^{\mathrm{i}\eta x_{\mathrm{rw}}}.$$

1 *is a normalized vacuum for the above Kähler structure.*

Remark 2.5.6 *If the dimension of* $\mathcal{X}$ *is finite, then the real wave representation is a special case of a weighted Schrödinger representation considered in Subsection* **??** *with*

$$m(x)=\pi^{-\frac{d}{4}}\mathrm{e}^{-\frac{1}{2}x^2}, \tag{2.5.20}$$

(2.5.20) is the pointwise positive ground state of $H=-\frac{1}{2}\Delta+\frac{1}{2}x^2-\frac{1}{2}$. *The Dirichlet form for (2.5.20) in the Hilbert space* $L^2(\mathcal{X},\pi^{-\frac{1}{2}d}\mathrm{e}^{-x^2}\mathrm{d}x)$ *equals*

$$-\frac{1}{2}\Delta+x\nabla.$$

Theorem 2.5.7 *Let* $z\in\mathcal{Z}\subset\mathbf{C}\mathcal{X}\oplus\mathbf{C}\mathcal{X}$. *In the real wave representation the creation and annihillation operators are equal*

$$\begin{aligned} z(a_{\mathrm{rw}}^*) &= \tfrac{1+\kappa}{2}z(2x-\nabla_x)=-\mathrm{e}^{x^2}\tfrac{1+\kappa}{2}z\nabla_x\mathrm{e}^{-x^2},\\ \overline{z}(a_{\mathrm{rw}}) &= \tfrac{1+\kappa}{2}\overline{z}\nabla_x. \end{aligned} \tag{2.5.21}$$

Proof. 1) An arbitrary $z\in\mathcal{Z}$ is of the form $z=(\eta,-\mathrm{i}\eta)$ for some $\eta\in\mathbf{C}\mathcal{X}$. We then have $\eta=\frac{1+\kappa}{2}z$. Therefore,

$$\begin{aligned} z(a_{\mathrm{rw}}^*) &= \eta x-\mathrm{i}\eta(D+\mathrm{i}x)\\ &= \eta(2x-\nabla_x)\\ &= (\tfrac{1+\kappa}{2}z)(2x-\nabla_x)=-\mathrm{e}^{x^2}\tfrac{1+\kappa}{2}z\nabla_x\mathrm{e}^{-x^2}. \end{aligned}$$

Likewise, $\overline{z}=(\overline{\eta},\mathrm{i}\overline{\eta})$, and therefore

$$\begin{aligned} \overline{z}(a_{\mathrm{rw}}) &= \overline{\eta}x+\mathrm{i}\overline{\eta}(D+\mathrm{i}x)\\ &= \overline{\eta}\nabla_x\\ &= (\tfrac{1+\kappa}{2}\overline{z})\nabla_x. \end{aligned}$$

□

For many purposes it is actually more convenient to consider a slightly different version of a real wave representation. The main difference is rescaling the Gaussian weight. This representation, denoted

$$\mathcal{X}\oplus\mathcal{X}\ni(\eta,q)\mapsto\mathrm{e}^{\mathrm{i}(\eta x_{\mathrm{rw1}}+qD_{\mathrm{rw1}})}\in U(L^2(\mathcal{X},(2\pi)^{-\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)),$$

is defined as

$$\eta x_{\mathrm{rw1}} + qD_{\mathrm{rw1}} = \tfrac{1}{\sqrt{2}}\eta x + \tfrac{\sqrt{2}}{\mathrm{i}}\nabla_x - \tfrac{1}{\mathrm{i}\sqrt{2}}x.$$

The corresponding creation and annihillation operators are equal

$$\begin{aligned} z(a_{\mathrm{rw1}}^*) &= \tfrac{1+\kappa}{\sqrt{2}}z(x-\nabla_x) = -\mathrm{e}^{\frac12 x^2}\tfrac{1+\kappa}{\sqrt{2}}z\nabla_x\mathrm{e}^{-\frac12 x^2}, \\ \overline{z}(a_{\mathrm{rw1}}) &= \tfrac{1+\kappa}{\sqrt{2}}\overline{z}\nabla_x. \end{aligned} \tag{2.5.22}$$

1 is again a normalized vacuum.

Clearly,

$$L^2(\mathcal{X},(2\pi)^{-\frac{d}{2}}\mathrm{e}^{-\frac12 x^2}\mathrm{d}x) \ni f \mapsto f(\sqrt{2}\cdot) \in L^2(\mathcal{X},\pi^{-\frac{d}{2}}\mathrm{e}^{-x^2}\mathrm{d}x),$$

intertwines the representation ϕ_{rw} and ϕ_{rw1}.

Remark 2.5.8 *If the dimension of $\mathcal{X}$ is finite, then the representation ϕ_{rw1} corresponds to*

$$m_1(x) = (2\pi)^{-\frac{d}{4}}\mathrm{e}^{-\frac14 x^2}, \tag{2.5.23}$$

the pointwise positive ground state of $H = \frac12(-\Delta + \frac14 x^2 - \frac12)$. Thus we lose the symmetry between the positions and momenta.

2.5.4 Complex wave representation for a symplectic space with conjugation

Suppose now that $\mathcal{X}$ is a real Hilbert space. Equip $\mathcal{X}\oplus\mathcal{X}$ with the Kähler structure as above. Consider the complex wave representation defined in Subsection **??** and denoted

$$\mathcal{X}\oplus\mathcal{X} \ni (\eta,q) \mapsto \mathrm{e}^{\mathrm{i}\phi_{\mathrm{cw}}(\eta,q)} \in U(\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{Z})),$$

where we have the identification

$$\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{Z}) \simeq L^2(\overline{\mathcal{Z}},(2\pi\mathrm{i})^{-\frac{d}{2}}\mathrm{e}^{-|w|^2}\mathrm{d}w\mathrm{d}\overline{w}).$$

Recall also that the creation/annihillation operators are equal

$$z(a_{\mathrm{cw}}^*) = z\overline{w}, \quad \overline{z}(a_{\mathrm{cw}}) = \overline{z}\nabla_{\overline{w}}.$$

Since our symplectic space is equipped with a conjugation, we can define the position and momentum operators in the complex wave representation:

$$\eta x_{\mathrm{cw}} := \phi(\eta,0), \quad qD_{\mathrm{cw}} := \phi_{\mathrm{cw}}(0,q), \quad \eta,q\in\mathbf{C}\mathcal{X}.$$

Theorem 2.5.9 *Let $\eta, q \in \mathbf{C}\mathcal{X}$. In the complex wave representation, the position and the momentum operators are equal*

$$\begin{aligned}\eta x_{\mathrm{cw}} &= 1_{\overline{\mathcal{Z}}}\eta(\nabla_{\overline{w}} + \kappa\overline{w}) = \mathrm{e}^{-\overline{w}\kappa\overline{w}}(1_{\overline{\mathcal{Z}}}\eta)\nabla_{\overline{w}}\mathrm{e}^{\overline{w}\kappa\overline{w}} \\ qD_{\mathrm{cw}} &= 1_{\overline{\mathcal{Z}}}\eta(\nabla_{\overline{w}} - \kappa\overline{w}) = \mathrm{e}^{\overline{w}\kappa\overline{w}}(1_{\overline{\mathcal{Z}}}q)\nabla_{\overline{w}}\mathrm{e}^{-\overline{w}\kappa\overline{w}}.\end{aligned} \tag{2.5.24}$$

Proof. Set $y = (\eta, q) \in \mathbf{C}(\mathcal{X} \oplus \mathcal{X})$ and compute in the complex wave representation

$$\begin{aligned}\phi_{\mathrm{rw}}(y) &= (1_{\mathcal{Z}}y)(a^*) + (1_{\overline{\mathcal{Z}}}y)(a) \\ &= (1_{\mathcal{Z}}y)\overline{w} + (1_{\overline{\mathcal{Z}}}y)\nabla_{\overline{w}} \\ &= (1_{\overline{\mathcal{Z}}}\kappa y)\kappa\overline{w} + (1_{\overline{\mathcal{Z}}}y)\nabla_{\overline{w}} \\ &= \mathrm{e}^{-\overline{w}\kappa\overline{w}}(1_{\overline{\mathcal{Z}}}\eta)\nabla_{\overline{w}}\mathrm{e}^{\overline{w}\kappa\overline{w}} + \mathrm{e}^{\overline{w}\kappa\overline{w}}(1_{\overline{\mathcal{Z}}}q)\nabla_{\overline{w}}\mathrm{e}^{-\overline{w}\kappa\overline{w}}.\end{aligned}$$

□

One should admit, that the above form of the complex wave representation is somewhat confusing – two distinct conjugations and two distinct antiinvolutions are used! Therefore, it is common in the literature to use a somewhat modified version of a complex wave representation, which we describe below.

We replace the space $\mathcal{Z}$ by $\mathbf{C}\mathcal{X}$. Its generic (complex) element will be called x and the Lebesgue measure on $\mathbf{C}\mathcal{X}_{\mathbf{R}}$ will be denoted $(2\mathrm{i})^{-1}\mathrm{d}x\mathrm{d}\overline{x}$. We will use the identification

$$\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathbf{C}\mathcal{X}) \simeq L^2(\overline{\mathcal{Z}}, (2\pi\mathrm{i})^{-\frac{d}{2}}\mathrm{e}^{-|x|^2}\mathrm{d}x\mathrm{d}\overline{x}).$$

We define the following regular representation of CCR:

$$\mathcal{X} \oplus \mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}\eta x_{\mathrm{rw1}} + qD_{\mathrm{rw1}}} \in U(\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathbf{C}\mathcal{X})),$$

by setting

$$\begin{aligned}\eta x_{\mathrm{cw1}} &:= \eta\tfrac{1}{\sqrt{2}}(\nabla_x + x) = \eta\tfrac{1}{\sqrt{2}}\mathrm{e}^{-\frac{1}{2}x^2}\nabla_x\mathrm{e}^{\frac{1}{2}x^2}, \\ qD_{\mathrm{cw1}} &:= q\tfrac{1}{\mathrm{i}\sqrt{2}}(\nabla_x - x) = q\tfrac{1}{\mathrm{i}\sqrt{2}}\mathrm{e}^{\frac{1}{2}x^2}\nabla_x\mathrm{e}^{-\frac{1}{2}x^2}.\end{aligned}$$

The creation and annihillation operators for $z \in \mathcal{Z}$ are now equal to

$$z(a_{\mathrm{cw1}}^*) = \frac{1+\kappa}{\sqrt{2}}zx, \quad \overline{z}(a_{\mathrm{cw1}}) = \frac{1+\kappa}{\sqrt{2}}z\nabla_x.$$

The representations $y \to \mathrm{e}^{\mathrm{i}\phi_{\mathrm{cw}}(y)}$ and $y \to \mathrm{e}^{\mathrm{i}\phi_{\mathrm{cw1}}(y)}$ are intertwined by the unitary operator

$$\Gamma(\tfrac{1+\kappa}{\sqrt{2}}) : \Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{Z}) \to \Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathbf{C}\mathcal{Y}),$$

with the inverse

$$\Gamma(\sqrt{2}1_{\mathcal{Z}}) : \Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathbf{C}\mathcal{X}) \to \Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathbf{C}\mathcal{Z}).$$

2.5.5 Wick operation

Let $\mathcal{X}$ be a real Hilbert space.

Theorem 2.5.10 *Let $\psi \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$, $t \in \mathbf{C}\backslash\{0\}$. Then*

$$\mathrm{e}^{\frac{t}{2}\nabla_x^2}\psi(x) = \mathrm{e}^{-\frac{1}{2t}x^2}\psi(t\nabla_x)\mathrm{e}^{\frac{1}{2t}x^2}. \tag{2.5.25}$$

Proof. Assume for the moment that $t > 0$. We have

$$\begin{aligned}
\mathrm{e}^{\frac{t}{2}\nabla_x^2}\psi(x) &= (2\pi t)^{-\frac{d}{2}}\int \mathrm{e}^{-\frac{1}{2t}(x-y)^2}\psi(y)\mathrm{d}y \\
&= (2\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{1}{2t}x^2}\psi(t\nabla_x)\int \mathrm{e}^{-\frac{1}{t}xy-\frac{1}{2t}y^2}\mathrm{d}y \\
&= (2\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{1}{2t}x^2}\psi(t\nabla_x)\mathrm{e}^{\frac{1}{2t}x^2}\int \mathrm{e}^{\frac{1}{2t}(x-y)^2}\mathrm{d}y = \mathrm{e}^{-\frac{1}{2t}x^2}\psi(t\nabla_x)\mathrm{e}^{\frac{1}{2t}x^2}.
\end{aligned}$$

For arbitrary $t \neq 0$ the identity (2.5.25) follows by analytic continuation. □

For $\psi \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$ define the "Wick operation"

$$:\psi(x) := \mathrm{e}^{-\frac{1}{2}\nabla_x^2}\psi(x) = \mathrm{e}^{\frac{1}{2}x^2}\psi(-\nabla_x)\mathrm{e}^{-\frac{1}{2}x^2} \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}).$$

The range of the map $\psi \mapsto :\psi:$ is $\mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$. If $\rho(x) =: \psi(x):$, then

$$\psi(x) = \mathrm{e}^{\frac{1}{2}\nabla_x^2}\rho(x) = \mathrm{e}^{-\frac{1}{2}x^2}\rho(\nabla_x)\mathrm{e}^{\frac{1}{2}x^2} \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X}).$$

Theorem 2.5.11 *The map $\psi \mapsto :\psi:$ extends to a unitary map*

$$\mathbf{C}\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{X}) \ni \psi \mapsto :\psi: \in L^2(\mathcal{X}, (2\pi)^{-\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x).$$

It intertwines the representations of CCR:

$$:\mathrm{e}^{\mathrm{i}\eta x_{\mathrm{cw1}}+qD_{\mathrm{cw1}}}\psi := \mathrm{e}^{\mathrm{i}\eta x_{\mathrm{rw1}}+qD_{\mathrm{rw1}}}:\psi:, \quad \psi \in \mathbf{C}\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{X}) \simeq \Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathbf{C}\mathcal{X}).$$

Proof. To prove this theorem one can choose an othonormal basis in $\mathcal{X}$ and reduce the problem to 1 dimension.

Thus assume that $\mathcal{X} = \mathbf{R}$ and introduce the operators $a^* = \frac{x}{2} - \nabla_x$, $a = \frac{x}{2} + \nabla_x$ on $L^2(\mathbf{R})$ and the vector $\Omega = (2\pi)^{-\frac{1}{4}}\mathrm{e}^{-\frac{1}{4}x^2}$. They satisfy

$$[a, a^*] = 1, \quad a\Omega = 0, \quad \|\Omega\| = 1.$$

$\mathbf{C}\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathbf{R})$ has an orthogonal basis given by monomials x^n normed by the condition $\|x^n\|^2 = n!$. Now

$$\begin{aligned}
:x^n:\mathrm{e}^{-\frac{1}{4}x^2} &= \mathrm{e}^{\frac{1}{4}x^2}(-\nabla_x)^n\mathrm{e}^{-\frac{1}{2}x^2} \\
&= (\tfrac{x}{2} - \nabla_x)^n\mathrm{e}^{-\frac{1}{4}x^2} = (2\pi)^{\frac{1}{4}}(a^*)^n\Omega).
\end{aligned}$$

Properties of representations of CCR imply that $((a^*)^n\Omega|(a^*)^m\Omega) = \delta_{n,m}n!$. □

Remark 2.5.12 *The Wick operation for monomials is the same as the Gramm-Schmidt orthogonalization procedure with the weight* $\mathrm{e}^{-\frac{1}{2}x^2}$*. In the case of a single variable, the polynomials* $:x^n:$ *are rescaled Hermite polynomials. More precisely, if one adopts the following definition of Hermite polynomials*

$$\mathrm{e}^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x),$$

then

$$:x^n: = \sqrt{2}^n H_n(\tfrac{x}{\sqrt{2}}).$$

Theorem 2.5.13 *If* $\rho \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$*, then*

$$(2\pi)^{-\frac{d}{2}} \int \rho(x) \mathrm{e}^{-\frac{1}{2}x^2} \mathrm{d}x = \left(\mathrm{e}^{\frac{1}{2}\nabla_x^2}\rho\right)(0). \tag{2.5.26}$$

Proof. We treat ρ as an element of $\mathbf{C}\Gamma_{\mathrm{s}}^{\mathrm{cw}}(\mathcal{X})$. We can write $\rho(x) =: \psi(x) :=: \mathrm{e}^{\frac{1}{2}\nabla_x^2}\rho(x) :$. By Theorem 2.5.11

$$(2\pi)^{-\frac{d}{2}} \int :\psi(x): \mathrm{e}^{-\frac{1}{2}x^2} \mathrm{d}x = (1|:\psi:)_{L^2(\mathcal{X},(2\pi)^{-\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)} = (\Omega|\psi)_{\mathrm{cw}} = \psi(0). \tag{2.5.27}$$

Note that the right hand side of (2.5.26) can be expanded in a finite sum and leads to the well known sum over all possible "pairings". This is the simplest version of what is usually called the Wick theorem.

A more complicated version of the Wick theorem, leading to a graphical interpretation in terms of diagrams is given below.

Theorem 2.5.14 *Let* $\psi_1, \dots, \psi_n \in \mathbf{C}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$*. Then*

$$:\psi_1(x): \cdots :\psi_n(x): =: \exp\Big(\sum_{i<j} \nabla_{x_i}\nabla_{x_j}\Big)\psi_1(x_1)\cdots\psi_n(x_n)\Big|_{x=x_1=\cdots=x_n} :. \tag{2.5.28}$$

$$(2\pi)^{-\frac{d}{2}} \int :\psi_1(x): \cdots :\psi_n(x): \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x = \exp\Big(\sum_{i<j} \nabla_{x_i}\nabla_{x_j}\Big)\psi_1(x_1)\cdots\psi_n(x_n)\Big|_{0=x_1=\cdots=x_n}. \tag{2.5.29}$$

Proof. Let us prove (2.5.28).

$$\begin{aligned}
&:\psi_1(x): \cdots :\psi_n(x): \\
&= \mathrm{e}^{-\frac{1}{2}\nabla_{x_1}^2}\psi_1(x_1)\cdots \mathrm{e}^{-\frac{1}{2}\nabla_{x_n}^2}\psi_n(x_n)\Big|_{x=x_1=\cdots=x_n} \\
&=: \mathrm{e}^{\frac{1}{2}\nabla_x^2}\Big(\mathrm{e}^{-\frac{1}{2}\nabla_{x_1}^2}\psi_1(x_1)\cdots \mathrm{e}^{-\frac{1}{2}\nabla_{x_n}^2}\psi_n(x_n)\Big|_{x=x_1=\cdots=x_n}\Big) : \\
&=: \mathrm{e}^{\frac{1}{2}(\nabla_{x_1}+\cdots+\nabla_{x_n})^2-\frac{1}{2}\nabla_{x_1}^2-\cdots-\frac{1}{2}\nabla_{x_n}^2}\psi_1(x_1)\cdots\psi_n(x_n)\Big|_{x=x_1=\cdots=x_n} :.
\end{aligned}$$

In the last step we used $\nabla_x = \nabla_{x_1} + \cdots + \nabla_{x_n}$.

(2.5.29) follows from (2.5.28) by (2.5.27). □

For completeness, let us describe the unitary operator intertwining $\phi_{\rm cw}$ and $\phi_{\rm rw}$. It is also a version of the Wick operation and we will denote it also by double dots.

For $\psi \in \overset{\circ}{\Gamma}_{\rm s}(\mathcal{Z})$ we define

$$:\psi(x): = {\rm e}^{x^2}\left(\Gamma(\tfrac{1+\kappa}{2})\psi\right)(-\nabla_x){\rm e}^{-x^2} = {\rm e}^{-\frac14\nabla_x^2}\Gamma(\tfrac{1+\kappa}{2})\psi(x), \tag{2.5.30}$$

The range of (2.5.30) equals $\mathbf{C}\overset{\circ}{\Gamma}_{\rm s}(\mathcal{X})$. To describe the inverse of (2.5.30), let $\rho(x) =: \psi(x) :\in \mathbf{C}\overset{\circ}{\Gamma}_{\rm s}(\mathcal{X})$. Then

$$\psi(\overline{w}) = {\rm e}^{-\overline{w}\kappa\overline{w}}\left(\Gamma(1_{\mathcal{Z}})\rho\right)(\nabla_{\overline{w}}){\rm e}^{\overline{w}\kappa\overline{w}} = {\rm e}^{\nabla_{\overline{w}}\kappa\nabla_{\overline{w}}}\Gamma(1_{\mathcal{Z}})\rho(\overline{w}). \tag{2.5.31}$$

Theorem 2.5.15 *The map $\psi \mapsto :\psi:$ extends to a unitary map*

$$\Gamma_{\rm s}^{\rm cw}(\mathcal{Z}) \ni \psi \mapsto :\psi: \in L^2(\mathcal{X}, 2\pi^{-\frac{d}{2}}{\rm e}^{-x^2}{\rm d}x).$$

It intertwines the representations of CCR:

$$:{\rm e}^{{\rm i}\eta x_{\rm cw} + qD_{\rm cw}}\psi := {\rm e}^{{\rm i}\eta x_{\rm rw}+qD_{\rm rw}} :\psi:, \quad \psi \in \Gamma_{\rm s}^{\rm cw}(\mathcal{Z}).$$

Proof. Let us show (2.5.30). It follows from (2.5.21) that, for $\psi \in \overset{\circ}{\Gamma}_{\rm s}(\mathcal{Z})$, in the real wave representation we have

$$\psi(a_{\rm rw}^*) = {\rm e}^{x^2}\left(\Gamma(\tfrac{1+\kappa}{2})\psi\right)(-\nabla_x){\rm e}^{-x^2}. \tag{2.5.32}$$

But the vacuum both in the complex and real wave representations is 1.

Let us show (2.5.31). By (2.5.24), if $\rho \in \mathbf{C}\overset{\circ}{\Gamma}_{\rm s}(\mathcal{X})$, in the complex wave representation we have

$$\rho(x_{\rm cw}) = {\rm e}^{-\overline{w}\kappa\overline{w}}\left(\Gamma(1_{\overline{\mathcal{Z}}})\rho\right)(\nabla_{\overline{w}}){\rm e}^{\overline{w}\kappa\overline{w}}\ .$$

Again, we use that the vacuum is 1. □

2.6 Translations in phase space

2.6.1 Translated representations

Let $\mathcal{Y}$ be a space with an antisymmetric form ω. Let $\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$ be a representation of CCR.

Let $v \in \mathcal{Y}^{/\mathrm{al}}$. Then

$$\mathcal{Y} \ni y \mapsto W_v(y) := W(y)\mathrm{e}^{\mathrm{i}vy} \in U(\mathcal{H}) \tag{2.6.33}$$

is also a representation of CCR. We say that this representation is the translation of the representation $y \mapsto W(y)$ by v. Note that the quantization wrt the new representation is related to the old quantization by the relation

$$\mathrm{Op}_v(h) = \mathrm{Op}(h(\cdot - v)).$$

Clearly, if we first translate our representation by v_1 and then by v_2, then we obtain the original representation translated by $v_1 + v_2$.

Theorem 2.6.1 *If $v \in \omega\mathcal{Y}$, (which is a subspace of $\mathcal{Y}^{/\mathrm{al}}$), then $y \mapsto W_v(y)$ and $y \mapsto W(y)$ are unitarily equivalent:*

$$W_v(y) = W(\omega^{-1}v)W(y)W(-\omega^{-1}v).$$

2.6.2 Translated Fock representations

Theorem 2.6.2 *Suppose that $\mathcal{Y}_0$ is a Kähler space. Let $\mathcal{Y}$ be its completion and $\mathcal{Z}$ the holomorphic subspace of $\mathbf{C}\mathcal{Y}$. Let $\mathcal{Y} \ni y \mapsto W(y) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ be the corresponding Fock representation of CCR and Ω the its vacuum, which, as we recall, satisfies*

$$(\Omega|W(y)\Omega) = \mathrm{e}^{-\frac{1}{4}y^2}$$

Let $v \in \mathcal{Y}_0^{/\mathrm{al}}$ and consider the representation of CCR

$$\mathcal{Y}_0 \ni y \mapsto W_v(y) := W(y)\mathrm{e}^{\mathrm{i}vy} \in U(\Gamma_{\mathrm{s}}(\mathcal{Z})).$$

Then the following conditions are equivalent:
1) There exists a unitary operator $U_v \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ such that

$$U_v W(y) = W_v(y) U_v, \quad y \in \mathcal{Y}_0;$$

2) $v \in \mathcal{Y}$. 2.3.49

Proof. 1)$\Leftarrow$2) follows from Theorem 2.6.1.

1)$\Rightarrow$2). First note that for $y \in \mathcal{Y}_0$, in the representation $y \mapsto W_v(y)$, the field operators are given by

$$\phi_v(y) = \phi(y) + yv.$$

Let $\mathcal{Z}_0$ denote the holomorphic subspace of $\mathbf{C}\mathcal{Y}_0$ (which is a dense subspace of $\mathcal{Z}$). We have $v = w + \overline{w}$ for some $w \in \mathcal{Z}_0^{/\mathrm{al}}$ For $z \in \mathcal{Z}_0$, the annihillation operators in the translated representation are given by

$$\overline{z}(a_v) = \overline{z}(a) + (z|w).$$

Suppose that 1) holds. Then there exists a vacuum for the annihilation operators a_v, that means a vector $\Omega_w \in \Gamma_{\mathrm{s}}(\mathcal{Z})$ such that

$$\overline{z}(a_v)\Omega_v = 0, \quad z \in \mathcal{Z}_0.$$

Both Ω and Ω_w are analytic vectors for $\overline{z}(a_v)$ (for Ω_w this follows from Theorem **??**, for Ω this is obvious). Therefore, the second identity below is justified:

$$0 = (\Omega|\overline{z_n}(a_v)\cdots\overline{z_1}(a_v)\Omega_w) = (z_1(a_v^*)\cdots z_n(a_v^*)\Omega|\Omega_w), \qquad z_1, \ldots, z_n \in \mathcal{Z}_0.$$

By induction, this implies that if $\lambda = (\Omega|\Omega_w)$, then

$$(z_1(a^*)\cdots z_n(a^*)\Omega|\Omega_w) = (-1)^n\lambda(z_1|w)\cdots(z_n|w).$$

In particular, for $z \in \mathcal{Z}_0$, this gives the following formula for the 1-particle component of Ω_v:

$$(z|\Omega_w) = -\lambda(z|w), \quad z \in \mathcal{Z}_0.$$

This can be extended to a bounded functional on $\mathcal{Z}$ only if either $w \in \mathcal{Z}$ or $\lambda = 0$. But $w \in \mathcal{Z}$ iff $v \in \mathcal{Y}$. □

Remark 2.6.3 *Clearly, if $v \in \mathcal{Y}$, and hence $w \in \mathcal{Z}$, then Ω_w are the familiar Gaussian coherent vector.*

2.6.3 Relative boundedness of field operators

Let $\mathcal{Y}_0$ be a dense subspace of $\mathcal{Y}$. Let $\mathcal{Z}_0$ be the corresponding holomorphic subspace of $\mathcal{Z}$ (which is dense in $\mathcal{Z}$). Clearly, $\mathcal{Y}$ is naturally embedded in $\mathcal{Y}_0^{/\mathrm{al}}$ and $\mathcal{Z}$ is naturally embedded in $\mathcal{Z}_0^{/\mathrm{al}}$.

Let $z \in \mathcal{Z}_0^{/\mathrm{al}}$. Then $\overline{z}(a)$ can be defined as an operator with domain, say, $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z}_0)$. $z(a^*)$ can be defined as a quadratic form with domain $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z}_0)$.

Likewise, if $y \in \mathcal{Y}^{/\mathrm{al}}$, Then $\phi(y)$ can be defined as a quadratic form with domain $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z}_0)$.

Let h be a postive operator on $\mathcal{Z}$. Then for $\alpha \geq 0$, $\mathcal{D}(h^\alpha)$ is a subspace of $\mathcal{Z}$ equal to $(1+h)^{-\alpha}\mathcal{Z}$ with the natural structure of a Hilbert space. It is natural to denote the space of continuous functionals on $\mathcal{D}(h^\alpha)$ by $(1+h)^\alpha\mathcal{Z}$. It is clearly a subspace of $\mathcal{Z}^{/\mathrm{al}}$ containing $\mathcal{Z}$.

Theorem 2.6.4 *Let h be a positive operator on $\mathcal{Z}$.*
1) For $z \in (1+h)^{\frac{1}{2}}\mathcal{Z}$ we have

$$\|\overline{z}(a)\psi\|^2 \leq (z|h^{-1}z)(\psi|\mathrm{d}\Gamma(h)\psi).$$

2) For $z \in \mathcal{Z}$ we have

$$\|z(a^*)\psi\|^2 \leq \|z\|^2\|\psi\|^2 + (z|h^{-1}z)(\psi|\mathrm{d}\Gamma(h)\psi).$$

3) Let $z \in (1+h)^{\frac{1}{2}}\mathcal{Z}$. Then for any $\epsilon > 0$, we have

$$|(\psi|\phi(z+\overline{z})\psi)| \leq \epsilon(z|h^{-1}z)(\psi|\mathrm{d}\Gamma(h)\psi) + \epsilon^{-1}\|\psi\|^2, \tag{2.6.34}$$

$$\|\phi(z+\overline{z})\psi\| \leq (|z| + \epsilon^{-1}(z|h^{-1}z)^{\frac{1}{2}})\|\psi\| + \epsilon\|\mathrm{d}\Gamma(h)\psi\|. \tag{2.6.35}$$

4)

$$-(z|h^{-1}z) \leq \mathrm{d}\Gamma(h) + \phi(z+\overline{z}).$$

Proof. First we show 1), which implies 2). From 1) we obtain

$$|(\psi|\phi(z+\overline{z})\psi)| \leq 2\|\psi\|(z|h^{-1}z)^{\frac{1}{2}}(\psi|\mathrm{d}\Gamma(h)\psi)^{\frac{1}{2}},$$

which implies 3) and 4). □

2.6.4 Van Hove model

Consider an operator on $\Gamma_{\mathrm{s}}(\mathcal{Z})$, given (at least formally) by the expression

$$H = \mathrm{d}\Gamma(h) + \phi(z+\overline{z}),$$

where h is a self-adjoint operator on $\mathcal{Z}$ and $z \in \mathcal{Z}$ (or is in some sense affilliated to $\mathcal{Z}$).

The ultraviolet problem

Suppose that h is unbounded and $h \geq c > 0$.
1) $z \in \mathcal{Z}$.

$\phi(z+\overline{z})$ is a relatively bounded perturbation of N, hence also of $\mathrm{d}\Gamma(h)$. Therefore, H is a well defined operator bounded from below, with the domain $\mathcal{D}(\mathrm{d}\Gamma(h))$.

Let us define the "dressing operator" $U := W(-\mathrm{i}h^{-1}z + \mathrm{i}\overline{h^{-1}z})$. Then

$$U\mathrm{d}\Gamma(h)U^* = \mathrm{d}\Gamma(h) - \phi(z+\overline{z}) + (z|h^{-1}z), \tag{2.6.36}$$

$$U\phi(z+\overline{z})U^* = \phi(z+\overline{z}) - 2(z|h^{-1}z) =: \phi^{\mathrm{as}}(z+\overline{z}), \quad z \in \mathcal{Z}, \tag{2.6.37}$$

and hence

$$UHU^* = \mathrm{d}\Gamma(h) - (z|h^{-1}z), \tag{2.6.38}$$

As a consequence of (2.6.38), H possesses a unique ground state (this means, the infimum of the spectrum of H is its non-degenerate eigenvalue).

2) $z \in h^{\frac{1}{2}}\mathcal{Z}\backslash\mathcal{Z}$

By (2.6.34), $\phi(z+\overline{z})$ is a densely defined quadratic form relatively bounded wrt $\mathrm{d}\Gamma(h)$. H is a well defined semibounded operator with the form domain $\mathcal{D}(\mathrm{d}\Gamma(h)^{\frac{1}{2}})$.

The dressing operator U is well defined. The relations (2.6.36), (2.6.38) and (2.6.37) hold. By (2.6.38), H possesses a unique ground state.

3) $z \in h\mathcal{Z}\backslash h^{\frac{1}{2}}\mathcal{Z}$.

$\phi(z+\overline{z})$ is not relatively form bounded wrt $\mathrm{d}\Gamma(h)$. H is not well defined. But U is well defined and one can define the "renormalized Hamiltonian" $H^{\mathrm{ren}} := U^*\mathrm{d}\Gamma(h)U + C$, where C is an arbitrary constant. Then (2.6.37) is still true, H^{ren} is a semibounded operator with a unique ground state.

The infrared problem

Suppose that h is bounded and $h \geq 0$.

1) $z \in h\mathcal{Z}$.

By Theorem 2.6.35), $\phi(z+\overline{z})$ is a relatively bounded perturbation of $\mathrm{d}\Gamma(h)$ with the infinitesimal bound. Therefore, H is well defined with the domain $\mathcal{D}(\mathrm{d}\Gamma(h))$ and bounded from below. U is well defined, (2.6.36), (2.6.38) and (2.6.37) are satisfied. By (2.6.38), the operator H has a unique ground state.

2) $z \in h^{\frac{1}{2}}\mathcal{Z}\backslash h\mathcal{Z}$.

By Theorem 2.6.4 3), $\phi(z+\overline{z})$ is a relatively bounded perturbation of $\mathrm{d}\Gamma(h)$ with an infinitesimal bound. Therefore, H is well defined with the domain $\mathcal{D}(\mathrm{d}\Gamma(h))$ and bounded from below. U is not defined, thus (2.6.36), (2.6.38), (2.6.37) do not make sense. For $z_i \in h^{\frac{1}{2}}\mathcal{Z}$, $\mathrm{e}^{\mathrm{i}\phi^{\mathrm{as}}(z_1+\overline{z_1})}$ are well defined but yield a representation of CCR which is not equivalent to the Fock representation. As we will show below, H has no ground states and the infimum of H equals $-(z|h^{-1}z)$.

Theorem 2.6.5 *Assume $z \in h^{\frac{1}{2}}\mathcal{Z}\backslash h\mathcal{Z}$. Then the operator H satisfies*

$$\sigma(H) = [-(z|h^{-1}z), \infty[, \quad -(z|h^{-1}z) \notin \sigma_{\mathrm{pp}}(H).$$

Proof. Step 1) For any n define $\mathcal{Z}_n := 1_{[\frac{1}{n},\infty[}(h)\mathcal{Z}$, $h_n := 1_{[\frac{1}{n},\infty[}(h)h$, $z_n := 1_{[\frac{1}{n},\infty[}(h)v$, $H_n := \mathrm{d}\Gamma(h_n) + \phi(z_n + \overline{z_n})$, $U_n := W(-\mathrm{i}h^{-1}z_n + \mathrm{i}\overline{h_n^{-1}z})$. Then $U_nH_nU_n^* = \mathrm{d}\Gamma(h_n) - (z_n|h^{-1}z_n)$. Hence

$$\sigma(H_n) = [-(z_n|h^{-1}z_n), \infty[. \tag{2.6.39}$$

All the groundstates of H_n satisfy the conditions

$$(\overline{w}(a) - (w|h^{-1}z_n))\psi = 0, \quad w \in \mathcal{Z}. \tag{2.6.40}$$

For further use let us write down the following corrollary of (2.6.40)

$$(\overline{w}(a) - (w|h^{-1}z))\psi = 0, \quad w \in \mathcal{Z}_n. \tag{2.6.41}$$

Step 2) Likewise, set $h_n^\perp := h - h_n$, $z_n^\perp := z - z_n$, $H_n^\perp := \mathrm{d}\Gamma(h_n^\perp) + \phi(z_n^\perp + \overline{z_n^\perp})$. By Theorem 2.6.4 4),

$$H_n^\perp \geq -(z_n^\perp|h^{-1}z_n^\perp). \tag{2.6.42}$$

Step 3) We have $\mathcal{D}(H_n^\perp) = \mathcal{D}(\mathrm{d}\Gamma(h_n^\perp)) \ni \Omega$. Hence

$$\inf H_n^\perp \leq (\Omega|H_n^\perp\Omega) = 0. \tag{2.6.43}$$

Step 4) $\Gamma_{\mathrm{s}}(\mathcal{Z})$ can be identified with $\Gamma_{\mathrm{s}}(\mathcal{Z}_n^\perp) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_n)$ and

$$H = H_n^\perp \otimes 1 + 1 \otimes H_n. \tag{2.6.44}$$

Therefore,

$$\sigma(H) = \sigma(H_n^\perp) + \sigma(H_n). \tag{2.6.45}$$

It follows from (2.6.39), (2.6.42), (2.6.43) and (2.6.45) that

$$[-(z|h^{-1}z), \infty[\subset \sigma(H) \subset [-(z_n|h^{-1}z_n), \infty[,$$

where we used $(z_n^\perp|h^{-1}z_n^\perp) + (z_n|h^{-1}z_n) = (z|h^{-1}z)$.
Step 5) We have

$$\lim_{n\to\infty} (z_n^\perp|h^{-1}z_n^\perp) = 0.$$

Hence

$$\sigma(H) = [-(z|h^{-1}z), \infty[.$$

Step 6) Let $\psi \in \Gamma_{\mathrm{s}}(\mathcal{Z})$ be a non-zero ground state of H. This means

$$H\psi = -(z|h^{-1}z)\psi.$$

By (2.6.44), ψ has to be a ground state of $1 \otimes H_n$ for any n. By (2.6.41), for $w \in \bigcup\limits_{n=1}^{\infty} \mathcal{Z}_n$

$$(\overline{w}(a) - (w|h^{-1}z))\psi = 0.$$

But $\bigcup\limits_{n=1}^{\infty} \mathcal{Z}_n$ is dense in $\mathcal{Z}$ and hence Theorem 2.3.49 implies that $h^{-1}z \in \mathcal{Z}$. $\square$

2.7 Symplectic covariance of CCR's

2.7.1 Quadratic Hamiltonians

Let $(\mathcal{Y},\omega)$ be a space with an antisymmetric form. Let us recall that the Poisson bracket is defined for $f,g\in C^\infty(\mathcal{Y}')$ as

$$\{f,g\}(v):=\nabla f(v)\omega\nabla_v g(v).$$

Recall that for $f\in C^\infty(\mathcal{Y}')$, $\mathrm{H}_f:=-\omega\nabla f(v)$. We have

$$\{f,g\}=\mathrm{H}_f g(v).$$

In particular, if $g(v)=\frac{1}{2}vbv+yv$, for $b\in L_{\mathrm{s}}(\mathcal{Y},\mathcal{Y}')$, $y\in\mathcal{Y}$, then $\mathrm{H}_g=-\omega(bv+y)$.

Let $\mathcal{Y}\ni y\mapsto\phi(y)\in L(\mathcal{D})$ be a Heisenberg-type representation of CCR.

Let $\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\leq n}(\mathcal{Y})=\oplus_{j=0}^{n}\overset{\circ}{\Gamma}{}^{n}(\mathcal{Y})$. Let $\mathrm{CCR}^{\leq n}$ denote the family of polynomials in $\phi(y)$ of degree at most n.

Theorem 2.7.1 *1) For $g\in\mathbf{C}\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\leq 2}(\mathcal{Y})$, $f\in\mathbf{C}\overset{\circ}{\Gamma}{}_{\mathrm{s}}(\mathcal{Y})$ we have*

$$-\mathrm{i}[\mathrm{Op}(g),\mathrm{Op}(f)]=\mathrm{Op}(\{g,f\}).$$

2) The map

$$\mathbf{C}\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\leq 2}(\mathcal{Y})\ni g\mapsto\mathrm{Op}(g)\in\mathrm{CCR}^{\leq 2}(\mathcal{Y})$$

is a $*$-isomorphism of Lie algebras, where $\mathbf{C}\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\leq 2}(\mathcal{Y})$ is eqipped with the Poisson bracket $\{\cdot,\cdot\}$ and $\mathrm{CCR}^{\leq 2}(\mathcal{Y})$ is equipped with the bracket $-\mathrm{i}[\cdot,\cdot]$.

3) If $g,f\in\mathbf{C}\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\leq 2}(\mathcal{Y})$, then

$$[\mathrm{ad}_{\mathrm{Op}(g)},\mathrm{ad}_{\mathrm{Op}(f)}]=\mathrm{ad}_{\mathrm{Op}(\{g,f\})}\tag{2.7.46}$$

where $\mathrm{ad}_G\in L(\mathrm{CCR}(\mathcal{Y}))$ is defined as $\mathrm{ad}_G F:=[G,F]$.

Proof. 1) follows from the definition of the Weyl-Wigner-Moyall product. 2) and 3) follow immediately from 1). □

2.7.2 Quadratic Hamiltonians in the Schrödinger representation

Suppose that the $\mathcal{X}$ is a vector space and space $\mathcal{Y}=\mathcal{X}'\oplus\mathcal{X}$ is eqipped with the canonical symplectic form.

Any $b\in\mathbf{C}\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{2}(\mathcal{Y})$ is of the form

$$\mathcal{X}\oplus\mathcal{X}'\ni(x,\xi)\mapsto b(x,\xi)=\frac{1}{2}x\alpha x+\xi cx+\frac{1}{2}\xi\beta\xi,$$

where $\alpha \in \mathring{L}_{\mathrm{s}}(\mathcal{X}, \mathcal{X}')$, $\beta \in \mathring{L}_{\mathrm{s}}(\mathcal{X}', \mathcal{X})$ and $c \in \mathring{L}(\mathcal{X})$. We have

$$\begin{aligned}\mathrm{Op}^{x,D}(b) &= \tfrac{1}{2}x\alpha x + xc'D + \tfrac{1}{2}D\beta D,\\ \mathrm{Op}(b) &= \mathrm{Op}^{x,D}(b) + \tfrac{\mathrm{i}}{2}\mathrm{Tr}c.\end{aligned}$$

2.7.3 Quadratic Hamiltonians in the Fock representation

Suppose that $\mathcal{Y}$ is a complete Kähler space. Let $\mathcal{T} \ni y \mapsto W(y) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ be the corresponding Fock representation. Every $b \in \mathbf{C}\mathring{\Gamma}_{\mathrm{s}}^{2}(\mathcal{Y})$ is of the form

$$\mathbf{C}\mathcal{Y} \ni (\overline{w}, w) \mapsto b(\overline{w}, w) = \frac{1}{2}w\overline{d_1}w + \overline{w}hw + \frac{1}{2}\overline{w}d_2\overline{w}, \tag{2.7.47}$$

where $d_1, d_2 \in \mathring{L}_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$, $h \in \mathring{L}(\mathcal{Z})$. We have

$$\begin{aligned}\mathrm{Op}^{a^*,a}(b) &= \tfrac{1}{2}a\overline{d_1}a + a^*ha + \tfrac{1}{2}a^*d_2a^*,\\ \mathrm{Op}(b) &= \mathrm{Op}^{a^*,a}(b) + \tfrac{1}{2}\mathrm{Tr}h.\end{aligned} \tag{2.7.48}$$

Remark 2.7.2 *Note that we have identifications $\mathring{L}_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z}) \simeq \mathring{\Gamma}_{\mathrm{s}}^{2}\mathcal{Z}$. Thus we have at least three possible notations for the same operators:*

$$\mathrm{Op}^{a^*,a}(\overline{d_1}) = \overline{d_1}(a) = a\overline{d_1}a,$$

$$\mathrm{Op}^{a^*,a}(d_2) = d_2(a^*) = a^*d_2a^*.$$

Likewise, we have the following notations

$$\mathrm{Op}^{a^*,a}(h) = \mathrm{d}\Gamma(h) = a^*ha.$$

In this section, in general we will adopt the last notation.

Clearly, d_1, d_2 and h do not have to be of finite rank for the above operators to be defined.

Theorem 2.7.3 *1) If $d \in l_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z}) \simeq \Gamma_{\mathrm{s}}^2(\mathcal{Z})$, then $a\overline{d}a$ and a^*da^* with the domain $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ are closable operators.*
2) If h is is a closed operator on $\mathcal{Z}$, then a^*ha with the domain $\mathring{\Gamma}_{\mathrm{s}}(\mathcal{D}(h)$ is closable.
3) Suppose that $d \in l_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z}) \simeq \Gamma_{\mathrm{s}}^2(\mathcal{Z})$ and $h \in B_{\mathrm{h}}(\mathcal{Z})$ (h is bounded and self-adjoint).

$$\mathrm{Op}^{a^*,a}(b) = \frac{\mathrm{i}}{2}a\overline{d}a + a^*ha - \frac{\mathrm{i}}{2}a^*da^* \tag{2.7.49}$$

In particular, (2.7.49) is essentially self-adjoint on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$.

Proof. To show 3) we check that $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ is an invariant subspace of analytic vectors of (2.7.49). □

If b is of the form (2.7.47) with $d_1, d_2 \in \Gamma_{\mathrm{s}}^2(\mathcal{Z})$ and $h \in l^1(\mathcal{Z})$, then we define the Weyl-Wigner quantization of b by the formula

$$\mathrm{Op}(h) = \frac{1}{2} a\overline{d_1} a + a^* h a + \frac{1}{2} a^* d_2 a^* + \frac{1}{2}\mathrm{Tr} h.$$

Clearly, this definition is an extension of the usual definition of the Weyl-Wigner quantization for the symbols in $\mathbf{C}\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{2}(\mathcal{Y})$.

2.7.4 1-parameter groups generated by quadratic Hamiltonians

Let $(\mathcal{Y}, \omega)$ be a space with an antisymmetric form. Let $\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$ be a regular representation of CCR.

Theorem 2.7.4 *Let $g \in \overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\leq 2}(\mathcal{Y})$. Let $\mathcal{Y}_1 \subset \mathcal{Y}$ be a finite dimensional subspace, such that $g \in \overset{\circ}{\Gamma}{}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}_1)$. Then $\mathrm{Op}(g)$ is essentially self-adjoint on $\mathcal{G}(\mathcal{Y}_1)$ and if $f \in \mathcal{S}'(\mathcal{Y}_1')$, then*

$$\mathrm{e}^{-\mathrm{i}\mathrm{Op}(g)} \mathrm{Op}(f) \mathrm{e}^{\mathrm{i}\mathrm{Op}(g)} = \mathrm{Op}(\mathrm{e}^{\mathrm{H}_g} f). \tag{2.7.50}$$

Proof. It suffices to assume that $\mathcal{Y}$ is finite dimensional and the form ω symplectic. Then the representation $\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$ is unitarily equivalent to the Fock representation in $\Gamma_{\mathrm{s}}(\mathcal{Z})$, so it is sufficient to consider the latter.

Just as in the proof of Theorem 2.7.3, we show that $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ is an invariant subspace of analytic vectors for $\mathrm{Op}(g)$. Therefore, $\mathrm{Op}(g)$ is essentially self-adjoint on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$.

Clearly,

$$\mathcal{S}(\mathcal{Y}') \ni f \mapsto \mathrm{e}^{t\mathrm{H}_g} f \in \mathcal{S}(\mathcal{Y}') \tag{2.7.51}$$

is a 1-parameter group with respect to $t \in \mathbf{R}$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{t\mathrm{H}_g} f\Big|_{t=0} = \mathrm{H}_g f.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Op}(\mathrm{e}^{t\mathrm{H}_g} f))\Big|_{t=0} = \mathrm{Op}(\mathrm{H}_g f).$$

Let $\mathcal{F}(\mathcal{Y})$ denote the set of operators of the form $\mathrm{Op}(f)$ with $f \in \mathcal{S}(\mathcal{Y}')$. Then

$$\mathcal{F}(\mathcal{Y}) \ni \mathrm{Op}(f) \mapsto \mathrm{e}^{-\mathrm{i}t\mathrm{Op}(g)} \mathrm{Op}(f) \mathrm{e}^{\mathrm{i}t\mathrm{Op}(g)} \in \mathcal{F}(\mathcal{Y}) \tag{2.7.52}$$

is a 1-parameter group with respect to $t \in \mathbf{R}$. By Theorem 2.7.1 1)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(g)}\mathrm{Op}(f)\mathrm{e}^{\mathrm{i}t\mathrm{Op}(g)}\Big|_{t=0} &= -\mathrm{i}[\mathrm{Op}(g), \mathrm{Op}(f)] \\ &= \mathrm{Op}(\{g, f\}) = \mathrm{Op}(\mathrm{H}_g f). \end{aligned}$$

Therefore, (2.7.51) equals (2.7.52). Hence (2.7.50) holds for $f \in \mathcal{S}(\mathcal{Y})$.

By duality, we extend (2.7.50) to $f \in \mathcal{S}'(\mathcal{Y}')$. □

In particular, if $b \in \overset{\circ}{\Gamma}{}^2_{\mathrm{s}}(\mathcal{Y})$, then

$$\mathrm{e}^{-\mathrm{i}\mathrm{Op}(\frac{1}{2}b)}\mathrm{Op}(f)\mathrm{e}^{\mathrm{i}\mathrm{Op}(\frac{1}{2}b)} = \mathrm{Op}(h(\mathrm{e}^{-\omega b}\cdot)),$$

or equivalently

$$\mathrm{e}^{-\mathrm{i}\mathrm{Op}(\frac{1}{2}b)}W(y)\mathrm{e}^{\mathrm{i}\mathrm{Op}(\frac{1}{2}b)} = W(\mathrm{e}^{b\omega}y), \quad y \in \mathcal{Y}.$$

2.7.5 Implementable symplectic transformations

Let $(\mathcal{Y}, \omega)$ be a symplectic space. Let $\mathcal{Y} \ni y \mapsto W_\pi(y) \in U(\mathcal{H})$ be a regular representation of CCR.

If $r \in Sp(\mathcal{Y})$, then

$$\mathcal{Y} \ni y \mapsto W_\pi(ry) \in U(\mathcal{H})$$

is also a representation of CCR. We say that an operator $A \in B(\mathcal{H})$ intertwines a map $r \in Sp(\mathcal{Y})$ if

$$AW_\pi(y) = W_\pi(ry)A.$$

If U is a unitary map that intertwines $r \in Sp(\mathcal{Y})$, then we say that U implements r. We say that $r \in Sp(\mathcal{Y})$ is implementable if there exists $U \in U(\mathcal{H})$ implementing r. The set of implementable symplectic transformations forms a group.

Theorem 2.7.5 *1) If $\mathcal{Y}$ is finite dimensional, then all the transformations $r \in Sp(\mathcal{Y})$ are implementable.*
2) All the elements of $\overset{\circ}{Sp}(\mathcal{Y})$ are implementable. Moreover, operators of the form $\mathrm{e}^{-\mathrm{i}\mathrm{Op}_\pi(b)}$ for $b \in \overset{\circ}{L}_{\mathrm{s}}(\mathcal{Y}', \mathcal{Y}) \simeq \overset{\circ}{\Gamma}{}^2_{\mathrm{s}}(\mathcal{Y})$ implement the map $\mathrm{e}^{b\omega} \in \overset{\circ}{Sp}(\mathcal{Y})$.
3) If the representation $y \mapsto W_\pi(y)$ is irreducible and $r \in Sp(\mathcal{Y})$, then the set of operators intertwining r equals either $\{0\}$ or $\{\lambda U \ : \lambda \in \mathbf{C}\}$ for some $U \in U(\mathcal{H})$.

Proof. 1) follows by the Stone – von-Neumann Theorem. It follows also from 2).

By Theorem **??**, we know that the group $\overset{\circ}{Sp}(\mathcal{Y})$ is generated by elements of the form $\mathrm{e}^{b\omega}$ for $b \in \overset{\circ}{L}_{\mathrm{s}}(\mathcal{Y}', \mathcal{Y}) \simeq \overset{\circ}{\Gamma}{}^2_{\mathrm{s}}(\mathcal{Y})$. Thus 2) follows by Theorem 2.7.4.

To prove 3), assume that $A \in B(\mathcal{H})$ intertwines r. Then A^* intertwines r^{-1}, and hence both AA^* and A^*A commute with $W_\pi(y)$, $y \in \mathcal{Y}$. By the irreducibility, $AA^* = \lambda_1$ and $A^*A = \lambda_2$, for some $\lambda_1, \lambda_2 \in \mathbf{R}$. Now

$$\lambda_1^2 = AA^*AA^* = A\lambda_2 A^* = \lambda_2\lambda_1. \tag{2.7.53}$$

If $\lambda_1 = 0$, then $A = 0$ and hence $\lambda_2 = 0$. Therefore, (2.7.53) implies $\lambda_1 = \lambda_2$, which means that A is proportional to a unitary operator.

If we have two operators $A_1, A_2 \in B(\mathcal{H})$ intertwining r, then a similar argument shows that $A_1A_2^*$ is proportional to identity. This means that A_1 is proportional to A_2. □

2.7.6 Unitary equivalence of Fock representations

Let $(\mathcal{Y}, \cdot, \omega, \mathbf{i})$ be a complete Kähler space. Let $\mathbf{j} \in B(\mathcal{Y})$ be an anti-involution such that $(\omega, \mathbf{j})$ is Kähler. That means

$$\mathbf{j}^2 = -1, \quad (\mathbf{ij})' = \mathbf{ij}, \quad \mathbf{ij} \le 0.$$

Recall from Subsection **??** that if we set $k := -\mathbf{ij} \in B(\mathcal{Y})$, then we obtain a second Kähler structure $(\mathcal{Y}, \cdot k, \omega, \mathbf{j})$. Recall that k satisfies

$$k = k', \quad \mathbf{i}^{-1}k\mathbf{i} = k^{-1}, \quad k > 0. \tag{2.7.54}$$

Recall also that one can define

$$c := 1_{\mathcal{Z}}(1-k)(1+k)^{-1} = (1-k)(1+k)^{-1}1_{\overline{\mathcal{Z}}} \in B(\overline{\mathcal{Z}}, \mathcal{Z}) \tag{2.7.55}$$

satisfying

$$c' = c, \quad c^*c < 1. \tag{2.7.56}$$

One can retrieve k from c by

$$k = (1 - c - \overline{c})(1 + c + \overline{c})^{-1} = \begin{bmatrix} (1+c^*c)^{-1} & (1+cc^*)^{-1}c \\ c^*(1+cc^*)^{-1} & (1+cc^*)^{-1} \end{bmatrix}. \tag{2.7.57}$$

Proposition 2.7.6 *The following statements are equivalent: 1)* $\mathbf{i} - \mathbf{j} \in l^2(\mathcal{Y})$*, 2)* $1 - k \in l^2(\mathcal{Y})$*, 3)* $c \in l^2(\overline{\mathcal{Z}}, \mathcal{Z})$*.*

Proof. The identity $-\mathbf{i}(\mathbf{i} - \mathbf{j}) = 1 - k$ and $\mathbf{i} \in O(\mathcal{Y})$ imply the equivalence of 1) and 2).

(2.7.55) and the boundedness of $(1 + k)^{-1}$ imply 2)⇒3).

Clearly, $c^*c < 1_{\mathcal{Z}}$ and $c \in l^2(\mathcal{Z}, \overline{\mathcal{Z}})$ imply that $(1 - c^*c)^{-1}$ and $(1 - cc^*)^{-1}$ are bounded. Hence 3)⇒2) follows from (2.7.57). □

Theorem 2.7.7 *Let $\mathcal{Z}$ be $\mathcal{Z}_{\mathbf{j}}$ be the holomorphic subspaces of $\mathbf{C}\mathcal{Y}$ corresponding to the Kähler anti-involutions $\mathbf{i}$ and $\mathbf{j}$. Let $\mathcal{Y} \ni y \mapsto W(y) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ and $\mathcal{Y} \ni y \mapsto W_{\mathbf{j}}(y) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathbf{j}}))$ be the corresponding Fock representations of CCR. Then the following statements are equivalent.*
1) There exists a unitary operator $U : \Gamma_{\mathrm{s}}(\mathcal{Z}) \to \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathbf{j}})$ such that

$$UW(y) = W_{\mathbf{j}}(y)U.$$

2) $\mathbf{i}-\mathbf{j}$ is Hilbert-Schmidt (or any of the equivalent conditions of Proposition 2.7.6 holds).

Remark 2.7.8 *The Hilbert-Schmidt property in condition 2) uses the real scalar product on $\mathcal{Y}$ that belongs to the Kähler structure $(\cdot, \omega, \mathbf{i})$. Therefore, condition 2) may seem not symmetric wrt the anti-involutions $\mathbf{i}$ and $\mathbf{j}$. Nevertheless, if the condition 2) is satisfied, then the scalar products $\cdot$ and $\cdot k$ are related with the operator k, which is bounded with a bounded inverse, hence the set of Hilbert-Schmidt operators wrt the scalar products $\cdot$ and $\cdot k$ coincide.*

Let $c \in \Gamma_{\mathrm{s}}^2(\mathcal{Z}) \simeq l_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Recall that $c(a^*) = a^* c a^*$ is a well defined closed operator with domain containing $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$.

Note that c^*c is trace class, so $\det(1 - c^*c)$ is well defined. If we assume also that $c^*c < 1$, then $\det(1 - c^*c) > 0$.

Lemma 2.7.9 *1) Suppose that, if $c \in l_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ and $c^*c < 1$. Then*

$$\Omega_c := \det(1 - c^*c)^{\frac{1}{4}} \mathrm{e}^{-\frac{1}{2}a^* c a^*} \Omega.$$

defines a normalized vector in $\Gamma_{\mathrm{s}}(\mathcal{Z})$.
2) Suppose that c is a densely defined operator from $\overline{\mathcal{Z}}$ to $\mathcal{Z}$ such that $c' = c$. Suppose that there exists $\psi \in \Gamma_{\mathrm{s}}(\mathcal{Z})$ satisfying

$$(\overline{z}(a) + c\overline{z}(a^*))\psi = 0, \quad \overline{z} \in \mathcal{D}(c) \subset \overline{\mathcal{Z}}.$$

Then $c \in l_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ and $c^*c < 1$. Moreover, ψ is proportional to Ω_c.

Proof. We have

$$\begin{aligned} 0 &= (\Omega|(\overline{z_1}(a) + c\overline{z_1}(a^*)) \cdots (\overline{z_n}(a) + c\overline{z_n}(a^*))\psi) \\ &= ((z_n(a^*) + c^* z_n(a)) \cdots (z_1(a^*) + c^* z_1(a))\Omega|\psi), \qquad \overline{z_1}, \dots, \overline{z_n} \in \mathcal{D}(c). \end{aligned}$$

By induction this shows that if $\lambda := (\Omega|\psi)$, then

$$\begin{aligned} &(z_1(a^*) \cdots z_{2m+1}(a^*)\Omega|\psi) = 0, \\ &(z_1(a^*) \cdots z_{2m}(a^*)\Omega|\psi) = \lambda(-1)^m \tfrac{(2m)!}{2^m m!} \langle z_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} z_{2m} | c^{\otimes_{\mathrm{s}} m} \rangle, \end{aligned}$$

where in the last expression we treat c as an element of $\overset{\circ}{\Gamma}{}^{2}_{\mathrm{s}}(\mathcal{Z})^{/\mathrm{al}}$ In particular, for $z_1, z_2 \in \mathcal{D}(c)$, this gives the following formula for the 2-particle component of ψ:

$$\begin{aligned} (z_1 \otimes_{\mathrm{s}} z_2|\psi) &= -\lambda\langle z_1 \otimes_{\mathrm{s}} z_2|c\rangle \\ &= -\lambda(z_1|cz_2), \end{aligned} \tag{2.7.58}$$

where in the first line we treat c as an element of $\overset{\circ}{\Gamma}{}^{2}_{\mathrm{s}}(\mathcal{Z})^{/\mathrm{al}}$ and in the second line as an operator from $\overline{\mathcal{Z}}$ to $\mathcal{Z}$. (2.7.58) can be extended to a bounded functional on $\Gamma_{\mathrm{s}}^2(\mathcal{Z})$ only if either $\lambda = 0$ or $c \in \Gamma_{\mathrm{s}}^2(\mathcal{Z})$. Now, $c \in \Gamma_{\mathrm{s}}^2(\mathcal{Z})$ is equivalent to $c \in l_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$.

Thus

$$\psi_{2m} = \lambda(-1)^m \frac{\sqrt{(2m)!}}{2^m m!} c^{\otimes_{\mathrm{s}} m} = (-1)^m \frac{1}{2^m m!}(a^* c a^*)^m \Omega, \qquad \psi_{2m+1} = 0, \qquad m = 0, 1, \dots.$$

For what follows, we can assume $\lambda = 1$.

Since $c^* c$ is trace class, we can find an orthonormal basis $\{\overline{z}_i, \ : \ i \in I\}$ such that $c^* c \overline{z_i} = \lambda_i \overline{z_i}$. Using the corresponding basis in $\Gamma_{\mathrm{s}}(\mathcal{Z})$, (see **??**), we obtain

$$\begin{aligned} \|\psi\|^2 &= \prod_{i\in I} \sum_{m=0}^{\infty} \tfrac{(2m)!\lambda_i^m}{2^{2m}(m!)^2} \\ &= \prod_{i\in I} \sum_{m=0}^{\infty} \tfrac{1}{m!}(- \lambda_i)^m (-\tfrac{1}{2})(-\tfrac{3}{2})(-m + \tfrac{1}{2}) \ . \\ &= \prod_{i\in I} \Big(1 - \lambda_i\Big)^{-\frac{1}{2}} = \det \Big(1 - c^* c\Big)^{-\frac{1}{2}}. \end{aligned}$$

This shows that the vector Ω_c is normalized.

If for some $i \in I$, $\lambda_i \geq 1$, then the above series is divergent and the vector ψ is nonnormalizable. This shows the necessity of the condition $c^* c < 1$. $\square$

Theorem 2.7.10 *1) Let $c \in B(\overline{\mathcal{Z}}, \mathcal{Z})$ satisfy (2.7.55) and let $\mathbf{j}$ be the Kähler anti-involution determined by c as in the beginning of this subsection. Then Ω_c is the unique vector satisfying the following conditions:*

$$\begin{aligned} &a) \quad \|\Omega_c\| = 1, \\ &b) \quad (\Omega_c|\Omega) > 0, \\ &c) \quad \Omega_c \textit{ is a vacuum for } \mathbf{j}. \end{aligned}$$

2) The statement 1c) is equivalent to

$$(\overline{z}(a) + c\overline{z}(a^*))\Omega_c = 0, \quad z \in \mathcal{Z}. \tag{2.7.59}$$

3) By Theorem 1.7.13 we can find a conjugation $\kappa \in B(\mathcal{Y})$ such that, if we use the decomposition $\mathcal{Y} = \mathcal{Y}_\kappa \oplus \mathcal{Y}_{-\kappa}$, then $\mathbf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $k = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}$, *where $m \in L_\mathrm{s}(\mathcal{Y}_\kappa)$, $m > 0$. In the corresponding Schrödinger representation (or real wave representation)*

$$\Omega_c = \pi^{-\frac{d}{4}} (\det m)^{\frac{1}{4}} \mathrm{e}^{-\frac{1}{2}xmx}.$$

Proof. We know that every element of $\overline{\mathcal{Z}_\mathbf{j}}$ is of the form $(1 + \mathrm{i}\mathbf{j})y_1$ for some $y_1 \in \mathcal{Y}$. But since $(1 + k)$ is bounded, every element of $\mathcal{Y}$ is of the form $y_1 = (1 + k)^{-1}y$. Now

$$\begin{aligned} (1 + \mathrm{i}\mathbf{j})(1 + k)^{-1}y &= (1 + \mathrm{i}\mathbf{i}k)(1 + k)^{-1}y \\ &= \left(1_{\overline{\mathcal{Z}}} + 1_{\mathcal{Z}}(1 - k)(1 + k)^{-1}\right)(1 + k)^{-1}y \\ &= \overline{z} + c\overline{z}, \end{aligned}$$

where $z = 1_{\mathcal{Z}}(1 + k)^{-1}y \in \mathcal{Z}$. Hence every element of $\overline{\mathcal{Z}_\mathbf{j}}$ is of the form $\overline{z} + c\overline{z}$. This implies the equivalence of 3) and 2c). Therefore, it follows from Lemma 2.7.9 that Ω_c is a vacuum for $\mathbf{j}$.

We know from Lemma 2.7.9 that $\|\Omega_c\| = 1$. $(\Omega_c|\Omega) > 0$ is obvious. Since the space of the vacua for $\mathbf{j}$ is 1-dimensional, the conditions a), b) and c) specify a unique vector. □

Proof of Theorem 2.7.7 1)$\Leftarrow$2). If 2) is satisfied, then the Hilbert space contains the vector Ω_c, which by Theorem 2.7.10 1) is a vacuum for the anti-involution $\mathbf{j}$. Moreover, the representation $\mathcal{Y} \ni y \mapsto W_\mathbf{j}(y) \in U(\Gamma_\mathrm{s}(\mathcal{Z}))$ is irreducible. Therefore, every non-zero vector of $\Gamma_\mathrm{s}(\mathcal{Z})$, in particular Ω_c, is cyclic. Hence we can apply Theorem 2.4.16.

1)$\Rightarrow$2). Assume 1). Then there exists a vacuum $\psi \in \Gamma_\mathrm{s}(\mathcal{Z})$ for the anti-involution $\mathbf{j}$. By Theorem 2.7.10 2), this means that

$$(\overline{z}(a) + c\overline{z}(a^*))\psi = 0.$$

By Lemma 2.7.9 such a vector exists only if $c \in l^2(\overline{\mathcal{Z}}, \mathcal{Z})$. By Proposition 2.7.6, this means that $\mathbf{i} - \mathbf{j}$ is Hilbert-Schmidt. □

Remark 2.7.11 *The proof of the unitary equivalence of $\ni y \mapsto W(y)$ and $\ni y \mapsto W_\mathbf{j}(y)$ that we gave follows the proof contained in the book of Berezin (based on formulas known already to Friedrichs). The original proof given by Shale is different – it uses Theorem 1.7.13 to find a conjugation adapted to the pair of antiinvolutions $\mathbf{i}$ and $\mathbf{j}$, and then the appropriate real wave representation. The advantage of Berezin's proof is that it uses only canonically defined objects (the above mentioned conjugation is not always uniquely determined).*

Theorem 2.7.12 *Let $c_1, c_2 \in B(\overline{\mathcal{Z}}, \mathcal{Z})$ satisfy (2.7.55). Then*

$$(\Omega_{c_1}|\Omega_{c_2}) = \det(1-c_1^*c_1)^{\frac{1}{4}}\det(1-c_2^*c_2)^{\frac{1}{4}}\det(1-c_1^*c_2)^{-\frac{1}{2}}.$$

Proof. It suffices to assume that $\mathcal{Z}$ is finite dimensional. In the complex wave representation, $\mathrm{e}^{-\frac{1}{2}a^*ca^*}\Omega$ equals $\mathrm{e}^{-\frac{1}{2}\overline{w}c\overline{w}}$. We will use the formula (2.4.8). We first use the integral over $\overline{\mathcal{Z}}$ and then we change the coordinates to $w \mapsto v := w + \overline{w} \in \mathcal{Y}$:

$$\begin{aligned}(\mathrm{e}^{-\frac{1}{2}a^*c_1a^*}\Omega|\mathrm{e}^{-\frac{1}{2}a^*c_2a^*}\Omega) &= (2\pi\mathrm{i})^{-d}\int_{\overline{\mathcal{Z}}}\mathrm{e}^{-|w|^2}\mathrm{e}^{\frac{1}{2}w\overline{c_1}w}\mathrm{e}^{\frac{1}{2}\overline{w}c_2\overline{w}}\mathrm{d}z\mathrm{d}\overline{w}\\ &= (2\pi)^{-d}\int_{\mathcal{Y}}\mathrm{e}^{-\frac{1}{2}v^2}\mathrm{e}^{\frac{1}{2}v\overline{c_1}v}\mathrm{e}^{\frac{1}{2}vc_2v}\mathrm{d}v\\ &= \det{}^{-\frac{1}{2}}\begin{bmatrix} 1 & -\overline{c}_1\\ -c_2 & 1\end{bmatrix} = \det{}^{-\frac{1}{2}}(1-\overline{c_1}c_2).\end{aligned}$$

(Recall that $c = c' \in L(\overline{\mathcal{Z}}, \mathcal{Z})$, $\overline{c} = c^* \in L(\mathcal{Z}, \overline{\mathcal{Z}})$, and we use the decomposition $\mathbf{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$ in the matrix above). □

2.7.7 Implementable symplectic transformations in the Fock representation

We consider a complete Kähler space $(\mathcal{Y}, \cdot, \omega, \mathbf{i})$. Let $r \in Sp(\mathcal{Y}) \cap B(\mathcal{Y})$. Recall that treating $\mathbf{C}\mathcal{Y}$ as the direct sum $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ we can write r as

$$r = \begin{bmatrix} p & q\\ \overline{q} & \overline{p}\end{bmatrix},$$

where $p \in L(\mathcal{Z}, \mathcal{Z})$, $q \in L(\overline{\mathcal{Z}}, \mathcal{Z})$ satisfy

$$\begin{aligned} p^*p - q'\overline{q} = 1, &\quad p'\overline{q} - q^*p = 0,\\ pp^* - qq^* = 1, &\quad pq' - qp' = 0.\end{aligned}$$

We have

$$pp^* \geq 1, \quad p^*p \geq 1,$$

hence p^{-1} exists.

We define the operators $c, d \in L_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$

$$c := \overline{p}^{-1}\overline{q} = q^*(p^*)^{-1}, \quad d := q\overline{p}^{-1} = (p^*)^{-1}q'.$$

Proposition 2.7.13 *Let $r \in Sp(\mathcal{Y}) \cap B(\mathcal{Y})$. The following conditions are equivalent:*

$$0)\ \mathbf{i} - r\mathbf{i}r^{-1} \in l^2(\mathcal{Y}), \quad 1)\ r\mathbf{i} - \mathbf{i}r \in l^2(\mathcal{Y}), \quad 2)\ \mathrm{Tr}q^*q < \infty, \quad 3)\ \mathrm{Tr}(pp^* - 1) < \infty,$$

$$4)\ d \in l^2(\overline{\mathcal{Z}}, \mathcal{Z}), \quad 5)\ c \in l^2(\overline{\mathcal{Z}}, \mathcal{Z}).$$

Define $Sp_2(\mathcal{Y})$ to be the set of $r \in Sp(\mathcal{Y}) \cap B(\mathcal{Y})$ satisfying the above conditions. Then $Sp_2(\mathcal{Y})$ is a group.

Proof. 0)$\Leftrightarrow$ 1) is obvious.

We have

$$r\mathbf{i}-\mathbf{i}r=\begin{bmatrix}0 & -2\mathrm{i}q\\ 2\mathrm{i}\overline{q} & 0\end{bmatrix},\qquad (r\mathbf{i}-\mathbf{i}r)^*=\begin{bmatrix}0 & -2\mathrm{i}q'\\ 2\mathrm{i}q^* & 0\end{bmatrix}$$

Hence

$$\begin{aligned}\mathrm{Tr}(r\mathbf{i}-\mathbf{i}r)^*(r\mathbf{i}-\mathbf{i}r) &= 4\mathrm{Tr}\begin{bmatrix}q'\overline{q} & 0\\ 0 & q^*q\end{bmatrix}\\ &= 4(\mathrm{Tr}q'\overline{q}+\mathrm{Tr}q^*q)=8\mathrm{Tr}q^*q=8\mathrm{Tr}(p^*p-1).\end{aligned}$$

This implies 1)$\Leftrightarrow$2)$\Leftrightarrow$3).

$d=(p^*)^{-1}q'$ and the boundedness of $(p^*)^{-1}$ implies 3)$\Rightarrow$4). $\overline{q}q'=d^*d(1-d^*d)^{-1}$ implies 4)$\Rightarrow$ 3).

The proof of the equivalence of 3)$\Leftrightarrow$5) is similar. □

Consider now the Fock representation $\mathcal{Y}\ni y\mapsto W(y)\in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$.

Theorem 2.7.14 *1) Let $r\in Sp(\mathcal{Y})\cap B(\mathcal{Y})$. Then the following conditions are equivalent:*
a) There exists $U\in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ such that

$$UW(y)=W(ry)U,\quad y\in\mathcal{Y}. \tag{2.7.60}$$

b) $r\in Sp_2(\mathcal{Y})$
2) If the above conditions are satisfied, define

$$U_r^{\mathbf{i}}=|\det pp^*|^{-\frac{1}{4}}\mathrm{e}^{-\frac{1}{2}a^*da^*}\Gamma((p^*)^{-1})\mathrm{e}^{\frac{1}{2}a\overline{c}a}.$$

Then $U_r^{\mathbf{i}}$ is is the unique unitary operator satisfying (2.7.60) and

$$(\Omega|U_r^{\mathbf{i}}\Omega)>0. \tag{2.7.61}$$

c)

$$Sp_2(\mathcal{Y})\ni r\mapsto[U_r^{\mathbf{i}}]\in U(\mathcal{H})/U(1)$$

is a homomorphism of groups. (Above, we use the notation $[U]=\{\lambda U\ :\ \lambda\in\mathbf{C},\ |\lambda|=1\}$).

Proof. $\mathcal{Y}\ni y\mapsto W(ry)\in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ is an irreducible representation of CCR and the vector Ω is its vaccum for the Kähler antiinvolution $\mathbf{j}:=r\mathbf{i}r^{-1}$. By Theorem 2.4.16, it is unitarily equivalent to the Fock representation $\mathcal{Y}\ni y\mapsto W_{\mathbf{j}}(y)\in U(\Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathbf{j}}))$. By Theorem 2.7.7, its unitary equivalence to the Fock representation $\mathcal{Y}\ni y\mapsto W(y)\in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ is equivalent to b). This shows 1).

The proof of 2) will be given later on.

3) is an immediate consequence of 2). □

2.7.8 Metaplectic group

Let $(\mathcal{Y}, \omega)$ be a finite dimensional symplectic space and $\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$ a regular irreducible representation of CCR.

Definition 2.7.15 *The metaplectic group $Mp(\mathcal{Y})$ is defined as the subgroup of $U(\mathcal{H})$ generated by operators of the form $\mathrm{e}^{\mathrm{iOp}(b)}$ for $b \in \Gamma^2_{\mathrm{s}}(\mathcal{Y})$.*

Theorem 2.7.16 *For any $U \in Mp(\mathcal{Y})$ there exists a unique $r_U \in Sp(\mathcal{Y})$ such that*

$$UW(y) = W(r_U y)U, \qquad y \in \mathcal{Y}.$$

The map

$$Mp(\mathcal{Y}) \ni U \mapsto r_U \in Sp(\mathcal{Y}) \tag{2.7.62}$$

is a surjective homomorphism whose kernel consists of $\{1, -1\}$. The group $Mp(\mathcal{Y})$ is connected.

Proof. The existence of a surjective homomorphism (2.7.62) follows from Theorem 2.7.4 (see Theorem 2.7.5 2)). It is more difficult to show the kernel of this homorphism is $\{1, -1\}$. We will prove it in the section devoted to the metaplectic group in the Fock representation. □

Remark 2.7.17 *The surjectivity of (2.7.62) means that, for any $r \in Sp(\mathcal{Y})$ there exist exactly two $U \in Mp(\mathcal{Y})$, differing by a sign, which implement r.*

Remark 2.7.18 *It is easy to prove that the homotopy group of $Sp(\mathcal{Y})$ is $\mathbf{Z}$. Thus, for any $n = 1, 2, \dots, \infty$ the group $Sp(\mathcal{Y})$ has a uniquely defined n-fold covering $Sp^{(n)}(\mathcal{Y})$, which is also a Lie group. The group $Mp(\mathcal{Y})$ is isomorphic to $Sp^{(2)}(\mathcal{Y})$.*

2.7.9 Metaplectic group in $L^2(\mathbf{R})$

This subsection has an illustrative character.

Let us consider the Schrödinger representation in $L^2(\mathbf{R})$ over $\mathbf{R} \oplus \mathbf{R}$. We will describe some examples of subgroups of the metaplectic group $Mp(\mathbf{R} \oplus \mathbf{R}) \subset U(L^2(\mathbf{R}))$.

Example 2.7.19 *Let $b(x, \xi) = x\xi$. Then $\mathrm{Op}(b) = \frac{1}{2}(xD + Dx)$, and the following operators belong to the metaplectic group:*

$$\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(b)}\psi(x) = \mathrm{e}^{-\frac{1}{2}t}\psi(\mathrm{e}^{-t}x).$$

Example 2.7.20 *The multiplication operator $\mathrm{e}^{-\frac{\mathrm{i}}{2}tx^2}$ is an element of the metaplectic group.*

Example 2.7.21 *The operator* $\mathrm{e}^{-\frac{\mathrm{i}}{2}tD^2}$ *is an element of the metaplectic group. Its integral kernel equals*

$$(2\pi \mathrm{i}t)^{-\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}}{2}\frac{(x-y)^2}{t}}.$$

Example 2.7.22 *Let* $b(x,\xi) := \frac{1}{2}\xi^2 + \frac{1}{2}x^2$. *Then* $\mathrm{Op}(b) = \frac{1}{2}D^2 + \frac{1}{2}x^2$.

First note that $\mathrm{e}^{-t\mathrm{Op}(b)}$ *has the Weyl symbol*

$$w(t,x,\xi) = (\mathrm{ch}\tfrac{t}{2})^{-1} \exp(-\mathrm{th}\tfrac{t}{2}(x^2+\xi^2)),$$

and the integral kernel

$$W(-t,x,y) = \pi^{-\frac{1}{2}}(\mathrm{sh}t)^{-\frac{1}{2}} \exp\left(\frac{-(x^2+y^2)\mathrm{ch}t + 2xy}{2\mathrm{sh}t}\right).$$

$\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(b)}$ *has the Weyl symbol*

$$w(\mathrm{i}t,x,\xi) = (\cos\tfrac{t}{2})^{-1} \exp\left(-\mathrm{i}\,\mathrm{tg}\tfrac{t}{2}(x^2+\xi^2)\right).$$

the integral kernel

$$W(\mathrm{i}t,x,y) = \pi^{-\frac{1}{2}}|\sin t|^{-\frac{1}{2}} \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}[\frac{t}{\pi}]} \exp\left(\frac{-(x^2+y^2)\cos t + 2xy}{2\mathrm{i}\sin t}\right)$$

Above, $[c]$ *denotes the integral part of* c.

Note that $W(\mathrm{it}+2\mathrm{i}\pi,x,y) = -W(\mathrm{it},x,y)$ *Note the special cases:*

$$\begin{aligned}
W(0,x,y) &= \delta(x-y),\\
W(\tfrac{\mathrm{i}\pi}{2},x,y) &= (2\pi)^{-\frac{1}{2}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\mathrm{e}^{-\mathrm{i}xy},\\
W(\mathrm{i}\pi,x,y) &= \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}}\delta(x+y),\\
W(\tfrac{\mathrm{i}3\pi}{2},x,y) &= (2\pi)^{-\frac{1}{2}}\mathrm{e}^{-\frac{\mathrm{i}3\pi}{4}}\mathrm{e}^{\mathrm{i}xy},
\end{aligned}$$

2.7.10 Metaplectic group in the Fock representation

Let $\mathcal{Y}$ be a complete Kähler space. Consider the Fock representation $\mathcal{Y} \ni y \mapsto W(y) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$.

Definition 2.7.23 *The metaplectic group of* $\mathcal{Y}$, *denoted* $Mp(\mathcal{Y})$ *is the subgroup of* $U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ *generated by* $\mathrm{e}^{\mathrm{i}\mathrm{Op}(b)}$ *with* $b \in \Gamma^2(\mathcal{Y})$ *of the form*

$$\mathcal{Y} \ni (\overline{w}, w) \mapsto b(\overline{w}, w) = \frac{\mathrm{i}}{2} w\overline{d}w + \overline{w}hw - \frac{\mathrm{i}}{2}\overline{w}d\overline{w},$$

where $d \in \Gamma_{\mathrm{s}}^2(\mathcal{Z})$ *and* $h \in l_{\mathrm{h}}^1(\mathcal{Z})$.

(Note that Theorem 2.7.3 and (2.7.48) imply that $\mathrm{Op}(b)$ is a well defined self-adjoint operator).

Lemma 2.7.24 *Let $v \in U(\mathcal{Y}^{\mathbf{C}})$, which means that*

$$v = \begin{bmatrix} u & 0 \\ 0 & \overline{u} \end{bmatrix},$$

with $u \in U(\mathcal{Z})$, then

$$(\det u)^{\frac{1}{2}}\Gamma(u) \tag{2.7.63}$$

(where both signs of the square root are possible) belongs to $Mp(\mathcal{Y})$ and implements v.

Proof. Let $h \in l^1(\mathcal{Z})$. Then

$$\mathrm{e}^{\mathrm{i}\mathrm{Op}(h)} = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Tr}h}\mathrm{e}^{\mathrm{i}\mathrm{Op}^{a^*,a}(h)} = \det \mathrm{e}^{\frac{\mathrm{i}}{2}h}\mathrm{e}^{\mathrm{i}\mathrm{d}\Gamma(h)} = (\det \mathrm{e}^{\mathrm{i}h})^{\frac{1}{2}}\Gamma(\mathrm{e}^{\mathrm{i}h}).$$

Setting $u = \mathrm{e}^{\mathrm{i}h}$ proves the lemma. □

Lemma 2.7.25 *Let $d \in \Gamma_{\mathrm{s}}^2(\mathcal{Z}) \simeq l_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Recall that $\mathrm{Op}(d) = a^*da^*$ and $\mathrm{Op}(\overline{d}) = a\overline{d}a$ are densely defined closable operators. $\mathrm{e}^{-\frac{1}{2}a^*da^*}$ and $\mathrm{e}^{\frac{1}{2}a\overline{d}a}$ are densely defined closable operator iff $d^*d < 1$ and we have*

$$\mathrm{e}^{-\frac{1}{2}a^*da^*}\overline{z}(a)\mathrm{e}^{\frac{1}{2}a^*da^*} = \overline{z}(a) + (d\overline{z})(a^*), \qquad z \in \mathcal{Z}. \tag{2.7.64}$$

$$\mathrm{e}^{\frac{1}{2}a\overline{d}a}z(a^*)\mathrm{e}^{-\frac{1}{2}a\overline{d}a} = z(a^*) - (\overline{d}z)(a), \qquad z \in \mathcal{Z}. \tag{2.7.65}$$

Lemma 2.7.26 *1) Let $r \in Sp(\mathcal{Y})$ satisfy $r = r'$, $r > 0$, which as we know is equivalent to*

$$r = \begin{bmatrix} p & q \\ q^* & \overline{p} \end{bmatrix} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p^{-1} & 0 \\ 0 & p \end{bmatrix}\begin{bmatrix} 1 & 0 \\ c^* & 1 \end{bmatrix},$$

for some $p \in L_{\mathrm{h}}(\mathcal{Z})$, $q \in L_{\mathrm{s}}(\mathcal{Z}, \overline{\mathcal{Z}})$ satisfying $p > 0$ and $p^2 - qq^* = 1$ and

$$c = p^{-1}q = qp^{-1} \in L_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z}).$$

Then

$$\exp(-\tfrac{1}{2}a^*ca^*)\det p^{-\frac{1}{2}}\Gamma(p^{-1})\exp(\tfrac{1}{2}a\overline{c}a) \tag{2.7.66}$$

belongs to $Mp(\mathcal{Y})$ and and implements r. 2) Choose a conjugation κ as in Theorem 1.7.13 and identify $\mathcal{Y}$ with $\mathcal{X} \oplus \mathcal{X}$ so that for $m \in L_{\mathrm{s}}(\mathcal{X})$, $m > 0$.

$$r = \begin{bmatrix} m^{-1} & 0 \\ 0 & m \end{bmatrix}.$$

Then the operator (2.7.66) in the Schrödinger representation on $L^2(\mathcal{X})$ (or, appropriately interpreted, in the real wave reperesentation) equals $(\det m)^{-\frac{1}{2}}m_*$.

Proof. Let $d \in \Gamma_{\mathrm{s}}^2(\mathcal{Z}) \simeq l_{\mathrm{s}}^2(\mathcal{Z}, \overline{\mathcal{Z}})$. Then

$$\mathrm{Op}(\tfrac{\mathrm{i}}{2}(d - \overline{d})) = \mathrm{Op}^{a^*,a}(\tfrac{\mathrm{i}}{2}(d - \overline{d})) = \tfrac{\mathrm{i}}{2}(a^* d a^* - a\overline{d}a). \tag{2.7.67}$$

is essentiallyself-adjoint on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$. We have

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\mathrm{Op}(\frac{\mathrm{i}}{2}(d-\overline{d}))} \\ &= \exp\Big(-\tfrac{1}{2} a^* d \tfrac{\mathrm{th}\sqrt{d^*d}}{\sqrt{d^*d}} a^* \Big) \Big(\det \mathrm{ch}\sqrt{d^*d} \Big)^{-\frac{1}{2}} \Gamma\Big(\mathrm{ch}\sqrt{d^*d} \Big)^{-1} \exp\Big(\tfrac{1}{2} a d^* \tfrac{\mathrm{th}\sqrt{dd^*}}{\sqrt{dd^*}} a \Big). \end{aligned} \tag{2.7.68}$$

Setting $p = \mathrm{ch}\sqrt{d^*d}$ and $c = d\frac{\mathrm{th}\sqrt{d^*d}}{\sqrt{d^*d}}$ gives 1).

Note that the most convenient way of computing (2.7.68) is probably using the Schrödinger representation, as in 2). □

Proof of Theorem 2.7.14. Recall that

$$\Gamma(p) z(a^*) \Gamma(p^{-1}) = pz(a^*), \quad \Gamma(p)\overline{z}(a)\Gamma(p^{-1}) = \overline{(p^*)^{-1}z}(a), \quad z \in \mathcal{Z} \tag{2.7.69}$$

Set $V := \mathrm{e}^{-\frac{1}{2}a^* d a^*} \Gamma((p^*)^{-1}) \mathrm{e}^{\frac{1}{2}a\overline{c}a}$. Using (2.7.69), (2.7.64) and (2.7.65), we check that V implements r. In fact,

$$Vz(a^*) = \Big(((p^*)^{-1}z - d\overline{pc}z)(a^*) + (\overline{pc}z)(a) \Big) V = (pz(a^*) + \overline{q}z(a)) V,$$

$$V\overline{z}(a) = ((\overline{pz})(a) + (d\overline{pz})(a^*)) V = (q\overline{z}(a) + \overline{pz}(a)) V.$$

Hence

$$V\phi(y) = \phi(ry)V, \quad y \in \mathcal{Y},$$

and thus (2.7.60) is satisfied.

The map $\mathcal{Y} \ni y \mapsto W(ry) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ is an irreducible representation of CCR. Hence any operator V satisfying (2.7.60) is proportional to a unitary operator.

Now

$$V\Omega = \mathrm{e}^{-\frac{1}{2}a^* d a^*}\Omega.$$

Hence, using Theorem 2.7.10 and (**??**) we obtain

$$\|V\Omega\|^2 = \det(1 - d^*d)^{-\frac{1}{2}} = (\det pp^*)^{-\frac{1}{2}}.$$

Hence $U_r^{\mathbf{i}} = (\det pp^*)^{-\frac{1}{4}} V$ is unitary. $(\Omega | U_r^{\mathbf{i}} \Omega) = (\det pp^*)^{-\frac{1}{2}} > 0$, hence $U_r^{\mathbf{i}}$ satisfies also the condition (2.7.61). Since there is a strong inequality in this condition, it fixes the choice of $U_r^{\mathbf{i}}$ uniquely. □

Theorem 2.7.27 *Define $Sp_1(\mathcal{Y})$ to be the set of $r \in Sp(\mathcal{Y})$ satisfying $1 - p \in l^1(\mathcal{Z})$. For $r \in Sp_1(\mathcal{Y})$, define*

$$U_r = \pm(\det p^*)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}a^* d a^*} \Gamma((p^*)^{-1}) \mathrm{e}^{\frac{1}{2}a \bar{c} a}. \tag{2.7.70}$$

(Here U_r denotes a pair of operators differing by a sign). Then
1) $Sp_1(\mathcal{Y})$ is a subgroup of $Sp(\mathcal{Y})$.
2) $U_r \in U(\mathcal{H}) \backslash \{1 - 1\}$ and $U_r W(y) = W(ry) U_r$.
3) The following map is a group homomorphism.

$$Sp_1(\mathcal{Y}) \ni r \mapsto U_r \in U(\mathcal{H})/\{1, -1\} \tag{2.7.71}$$

4) The range of (2.7.71) is $Mp(\mathcal{Y})/\{1, -1\}$.

Proof. Let us prove 1). It follows from $1 - p \in l^1(\mathcal{Z})$, that $1 - p^*p \in l^1(\mathcal{Z})$. But $q^*q = p^*p - 1$. Hence $q \in l^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Hence also $c, d \in l^2(\overline{\mathcal{Z}}, \mathcal{Z})$.

Let $r_1, r_2 \in Sp_1(\mathcal{Y})$. Let p_i, q_i, c_i, d_i be defined as usual from r_i for $i = 1, 2$. Let $r = r_1 r_2$ and define p, q, c, d the usual way. We have

$$p = p_1 p_2 + q_1 \overline{q_2} = p_1(1 + c_1 d_2^*) p_2.$$

We have $1 - p_1, 1 - p_2, c_1 d_2^* \in l^1(\mathcal{Z})$. Hence $1 - p \in l^1(\mathcal{Z})$. This proves 1)

The operators U_r differ by a phase factor from $U_r^{\mathbf{i}}$ from Theorem 2.7.14. Therefore they are unitary and implement r. This proves 2).

Let us prove 3). We know that

$$(\Omega | U_{r_1 r_2} \Omega) = \pm(\det p^*)^{-\frac{1}{2}} = \pm\left(\det(p_1 p_2 + q_1 \overline{q}_2)^*\right)^{-\frac{1}{2}}. \tag{2.7.72}$$

Moreover,

$$\begin{aligned}
(\Omega | U_{r_1} U_{r_2} \Omega) &= (\mathrm{e}^{\frac{1}{2}a^* c_1 a^*} \Omega | \mathrm{e}^{-\frac{1}{2}a^* d_2 a^*} \Omega)(\det p_1^*)^{-\frac{1}{2}} (\det p_2^*)^{-\frac{1}{2}} \\
&= \det(1 + d_2 c_1^*)^{-\frac{1}{2}} (\det p_1^*)^{-\frac{1}{2}} (\det p_2^*)^{-\frac{1}{2}} \\
&= (\det(p_1 p_2 + q_1 \overline{q}_2)^*)^{-\frac{1}{2}}.
\end{aligned}$$

Hence

$$(\Omega | U_{r_1} U_{r_2} \Omega) = \pm(\Omega | U_{r_1 r_2} \Omega).$$

We know that $U_{r_1} U_{r_2}$ and $U_{r_1 r_2}$ implement $r_1 r_2$ and the representation is irreducible. Hence $U_{r_1} U_{r_2} = \pm U_{r_1 r_2}$. This ends the proof of 3).

Then we note that any $r \in Sp(\mathcal{Y})$ can be written as $r = v r_0$, where $v \in U(\mathcal{Y}^{\mathbf{C}})$ and r_0 is positive and self-adjoint. By Example 2.7.24, U_v belongs to $Mp(\mathcal{Y})$. By example 2.7.26, $U_{r_0} \in Mp(\mathcal{Y})$. This proves 4). □

2.7.11 Metaplectic group in the Schrödinger representation

Let $\mathcal{X}$ be a finite dimensional vector space. We consider the Schrödinger representation of CCR over $\mathcal{X}' \oplus \mathcal{X}$.

Lemma 2.7.28 *If* $m \in Gl(\mathcal{X})$ *with* $\det m > 0$, *then*

$$(\det m)^{-\frac{1}{2}} m_* \in Mp(\mathcal{X}' \oplus \mathcal{X})$$

implements $\begin{bmatrix} (m')^{-1} & 0 \\ 0 & m \end{bmatrix} \in Sp(\mathcal{X}' \oplus \mathcal{X})$.

Proof. Let $c \in \mathcal{X} \otimes_{\mathrm{s}} \mathcal{X}' \simeq \mathcal{X} \otimes \mathcal{X}' \simeq gl(\mathcal{X})$. Recall that

$$\mathrm{Op}(c) = xc'D + \frac{\mathrm{i}}{2}\mathrm{Tr}c.$$

Thus

$$\mathrm{e}^{\mathrm{i}\mathrm{Op}(c)} = \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}c}\mathrm{e}^{\mathrm{i}\mathrm{Op}^{x,D}(c)} = \det \mathrm{e}^{-\frac{1}{2}c}\mathrm{e}_*^{-c}.$$

Lemma 2.7.29 *Let* $\alpha \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}')$. *Then* $\mathrm{e}^{\frac{\mathrm{i}}{2}x\alpha x} \in Mp(\mathcal{X}' \oplus \mathcal{X})$.

Lemma 2.7.30 *Likewise, if* $\beta \in L_{\mathrm{s}}(\mathcal{X}', \mathcal{X})$. *Then* $\mathrm{e}^{\frac{\mathrm{i}}{2}D\beta D} \in Mp(\mathcal{X}' \oplus \mathcal{X})$. *Its integral kernel equals*

$$(2\pi\mathrm{i})^{-\frac{d}{2}}(\det \beta)^{-\frac{1}{2}}\mathrm{e}^{-\frac{\mathrm{i}}{2}(x_1-x_2)\beta^{-1}(x_1-x_2)}.$$

Let us now describe the case of an (almost) arbitrary $r \in Sp(\mathcal{X}' \oplus \mathcal{X})$. We can write $r = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Recall that if b is invertible, then we can define

$$e = db^{-1} = b'^{-1}d' \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}'), \qquad f = b^{-1}a = a'b'^{-1} \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}'),$$

and the so-called generating funtion

$$\mathcal{X} \times \mathcal{X} \ni (x_1, x_2) \mapsto S(x_1, x_2) := \frac{1}{2}x_1 e x_1 - (b^{-1}x_2)x_1 + \frac{1}{2}x_2 f x_2.$$

Theorem 2.7.31 *Let* $r \in Sp(\mathcal{X}' \oplus \mathcal{X})$ *be such that* b *is invertible. Then the operators* $U_r \in Mp(\mathcal{X}' \oplus \mathcal{X})$ *implementing* r *have the integral kernel equal to*

$$U_r(x_1, x_2) = \pm(2\pi\mathrm{i})^{-\frac{1}{2}d}\sqrt{\det \nabla_{x_1}\nabla_{x_2}S}\,\mathrm{e}^{\mathrm{i}S(x_1,x_2)}.$$

2.7.12 Weyl-Wigner symbols of elements of the metaplectic group

Assume that $\mathcal{Y}$ is finite dimensional. We fix a regular representation of CCR over $\mathcal{Y}$.

Let $r \in Sp(\mathcal{Y})$. Assume that $r + 1$ is invertible. Then we can define

$$\gamma := (r-1)(r+1)^{-1}\omega^{-1} \in L_{\mathrm{s}}(\mathcal{Y}', \mathcal{Y}) \simeq \Gamma_{\mathrm{s}}^{2}(\mathcal{Y}).$$

Theorem 2.7.32 *Let $r \in Sp(\mathcal{Y})$ such that $r+1$ is invertible. Then the elements of $Mp(\mathcal{Y})$ implementing r have the form*

$$U_r = \pm \det(1+\gamma\omega)^{\frac{1}{2}}\mathrm{Op}(\mathrm{e}^{\mathrm{i}\gamma}).$$

Proof. Instead of proving the theorem we will check a weaker statement – that the above defined operators implement r and are unitary.

Using

$$\mathrm{Op}(\mathrm{e}^{\mathrm{i}\gamma})\phi(y) = \frac{1}{2}\mathrm{Op}(\mathrm{i}^{-1}\{\mathrm{e}^{\mathrm{i}\gamma}, y\}) + \mathrm{Op}(\mathrm{e}^{\mathrm{i}\gamma}y),$$

$$\phi(ry)\mathrm{Op}(\mathrm{e}^{\mathrm{i}\gamma}) = \frac{1}{2}\mathrm{Op}(\mathrm{i}\{ry, \mathrm{e}^{\mathrm{i}\gamma}\}) + \mathrm{Op}(\mathrm{e}^{\mathrm{i}\gamma}ry),$$

we get

$$y + \gamma\omega y = ry - \gamma\omega ry.$$

Thus $\mathrm{Op}(\mathrm{e}^{\mathrm{i}\gamma})$ implements r.

Next we compute that $\mathrm{Op}(\mathrm{e}^{-\mathrm{i}\gamma})\mathrm{Op}(\mathrm{e}^{\mathrm{i}\gamma})$ has the symbol

$$\begin{aligned}&\pi^{-d}\int \mathrm{e}^{-\mathrm{i}v_1\gamma v_1 + \mathrm{i}v_2\gamma v_2 + 2\mathrm{i}v_1\omega^{-1}v + 2\mathrm{i}v\omega^{-1}v_2 - 2\mathrm{i}v_1\omega^{-1}v_2}\mathrm{d}v_1\mathrm{d}v_2\\ &= (2\pi)^{-d}\int \mathrm{e}^{-\mathrm{i}t\gamma s - \mathrm{i}t\omega^{-1}s + 2\mathrm{i}s\omega^{-1}v}\mathrm{d}t\mathrm{d}s = \det(\gamma+\omega^{-1})^{-1} = \det(\gamma\omega+1)^{-1}.\end{aligned}$$

Thus $\pm\det(1+\gamma\omega)^{\frac{1}{2}}\mathrm{Op}(\mathrm{e}^{\mathrm{i}\gamma})$ are unitary. □