# From the conformal group to symmetries of hypergeometric type functions

Jan Dereziński
Dep. of Math. Methods in Physics,

FACULTY OF PHYSICS
UNIVERSITY OF WARSAW

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# Hypergeometric type

# equations and functions

For $a_1, \ldots, a_k, c_1, \ldots, c_m \in \mathbb{C}$, we define the (generalized) hypergeometric series of type ${}_kF_m$:

$$
{}_kF_m(a_1, \ldots, a_k; c_1, \ldots, c_m; z) := \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_k)_j z^j}{(c_1)_j \cdots (c_m)_j j!}.
$$

Notice that

(1) if $m + 1 > k$, then it is convergent for $z \in \mathbb{C}$;

(2) if $m + 1 = k$, then it is convergent for $|z| < 1$;

(3) if $m + 1 < k$, then it is divergent (however sometimes we can give a meaning to the function ${}_kF_m$).

The function $F$ solves the equation

$$(c_1 + z\partial_z)\cdots(c_m + z\partial_z)\partial_z F$$
$$(a_1 + z\partial_z)\cdots(a_k + z\partial_z)F.$$

Note that this equation is of the order $\max(k, m+1)$. Below we list all equations and hypergeometric functions with equations of the order at most $2$.

## Hypergeometric function or the ${}_2F_1$ function

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n.$$

The series is convergent for $|z| < 1$, it extends to a multivalued function on a covering of $\mathbb{C}\backslash\{0, 1\}$.

The function is a solution of the hypergeometric equation

$$\left(z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab\right) u(z) = 0$$

that is analytic around $0$ and equals there $1$.

## Confluent function or the ${}_1F_1$ function

$$F(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(c)_n} z^n.$$

The series is convergent for all $z \in \mathbb{C}$. It defines a solution analytic around $0$ and equal there $1$ of the confluent equation

$$(z\partial_z^2 + (c-z)\partial_z - a)u(z) = 0,$$

## The ${}_0F_1$ function

$$F(-;c;z) = F(c;z) = \sum_{n=0}^{\infty} \frac{1}{n!(c)_n} z^n.$$

The series is convergent for all $z \in \mathbb{C}$. It defines a solution analytic around $0$ and equal there $1$ of the ${}_0F_1$ equation (related to the Bessel equation)

$$(z\partial_z^2 + c\partial_z - 1)u(z) = 0.$$

## The ${}_2F_0$ function

For $\arg z \neq 0$ we define

$$F(a,b;-;z) := \lim_{c\to\infty} F(a,b;c;cz).$$

It extends to an analytic function on the universal cover of $\mathbb{C}\backslash\{0\}$ with a branch point of an infinite order at 0. It has the following asymptotic expansion:

$$F(a,b;-;z) \sim \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!} z^n, \ |\arg z - \pi| < \pi - \epsilon.$$

This function has a branch point at zero. Hence it cannot be defined with a series around zero. It solves the ${}_2F_0$ equation (related to the confluent equation)

$$\left(z^2\partial_z^2 + (-1 + (a+b+1)z)\partial_z + ab\right) u(z) = 0.$$

## Power function or the ${}_1F_0$ function

$$F(a; -; z) = (1 - z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n$$

The series is convergent for $|z| < 1$, it extends to a multivalued function on a covering of $\mathbb{C}\backslash\{1\}$. It is a solution of

$$((z - 1)\partial_z - a)u(z) = 0.$$

## Exponential function or the ${}_0F_0$ function

$$F(-;-;z) = \mathrm{e}^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

It solves

$$(\partial_z - 1)u(z) = 0.$$

Following Nikiforov–Uvarov, we adopt the following terminology. Equations of the form

$$\left(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta\right) f(z) = 0,$$

where $\sigma$ is a polynomial of degree $\leq 2$,

$\tau$ is a polynomial of degree $\leq 1$,

$\eta$ is a number,

will be called hypergeometric type equations, and their solutions —hypergeometric type functions. Differential operators of the form

$$\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta$$

will be called hypergeometric type operators.

Let us review basic classes of hypergeometric type equations.

(1) The ${}_2F_1$ or hypergeometric equation

$$z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab.$$

(2) The ${}_2F_0$ equation

$$z^2\partial_z^2 + (-1 + (1+a+b)z)\partial_z + ab.$$

(3) The ${}_1F_1$ or confluent equation

$$z\partial_z^2 + (c-z)\partial_z - a.$$

(4) The ${}_0F_1$ equation

$$z\partial_z^2 + c\partial_z - 1.$$

(5) The Gegenbauer equation

$$(1 - z^2)\partial_z^2 - (a + b + 1)z\partial_z - ab.$$

(6) The Hermite equation

$$\partial_z^2 - 2z\partial_z - 2a.$$

(7) 2nd order Euler equation

$$\left(z^2\partial_z^2 + bz\partial_z + a\right) f(z) = 0.$$

(8) 1st order Euler equation for the derivative

$$z\partial_z^2 + c\partial_z.$$

(9) 2nd order equation with constant coefficients

$$\partial_z^2 + c\partial_z + a.$$

(5) and (6) are reflection invariant.

(7), (8) and (9) are solvable in elementary functions. Therefore, it will not be considered in what follows.

The ${}_2F_0$ and ${}_1F_1$ equation are equivalent by a simple substitution, therefore they can be discussed together.

Up to an affine transformation, (5) is a subclass of (1). However, it has additional properties, therefore it is useful to discuss it separately.

Identities for hypergeometric type operators and functions have a high degree of symmetry. Therefore, it is not surprising that behind each of these operators there is a group-theoretical structure.

Each hypergeometric type equation can be obtained by a separation of variables of a certain 2nd order PDE with constant coefficients. On the highest level, which we call the extended space, the symmetries of the equation are very straightforward. One can introduce the Lie algebra of generalized symmetries of this PDE. In this Lie algebra we fix a certain maximal commutative algebra, which we will call the "Cartan algebra". Operators that are eigenvectors of the adjoint action of the "Cartan algebra" will be called "root operators". Elements of the group that yield automorphisms leaving invariant the "Cartan algebra" will be called "Weyl symmetries".

The parameters of hypergeometric type equation can be interpreted as the eigenvalues of elements of the "Cartan algebra". The "root operators" and "Weyl symmetries" commute with the underlying 2nd order operator and transform the "Cartan algebra" in a simple way. Therefore, after the dimensional reduction and a change of coordinates they lead to to certain transmutation relations and discrete symmetries for the corresponding hypergeomentric type equations.

We can distinguish 3 kinds of PDE's with constant coefficients:

(1) The Helmholtz equation on $\mathbb{C}^n$ given by $\Delta_n + 1$, whose Lie algebra of symmetries is the affine orthogonal Lie algebra $\mathbb{C}^n \times \mathrm{so}(n)$;

(2) The Laplace equation on $\mathbb{C}^n$ given by $\Delta_n$, whose Lie algebra of generalized symmetries is the orthogonal Lie algebra $\mathrm{so}(n+2)$. One can derive the generalized symmetries from the Laplacian $\Delta_{n+2}$ on the extended space $\mathbb{C}^{n+2}$.

(3) The heat equation on $\mathbb{C}^n \oplus \mathbb{C}$ given by $\Delta_n + 2\partial_t$, whose Lie algebra of generalized symmetries is the so-called Schrödinger Lie algebra $\mathrm{sch}(n)$, One can derive the generalized symmetries from the Laplacian $\Delta_{n+4}$ on the extended space $\mathbb{C}^{n+4}$.

Separating the variables in these equations usually leads to differential equations with many variables. Only in a few cases it leads to ordinary differential equations, which turn out to be of hypergeometric type. Here is the table of these cases:

| PDE | Lie algebra | dimension of Cartan algebra | discrete symmetries | equation |
|---|---|---|---|---|
| $\Delta_2 + 1$ | $\mathbb{C}^2 \rtimes \mathrm{so}(2)$ | $1$ | $\mathbb{Z}_2$ | ${}_0F_1$; |
| $\Delta_4$ | $\mathrm{so}(6)$ | $3$ | cube | ${}_2F_1$; |
| $\Delta_3$ | $\mathrm{so}(5)$ | $2$ | square | Gegenbauer; |
| $\Delta_2 + 2\partial_t$ | $\mathrm{sch}(2)$ | $2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | ${}_1F_1$ or ${}_2F_0$; |
| $\Delta_1 + \partial_s$ | $\mathrm{sch}(1)$ | $1$ | $\mathbb{Z}_4$ | Hermite. |

# Orthogonal group and Lie algebra

Suppose that

$$\mathbb{R}^n \ni y, x \mapsto \langle y|x\rangle = \sum_{i,j} g_{i,j} y^i x^j$$

is a scalar product on $\mathbb{R}^n$ or $\mathbb{C}^n$. Define

$$\begin{aligned} \mathrm{O}(n) &:= \{\alpha \in \mathrm{GL}(n) \;:\; \langle \alpha y|\alpha x\rangle = \langle y|x\rangle\}, \\ \mathrm{SO}(n) &:= \{\alpha \in \mathrm{O}(n) \;:\; \det \alpha = 1\}. \end{aligned}$$

The Lie algebra $\mathrm{gl}(n)$ can be represented by vector fields and is spanned by $y^i \partial_{y^j}$.

The Lie algebra $\mathrm{so}(n)$, represented by vector fields on $\mathbb{R}^n$, is

$$\mathrm{so}(n) := \{B \in \mathrm{gl}(n) \ : \ B\langle y|y\rangle = 0\}.$$

For $i, j = 1, \ldots, n$, define

$$B_{i,j} := \sum_{j,k} (g_{j,k} y^j \partial_{y^k} - g_{i,k} y^k \partial_{y^j}).$$

$\{B_{i,j} \ : \ i < j\}$ span $\mathrm{so}(n)$.

We define

$$\text{the Laplacian } \Delta_n := \sum_{i,j} g^{i,j}\partial_{y^i}\partial_{y^j},$$

$$\text{the Casimir operator } \mathcal{C}_n := \frac{1}{2}\sum_{i,j,k,l} g^{i,k}g^{j,l}B_{i,j}B_{k,l}.$$

The above definitions do not depend on the choice of a basis. $\Delta_n$ commutes with $\mathbb{C}^n \times \mathrm{O}(n)$ and $\mathbb{C}^n \times \mathrm{so}(n)$. $\mathcal{C}_n$ commutes with $\mathrm{O}(n)$ and $\mathrm{so}(n)$.

Note the identity

$$\langle y|y\rangle\Delta_n = A_n^2 + (n-2)A_n + \mathcal{C}_n,$$

where

$$A_n := \sum_i y^i \partial_{y^i}.$$

Suppose that $2m = n$. Every scalar product of signature $(m, m)$ can be expressed in even split coordinates

$$\langle y|y\rangle = \sum_{i=1}^{m} 2y_{-i}y_i.$$

Then $so(\mathbb{R}^n)$ has a basis consisting of

$$\begin{aligned} N_i := B_{-i,i} &= -y_{-i}\partial_{y_{-i}} + y_i\partial_{y_i}, \\ B_{i,j} &= y_{-i}\partial_{y_j} - y_{-j}\partial_{y_i}, \;\; |i| < |j|. \end{aligned}$$

The subalgebra of $\mathrm{so}(n)$ spanned by $N_i$ is its Cartan algebra. $B_{i,j}$ are its root operators:

$$[N_k, B_{i,j}] = -(\mathrm{sgn}(i)\delta_{k,|i|} + \mathrm{sgn}(j)\delta_{k,|j|})B_{i,j}.$$

We have

$$\Delta_n = \sum_{i=1}^{m} 2\partial_{y_{-i}}\partial_{y_j},$$

$$\mathcal{C}_n = \sum_{1\leq|i|<|j|\leq m} B_{ij}B_{-i-j} - \sum_{i=1}^{m} N_i^2.$$

Suppose that $2m+1=n$. Every scalar product of signature $(m,m+1)$ can be expressed in odd split coordinates

$$\langle y|y\rangle = y_0^2 + \sum_{i=1}^{m} 2y_{-i}y_i.$$

Then $so(n)$ has a basis consisting of the basis of $so(2m)$ and

$$B_{0j} = y_0\partial_{y_j} - y_{-j}\partial_{y_0}, \quad |j| = 1,\ldots,m.$$

The additional roots satisfy

$$[N_k, B_{0,j}] = -\mathrm{sgn}(j)\delta_{k,|j|}B_{i,j}.$$

We have

$$\Delta_n = \partial_{y_0}^2 + \sum_{i=-1}^{m} 2\partial_{y_{-i}}\partial_{y_j},$$

$$\mathcal{C}_n = \sum_{|i|=1}^{m} B_{0i}B_{0-i}$$

$$+ \sum_{1\le|i|<|j|\le m} B_{ij}B_{-i-j} - \sum_{i=1}^{m} N_i^2.$$

We will treat the split signature $(m, m)$ and $(m, m+1)$ and the split scalar product as the standard one.

Consider the split scalar product in dimension $2m$. Permutations of $\{1,\ldots,m\}\cup\{-1,\ldots,-m\}$ that preserve the pairs $\{1,-1\},\ldots\{-m,m\}$ are orthogonal. They form a group, that we will denote $W(2m)$. It is isomorphic to $\mathbb{Z}_2^m \times\!| S_m$.

The subgroup isomorphic to $\mathbb{Z}_2^m$ consists of elements that preserve each pair $(-i,i)$.

$$\tau_i N_j \tau_i^{-1} = (-1)^{\delta_{ij}} N_j.$$

The subgroup isomorphic to $S_m$, permutes pairs $(-i,i)$. For $\pi \in S_m$, the corresponding permutation operator will be denoted $\sigma_\pi$. It belongs to $\mathrm{O}(n)$ and

$$\sigma_\pi N_j \sigma_\pi^{-1} = N_{\pi_j}.$$

We say that a function $F$ on $\mathbb{R}^n$ is harmonic if

$$\Delta_n F = 0.$$

Let $e_1, \dots e_k \in \mathbb{R}^n$ satisfy

$$\langle e_i | e_j \rangle = 0, \quad 1 \leq i, j \leq k.$$

In other words, assume that $e_1, \dots, e_k$ span an isotropic subspace of $\mathbb{R}^n$. Then

$$F(z) := f\big(\langle e_1 | z \rangle, \dots, \langle e_k | z \rangle\big)$$

is harmonic.

For instance, consider $\mathbb{R}^n$ with a split scalar product (where $n$ is even or odd). Then for any $\alpha_1, \dots, \alpha_m$

$$F_{\alpha_1, \dots \alpha_m} := z_1^{\alpha_1} \cdots z_m^{\alpha_m}$$

is harmonic. Besides, it satisfies

$$N_j F_{\alpha_1, \dots \alpha_m} = \alpha_j F_{\alpha_1, \dots \alpha_m}.$$

Suppose that for $z = (x, y, z') \in \mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{R}^{n-2}$

$$\langle x, y, z' | x, y, z' \rangle = x^2 + y^2 + \langle z' | z' \rangle.$$

Set

$$N_1 := -\mathrm{i}(x\partial_y - y\partial_x).$$

Consider a function $f(x, y, z')$. We introduce a wave packet of angular momentum $m \in \mathbb{Z}$ made out of rotated $f$:

$$F_m(x, y, z') := \int_0^{2\pi} f(\cos\phi x - \sin\phi y, \sin\phi x + \cos\phi y) \mathrm{e}^{-\mathrm{i}m\phi} \frac{\mathrm{d}\phi}{2\pi},$$

$$N_1 F_m(x, y, z') = m F_m(x, y, z').$$

Introduce complex coordinates

$$z_{\pm 1} := \frac{1}{\sqrt{2}}(x \pm \mathrm{i}y).$$

We will write $f(z_{-1}, z_1, z') = f(x, y, z')$, $F_m(z_{-1}, z_1, z') = F_m(x, y, z')$. The operator $N_1$ and the metric take the familiar form

$$N_1 = z_{-1}\partial_{z_{-1}} - z_1\partial_{z_1},$$
$$\langle z_{-1}, z_1, z' | z_{-1}, z_1, z'\rangle = 2z_{-1}z_1 + \langle z'|z'\rangle.$$

We obtain

$$F_m(z_{-1}, z_1, z') := \int_\gamma f(\tau^{-1}z_{-1}, \tau z_1, z')\tau^{-m-1}\frac{\mathrm{d}\tau}{\mathrm{i}2\pi},$$
$$N_1 F_m(z_{-1}, z_1, z') = mF_m(z_{-1}, z_1, z').$$

where $\gamma$ is the closed contour $[0, 2\pi[ \ni \phi \mapsto \tau(\phi) = \mathrm{e}^{\mathrm{i}\phi}$.

We again consider $\mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{R}^{n-2}$, but we change the signature of the metric. We assume that the scalar product is given by

$$\langle z_{-1}, z_1, z' | z_{-1}, z_1, z' \rangle = 2z_{-1}z_1 + \langle z' | z' \rangle.$$

We start from a function $f(z_{-1}z_1, z')$ that satisfies

$$f(\tau^{-1}z_{-1}, \tau z_1, z')\tau^{-\alpha}\Big|_{\tau=0}^{\tau=\infty} = 0.$$

and we set

$$\begin{aligned} F_\alpha &:= \frac{1}{2\pi \mathrm{i}} \int_0^\infty f(\tau^{-1}z_{-1}, \tau z_1, z')\tau^{-\alpha-1}\mathrm{d}\tau \\ &= \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^\infty f(\mathrm{e}^{-\psi}z_{-1}, \mathrm{e}^{\psi}z_1, z')\mathrm{e}^{-\psi\alpha}\mathrm{d}\psi. \end{aligned}$$

Then

$$N_1 F_\alpha(z_{-1}, z_1, z') = \alpha F_\alpha(z_{-1}, z_1, z').$$

Assume now that $z_{-1}, z_1, z'$ are complex variables and $f$ is holomorphic. The following result summarizes the above constructions:

Suppose that $]0,1[\ni s \overset{\gamma}{\mapsto} \tau(s)$ is a contour on the Riemann surface of

$$\tau \mapsto f(\tau^{-1}z_{-1}, \tau z_1, z')\tau^{-\alpha}$$

that satisfies

$$f(\tau^{-1}z_{-1}, \tau z_1, z')\tau^{-\alpha}\Big|_{\tau(0)}^{\tau(1)} = 0.$$

Then

$$F_\alpha := \int_0^\infty f(\tau^{-1}z_{-1}, \tau z_1, z')\tau^{-\alpha-1}\frac{\mathrm{d}\tau}{2\pi\mathrm{i}}$$

solves

$$N_1 F_\alpha = \alpha F_\alpha.$$

Suppose that $\mathbb{R}^n$ is equipped with a scalar product

$$\langle z_{-1}, z_1, z' | z_{-1}, z_1, z' \rangle_n = 2z_{-1}z_1 + \langle z' | z' \rangle_{n-2}.$$

Introduce new variables (essentially, polar coordinates for $z_{-1}, z_1$):

$$z_0 := \sqrt{2z_{-1}z_1}, \quad u := \sqrt{\frac{z_1}{z_{-m}}}.$$

In the new variables,

$$\begin{aligned}\Delta_n &= \partial_{z_0}^2 + \frac{1}{z_0}\partial_{z_0} - \frac{1}{z_0^2}(u\partial_u)^2 + \Delta_{n-2} \\ &= \left(\partial_{z_0} + \frac{1}{2z_0}\right)^2 - \frac{1}{z_0^2}\left(u\partial_u - \frac{1}{2}\right)\left(u\partial_u + \frac{1}{2}\right) + \Delta_{n-2}.\end{aligned}$$

Therefore, if we set

$$\Delta_{n-1} := \partial_{z_0}^2 + \Delta_{n-2},$$

$$F_\pm(z', z_0, u) = f_\pm(z', z_0) u^{\pm\frac{1}{2}} z_0^{-\frac{1}{2}},$$

then

$$N_1 F_\pm = \pm\frac{1}{2} F_\pm,$$

$$z_0^{\frac{1}{2}} u^{\mp\frac{1}{2}} \Delta_n F_\pm = \Delta_{n-1} f_\pm.$$

Therefore, the $n-1$-dimensional Laplace equation $\Delta_{n-1} f = 0$ is essentially equivalent to the $n$-dimensional Laplace equation $\Delta_n F = 0$ restricted to the eigenspace of $N_1 = \pm\frac{1}{2}$.

# Conformal invariance of the Laplacian

We say that a manifold $\mathcal{Y}$ is pseudo-Riemannian if it is equipped with a nondegenerate symmetric covariant 2-tensor

$$\mathcal{Y} \ni y \mapsto g(y) = [g_{ij}(y)],$$

called the metric tensor. We say that a transformation $\alpha$ is isometric if $\alpha g = g$. We say that a vector field $X$ is Killing if $Xg = 0$.

We say that the metric tensor $g_1$ is conformally equivalent to $g$ if there exists a positive function $m \in C_+^\infty(\mathcal{Y})$ such that

$$m(y)g(y) = g_1(y).$$

We say that a manifold $\mathcal{Y}$ is equipped with a conformal structure, if it is equipped with an equivalence class of conformally equivalent metric tensors.

We say that a transformation $\alpha$ is conformal if it preserves the conformal class. We say that a vector field $X$ is conformal Killing if

$$Xg = Mg.$$

Consider a pseudo-Euclidean vector space $(\mathbb{R}^{n+2}, g)$ of signature $(q+1, p+1)$.

$$\mathcal{V} := \{z \in \mathbb{R}^{n+2} \ : \ \langle z|z\rangle = 0, \quad z \neq 0\}.$$

is the null quadric. Obviously, $\mathrm{O}(n+2)$ and $\mathrm{so}(n+2)$ preserve $\mathcal{V}$.

$\mathbb{R}^{\times}$ also preserves on $\mathcal{V}$. Let $\mathcal{Y} := \mathcal{V}/\mathbb{R}^{\times}$ be the projective null quadric. We obtain a line bundle

$$\mathcal{V} \mapsto \mathcal{Y}$$

with the base $\mathcal{Y}$ and the fiber $\mathbb{R}^{\times}$.

Let $\mathcal{Y}_i$ be an open subset of $\mathcal{Y}$ and $\mathcal{V}_i$ the corresponding open subset of $\mathcal{V}$. Let

$$\mathcal{Y}_i \ni y \mapsto \gamma_i(y) \in \mathcal{V}_i$$

be a section of this bundle. Let $g_{\gamma_i}$ be the metric tensor $g$ restricted to $\gamma_i(\mathcal{Y}_i)$ transported to $\mathcal{Y}_i$. Then $g_{\gamma_i}$ is a metric tensor on $\mathcal{Y}_i$ of signature $(q, p)$.

The metrics $g_{\gamma_1}$ and $g_{\gamma_2}$ on $\mathcal{Y}_1 \cap \mathcal{Y}_2$ are conformally equivalent.

Thus $\mathcal{Y}$ is equipped with a conformal structure.

We have representations

$$\mathrm{so}(n+2) \ni B \mapsto B^{\diamond}$$
$$\mathrm{O}(n+2) \ni \alpha \mapsto \alpha^{\diamond}$$

in conformal Killing vector fields/conformal transformations on $\mathcal{Y}$.

Consider a pseudo-Euclidean space $(\mathbb{R}^{n+2}, g_{n+2})$ of signature $(q+1, p+1)$, with the square of a vector $(y, r_-, r_+) \in \mathbb{R}^{n+2} = \mathbb{R}^n \oplus \mathbb{R}^2$ defined as

$$\langle y, r_-, r_+ | y, r_-, r_+ \rangle_{n+2} := \langle y | y \rangle_n + 2r_+ r_-.$$

Thus the pseudo-Euclidean space $(\mathbb{R}^n, g_n)$ of signature $(q, p)$ is embedded in the pseudo-Euclidean space $(\mathbb{R}^{n+2}, g_{n+2})$

Recall that $\mathcal{V}$, resp. $\mathcal{Y}$ are the null quadric, resp. the projective null quadric in $\mathbb{R}^{n+2}$. Set

$$\mathcal{V}_0 := \{(y, r_-, r_+) \in \mathcal{V} \ : \ r_- \neq 0\}, \quad \mathcal{Y}_0 := \mathcal{V}_0/\mathbb{R}^\times.$$

$\mathcal{Y}_0$ is dense and open in $\mathcal{Y}$.

We have a bijection and a section

$$\mathcal{Y}_0 \ni \mathbb{R}^\times \begin{bmatrix} y \\ 1 \\ -\frac{\langle y|y\rangle_n}{2} \end{bmatrix} \leftrightarrow \underset{\mathbb{R}^n}{\overset{y}{\cap}} \mapsto \begin{bmatrix} y \\ 1 \\ -\frac{\langle y|y\rangle_n}{2} \end{bmatrix} \in \mathcal{V}_0.$$

We have thus embedded $\mathbb{R}^n$ as a dense open subset of the manifold $\mathcal{Y}$.

Consider a Euclidean space $(\mathbb{R}^{n+1}, g_{n+1})$. Let us embed it in $(\mathbb{R}^{n+2}, g_{n+2})$, so that for $(y, r) \in \mathbb{R}^{n+1} \oplus \mathbb{R} = \mathbb{R}^{n+2}$

$$\langle y, r | y, r \rangle_{n+2} = \langle y | y \rangle_{n+1} - r^2.$$

Clearly, the signature of $\mathbb{R}^{n+2}$ is $(n+1, n)$.

We have a bijection and a section

$$\mathcal{Y} \ni \mathbb{R}^\times \begin{bmatrix} y \\ 1 \end{bmatrix} \leftrightarrow \underset{\mathbb{S}^n}{\underset{\cap}{y}} \mapsto \begin{bmatrix} y \\ 1 \end{bmatrix} \in \mathcal{V}.$$

Thus $\mathbb{S}^n$ can be identified with $\mathcal{Y}$.

Consider now the space $\mathbb{R}^{n+2}$ of signature $(q+1, p+1)$, so that for $(\vec{t}, \vec{x}) = (t_0, \ldots, t_q, x_0, \ldots, x_p)$

$$\langle \vec{t}, \vec{x} | \vec{t}, \vec{x} \rangle := -t_0^2 - \cdots - t_q^2 + x_0^2 + \cdots + x_p^2.$$

We have a double covering and a section

$$\mathbb{R}^{\times}(\vec{\rho}, \vec{\omega}) \in \mathcal{Y} \leftrightarrow \underset{\mathbb{S}^q \times \mathbb{S}^p}{\underset{\Cap}{(\vec{\rho}, \vec{\omega})}} \mapsto (\vec{\rho}, \vec{\omega}) \in \mathcal{V}.$$

We have thus identified $\mathcal{V}$ with $\mathbb{S}^q \times \mathbb{S}^p / \mathbb{Z}_2$. Its metric tensor is minus the standard metric tensor on $\mathbb{S}^q$ plus the standard metric tensor on $\mathbb{S}^p$. Its signature is $(q, p)$.

Let us describe the projective quadric in the lowest dimensions, where everything is very explicit. We start with dimension $n = 1$.

Consider $\mathbb{R}^{1+2}$ with the scalar product

$$\langle z|z\rangle = z_0^2 + 2z_{-1}z_1.$$

We have

$$\mathcal{Y}^1 \simeq \mathbb{S}^1 \simeq \mathbb{R} \cup \{\infty\} = P^1\mathbb{R}.$$

The Lie algebra $\mathrm{so}(3)$ is spanned by

$$B_{0,1},\ B_{0,-1},\ N_1.$$

The Casimir operator is

$$\begin{aligned}\mathcal{C}_3 &= 2B_{0,1}B_{0,-1} - N_1^2 + N_1 \\ &= 2B_{0,-1}B_{0,1} - N_1^2 - N_1.\end{aligned}$$

Consider dimension 2. Equip $\mathbb{R}^{2+2}$ with the scalar product

$$\langle z|z\rangle = 2z_{-1}z_1 + 2z_{-2}z_2.$$

We have

$$\mathcal{Y}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1.$$

The Lie algebra $\mathrm{so}(4)$ is spanned by

$$N_1,\ N_2,\ B_{1,2},\ B_{1,-2},\ B_{-1,2},\ B_{-1,-2}.$$

Its Casimir operator is

$$\mathcal{C}_4 = 2B_{1,2}B_{-1,-2} + 2B_{1,-2}B_{-1,2} - N_1^2 - N_2^2 - 2N_1.$$

$\mathrm{so}(4)$ decomposes into a direct sum of two copies of $\mathrm{so}(3)$:

$$\mathrm{so}(4) = \mathrm{so}^+(3) \oplus \mathrm{so}^-(3),$$

where $\mathrm{so}^+(3)$, resp. $\mathrm{so}^-(3)$ are spanned by

$B_{1,2},\ B_{-1,-2},\ N_1 + N_2$; resp. $B_{1,-2},\ B_{-1,2},\ N_1 - N_2$.

The corresponding Casimir operators are

$$\begin{aligned}
\mathcal{C}_3^+ &= 2B_{1,2}B_{-1,-2} - \frac{1}{2}(N_1 + N_2)^2 + N_1 + N_2 \\
&= 2B_{-1,-2}B_{1,2} - \frac{1}{2}(N_1 + N_2)^2 - N_1 - N_2, \\
\mathcal{C}_3^- &= 2B_{1,-2}B_{-1,2} - \frac{1}{2}(N_1 - N_2)^2 + N_1 - N_2 \\
&= 2B_{-1,2}B_{1,-2} - \frac{1}{2}(N_1 - N_2)^2 - N_1 + N_2.
\end{aligned}$$

Thus

$$\mathcal{C}_4 = \mathcal{C}_3^+ + \mathcal{C}_3^-.$$

Inside the associative algebra of differential operators on $\mathbb{R}^4$ we have the identity

$$\mathcal{C}_3^+ = \mathcal{C}_3^-$$

Therefore, represented in the algebra of differential operators

$$\begin{aligned}
\mathcal{C}_4 &= 4B_{1,2}B_{-1,-2} - (N_1 + N_2)^2 + 2N_1 + 2N_2 \\
&= 4B_{-1,-2}B_{1,2} - (N_1 + N_2)^2 - 2N_1 - 2N_2 \\
&= 4B_{1,-2}B_{-1,2} - (N_1 - N_2)^2 + 2N_1 - 2N_2 \\
&= 4B_{-1,2}B_{1,-2} - (N_1 - N_2)^2 - 2N_1 + 2N_2.
\end{aligned}$$

Let $\eta \in \mathbb{C}$. We define $\Lambda^\eta(\mathcal{V})$ to be the set of smooth functions on $\mathcal{V}$ homogeneous of degree $\eta$, that is satisfying

$$f(ty) = t^\eta f(y), \quad t \neq 0, \quad y \in \mathcal{V}.$$

Clearly, $B \in \mathrm{so}(n+2)$, resp. $\alpha \in \mathrm{O}(n+2)$ preserve $\Lambda^\eta(\mathcal{V})$. We will denote by $B^{\diamond,\eta}$, resp. $\alpha^{\diamond,\eta}$ the restriction of $B$, resp. $\alpha$ to $\Lambda^\eta(\mathcal{V})$. Thus we have representations

$$\mathrm{so}(n+2) \ni B \mapsto B^{\diamond,\eta},$$
$$\mathrm{O}(n+2) \ni \alpha \mapsto \alpha^{\diamond,\eta}.$$

Let $\mathcal{Y}_0 \subset \mathcal{Y}$ be open and $\mathcal{V}_0 := \mathbb{R}^\times \mathcal{Y}_0$.

Let $\gamma$ be a section based on $\mathcal{Y}_0$. We then have an obvious map $\psi^{\gamma,\eta} : \Lambda^\eta(\mathcal{V}_0) \to C^\infty(\mathcal{Y}_0)$:

$$\big(\psi^{\gamma,\eta} k\big)(y) := k\big(\gamma(y)\big), \quad y \in \mathcal{Y}_0.$$

$\psi^{\gamma,\eta}$ is bijective and we can introduce its inverse, denoted $\phi^{\gamma,\eta}$:

$$\big(\phi^{\gamma,\eta} f\big)\big(s\gamma(y)\big) = s^\eta f(y), \quad s \in \mathbb{C}^\times, \quad y \in \mathcal{Y}_0.$$

We can transport $\mathrm{so}(\mathbb{R}^{n+2})$ and $\mathrm{O}(\mathbb{R}^{n+2})$ on $\mathcal{Y}_0$:

$$B^{\gamma,\eta} := \psi^{\gamma,\eta} B \phi^{\gamma,\eta},$$
$$\alpha^{\gamma,\eta} := \psi^{\gamma,\eta} \alpha \phi^{\gamma,\eta}.$$

We thus obtain representations

$$\mathrm{so}(n+2) \ni B \mapsto B^{\gamma,\eta},$$
$$\mathrm{O}(n+2) \ni \alpha \mapsto \alpha^{\gamma,\eta}.$$

The following theorem apparently goes back to Dirac. We find it curious because it allows in some situations to restrict a second order differential operator to a submanifold.

**Theorem.** Let $\Omega \subset \mathbb{R}^{n+2}$ be an open conical set. Let $K \in C^\infty(\Omega)$ be homogeneous of degree $1 - \frac{n}{2}$ such that

$$K\Big|_{\mathcal{V}\cap\Omega} = 0.$$

Then

$$\Delta_{n+2} K\Big|_{\mathcal{V}\cap\Omega} = 0.$$

Let $k \in \Lambda^{1-\frac{n}{2}}(\mathcal{V})$. Let $\Omega$ be a conical neighborhood of $\mathcal{V}$, $K \in C^\infty(\Omega)$ be homogeneous of degree $1-\frac{n}{2}$ and

$$k = K\Big|_{\mathcal{V}}.$$

(We can always find such $\Omega$ and $K$). We set

$$\Delta^\diamond_{n+2} k := \Delta_{n+2} K\Big|_{\mathcal{V}}.$$

By the previous theorem the above definition does not depend on the choice of $\Omega$ and $K$ and $\Delta_{n+2}K$ is homogeneous of degree $-1-\frac{n}{2}$. We have thus defined a map

$$\Delta^\diamond_{n+2} : \Lambda^{1-\frac{n}{2}}(\mathcal{V}) \to \Lambda^{-1-\frac{n}{2}}(\mathcal{V}).$$

Obviously,

$$\begin{aligned} B\Delta_{n+2} &= \Delta_{n+2}B, \quad B \in \mathrm{so}(\mathbb{R}^{n+2}), \\ \alpha\Delta_{n+2} &= \Delta_{n+2}\alpha, \quad \alpha \in \mathrm{O}(\mathbb{R}^{n+2}). \end{aligned}$$

Restricting this to $\Lambda^{1-\frac{n}{2}}(\mathcal{V})$ we obtain

$$\begin{aligned} B^{\diamond,-1-\frac{n}{2}}\Delta^{\diamond}_{n+2} &= \Delta^{\diamond}_{n+2}B^{\diamond,1-\frac{n}{2}}, \quad B \in \mathrm{so}(\mathbb{R}^{n+2}), \\ \alpha^{\diamond,-1-\frac{n}{2}}\Delta^{\diamond}_{n+2} &= \Delta^{\diamond}_{n+2}\alpha^{\diamond,1-\frac{n}{2}}, \quad \alpha \in \mathrm{O}(\mathbb{R}^{n+2}). \end{aligned}$$

In the first proof of Dirac's Theorem, we use the decomposition $\mathbb{R}^{n+2} = \mathbb{R}^n \oplus \mathbb{R}^2$. We denote the square of a vector, the Laplacian, the Casimir, resp. the generator of dilations on $\mathbb{R}^{n+2}$ by $R_{n+2}$, $\Delta_{n+2}$, $\mathcal{C}_{n+2}$, resp. $A_{n+2}$. Similarly, we denote the square of a vector, the Laplacian, the Casimir, resp. the generator of dilations on $\mathbb{R}^n$ by $R_n$, $\Delta_n$, $\mathcal{C}_n$ resp. $A_n$. We will also write

$$N_{m+1} := r_+\partial_{r_+} - r_-\partial_{r_-}.$$

We have

$$R_{n+2} = R_n + 2r_+r_-,$$
$$\Delta_{n+2} = \Delta_n + 2\partial_{r_+}\partial_{r_-},$$
$$A_{n+2} = A_n + r_+\partial_{r_+} + r_-\partial_{r_-}.$$

Using the identities

$$R_n\Delta_n = \left(A_n - 1 + \frac{n}{2}\right)^2 - \left(\frac{n}{2} - 1\right)^2 + \mathcal{C}_n,$$
$$4r_+r_-\partial_{r_+}\partial_{r_-} = (r_+\partial_{r_+} + r_-\partial_{r_-})^2 - N_{m+1}^2,$$

we obtain

$$
\begin{aligned}
R_n\Delta_{n+2} &= R_n\Delta_n + \big(R_{n+2} - 2r_+r_-\big)2\partial_{r_+}\partial_{r_-} \\
&= \mathcal{C}_n + \left(A_n - 1 + \frac{n}{2}\right)^2 - \left(\frac{n}{2} - 1\right)^2 \\
&\quad + R_{n+2}2\partial_{r_+}\partial_{r_-} - (r_+\partial_{r_+} + r_-\partial_{r_-})^2 + N_{m+1}^2 \\
&= R_{n+2}2\partial_{r_+}\partial_{r_-} \\
&\quad + \left(A_n - 1 + \frac{n}{2} - r_+\partial_{r_+} - r_-\partial_{r_-}\right)\left(A_{n+2} - 1 + \frac{n}{2}\right) \\
&\quad - \left(\frac{n}{2} - 1\right)^2 + \mathcal{C}_n + N_{m+1}^2.
\end{aligned}
$$

All operators in the last line can be restricted to $\mathcal{V}$. The operator $A_{n+2} - 1 + \frac{n}{2}$ vanishes on functions in $\Lambda^{1-\frac{n}{2}}(\Omega)$. The operator $R_{n+2} 2\partial_{r_+}\partial_{r_-}$ is zero when restricted to $\mathcal{V}$.

**Corollary**

$$R_n \Delta^{\diamond}_{n+2} = -\Big(\frac{n}{2} - 1\Big)^2 + \mathcal{C}_n + N^2_{m+1}.$$

In the second proof of Dirac's Theorem we use the decomposition $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \oplus \mathbb{R}$. We denote the square of a vector, the Laplacian, the Casimir, resp. the generator of dilations on $\mathbb{R}^{n+1}$ by $R_{n+1}$, $\Delta_{n+1}$, $\mathcal{C}_{n+1}$, resp. $A_{n+1}$. We have

$$R_{n+1} = R_n + r^2,$$
$$A_{n+2} = A_{n+1} + r\partial_r,$$
$$\Delta_{n+2} = \Delta_{n+1} + \partial_r^2.$$

We use the following identity

$$
\begin{aligned}
R_{n+1}\Delta_{n+2} &= R_{n+1}\Delta_{n+1} + \big(R_{n+2} - r^2\big)\partial_r^2 \\
&= \mathcal{C}_{n+1} + \Big(A_{n+1} + \frac{n-1}{2}\Big)^2 - \Big(\frac{n-1}{2}\Big)^2 \\
&\quad + R_{n+2}\partial_r^2 - \Big(r\partial_r - \frac{1}{2}\Big)^2 + \Big(\frac{1}{2}\Big)^2 \\
&= R_{n+2}\partial_r^2 + \Big(A_{n+1} + \frac{n}{2} - r\partial_r\Big)\Big(A_{n+2} + \frac{n}{2} - 1\Big) \\
&\quad - \Big(\frac{n}{2} - 1\Big)\frac{n}{2} + \mathcal{C}_{n+1}.
\end{aligned}
$$

**Corollary.**

$$
R_{n+1}\Delta_{n+2}^{\diamond} = -\Big(\frac{n}{2} - 1\Big)\frac{n}{2} + \mathcal{C}_{n+1}.
$$

Below we sum up information about conformal symmetries on the level of the extended space $\mathbb{R}^{n+2}$ and the space $\mathbb{R}^n$. We will use the subscript fl, for flat.

We will use the split coordinates,

$$\langle z|z\rangle = \sum_j z_{-j}z_j, \quad z \in \mathbb{R}^{n+2}$$

$$\langle y|y\rangle = \sum_j y_{-j}y_j, \quad y \in \mathbb{R}^n.$$

As a rule, if an operator does not depend on $\eta$, we will omit $\eta$.

## Cartan algebra of $\mathrm{so}(n+2)$

Cartan operators of $\mathrm{so}(\mathbb{R}^n)$, $i = 1, \ldots, m$:

$$N_i = -z_{-i}\partial_{z_{-i}} + z_i\partial_{z_i},$$
$$N_i^{\mathrm{fl}} = -y_{-i}\partial_{y_{-i}} + y_i\partial_{y_i}.$$

Generator of dilations:

$$N_{m+1} = -z_{-m-1}\partial_{z_{-m-1}} + z_{m+1}\partial_{z_{m+1}},$$
$$N_{m+1}^{\mathrm{fl},\eta} = \sum_i y_i\partial_{y_i} - \eta = A_n - \eta.$$

## Root operators

Roots of $\mathrm{so}(\mathbb{R}^n)$, $|i| < |j|$

$$B_{i,j} = z_{-i}\partial_{z_j} - z_{-j}\partial_{z_i},$$
$$B^{\mathrm{fl}}_{i,j} = y_{-i}\partial_{y_j} - y_{-j}\partial_{y_i}.$$

Generators of translations, $|j| \le n$,

$$B_{m+1,j} = z_{-m-1}\partial_{z_j} - z_{-j}\partial_{z_{m+1}},$$
$$B^{\mathrm{fl}}_{m+1,j} = \partial_{y_j}.$$

Generators of special conformal transformations, $|j| \le n$:

$$B_{-m-1,j} = z_{m+1}\partial_{z_j} - z_{-j}\partial_{z_{-m-1}},$$
$$B^{\mathrm{fl},\eta}_{-m-1,j} = -\frac{1}{2}\langle y|y\rangle\partial_{y_j} + y_{-j}\sum_{i\in I_n} y_i\partial_{y_i} - \eta y_{-j}.$$

## Weyl symmetries.

We will write $K$ for a function on $\mathbb{R}^{n+2}$ and $f$ for a function on $\mathbb{R}^n$.

Reflection:

$$\begin{aligned} \tau_0 K(z_0, \dots) &= K(-z_0, \dots), \\ \tau_0^{\mathrm{fl}} f(y_0, \dots) &= f(-y_0, \dots). \end{aligned}$$

Flips, $j = 1, \dots, m$:

$$\begin{aligned} \tau_j K(\dots, z_{-j}, z_j, \dots) &= K(\dots, z_j, z_{-j}, \dots), \\ \tau_j^{\mathrm{fl}} f(\dots, y_{-j}, y_j, \dots) &= f(\dots y_j, y_{-j}, \dots). \end{aligned}$$

Inversion:

$$\begin{aligned} \tau_{m+1} K(\dots, z_{-m-1}, z_{m+1}) &= K(\dots, z_{m+1}, z_{-m-1}), \\ \tau_{m+1}^{\mathrm{fl},\eta} f(y) &= \Big(\frac{\langle y|y\rangle}{2}\Big)^{\eta} f\Big(-\frac{2y}{\langle y|y\rangle}\Big). \end{aligned}$$

Permutations, $\pi \in S_m$:

$$\begin{aligned} \sigma_\pi K(\ldots, z_{-j}, z_j, \ldots) &= K(\ldots, z_{-\pi_j}, z_{\pi_j}, \ldots), \\ \sigma_\pi^{\mathrm{fl}} f(\ldots, y_{-j}, y_j, \ldots) &= f(\ldots y_{-\pi_j}, y_{\pi_j}, \ldots). \end{aligned}$$

Special conformal transformations, $j = 1, \ldots, m$:

$$\begin{aligned} &\sigma_{(j,m+1)} K(\ldots, z_{-j}, z_j, \ldots, z_{-m-1}, z_{m+1}) \\ = \ &K(\ldots, z_{-m-1}, z_{m+1}, \ldots, z_{-j}, z_j), \end{aligned}$$

$$\begin{aligned} &\sigma^{\text{fl},\eta}_{(j,m+1)} f(y_{-1}, y_1, \ldots, y_{-j}, y_j, \ldots) \\ = \ &y^{\eta}_{-j} f\Big(\frac{y_{-1}}{y_{-j}}, \frac{y_1}{y_{-j}}, \ldots, \frac{1}{y_{-j}}, -\frac{\langle y|y\rangle}{2y_{-j}} \ldots\Big). \end{aligned}$$

## Laplacian

$$\Delta_{n+2} = \sum_i \partial_{z_i}\partial_{z_{-i}},$$

$$\Delta^{\text{fl}}_{n+2} = \sum_i \partial_{y_i}\partial_{y_{-i}} = \Delta_n.$$

We have the representations

$$\mathrm{so}(n+2) \ni B \mapsto B^{\mathrm{fl},\eta},$$
$$\mathrm{O}(n+2) \ni \alpha \mapsto \alpha^{\mathrm{fl},\eta}$$

They yield a generalized symmetry:

$$B^{\mathrm{fl},\frac{-2-n}{2}}\Delta_n = \Delta_n B^{\mathrm{fl},\frac{2-n}{2}}, \quad B \in \mathrm{so}(n+2),$$
$$\alpha^{\mathrm{fl},\frac{-2-n}{2}}\Delta_n = \Delta_n \alpha^{\mathrm{fl},\frac{2-n}{2}}, \quad \alpha \in \mathrm{O}(n+2).$$

# Laplacian in 4 dimensions

# and the hypergeometric equation

We consider the extended space $\mathbb{R}^6$ with the split coordinates

$$z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3$$

and the scalar product given by

$$\langle z|z\rangle = 2z_{-1}z_1 + 2z_{-2}z_2 + 2z_{-3}z_3.$$

**Lie algebra** $\mathrm{so}(6)$**.** Cartan algebra:

$$N_1 = -z_{-1}\partial_{z_{-1}} + z_1\partial_{z_1},$$
$$N_2 = -z_{-2}\partial_{z_{-2}} + z_2\partial_{z_2},$$
$$N_3 = -z_{-3}\partial_{z_{-3}} + z_3\partial_{z_3}.$$

Root operators:

$$
\begin{aligned}
B_{-2,-1} &= z_2\partial_{z_{-1}} - z_1\partial_{z_{-2}},\\
B_{2,1} &= z_{-2}\partial_{z_1} - z_{-1}\partial_{z_2},\\
B_{2,-1} &= z_{-2}\partial_{z_{-1}} - z_1\partial_{z_2},\\
B_{-2,1} &= z_2\partial_{z_1} - z_{-1}\partial_{z_{-2}};
\end{aligned}
$$

$$
\begin{aligned}
B_{-3,-2} &= z_3\partial_{z_{-2}} - z_2\partial_{z_{-3}},\\
B_{3,2} &= z_{-3}\partial_{z_2} - z_{-2}\partial_{z_3},\\
B_{3,-2} &= z_{-3}\partial_{z_{-2}} - z_2\partial_{z_3},\\
B_{-3,2} &= z_3\partial_{z_2} - z_{-2}\partial_{z_{-3}};
\end{aligned}
$$

$$
\begin{aligned}
B_{-3,-1} &= z_3\partial_{z_{-1}} - z_1\partial_{z_{-3}},\\
B_{3,1} &= z_{-3}\partial_{z_1} - z_{-1}\partial_{z_3},\\
B_{3,-1} &= z_{-3}\partial_{z_{-1}} - z_1\partial_{z_3},\\
B_{-3,1} &= z_3\partial_{z_1} - z_{-1}\partial_{z_{-3}}.
\end{aligned}
$$

**Weyl symmetries.**

$$
\begin{aligned}
\sigma_{123}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3),\\
\sigma_{-12-3}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_1,z_{-1},z_{-2},z_2,z_3,z_{-3}),\\
\sigma_{1-2-3}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-1},z_1,z_2,z_{-2},z_{-3},z_3),\\
\sigma_{-1-23}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-1},z_1,z_{-2},z_2,z_3,z_{-3});\\
\sigma_{213}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-2},z_2,z_{-1},z_1,z_{-3},z_3),\\
\sigma_{-21-3}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_2,z_{-2},z_{-1},z_1,z_3,z_{-3}),\\
\sigma_{2-1-3}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-2},z_2,z_1,z_{-1},z_3,z_{-3}),\\
\sigma_{-2-13}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_2,z_{-2},z_1,z_{-1},z_{-3},z_3),\\
\sigma_{321}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-3},z_3,z_{-2},z_2,z_{-1},z_1),\\
\sigma_{-32-1}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_3,z_{-3},z_{-2},z_2,z_1,z_{-1}),\\
\sigma_{3-2-1}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_{-3},z_3,z_2,z_{-2},z_1,z_{-1}),\\
\sigma_{-3-21}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) &= K(z_3,z_{-3},z_2,z_{-2},z_{-1},z_1),
\end{aligned}
$$

$$\begin{aligned}
\sigma_{312}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_{-3},z_3,z_{-1},z_1,z_{-2},z_2),\\
\sigma_{-31-2}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_3,z_{-3},z_{-1},z_1,z_2,z_{-2}),\\
\sigma_{3-1-2}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_{-3},z_3,z_1,z_{-1},z_2,z_{-2}),\\
\sigma_{-3-12}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_3,z_{-3},z_1,z_{-1},z_{-2},z_2),\\
\sigma_{231}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_{-2},z_2,z_{-3},z_3,z_{-1},z_1),\\
\sigma_{-23-1}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_2,z_{-2},z_{-3},z_3,z_1,z_{-1}),\\
\sigma_{2-3-1}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_{-2},z_2,z_3,z_{-3},z_1,z_{-1}),\\
\sigma_{-2-31}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_2,z_{-2},z_3,z_{-3},z_{-1},z_1),\\
\sigma_{132}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_{-1},z_1,z_{-3},z_3,z_{-2},z_2),\\
\sigma_{-13-2}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_1,z_{-1},z_{-3},z_3,z_2,z_{-2}),\\
\sigma_{1-3-2}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_{-1},z_1,z_3,z_{-3},z_2,z_{-2}),\\
\sigma_{-1-32}K(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3) =&K(z_1,z_{-1},z_3,z_{-3},z_{-2},z_2).
\end{aligned}$$

## Laplacian.

$$\Delta_6 = 2\partial_{z_{-1}}\partial_{z_1} + 2\partial_{z_{-2}}\partial_{z_2} + 2\partial_{z_{-3}}\partial_{z_3}.$$

We have

$$B\Delta_6 = \Delta_6 B, \quad B \in so(6),$$
$$\alpha\Delta_6 = \Delta_6\alpha, \quad \alpha \in O(6).$$

Consider the section of the null quadric

$$\mathcal{V}^5 := \{z \in \mathbb{R}^6 \;:\; 2z_{-1}z_1 + 2z_{-2}z_2 + 2z_{-3}z_3 = 0\} \setminus \{(0,0,0,0,0,0)\}$$

given by equations

$$4 = 2\left(z_{-1}z_1 + z_{-2}z_2\right) = -2z_3z_{-3}.$$

We will call it the spherical section, because it coincides with $\mathcal{S}^3(4) \times \mathcal{S}^1(-4)$.

Let us use the coordinates

$$r = \sqrt{2\,(z_{-1}z_1 + z_{-2}z_2)}\,, \qquad w = \frac{z_{-1}z_1}{z_{-1}z_1 + z_{-2}z_2}$$

$$u_1 = \frac{z_1}{\sqrt{z_{-1}z_1 + z_{-2}z_2}}\,, \qquad u_2 = \frac{z_2}{\sqrt{z_{-1}z_1 + z_{-2}z_2}},$$

$$p = \sqrt{-2z_3z_{-3}} \qquad u_3 = \sqrt{-\frac{z_3}{z_{-3}}}\,.$$

The null quadric in these coordinates is given by $r^2 = p^2$. We will restrict ourselves to the sheet $r = p$. The generator of dilations is

$$A_6 = r\,\partial_r + p\,\partial_p.$$

The spherical section is given by the condition $r^2 = 4$.

**Lie algebra** $\mathrm{so}(6)$**.** Cartan operators:

$$N_1^{\mathrm{sph}} = u_1\,\partial_{u_1}\,,$$
$$N_2^{\mathrm{sph}} = u_2\,\partial_{u_2}\,,$$
$$N_3^{\mathrm{sph}} = u_3\,\partial_{u_3}\,.$$

Roots:

$$B^{\mathrm{sph}}_{-2,-1} = u_1 u_2 \partial_w,$$
$$B^{\mathrm{sph}}_{2,1} = \frac{1}{u_1 u_2}\left((1-w)w\partial_w + (1-w)N_1^{\mathrm{sph}} - wN_2^{\mathrm{sph}}\right)\,,$$
$$B^{\mathrm{sph}}_{2,-1} = \frac{u_1}{u_2}\left((1-w)\partial_w - N_2^{\mathrm{sph}}\right)\,,$$
$$B^{\mathrm{sph}}_{-2,1} = \frac{u_2}{u_1}\left(w\partial_w + N_1^{\mathrm{sph}}\right)\,,$$

$$B^{\mathrm{sph},\eta}_{-3,-2} = u_2 u_3 \left( w\partial_w + \frac{1}{2}\big(N^{\mathrm{sph}}_1 + N^{\mathrm{sph}}_2 + N^{\mathrm{sph}}_3 - \eta\big) \right) ,$$

$$B^{\mathrm{sph},\eta}_{3,2} = \frac{1}{u_2 u_3} \left( w(w-1)\partial_w + \frac{(w-1)}{2}(N^{\mathrm{sph}}_1 + N^{\mathrm{sph}}_2 - N^{\mathrm{sph}}_3 - \eta) + N^{\mathrm{sph}}_2 \right) ,$$

$$B^{\mathrm{sph},\eta}_{3,-2} = \frac{u_2}{u_3} \left( w\partial_w + \frac{1}{2}\big(N^{\mathrm{sph}}_1 + N^{\mathrm{sph}}_2 - N^{\mathrm{sph}}_3 - \eta\big) \right) ,$$

$$B^{\mathrm{sph},\eta}_{-3,2} = \frac{u_3}{u_2} \left( w(w-1)\partial_w + \frac{(w-1)}{2}(N^{\mathrm{sph}}_1 + N^{\mathrm{sph}}_2 + N^{\mathrm{sph}}_3 - \eta) + N^{\mathrm{sph}}_2 \right) ,$$

$$B^{\mathrm{sph},\eta}_{-3,-1} = u_1 u_3 \left( (w-1)\partial_w + \frac{1}{2}\big(N^{\mathrm{sph}}_1 + N^{\mathrm{sph}}_2 + N^{\mathrm{sph}}_3 - \eta\big) \right) ,$$

$$B^{\mathrm{sph},\eta}_{3,1} = \frac{1}{u_1 u_3} \left( w(w-1)\partial_w + \frac{w}{2}\big(N^{\mathrm{sph}}_1 + N^{\mathrm{sph}}_2 - N^{\mathrm{sph}}_3 - \eta\big) - N^{\mathrm{sph}}_1 \right) ,$$

$$B^{\mathrm{sph},\eta}_{3,-1} = \frac{u_1}{u_3} \left( (w-1)\partial_w + \frac{1}{2}\big(N^{\mathrm{sph}}_1 + N^{\mathrm{sph}}_2 - N^{\mathrm{sph}}_3 - \eta\big) \right) ,$$

$$B^{\mathrm{sph},\eta}_{-3,1} = \frac{u_3}{u_1} \left( w(w-1)\partial_w + \frac{w}{2}\big(N^{\mathrm{sph}}_1 + N^{\mathrm{sph}}_2 + N^{\mathrm{sph}}_3 - \eta\big) - N^{\mathrm{sph}}_1 \right) .$$

# Weyl symmetries.

$$
\begin{aligned}
\sigma^{\mathrm{sph},\eta}_{123} f(w,u_1,u_2,u_3) &= f\left(w,u_1,u_2,u_3\right), \\
\sigma^{\mathrm{sph},\eta}_{-12-3} f(w,u_1,u_2,u_3) &= f\left(w,\frac{w}{u_1},u_2,\frac{1}{u_3}\right), \\
\sigma^{\mathrm{sph},\eta}_{1-2-3} f(w,u_1,u_2,u_3) &= f\left(w,u_1,\frac{1-w}{u_2},\frac{1}{u_3}\right), \\
\sigma^{\mathrm{sph},\eta}_{-1-23} f(w,u_1,u_2,u_3) &= f\left(w,\frac{w}{u_1},\frac{1-w}{u_2},u_3\right); \\
\sigma^{\mathrm{sph},\eta}_{213} f(w,u_1,u_2,u_3) &= f\left(1-w,u_2,u_1,u_3\right), \\
\sigma^{\mathrm{sph},\eta}_{-21-3} f(w,u_1,u_2,u_3) &= f\left(1-w,\frac{1-w}{u_2},u_1,\frac{1}{u_3}\right), \\
\sigma^{\mathrm{sph},\eta}_{2-1-3} f(w,u_1,u_2,u_3) &= f\left(1-w,u_2,\frac{w}{u_1},\frac{1}{u_3}\right), \\
\sigma^{\mathrm{sph},\eta}_{-2-13} f(w,u_1,u_2,u_3) &= f\left(1-w,\frac{1-w}{u_2},\frac{w}{u_1},u_3\right); \\
\sigma^{\mathrm{sph},\eta}_{321} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{1}{w},\frac{u_3}{\sqrt{-w}},\frac{u_2}{\sqrt{-w}},\frac{u_1}{\sqrt{-w}}\right), \\
\sigma^{\mathrm{sph},\eta}_{-32-1} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{1}{w},\frac{\sqrt{-w}}{wu_3},\frac{u_2}{\sqrt{-w}},\frac{\sqrt{-w}}{u_1}\right), \\
\sigma^{\mathrm{sph},\eta}_{3-2-1} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{1}{w},\frac{u_3}{\sqrt{-w}},\frac{(w-1)\sqrt{-w}}{wu_2},\frac{\sqrt{-w}}{u_1}\right), \\
\sigma^{\mathrm{sph},\eta}_{-3-21} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{1}{w},\frac{\sqrt{-w}}{wu_3},\frac{(w-1)\sqrt{-w}}{wu_2},\frac{u_1}{\sqrt{-w}}\right);
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathrm{sph},\eta}_{312} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{1}{1-w},\frac{u_3}{\sqrt{w-1}},\frac{u_1}{\sqrt{w-1}},\frac{u_2}{\sqrt{w-1}}\right),\\
\sigma^{\mathrm{sph},\eta}_{-31-2} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{1}{1-w},\frac{\sqrt{w-1}}{(w-1)u_3},\frac{u_1}{\sqrt{w-1}},\frac{\sqrt{w-1}}{u_2}\right),\\
\sigma^{\mathrm{sph},\eta}_{3-1-2} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{1}{1-w},\frac{u_3}{\sqrt{w-1}},\frac{w\sqrt{w-1}}{(w-1)u_1},\frac{\sqrt{w-1}}{u_2}\right),\\
\sigma^{\mathrm{sph},\eta}_{-3-12} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{1}{1-w},\frac{\sqrt{w-1}}{(w-1)u_3},\frac{w\sqrt{w-1}}{(w-1)u_1},\frac{u_2}{\sqrt{w-1}}\right);\\
\sigma^{\mathrm{sph},\eta}_{231} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{w-1}{w},\frac{u_2}{\sqrt{-w}},\frac{u_3}{\sqrt{-w}},\frac{u_1}{\sqrt{-w}}\right),\\
\sigma^{\mathrm{sph},\eta}_{-23-1} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{w-1}{w},\frac{(w-1)\sqrt{-w}}{wu_2},\frac{u_3}{\sqrt{-w}},\frac{\sqrt{-w}}{u_1}\right),\\
\sigma^{\mathrm{sph},\eta}_{2-3-1} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{w-1}{w},\frac{u_2}{\sqrt{-w}},\frac{\sqrt{-w}}{wu_3},\frac{\sqrt{-w}}{u_1}\right),\\
\sigma^{\mathrm{sph},\eta}_{-2-31} f(w,u_1,u_2,u_3) &= \left(\sqrt{-w}\right)^{\eta} f\left(\frac{w-1}{w},\frac{(w-1)\sqrt{-w}}{wu_2},\frac{\sqrt{-w}}{wu_3},\frac{u_1}{\sqrt{-w}}\right);\\
\sigma^{\mathrm{sph},\eta}_{132} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{w}{w-1},\frac{u_1}{\sqrt{w-1}},\frac{u_3}{\sqrt{w-1}},\frac{u_2}{\sqrt{w-1}}\right),\\
\sigma^{\mathrm{sph},\eta}_{-13-2} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{w}{w-1},\frac{w\sqrt{w-1}}{(w-1)u_1},\frac{u_3}{\sqrt{w-1}},\frac{\sqrt{w-1}}{u_2}\right),\\
\sigma^{\mathrm{sph},\eta}_{1-3-2} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{w}{w-1},\frac{u_1}{\sqrt{w-1}},\frac{\sqrt{w-1}}{(1-w)u_3},\frac{\sqrt{w-1}}{u_2}\right),\\
\sigma^{\mathrm{sph},\eta}_{-1-32} f(w,u_1,u_2,u_3) &= \left(\sqrt{w-1}\right)^{\eta} f\left(\frac{w}{w-1},\frac{w\sqrt{w-1}}{(w-1)u_1},\frac{\sqrt{w-1}}{(1-w)u_3},\frac{u_2}{\sqrt{w-1}}\right).
\end{aligned}
$$

## Laplacian:

After the change of coordinates we obtain

$$\Delta_6 = \frac{1}{r^2}\Big(4w(1-w)\partial_w^2 - 4\big((1+N_1)(w-1)+(1+N_2)w\big)\partial_w - (N_1+N_2+1)^2 + (r\partial_r)^2 + 2r\partial_r\Big) + \frac{1}{p^2}\Big((N_3)^2 - (p\partial_p)^2\Big).$$

Next we use $p^2 = r^2$ and $r\partial_r + p\partial_p = -1$ to obtain

$$\Delta_6^\diamond = \frac{1}{r^2}\Big(4w(1-w)\partial_w^2 - 4\big((1+N_1^\diamond)(w-1)+(1+N_2^\diamond)w\big)\partial_w - (N_1^\diamond+N_2^\diamond+1)^2 + (N_3^\diamond)^2$$

Finally, we restrict to the spherical section, which amounts to setting $r^2 = 4$:

$$\Delta_6^{\text{sph}} = \quad w(1-w)\partial_w^2 - \big((1+N_1^{\text{sph}})(w-1)+(1+N_2^{\text{sph}})w\big)\partial_w \\ - \frac{1}{4}(N_1^{\text{sph}}+N_2^{\text{sph}}+1)^2 + \frac{1}{4}(N_3^{\text{sph}})^2.$$

We have the generalized symmetries

$$B^{\text{sph},-3}\Delta_6^{\text{sph}} = \Delta_6^{\text{sph}} B^{\text{sph},-1}, \quad B \in so(6),$$
$$\alpha^{\text{sph},-3}\Delta_6^{\text{sph}} = \Delta_6^{\text{sph}} \alpha^{\text{sph},-1}, \quad \alpha \in O(6).$$

Let us make an ansatz

$$f(u_1,u_2,u_3,w) = u_1^\alpha u_2^\beta u_3^\mu F(w).$$

Clearly,

$$N_1^{\mathrm{sph}} f = \alpha f, \quad N_2^{\mathrm{sph}} f = \beta f, \quad N_3^{\mathrm{sph}} f = \mu f.$$

Therefore, on functions of this form we have

$$\Delta_6^{\mathrm{sph}} u_1^\alpha u_2^\beta u_3^\mu F(w) = u_1^\alpha u_2^\beta u_3^\mu \mathcal{F}_{\alpha,\beta,\mu}(w,\partial_w) F(w),$$

where

$$\begin{aligned}\mathcal{F}_{\alpha,\beta,\mu}(w,\partial_w) :=& w(1-w)\partial_w^2 - \big((1+\alpha)(w-1)+(1+\beta)w\big)\partial_w \\ & - \frac{1}{4}(\alpha+\beta+1)^2 + \frac{1}{4}\mu^2\end{aligned}$$

is the well-known ${}_2F_1$ hypergeometric operator

The root operators of $\mathrm{so}(6)$ lead to the following transmutation relations:

$$\begin{aligned}
\partial_z \mathcal{F}_{\alpha,\beta,\mu} &= \mathcal{F}_{\alpha+1,\beta+1,\mu}\partial_z, \\
(z(1-z)\partial_z + (1-z)\alpha - z\beta)\mathcal{F}_{\alpha,\beta,\mu} &= \mathcal{F}_{\alpha-1,\beta-1,\mu}(z(1-z)\partial_z + (1-z)\alpha - z\beta), \\
((1-z)\partial_z - \beta)\mathcal{F}_{\alpha,\beta,\mu} &= \mathcal{F}_{\alpha+1,\beta-1,\mu}((1-z)\partial_z - \beta), \\
(z\partial_z + \alpha)\mathcal{F}_{\alpha,\beta,\mu} &= \mathcal{F}_{\alpha-1,\beta+1,\mu}(z\partial_z + \alpha);
\end{aligned}$$

$$
\begin{aligned}
&\Big(z\partial_z + \frac{1}{2}(\alpha+\beta+\mu+1)\Big) z\mathcal{F}_{\alpha,\beta,\mu} \\
&\qquad = \qquad z\mathcal{F}_{\alpha,\beta+1,\mu+1}\Big(z\partial_z + \frac{1}{2}(\alpha+\beta+\mu+1)\Big), \\
&[1ex]\Big(z(1-z)\partial_z + \frac{1}{2}(1-z)(\alpha+\beta-\mu+1) - \beta\Big) z\mathcal{F}_{\alpha,\beta,\mu} \\
&\qquad = \qquad z\mathcal{F}_{\alpha,\beta-1,\mu-1}\Big(z(1-z)\partial_z + \frac{1}{2}(1-z)(\alpha+\beta-\mu+1) - \beta\Big), \\
&\Big(z\partial_z + \frac{1}{2}(\alpha+\beta-\mu+1)\Big) z\mathcal{F}_{\alpha,\beta,\mu} \\
&\qquad = \qquad z\mathcal{F}_{\alpha,\beta+1,\mu-1}\Big(z\partial_z + \frac{1}{2}(\alpha+\beta-\mu+1\Big), \\
&\Big(z(z-1)\partial_z - \frac{1}{2}(1-z)(\alpha+\beta+\mu+1) + \beta\Big) z\mathcal{F}_{\alpha,\beta,\mu} \\
&\qquad = \qquad z\mathcal{F}_{\alpha,\beta-1,\mu+1}\Big(z(z-1)\partial_z - \frac{1}{2}(1-z)(\alpha+\beta+\mu+1) + \beta\Big); \\
&\Big((z-1)\partial_z + \frac{1}{2}(\alpha+\beta+\mu+1)\Big)(1-z)\mathcal{F}_{\alpha,\beta,\mu} \\
&\qquad = \qquad (1-z)\mathcal{F}_{\alpha+1,\beta,\mu+1}\Big((z-1)\partial_z + \frac{1}{2}(\alpha+\beta+\mu+1\Big), \\
&\Big(z(1-z)\partial_z - \frac{1}{2}z(\alpha+\beta-\mu+1) + \alpha\Big)(1-z)\mathcal{F}_{\alpha,\beta,\mu} \\
&\qquad = \qquad (1-z)\mathcal{F}_{\alpha-1,\beta,\mu-1}\Big(z(1-z)\partial_z - \frac{1}{2}z(\alpha+\beta-\mu+1) + \alpha\Big), \\
&\Big((z-1)\partial_z + \frac{1}{2}(\alpha+\beta-\mu+1)\Big)(1-z)\mathcal{F}_{\alpha,\beta,\mu} \\
&\qquad = \qquad (1-z)\mathcal{F}_{\alpha+1,\beta,\mu-1}\Big((z-1)\partial_z + \frac{1}{2}(\alpha+\beta-\mu+1)\Big), \\
&\Big(z(z-1)\partial_z + \frac{1}{2}z(\alpha+\beta+\mu+1) - \alpha\Big)(1-z)\mathcal{F}_{\alpha,\beta,\mu} \\
&\qquad = \qquad (1-z)\mathcal{F}_{\alpha-1,\beta,\mu+1}\Big(z(z-1)\partial_z + \frac{1}{2}z(\alpha+\beta+\mu+1) - \alpha\Big).
\end{aligned}
$$

Here are the consequence of the Weyl symmetries:

All the operators below equal $\mathcal{F}_{\alpha,\beta,\mu}(w,\partial_w)$ for the corresponding $w$:

$$
\begin{array}{rrcl}
w=z: & & & \\
& & \mathcal{F}_{\alpha,\beta,\mu}(z,\partial_z), & \\
& (-z)^{-\alpha}(z-1)^{-\beta} & \mathcal{F}_{-\alpha,-\beta,\mu}(z,\partial_z) & (-z)^{\alpha}(z-1)^{\beta} \\
& (z-1)^{-\beta} & \mathcal{F}_{\alpha,-\beta,-\mu}(z,\partial_z) & (z-1)^{\beta}, \\
& (-z)^{-\alpha} & \mathcal{F}_{-\alpha,\beta,-\mu}(z,\partial_z) & (-z)^{\alpha}; \\
w=1-z: & & & \\
& & \mathcal{F}_{\beta,\alpha,\mu}(z,\partial_z), & \\
& (z-1)^{-\alpha}(-z)^{-\beta} & \mathcal{F}_{-\beta,-\alpha,\mu}(z,\partial_z) & (z-1)^{\alpha}(-z)^{\beta}, \\
& (z-1)^{-\alpha} & \mathcal{F}_{\beta,-\alpha,-\mu}(z,\partial_z) & (z-1)^{\alpha}, \\
& (-z)^{-\beta} & \mathcal{F}_{-\beta,\alpha,-\mu}(z,\partial_z) & (-z)^{\beta}; \\
w=\frac{1}{z}: & & & \\
& (-z)^{\frac12(\alpha+\beta+\mu+1)} & (-z)\mathcal{F}_{\mu,\beta,\alpha}(z,\partial_z) & (-z)^{\frac12(-\alpha-\beta-\mu-1)}, \\
& (-z)^{\frac12(\alpha+\beta-\mu+1)}(z-1)^{-\beta} & (-z)\mathcal{F}_{-\mu,-\beta,\alpha}(z,\partial_z) & (-z)^{\frac12(-\alpha-\beta+\mu-1)}(z-1)^{\beta}, \\
& (-z)^{\frac12(\alpha+\beta+\mu+1)}(z-1)^{-\beta} & (-z)\mathcal{F}_{\mu,-\beta,-\alpha}(z,\partial_z) & (-z)^{\frac12(-\alpha-\beta-\mu-1)}(z-1)^{\beta}, \\
& (-z)^{\frac12(\alpha+\beta-\mu+1)} & (-z)\mathcal{F}_{-\mu,\beta,-\alpha}(z,\partial_z) & (-z)^{\frac12(-\alpha-\beta+\mu-1)};
\end{array}
$$

$w = \frac{z-1}{z}$ :

$$\begin{array}{rcl} (-z)^{\frac{1}{2}(\alpha+\beta+\mu+1)} & (-z)\mathcal{F}_{\mu,\alpha,\beta}(z,\partial_z) & (-z)^{\frac{1}{2}(-\alpha-\beta-\mu-1)}, \\ (-z)^{\frac{1}{2}(\alpha+\beta-\mu+1)}(z-1)^{-\alpha} & (-z)\mathcal{F}_{-\mu,-\alpha,\beta}(z,\partial_z) & (-z)^{\frac{1}{2}(-\alpha-\beta+\mu-1)}(z-1)^{\alpha}, \\ (-z)^{\frac{1}{2}(\alpha+\beta+\mu+1)}(z-1)^{-\alpha} & (-z)\mathcal{F}_{\mu,-\alpha,-\beta}(z,\partial_z) & (-z)^{\frac{1}{2}(-\alpha-\beta-\mu-1)}(z-1)^{\alpha}, \\ (-z)^{\frac{1}{2}(\alpha+\beta-\mu+1)} & (-z)\mathcal{F}_{-\mu,\alpha,-\beta}(z,\partial_z) & (-z)^{\frac{1}{2}(-\alpha-\beta+\mu-1)}; \end{array}$$

$w = \frac{1}{1-z}$ :

$$\begin{array}{rcl} (z-1)^{\frac{1}{2}(\alpha+\beta+\mu+1)} & (z-1)\mathcal{F}_{\beta,\mu,\alpha}(z,\partial_z) & (z-1)^{\frac{1}{2}(-\alpha-\beta-\mu-1)}, \\ (-z)^{-\beta}(z-1)^{\frac{1}{2}(\alpha+\beta-\mu+1)} & (z-1)\mathcal{F}_{-\beta,-\mu,\alpha}(z,\partial_z) & (-z)^{\beta}(z-1)^{\frac{1}{2}(-\alpha-\beta+\mu-1)}, \\ (z-1)^{\frac{1}{2}(\alpha+\beta-\mu+1)} & (z-1)\mathcal{F}_{\beta,-\mu,-\alpha}(z,\partial_z) & (z-1)^{\frac{1}{2}(-\alpha-\beta+\mu-1)}, \\ (-z)^{-\beta}(z-1)^{\frac{1}{2}(\alpha+\beta+\mu+1)} & (z-1)\mathcal{F}_{-\beta,\mu,-\alpha}(z,\partial_z) & (-z)^{\beta}(z-1)^{\frac{1}{2}(-\alpha-\beta-\mu-1)}; \end{array}$$

$w = \frac{z}{z-1}$ :

$$\begin{array}{rcl} (z-1)^{\frac{1}{2}(\alpha+\beta+\mu+1)} & (z-1)\mathcal{F}_{\alpha,\mu,\beta}(z,\partial_z) & (z-1)^{\frac{1}{2}(-\alpha-\beta-\mu-1)}, \\ (-z)^{-\alpha}(z-1)^{\frac{1}{2}(\alpha+\beta-\mu+1)} & (z-1)\mathcal{F}_{-\alpha,-\mu,\beta}(z,\partial_z) & (-z)^{\alpha}(z-1)^{\frac{1}{2}(-\alpha-\beta+\mu-1)}, \\ (z-1)^{\frac{1}{2}(\alpha+\beta-\mu+1)} & (z-1)\mathcal{F}_{\alpha,-\mu,-\beta}(z,\partial_z) & (z-1)^{\frac{1}{2}(-\alpha-\beta+\mu-1)}, \\ (-z)^{-\alpha}(z-1)^{\frac{1}{2}(\alpha+\beta+\mu+1)} & (z-1)\mathcal{F}_{-\alpha,\mu,-\beta}(z,\partial_z) & (-z)^{\alpha}(z-1)^{\frac{1}{2}(-\alpha-\beta-\mu-1)}. \end{array}$$

The following analysis will lead to factorizations of the hypergeometric equation.

In the Lie algebra $\mathrm{so}(6)$ represented on $\mathbb{R}^6$ we have 3 distinguished Lie subalgebras isomorphic to $\mathrm{so}(4)$: in an obvious notation,

$$\mathrm{so}_{12}(4),\ \ \mathrm{so}_{23}(4),\ \ \mathrm{so}_{13}(4).$$

The corresponding Casimir operators are

$$\begin{aligned}
\mathcal{C}_{12} &= 4B_{1,2}B_{-1,-2} - (N_1 + N_2 + 1)^2 + 1 \\
&= 4B_{-1,-2}B_{1,2} - (N_1 + N_2 - 1)^2 + 1 \\
&= 4B_{1,-2}B_{-1,2} - (N_1 - N_2 + 1)^2 + 1 \\
&= 4B_{-1,2}B_{1,-2} - (N_1 - N_2 - 1)^2 + 1; \\
\mathcal{C}_{23} &= 4B_{2,3}B_{-2,-3} - (N_2 + N_3 + 1)^2 + 1 \\
&= 4B_{-2,-3}B_{2,3} - (N_2 + N_3 - 1)^2 + 1 \\
&= 4B_{2,-3}B_{-2,3} - (N_2 - N_3 + 1)^2 + 1 \\
&= 4B_{-2,3}B_{2,-3} - (N_2 - N_3 - 1)^2 + 1, \\
\mathcal{C}_{13} &= 4B_{1,3}B_{-1,-3} - (N_1 + N_3 + 1)^2 + 1 \\
&= 4B_{-1,-3}B_{1,3} - (N_1 + N_3 - 1)^2 + 1 \\
&= 4B_{1,-3}B_{-1,3} - (N_1 - N_3 + 1)^2 + 1 \\
&= 4B_{-1,3}B_{1,-3} - (N_1 - N_3 - 1)^2 + 1.
\end{aligned}$$

Of course, for any $\eta$ we can append the superscript ${}^{\diamond,\eta}$ to all the operators above.

After the reduction, we obtain the identities

$$(2z_{-1}z_1 + 2z_{-2}z_2)\Delta_6^{\diamond} = -1 + \mathcal{C}_{12}^{\diamond,-1} + (N_3^{\diamond,-1})^2,$$
$$(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_6^{\diamond} = -1 + \mathcal{C}_{23}^{\diamond,-1} + (N_1^{\diamond,-1})^2,$$
$$(2z_{-1}z_1 + 2z_{-3}z_3)\Delta_6^{\diamond} = -1 + \mathcal{C}_{13}^{\diamond,-1} + (N_2^{\diamond,-1})^2.$$

We obtain

$$
\begin{aligned}
&(2z_{-1}z_1 + 2z_{-2}z_2)\Delta_6^{\diamond} \\
= \ & 4B_{1,2}B_{-1,-2} - (N_1 + N_2 + N_3 + 1)(N_1 + N_2 - N_3 + 1) \\
= \ & 4B_{-1,-2}B_{1,2} - (N_1 + N_2 + N_3 - 1)(N_1 + N_2 - N_3 - 1) \\
= \ & 4B_{1,-2}B_{-1,2} - (N_1 - N_2 + N_3 + 1)(N_1 - N_2 - N_3 + 1) \\
= \ & 4B_{-1,2}B_{1,-2} - (N_1 - N_2 + N_3 - 1)(N_1 - N_2 - N_3 - 1); \\
&(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_6^{\diamond} \\
= \ & 4B_{2,3}B_{-2,-3} - (N_1 + N_2 + N_3 + 1)(-N_1 + N_2 + N_3 + 1) \\
= \ & 4B_{-2,-3}B_{2,3} - (N_1 + N_2 + N_3 - 1)(-N_1 + N_2 + N_3 - 1) \\
= \ & 4B_{2,-3}B_{-2,3} - (N_1 + N_2 - N_3 + 1)(-N_1 + N_2 - N_3 + 1) \\
= \ & 4B_{-2,3}B_{2,-3} - (N_1 + N_2 - N_3 - 1)(-N_1 + N_2 - N_3 - 1); \\
&(2z_{-1}z_1 + 2z_{-3}z_3)\Delta_6^{\diamond} \\
= \ & 4B_{1,3}B_{-1,-3} - (N_1 + N_2 + N_3 + 1)(N_1 - N_2 + N_3 + 1) \\
= \ & 4B_{-1,-3}B_{1,3} - (N_1 + N_2 + N_3 - 1)(N_1 - N_2 + N_3 - 1) \\
= \ & 4B_{1,-3}B_{-1,3} - (N_1 + N_2 - N_3 + 1)(N_1 - N_2 - N_3 + 1) \\
= \ & 4B_{-1,3}B_{1,-3} - (N_1 + N_2 - N_3 - 1)(N_1 - N_2 - N_3 - 1),
\end{aligned}
$$

where all the operators $B$ and $N$ need to be equipped with the superscript ${}^{\diamond,-1}$.

If we use the spherical section, we can rewrite this by making the replacements

$$\begin{aligned} 2z_{-1}z_1 + 2z_{-2}z_2 \to &\quad 1, \\ 2z_{-2}z_2 + 2z_{-3}z_3 \to &\quad -w, \\ 2z_{-1}z_1 + 2z_{-3}z_3 \to &\quad w-1, \end{aligned}$$

as well as replacing the superscript ${}^{\diamond}$ with ${}^{\mathrm{sph}}$.

The hypergeometric operator can be factorized in several ways:

$$\begin{aligned}
\mathcal{F}_{\alpha,\beta,\mu} &= \Big(z(1-z)\partial_z + \big((1+\alpha)(1-z)-(1+\beta)z\big)\Big)\partial_z \\
&\quad -\frac{1}{4}(\alpha+\beta+\mu+1)(\alpha+\beta-\mu+1), \\
&= \partial_z\Big(z(1-z)\partial_z + \big(\alpha(1-z)-\beta z\big)\Big) \\
&\quad -\frac{1}{4}(\alpha+\beta+\mu-1)(\alpha+\beta-\mu-1), \\
&= \Big(z\partial_z+\alpha+1\Big)\Big((1-z)\partial_z-\beta\Big) \\
&\quad -\frac{1}{4}(\alpha-\beta+\mu+1)(\alpha-\beta-\mu+1), \\
&= \Big((1-z)\partial_z-\beta-1\Big)\Big(z\partial_z+\alpha\Big) \\
&\quad -\frac{1}{4}(\alpha-\beta+\mu-1)(\alpha-\beta-\mu-1);
\end{aligned}$$

$$
\begin{aligned}
z\mathcal{F}_{\alpha,\beta,\mu} &= \Big(z\partial_z + \frac{1}{2}(\alpha+\beta+\mu-1)\Big)\Big(z(1-z)\partial_z + \frac{1}{2}(1-z)(\alpha+\beta-\mu+1) - \beta\Big) \\
&\quad -\frac{1}{4}(\alpha+\beta+\mu-1)(\alpha-\beta-\mu+1), \\
&= \Big(z(1-z)\partial_z + \frac{1}{2}(1-z)(\alpha+\beta-\mu+1) - \beta - 1\Big)\Big(z\partial_z + \frac{1}{2}(\alpha+\beta+\mu+1)\Big) \\
&\quad -\frac{1}{4}(\alpha+\beta+\mu+1)(\alpha-\beta-\mu-1), \\
&= \Big(z\partial_z + \frac{1}{2}(\alpha+\beta-\mu-1)\Big)\Big(z(1-z)\partial_z + \frac{1}{2}(1-z)(\alpha+\beta+\mu+1) - \beta\Big) \\
&\quad -\frac{1}{4}(\alpha+\beta-\mu-1)(\alpha-\beta+\mu+1), \\
&= \Big(z(1-z)\partial_z + \frac{1}{2}(1-z)(\alpha+\beta+\mu+1) - \beta - 1\Big)\Big(z\partial_z + \frac{1}{2}(\alpha+\beta-\mu+1)\Big) \\
&\quad -\frac{1}{4}(\alpha+\beta-\mu+1)(\alpha-\beta+\mu-1);
\end{aligned}
$$

$$
\begin{aligned}
(z-1)\mathcal{F}_{\alpha,\beta,\mu} &= \Big(z(z-1)\partial_z + \frac{1}{2}z(\alpha+\beta-\mu+1) - \alpha - 1\Big)\Big((z-1)\partial_z + \frac{1}{2}(\alpha+\beta+\mu+1)\Big) \\
&\quad -\frac{1}{4}(\alpha+\beta+\mu+1)(\alpha-\beta+\mu+1), \\
&= \Big((z-1)\partial_z + \frac{1}{2}(\alpha+\beta+\mu-1)\Big)\Big(z(z-1)\partial_z + \frac{1}{2}z(\alpha+\beta-\mu+1) - \alpha\Big) \\
&\quad -\frac{1}{4}(\alpha+\beta+\mu-1)(\alpha-\beta+\mu-1), \\
&= \Big(z(z-1)\partial_z + \frac{1}{2}z(\alpha+\beta+\mu+1) - \alpha - 1\Big)\Big((z-1)\partial_z + \frac{1}{2}(\alpha+\beta-\mu+1)\Big) \\
&\quad -\frac{1}{4}(\alpha+\beta-\mu+1)(\alpha-\beta-\mu+1), \\
&= \Big((z-1)\partial_z + \frac{1}{2}(\alpha+\beta-\mu-1)\Big)\Big(z(z-1)\partial_z + \frac{1}{2}z(\alpha+\beta+\mu+1) - \alpha\Big) \\
&\quad -\frac{1}{4}(\alpha+\beta-\mu-1)(\alpha-\beta-\mu-1).
\end{aligned}
$$

Traditionally, the hypergeometric equation is given by the operator

$$\mathcal{F}(a,b;c;z,\partial_z) := z(1-z)\partial_z^2 + \big(c-(a+b+1)z\big)\partial_z - ab,$$

where $a, b, c \in \mathbb{C}$.

Here is the relationship between the classical parameters $a, b, c$ and $\alpha, \beta, \mu \in \mathbb{C}$, which we will call Lie-algebraic:

$$\alpha := c-1, \quad \beta := a+b-c, \quad \mu := a-b;$$

$$a = \tfrac{1+\alpha+\beta+\mu}{2}, \quad b = \tfrac{1+\alpha+\beta-\mu}{2}, \quad c = 1+\alpha.$$

$0$ is a regular singular point of the hypergeometric equation. Its indices are $0$ and $1-c$. The Frobenius method implies that, for $c \neq 0, -1, -2, \ldots$, the unique solution of the hypergeometric equation equal to $1$ at $0$ is given by the series

$$F(a,b;c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!},$$

convergent for $|z| < 1$. The function extends to the whole complex plane cut at $[1, \infty[$ and is called the hypergeometric function.

Sometimes it is more convenient to consider the function

$$\mathbf{F}(a,b;c;z) := \frac{F(a,b,c,z)}{\Gamma(c)} = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{\Gamma(c+j)} \frac{z^j}{j!}$$

defined for all $a, b, c \in \mathbb{C}$. Another useful function proportional to ${}_2F_1$ is

$$\begin{aligned}\mathbf{F}^{\mathrm{I}}(a,b;c;z) &:= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F(a,b;c;z) \\ &= \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(c-a)(b)_j}{\Gamma(c+j)} \frac{z^j}{j!}.\end{aligned}$$

We will often use the Lie-algebraic parameters instead of the classical parameters:

$$\begin{aligned}
F_{\alpha,\beta,\mu}(z) &= F\Big(\frac{1+\alpha+\beta+\mu}{2}, \frac{1+\alpha+\beta-\mu}{2}; 1+\alpha; z\Big), \\
\mathbf{F}_{\alpha,\beta,\mu}(z) &= \mathbf{F}\Big(\frac{1+\alpha+\beta+\mu}{2}, \frac{1+\alpha+\beta-\mu}{2}; 1+\alpha; z\Big) \\
&= \frac{1}{\Gamma(\alpha+1)} F_{\alpha,\beta,\mu}(z), \\
\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) &= \mathbf{F}^{\mathrm{I}}\Big(\frac{1+\alpha+\beta+\mu}{2}, \frac{1+\alpha+\beta-\mu}{2}; 1+\alpha; z\Big) \\
&= \frac{\Gamma\big(\frac{1+\alpha+\beta+\mu}{2}\big)\Gamma\big(\frac{1+\alpha-\beta-\mu}{2}\big)}{\Gamma(\alpha+1)} F_{\alpha,\beta,\mu}(z).
\end{aligned}$$

# Kummer's table of standard solutions:

$$\begin{aligned}
\text{Solution} \sim 1 \text{ at } 0:\quad & F_{\alpha,\beta,\mu}(z) \\
=&(1-z)^{-\beta} F_{\alpha,-\beta,-\mu}(z) \\
=&(1-z)^{\frac{-1-\alpha-\beta+\mu}{2}} F_{\alpha,-\mu,-\beta}(\frac{z}{z-1}) \\
=&(1-z)^{\frac{-1-\alpha-\beta-\mu}{2}} F_{\alpha,\mu,\beta}(\frac{z}{z-1}); \\
\text{Solution} \sim z^{-\alpha} \text{ at } 0:\quad & z^{-\alpha} F_{-\alpha,\beta,-\mu}(z) \\
=&z^{-\alpha}(1-z)^{-\beta} F_{-\alpha,-\beta,\mu}(z) \\
=&z^{-\alpha}(1-z)^{\frac{-1+\alpha-\beta+\mu}{2}} F_{-\alpha,-\mu,\beta}(\frac{z}{z-1}) \\
=&z^{-\alpha}(1-z)^{\frac{-1+\alpha-\beta-\mu}{2}} F_{-\alpha,\mu,-\beta}(\frac{z}{z-1});
\end{aligned}$$

$$
\begin{aligned}
\text{Solution} \sim 1 \text{ at } 1: \quad & F_{\beta,\alpha,\mu}(1-z) \\
=& z^{-\alpha} F_{\beta,-\alpha,-\mu}(1-z) \\
=& z^{\frac{-1-\alpha-\beta+\mu}{2}} F_{\beta,-\mu,-\alpha}(1-z^{-1}) \\
=& z^{\frac{-1-\alpha-\beta-\mu}{2}} F_{\beta,\mu,\alpha}(1-z^{-1}); \\
\text{Solution} \sim (1-z)^{-\beta} \text{ at } 1: \quad & (1-z)^{-\beta} F_{-\beta,\alpha,-\mu}(1-z) \\
=& z^{-\alpha}(1-z)^{-\beta} F_{-\beta,-\alpha,\mu}(1-z) \\
=& z^{\frac{-1-\alpha+\beta-\mu}{2}} (1-z)^{-\beta} F_{-\beta,\mu,-\alpha}(1-z^{-1}) \\
=& z^{\frac{-1-\alpha+\beta+\mu}{2}} (1-z)^{-\beta} F_{-\beta,-\mu,\alpha}(1-z^{-1}); \\
\text{Solution} \sim z^{-a} \text{ at } \infty: \quad & (-z)^{\frac{-1-\alpha-\beta-\mu}{2}} F_{\mu,\beta,\alpha}(z^{-1}) \\
=& (-z)^{\frac{-1-\alpha+\beta-\mu}{2}} (1-z)^{-\beta} F_{\mu,-\beta,-\alpha}(z^{-1}) \\
=& (1-z)^{\frac{-1-\alpha-\beta-\mu}{2}} F_{\mu,\alpha,\beta}((1-z)^{-1}) \\
=& (-z)^{-\alpha}(1-z)^{\frac{-1+\alpha-\beta-\mu}{2}} F_{\mu,-\alpha,-\beta}((1-z)^{-1}); \\
\text{Solution} \sim z^{-b} \text{ at } \infty: \quad & (-z)^{\frac{-1-\alpha-\beta+\mu}{2}} F_{-\mu,\beta,-\alpha}(z^{-1}) \\
=& (-z)^{\frac{-1-\alpha+\beta+\mu}{2}} (1-z)^{-\beta} F_{-\mu,-\beta,\alpha}(z^{-1}) \\
=& (1-z)^{\frac{-1-\alpha-\beta+\mu}{2}} F_{-\mu,\alpha,-\beta}((1-z)^{-1}) \\
=& (-z)^{-\alpha}(1-z)^{\frac{-1+\alpha-\beta+\mu}{2}} F_{-\mu,-\alpha,\beta}((1-z)^{-1}).
\end{aligned}
$$

The recurrence relations follow easily from factorizations, or the integral representation.

$$
\begin{aligned}
\partial_z \mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) &= \frac{1+\alpha+\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha+1,\beta+1,\mu}(z),\\
(z(1-z)\partial_z+\alpha(1-z)-\beta z)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) &= \frac{-1+\alpha+\beta-\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha-1,\beta-1,\mu}(z),\\
((1-z)\partial_z-\beta)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) &= \frac{1+\alpha-\beta-\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha+1,\beta-1,\mu}(z),\\
(z\partial_z+\alpha)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) &= \frac{-1+\alpha-\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha-1,\beta+1,\mu}(z),
\end{aligned}
$$

$$\left(z\partial_z + \frac{1+\alpha+\beta+\mu}{2}\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) = \frac{1+\alpha+\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha,\beta+1,\mu+1}(z),$$

$$\left(z(z-1)\partial_z+\beta+\frac{1+\alpha+\beta-\mu}{2}(z-1)\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) = \frac{-1-\alpha+\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha,\beta-1,\mu-1}(z),$$

$$\left(z\partial_z + \frac{1+\alpha+\beta-\mu}{2}\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) = \frac{-1+\alpha-\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha,\beta+1,\mu-1}(z),$$

$$\left(z(z-1)\partial_z+\beta+\frac{1+\alpha+\beta+\mu}{2}(z-1)\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) = \frac{1-\alpha-\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha,\beta-1,\mu+1}(z),$$

$$\left((z-1)\partial_z + \frac{1+\alpha+\beta+\mu}{2}\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) = \frac{1+\alpha+\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha+1,\beta,\mu+1}(z),$$

$$\left(z(z-1)\partial_z-\alpha+\frac{1+\alpha+\beta-\mu}{2}z\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) = \frac{1-\alpha+\beta-\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha-1,\beta,\mu-1}(z),$$

$$\left((z-1)\partial_z + \frac{1+\alpha+\beta-\mu}{2}\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) = \frac{1+\alpha-\beta-\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha+1,\beta,\mu-1}(z),$$

$$\left(z(z-1)\partial_z-\alpha+\frac{1+\alpha+\beta+\mu}{2}z\right)\mathbf{F}^{\mathrm{I}}_{\alpha,\beta,\mu}(z) = \frac{1-\alpha-\beta+\mu}{2}\mathbf{F}^{\mathrm{I}}_{\alpha-1,\beta,\mu+1}(z).$$

Clearly, the following function is harmonic

$$K := z_1^{\alpha} z_2^{\beta} z_3^{\mu},$$

and solves

$$N_1 K = \alpha K, \quad N_2 K = \beta K, \quad N_3 K = \mu K.$$

However, in order to do the reduction we need

$$\alpha + \beta + \mu = -1,$$

which is two restrictive.

The following function is also harmonic:

$$K := (z_1 - \tau^{-1} z_{-2})^{\alpha} (z_2 + \tau^{-1} z_{-1})^{\beta} z_3^{\mu}.$$

However it solves only

$$N_3 K = \mu K,$$

and is not an eigenvector of $N_1$ or $N_2$.

Let the contour $]0,1[\ni s\mapsto \tau(s)$ satisfy

$$(z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\alpha_2+\nu}\tau^{\nu}\Big|_{\tau(0)}^{\tau(1)}=0,$$

$$(z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\alpha_2+\nu}\tau^{\nu-1}\Big|_{\tau(0)}^{\tau(1)}=0.$$

Set

$$\begin{aligned}&K_{\alpha,\alpha_2,\alpha_3,\nu}(z_{-1},z_1,z_{-2},z_2,z_{-3},z_3)\\ =&\int_\gamma (z_1-\tau^{-1}z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\beta+\nu}z_3^{\mu}\tau^{\nu-1}\frac{\mathrm{d}\tau}{2\pi \mathrm{i}}.\end{aligned}$$

Then we have

$$\begin{aligned}
\Delta_6 K_{\alpha,\beta,\mu,\nu} &= 0, \\
N_1 K_{\alpha,\beta,\mu,\nu} &= \alpha K_{\alpha,\beta,\mu,\nu}, \\
N_2 K_{\alpha,\beta,\mu,\nu} &= \beta K_{\alpha,\beta,\mu,\nu}, \\
N_3 K_{\alpha,\beta,\mu,\nu} &= \mu K_{\alpha,\beta,\mu,\nu}, \\
B_{-12} K_{\alpha,\beta,\mu,\nu} &= (\beta+\nu) K_{\alpha+1,\beta-1,\mu,\nu}, \\
B_{1-2} K_{\alpha,\beta,\mu,\nu} &= -(\alpha+\nu) K_{\alpha-1,\beta+1,\mu,\nu}, \\
B_{12} K_{\alpha,\beta,\mu,\nu} &= (\nu-1) K_{\alpha-1,\beta-1,\mu,\nu+1}, \\
B_{-1-2} K_{\alpha,\beta,\mu,\nu} &= -(\alpha+\beta+\nu+1) K_{\alpha+1,\beta+1,\mu,\nu-1}, \\
B_{1-3} K_{\alpha,\beta,\mu,\nu} &= -(\alpha+\nu) K_{\alpha-1,\beta,\mu+1,\nu}, \\
B_{-1-3} K_{\alpha,\beta,\mu,\nu} &= -(\beta+\nu) K_{\alpha+1,\beta,\mu+1,,\nu-1}.
\end{aligned}$$

To see that $K$ is an eigenvector of $N_1$ and $N_2$, we write

$$\begin{aligned}
&K_{\alpha,\beta,\mu,\nu}(z)\\
&=\int_\gamma (\tau z_1 - z_{-2})^{\alpha+\nu}(z_2+\tau^{-1}z_{-1})^{\beta+\nu}\tau^{-\alpha-1}\frac{\mathrm{d}\tau}{2\pi\mathrm{i}}\\
&=\int_\gamma (z_1 - \tau^{-1}z_{-2})^{\alpha+\nu}(\tau z_2+z_{-1})^{\beta+\nu}\tau^{-\beta-1}\frac{\mathrm{d}\tau}{2\pi\mathrm{i}},
\end{aligned}$$

If in addition
$$\nu = \frac{-\alpha - \beta - \mu - 1}{2},$$
then $K$ is homogeneous of degree $-1$, so that the Laplacian can be reduced to 4 dimensions. Let us substitute the coordinates $w, r, p, u_1, u_2, u_3$, and then set $\tau = \frac{t-w}{u_1 u_2}$
$$K(u_1, u_2, u_3, r, p, w) = 2^{-\frac{\mu}{2}+\frac{1}{2}} u_1^{\alpha} u_2^{\beta} u_3^{\mu} r^{-\mu-1} p^{\mu} F(w),$$
where $$F(w) = \int_{\gamma} (t-1)^{\frac{\alpha-\beta-\mu-1}{2}} t^{\frac{-\alpha+\beta-\mu-1}{2}} (t-w)^{\frac{-\alpha-\beta+\mu-1}{2}} \mathrm{d}t.$$
Therefore, the function $F$ satisfies the hypergeometric equation
$$\mathcal{F}_{\alpha,\beta,\mu} F = 0.$$

Of course, we can prove that representation without going to extra dimensions:

**Theorem.** Let $[0,1] \ni t \mapsto \gamma(t)$ satisfy

$$t^{b-c+1}(1-t)^{c-a}(t-z)^{-b-1}\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$\mathcal{F}(a,b;c;z,\partial_z)\int_\gamma t^{b-c}(1-t)^{c-a-1}(t-z)^{-b}\mathrm{d}t = 0.$$

**Proof.** We check that for any contour $\gamma$ the abbove expression is

$$-b\int_\gamma \Big(\partial_t t^{b-c+1}(1-t)^{c-a}(t-z)^{-b-1}\Big)\mathrm{d}t.$$

□

The hypergeometric function with the type I normalization has the integral representation

$$\int_1^\infty t^{b-c}(t-1)^{c-a-1}(t-z)^{-b}\mathrm{d}t$$
$$= \mathbf{F}^{\mathrm{I}}(a,b;c;z), \qquad \mathrm{Re}(c-a) > 0,\ \mathrm{Re}a > 0, \quad z \notin [1,\infty[.$$

Indeed, the left hand side is annihilated by the hypergeometric operator. Besides, by Euler's identity it equals $\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}$ at $0$. So does the right hand side. Then we apply the uniqueness of the solution by the Frobenius method.

The integrand has four singularities: $\{0, 1, \infty, z\}$. It is natural to chose $\gamma$ as the interval joining a pair of singularities. This choice leads to $6$ standard solutions with the $\mathrm{I}$-type normalization.

$$\begin{aligned}
\text{Solution} \sim 1 \text{ at } 0: &\quad [1, \infty]; \\
\text{Solution} \sim z^{-\alpha} \text{ at } 0: &\quad [0, z]; \\
\text{Solution} \sim 1 \text{ at } 1: &\quad [0, \infty]; \\
\text{Solution} \sim (1-z)^{-\beta} \text{ at } 1: &\quad [1, z]; \\
\text{Solution} \sim z^{-a} \text{ at } \infty: &\quad [z, \infty]; \\
\text{Solution} \sim z^{-b} \text{ at } \infty: &\quad [0, 1].
\end{aligned}$$

# Laplacian in 3 dimensions

# and the Gegenbauer equation

We consider $\mathbb{R}^5$ with the coordinates

$$z_0, z_{-2}, z_2, z_{-3}, z_3$$

and the scalar product given by

$$\langle z|z\rangle = z_0^2 + 2z_{-2}z_2 + 2z_{-3}z_3.$$

Note that we omit the indices $-1, 1$; this makes it easier to compare $\mathbb{R}^5$ with $\mathbb{R}^6$.

**Lie algebra $\mathrm{so}(5)$.** Cartan algebra:

$$N_2 = -z_{-2}\partial_{z_{-2}} + z_2\partial_{z_2},$$
$$N_3 = -z_{-3}\partial_{z_{-3}} + z_3\partial_{z_3}.$$

Root operators:

$$\begin{aligned} B_{0,-2} &= z_0 \partial_{z_{-2}} - z_2 \partial_{z_0}, \\ B_{0,2} &= z_0 \partial_{z_2} - z_{-2} \partial_{z_0}, \\ B_{0,-3} &= z_0 \partial_{z_{-3}} - z_3 \partial_{z_0}, \\ B_{0,3} &= z_0 \partial_{z_3} - z_{-3} \partial_{z_0}; \end{aligned}$$

$$\begin{aligned} B_{-3,-2} &= z_3 \partial_{z_{-2}} - z_2 \partial_{z_{-3}}, \\ B_{3,2} &= z_{-3} \partial_{z_2} - z_{-2} \partial_{z_3}, \\ B_{3,-2} &= z_{-3} \partial_{z_{-2}} - z_2 \partial_{z_3}, \\ B_{-3,2} &= z_3 \partial_{z_2} - z_{-2} \partial_{z_{-3}}; \end{aligned}$$

**Weyl symmetries.**

$$
\begin{aligned}
\sigma_{23}K(z_0,z_{-2},z_2,z_{-3},z_3) =&K(z_0,z_{-2},z_2,z_3,z_{-3}),\\
\tau_{2-3}K(z_0,z_{-2},z_2,z_{-3},z_3) =&K(-z_0,z_{-2},z_2,z_3,z_{-3}),\\
\sigma_{-2-3}K(z_0,z_{-2},z_2,z_{-3},z_3) =&K(z_0,z_2,z_{-2},z_{-3},z_3),\\
\tau_{-23}K(z_0,z_{-2},z_2,z_{-3},z_3) =&K(-z_0,z_2,z_{-2},z_3,z_{-3}),\\
\sigma_{32}K(z_0,z_{-2},z_2,z_{-3},z_3) =&K(z_0,z_{-3},z_3,z_2,z_{-2}),\\
\tau_{3-2}K(z_0,z_{-2},z_2,z_{-3},z_3) =&K(-z_0,z_{-3},z_3,z_2,z_{-2}),\\
\sigma_{-3-2}K(z_0,z_{-2},z_2,z_{-3},z_3) =&K(z_0,z_3,z_{-3},z_{-2},z_2),\\
\tau_{-32}K(z_0,z_{-2},z_2,z_{-3},z_3) =&K(-z_0,z_3,z_{-3},z_2,z_{-2})
\end{aligned}
$$

**Laplacian:**

$$\Delta_5 = \partial_{z_0}^2 + 2\partial_{z_{-2}}\partial_{z_2} + 2\partial_{z_{-3}}\partial_{z_3}.$$

It satisfies

$$\begin{aligned} B\Delta_5 &= \Delta_5 B, \qquad B \in so(5); \\ \alpha\Delta_5 &= \Delta_5 \alpha, \quad \alpha \in O(5). \end{aligned}$$

We consider the section of the quadric

$$\mathcal{V}^4 := \{z \in \mathbb{R}^5 \;:\; z_0^2 + 2z_{-2}z_2 + 2z_{-3}z_3 = 0\}$$

given by equations

$$1 = z_0^2 + 2z_{-2}z_2 = -2z_3z_{-3}.$$

We will call it the spherical section, because it is $\mathcal{S}^2(1) \times \mathcal{S}^1(-1)$. The superscript used for this section will be "sph" for spherical.

Introduce the following coordinates in $\mathbb{R}^5$:

$$r = \sqrt{z_0^2 + 2z_{-2}z_2}\,, \qquad p = \sqrt{-2z_3z_{-3}}\,,$$

$$w = \frac{z_0}{\sqrt{2z_{-2}z_2 + z_0^2}}\,, \quad u_2 = \frac{\sqrt{2}z_2}{\sqrt{z_0^2 + 2z_{-2}z_2}} \quad u_3 = \sqrt{-\frac{z_{-3}}{z_3}}\,.$$

Similarly as in the previous section, the null quadric in these coordinates is given by $r^2 = p^2$. We choose the sheet $r = p$. The generator of dilations is

$$A_5 = r\,\partial_r + p\,\partial_p.$$

The spherical section is given by the condition $r^2 = 1$.

**Lie algebra** $\mathrm{so}(5)$**.** Cartan operators:

$$N_2^{\mathrm{sph}} = u_2\,\partial_{u_2}\,,$$
$$N_3^{\mathrm{sph}} = u_3\,\partial_{u_3}\,.$$

# Roots:

$$B^{\mathrm{sph}}_{0,-2} = -\frac{u_2}{\sqrt{2}}\partial_w,$$
$$B^{\mathrm{sph}}_{0,2} = \frac{1}{\sqrt{2}u_2}\Big((w^2-1)\partial_w + 2wu_2\partial_{u_2}\Big),$$
$$B^{\mathrm{sph},\eta}_{0,-3} = \frac{u_3}{\sqrt{2}}\Big((w^2-1)\partial_w + wu_2\partial_{u_2} + wu_3\partial_{u_3} - w\eta\Big),$$
$$B^{\mathrm{sph},\eta}_{0,-3} = \frac{1}{\sqrt{2}u_3}\Big((1-w^2)\partial_w - wu_2\partial_{u_2} + wu_3\partial_{u_3} + w\eta\Big),$$

$$B^{\mathrm{sph},\eta}_{-3,-2} = \frac{u_2u_3}{2}\left(-w\partial_w - u_2\partial_{u_2} - u_3\partial_{u_3} - \eta\right),$$
$$B^{\mathrm{sph},\eta}_{3,2} = \frac{1}{2u_2u_3}\left(w(1-w^2)\partial_w - (1+w^2)u_2\partial_{u_2} + (w^2-1)u_3\partial_{u_3} + (w^2-1)\eta\right),$$
$$B^{\mathrm{sph},\eta}_{3,-2} = \frac{u_2}{2u_3}\left(w\partial_w + u_2\partial_{u_2} - u_3\partial_{u_3} - \eta\right),$$
$$B^{\mathrm{sph},\eta}_{-3,2} = \frac{u_3}{2u_2}\left(w(w^2-1)\partial_w + (1+w^2)u_2\partial_{u_2} + (w^2-1)u_3\partial_{u_3} + (1-w^2)\eta\right).$$

The Laplacian

$$\Delta_5^{\mathrm{sph}} = (1-w^2)\partial_w - 2(1+u_2\partial_{u_2})w\partial_w - \left(u_2\partial_{u_2}+\frac{1}{2}\right)^2 + (u_3\partial_{u_3})^2.$$

It satisfies

$$\begin{aligned} B^{\mathrm{sph},-3}\Delta_5^{\mathrm{sph}} &= \Delta_5^{\mathrm{sph}} B^{\mathrm{sph},-1}, \qquad B \in so(5); \\ \alpha^{\mathrm{sph},-3}\Delta_5^{\mathrm{sph}} &= \Delta_5^{\mathrm{sph}} \alpha^{\mathrm{sph},-1}, \quad \alpha \in O(5). \end{aligned}$$

Let us change the variables in the Laplacian:

$$\begin{aligned}\Delta_5 =&\frac{1}{r^2}\Big((1-w^2)\partial_w^2 - 2(1+u_2\partial_{u_2})w\partial_w \\ &- (u_2\partial_{u_2})^2 - u_2\partial_{u_2} + (r\partial_r)^2 + r\partial_r\Big) \\ &+ \frac{1}{p^2}\Big(-(p\partial_p)^2 + (u_3\partial_{u_3})^2\Big).\end{aligned}$$

Using $r^2 = p^2$, $r\partial_r + p\partial_p = -\frac{1}{2}$ we obtain

$$\Delta_5^\diamond = \frac{1}{r^2}\left((1-w^2)^2\partial_w^2 - 2(1+u_2\partial_{u_2})w\partial_w - \left(u_2\partial_{u_2} + \frac{1}{2}\right)^2 + (u_3\partial_{u_3})^2\right).$$

To obtain the Laplacian at the spherical section we drop $\frac{1}{r^2}$.

Let us make an ansatz

$$f(u_2,u_3,w)=u_2^{\alpha}u_3^{\lambda}F(w).$$

Clearly,

$$\begin{aligned} N_2^{\mathrm{sph}}f &= \alpha f, \\ N_3^{\mathrm{sph}}f &= \lambda f. \end{aligned}$$

Therefore, on functions of this form, we have

$$\Delta_5^{\mathrm{sph}}u_2^{\alpha}u_3^{\lambda}F(w)=u_2^{\alpha}u_3^{\lambda}\mathcal{S}_{\alpha,\lambda}(w,\partial_w)F(w),$$

where

$$\mathcal{S}_{\alpha,\lambda}(z,\partial_z) := (1-z^2)\partial_z^2-2(1+\alpha)z\partial_z+\lambda^2-\Big(\alpha+\frac{1}{2}\Big)^2.$$

Here is another parametrization of the Gegenbauer operator, which we call classical:

$$\mathcal{S}(a,b;z,\partial_z):=(1-z^2)\partial_z^2-(a+b+1)z\partial_z-ab.$$

Here is the relationship between the traditional and Lie-algebraic parameters:

$$\alpha := \tfrac{a+b-1}{2},\ \lambda := \tfrac{b-a}{2},$$

$$a = \tfrac{1}{2} + \alpha - \lambda,\ b = \tfrac{1}{2} + \alpha + \lambda.$$

The Gegenbauer equation is equivalent to certain subclasses of the hypergeometric equation by a number of different substitutions.

Therefore, we can reduce the Gegenbauer equation to the hypergeometric equation by two affine transformations. They move the singular points from $-1$, $1$ to $0$, $1$ or $1$, $0$:

$$\mathcal{S}(a,b;z,\partial_z) = \mathcal{F}(a,b;\tfrac{a+b+1}{2};u,\partial_u),$$

where

$$u = \tfrac{1-z}{2}, \quad z = 1-2u,$$
$$\text{or } u = \tfrac{1+z}{2}, \quad z = -1+2u.$$

In the Lie-algebraic parameters

$$\mathcal{S}_{\alpha,\lambda}(z,\partial_z) = \mathcal{F}_{\alpha,\alpha,2\lambda}(u,\partial_u).$$

Another pair of substitutions is a consequence of the reflection invariance of the Gegenbauer equation:

$$\mathcal{S}(a,b;z,\partial_z) = 4\mathcal{F}(\tfrac{a}{2},\tfrac{b}{2};\tfrac{1}{2};w,\partial_w),$$

$$z^{-1}\mathcal{S}(a,b;z,\partial_z)z = 4\mathcal{F}(\tfrac{a+1}{2},\tfrac{b+1}{2};\tfrac{3}{2};w,\partial_w),$$

where

$$w = z^2, \qquad z = \sqrt{w}.$$

In the Lie-algebraic parameters

$$\mathcal{S}_{\alpha,\lambda}(z,\partial_z) = \mathcal{F}_{-\frac{1}{2},\alpha,\lambda}(w,\partial_w), \tag{0.7}$$

$$z^{-1}\mathcal{S}_{\alpha,\lambda}(z,\partial_z)z = \mathcal{F}_{\frac{1}{2},\alpha,\lambda}(w,\partial_w). \tag{0.8}$$

## Transmutation relations

$$
\begin{aligned}
\partial_z \; & \mathcal{S}_{\alpha,\lambda} \\
= \quad \mathcal{S}_{\alpha+1,\lambda} \; & \partial_z,
\end{aligned}
$$

$$
\begin{aligned}
((1-z^2)\partial_z - 2\alpha z) \; & \mathcal{S}_{\alpha,\lambda} \\
= \quad \mathcal{S}_{\alpha-1,\lambda} \; & ((1-z^2)\partial_z - 2\alpha z),
\end{aligned}
$$

$$
\begin{aligned}
((1-z^2)\partial_z - (\alpha+\lambda+\tfrac{1}{2})z) \; & (1-z^2)\mathcal{S}_{\alpha,\lambda} \\
= \quad (1-z^2)\mathcal{S}_{\alpha,\lambda+1} \; & ((1-z^2)\partial_z - (\alpha+\lambda+\tfrac{1}{2})z),
\end{aligned}
$$

$$
\begin{aligned}
((1-z^2)\partial_z - (\alpha-\lambda+\tfrac{1}{2})z) \; & (1-z^2)\mathcal{S}_{\alpha,\lambda} \\
= \quad (1-z^2)\mathcal{S}_{\alpha,\lambda-1} \; & ((1-z^2)\partial_z - (\alpha-\lambda+\tfrac{1}{2})z);
\end{aligned}
$$

$$
\begin{aligned}
(z\partial_z + \alpha - \lambda + \tfrac{1}{2}) \; & z^2\mathcal{S}_{\alpha,\lambda} \\
= \quad z^2\mathcal{S}_{\alpha+1,\lambda-1} \; & (z\partial_z + \alpha - \lambda + \tfrac{1}{2}),
\end{aligned}
$$

$$
\begin{aligned}
(z(1-z^2)\partial_z - \alpha + \lambda + \tfrac{1}{2} - (\alpha+\lambda+\tfrac{1}{2})z^2) \; & z^2\mathcal{S}_{\alpha,\lambda} \\
= \quad z^2\mathcal{S}_{\alpha-1,\lambda+1} \; & (z(1-z^2)\partial_z - \alpha + \lambda + \tfrac{1}{2} - (\alpha+\lambda+\tfrac{1}{2})z^2),
\end{aligned}
$$

$$
\begin{aligned}
(z\partial_z + \alpha + \lambda + \tfrac{1}{2}) \; & z^2\mathcal{S}_{\alpha,\lambda} \\
= \quad z^2\mathcal{S}_{\alpha+1,\lambda+1} \; & (z\partial_z + \alpha + \lambda + \tfrac{1}{2}),
\end{aligned}
$$

$$
\begin{aligned}
(z(1-z^2)\partial_z - \alpha - \lambda + \tfrac{1}{2} - (\alpha-\lambda+\tfrac{1}{2})z^2) \; & z^2\mathcal{S}_{\alpha,\lambda} \\
= \quad z^2\mathcal{S}_{\alpha-1,\lambda-1} \; & (z(1-z^2)\partial_z - \alpha - \lambda + \tfrac{1}{2} - (\alpha-\lambda+\tfrac{1}{2})z^2).
\end{aligned}
$$

## Discrete symmetries

The operators below equal $\mathcal{S}_{\alpha,\lambda}(w,\partial_w)$ for an appropriate $w$:

$w = \pm z:$

$$\mathcal{S}_{\alpha,\pm\lambda},$$

$w = \pm z:$

$$(z^2-1)^{-\alpha}\ \mathcal{S}_{-\alpha,\pm\lambda}\ (z^2-1)^{\alpha},$$

$w = \frac{\pm z}{(z^2-1)^{\frac{1}{2}}}:$

$$(z^2-1)^{\frac{1}{2}(\alpha+\lambda+\frac{5}{2})}\ \ \mathcal{S}_{\lambda,\pm\alpha}\ \ (z^2-1)^{\frac{1}{2}(-\alpha-\lambda-\frac{1}{2})},$$

$w = \frac{\pm z}{(z^2-1)^{\frac{1}{2}}}:$

$$(z^2-1)^{\frac{1}{2}(\alpha-\lambda+\frac{5}{2})}\ \mathcal{S}_{-\lambda,\pm\alpha}\ (z^2-1)^{\frac{1}{2}(-\alpha+\lambda-\frac{1}{2})}.$$

Let us now discuss factorizations on the level of the Laplacian. In the Lie algebra $\mathrm{so}(5)$ with the coordinates $z_0, z_{-2}, z_2, z_{-3}, z_3$ we have 3 distinguished Lie subalgebras: two isomorphic to $\mathrm{so}(3)$ and one isomorphic to $\mathrm{so}(4)$. In an obvious notation,

$$\mathrm{so}_{02}(3), \quad \mathrm{so}_{03}(3), \quad \mathrm{so}_{23}(4).$$

The corresponding Casimir operators are

$$\begin{aligned}
\mathcal{C}_{02} &= 2B_{0,-2}B_{0,2} - \left(N_2 + \frac{1}{2}\right)^2 + \frac{1}{4} \\
&= 2B_{0,2}B_{0,-2} - \left(N_2 - \frac{1}{2}\right)^2 + \frac{1}{4}; \\
\mathcal{C}_{03} &= 2B_{0,-3}B_{0,3} - \left(N_3 + \frac{1}{2}\right)^2 + \frac{1}{4} \\
&= 2B_{0,3}B_{0,-3} - \left(N_3 - \frac{1}{2}\right)^2 + \frac{1}{4} \\
\mathcal{C}_{23} &= 4B_{2,3}B_{-2,-3} - (N_2 + N_3 + 1)^2 + 1 \\
&= 4B_{-2,-3}B_{2,3} - (N_2 + N_3 - 1)^2 + 1 \\
&= 4B_{2,-3}B_{-2,3} - (N_2 - N_3 + 1)^2 + 1 \\
&= 4B_{-2,3}B_{2,-3} - (N_2 - N_3 - 1)^2 + 1.
\end{aligned}$$

After the reduction, we obtain the identities

$$(z_0^2 + 2z_{-2}z_2)\Delta_5^{\diamond} = -\frac{1}{4} + \mathcal{C}_{02}^{\diamond,-\frac{1}{2}} + (N_3^{\diamond,-\frac{1}{2}})^2,$$

$$(z_0^2 + 2z_{-3}z_3)\Delta_5^{\diamond} = -\frac{1}{4} + \mathcal{C}_{03}^{\diamond,-\frac{1}{2}} + (N_2^{\diamond,-\frac{1}{2}})^2,$$

$$(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_5^{\diamond} = -\frac{3}{4} + \mathcal{C}_{23}^{\diamond,-\frac{1}{2}}.$$

We obtain

$$\begin{aligned}
(z_0^2+2z_{-2}z_2)\Delta_5^\diamond &= 2B_{0,-2}B_{0,2}-\Big(N_2+N_3+\frac{1}{2}\Big)\Big(N_2-N_3+\frac{1}{2}\Big)\\
&= 2B_{0,2}B_{0,-2}-\Big(N_2+N_3-\frac{1}{2}\Big)\Big(N_2-N_3-\frac{1}{2}\Big),\\
(z_0^2+2z_{-3}z_3)\Delta_5^\diamond &= 2B_{0,-3}B_{0,3}-\Big(N_2+N_3+\frac{1}{2}\Big)\Big(-N_2+N_3+\frac{1}{2}\Big)\\
&= 2B_{0,3}B_{0,-3}-\Big(N_2+N_3-\frac{1}{2}\Big)\Big(-N_2+N_3-\frac{1}{2}\Big),\\
(2z_{-2}z_2+2z_{-3}z_3)\Delta_5^\diamond &= 4B_{2,3}B_{-2,-3}-\Big(N_2+N_3+\frac{3}{2}\Big)\Big(N_2+N_3+\frac{1}{2}\Big)\\
&= 4B_{-2,-3}B_{2,3}-\Big(N_2+N_3-\frac{3}{2}\Big)\Big(N_2+N_3-\frac{1}{2}\Big)\\
&= 4B_{2,-3}B_{-2,3}-\Big(N_2-N_3+\frac{3}{2}\Big)\Big(N_2-N_3+\frac{1}{2}\Big)\\
&= 4B_{-2,3}B_{2,-3}-\Big(N_2-N_3-\frac{3}{2}\Big)\Big(N_2-N_3-\frac{1}{2}\Big),
\end{aligned}$$

where all the $B$ and $N$ operators need to have the superscript $\diamond,-\frac{1}{2}$.

If we use the spherical section, we need to make the replacements

$$\begin{aligned} z_0^2 + 2z_{-2}z_2 &\to 1 \\ z_0^2 + 2z_{-3}z_3 &\to w^2 - 1, \\ 2z_{-2}z_2 + 2z_{-3}z_3 &\to -w^2, \end{aligned}$$

and to replace the superscript $^\diamond$ with $^{\rm sph}$.

This leads to factorizations of the Gegenbauer operator:

$$
\begin{aligned}
\mathcal{S}_{\alpha,\lambda} &= \Big((1-z^2)\partial_z - 2(1+\alpha)z\Big)\partial_z \\
&\quad +\Big(\alpha+\lambda+\frac{1}{2}\Big)\Big(-\alpha+\lambda-\frac{1}{2}\Big) \\
&= \partial_z\Big((1-z^2)\partial_z - 2\alpha z\Big) \\
&\quad +\Big(\alpha+\lambda-\frac{1}{2}\Big)\Big(-\alpha+\lambda+\frac{1}{2}\Big), \\
(1-z^2)\mathcal{S}_{\alpha,\lambda} &= \Big((1-z^2)\partial_z + (\alpha-\lambda+\frac{3}{2})z\Big)\Big((1-z^2)\partial_z + (\alpha+\lambda+\frac{1}{2})z\Big) \\
&\quad -\Big(\alpha+\lambda+\frac{1}{2}\Big)\Big(\alpha-\lambda+\frac{3}{2}\Big) \\
&= \Big((1-z^2)\partial_z + (\alpha+\lambda+\frac{3}{2})z\Big)\Big((1-z^2)\partial_z + (\alpha-\lambda+\frac{1}{2})z\Big) \\
&\quad -\Big(\alpha-\lambda+\frac{1}{2}\Big)\Big(\alpha+\lambda+\frac{3}{2}\Big);
\end{aligned}
$$

$$
\begin{aligned}
z^2\mathcal{S}_{\alpha,\lambda} &= \Big(z(1-z^2)\partial_z - \alpha - \lambda - \frac{3}{2} + \big(-\alpha + \lambda - \frac{1}{2}\big)z^2\Big)\Big(z\partial_z + \alpha + \lambda + \frac{1}{2}\Big) \\
&\quad + \Big(\alpha + \lambda + \frac{1}{2}\Big)\Big(\alpha + \lambda + \frac{3}{2}\Big) \\
&= \Big(z\partial_z + \alpha + \lambda - \frac{3}{2}\Big)\Big(z(1-z^2)\partial_z - \alpha - \lambda + \frac{1}{2} + \big(-\alpha + \lambda - \frac{1}{2}\big)z^2\Big) \\
&\quad + \Big(\alpha + \lambda - \frac{1}{2}\Big)\Big(\alpha + \lambda - \frac{3}{2}\Big) \\
&= \Big(z(1-z^2)\partial_z - \alpha + \lambda - \frac{3}{2} + \big(-\alpha - \lambda - \frac{1}{2}\big)z^2\Big)\Big(z\partial_z + \alpha - \lambda + \frac{1}{2}\Big) \\
&\quad + \Big(\alpha - \lambda + \frac{1}{2}\Big)\Big(\alpha - \lambda + \frac{3}{2}\Big) \\
&= \Big(z\partial_z + \alpha - \lambda - \frac{3}{2}\Big)\Big(z(1-z^2)\partial_z - \alpha + \lambda + \frac{1}{2} + \big(-\alpha - \lambda - \frac{1}{2}\big)z^2\Big) \\
&\quad + \Big(\alpha - \lambda - \frac{1}{2}\Big)\Big(\alpha - \lambda - \frac{3}{2}\Big).
\end{aligned}
$$

As usual, by standard solutions we mean solutions with a simple behavior around singular points. The singular points of the Gegenbauer equation are at $\{1, -1, \infty\}$. The discussion of the point $-1$ can be easily reduced to that of $1$. Therefore, it is enough to discuss $2 \times 2$ solutions corresponding to two indices at $1$ and $\infty$.

All of the standard solutions can be expressed in terms of the function

$$\begin{aligned} S_{\alpha,\lambda}(z) = S(a, b; z) &:= F\Big(a, b; \frac{a+b+1}{2}; \frac{1-z}{2}\Big) \\ &= F\Big(\frac{a}{2}, \frac{b}{2}; \frac{a+b+1}{2}; 1-z^2\Big). \end{aligned}$$

On the next slide we give the 4 standard solutions. We consistently use the Lie-algebraic parameters.

$$\begin{aligned}
&\text{Solution} \sim 1 \text{ at } 1: && S_{\alpha,\lambda}(z): && = F_{\alpha,\alpha,2\lambda}\Big(\frac{1-z}{2}\Big)\\
& && && = F_{\alpha,-\frac12,\lambda}(1-z^2),\\
&\text{Solution} \sim \frac{1}{2^\alpha(1-z)^\alpha} \text{ at } 1: && \frac{1}{2^\alpha(1-z^2)^\alpha}S_{-\alpha,-\lambda}(z) && = 2^{-\alpha}(1-z)^{-\alpha}F_{-\alpha,\alpha,-2\lambda}\Big(\frac{1-z}{2}\Big)\\
& && && = (1-z^2)^{-\alpha}F_{-\alpha,-\frac12,-\lambda}(1-z^2)\\
&\text{Solution} \sim z^{-a} \text{ at } \infty: && (z^2-1)^{\frac{-1-2\alpha+2\lambda}{4}}S_{-\lambda,-\alpha}\Big(\frac{z}{\sqrt{z^2-1}}\Big) && = (1+z)^{-\frac12-\alpha+\lambda}F_{-2\lambda,\alpha,-\alpha}\Big(\frac{2}{1+z}\Big)\\
& && && = z^{-\frac12-\alpha+\lambda}F_{-\lambda,\alpha,\frac12}(z^{-2}),\\
&\text{Solution} \sim z^{-b} \text{ at } \infty: && (z^2-1)^{\frac{-1-2\alpha-2\lambda}{4}}S_{\lambda,\alpha}\Big(\frac{z}{\sqrt{z^2-1}}\Big) && = (1+z)^{-\frac12-\alpha-\lambda}F_{2\lambda,\alpha,\alpha}\Big(\frac{2}{1+z}\Big)\\
& && && = z^{-\frac12-\alpha-\lambda}F_{\lambda,\alpha,\frac12}(z^{-2}).
\end{aligned}$$

Sometimes it is convenient to use the following normalization:

$$\begin{aligned}\mathbf{S}_{\alpha,\lambda}(z) &:= \frac{1}{\Gamma(\alpha+1)} S_{\alpha,\lambda}(z) \\ &= \frac{1}{\Gamma(\frac{a+b+1}{2})} F\Big(a, b; \frac{a+b+1}{2}; \frac{1-z}{2}\Big) \\ &= \mathbf{F}_{\alpha,\alpha,2\lambda}\Big(\frac{1-z}{2}\Big).\end{aligned}$$

It will be used in the recurrence relations, which we give on the next slide.

$$
\begin{aligned}
\partial_z \mathbf{S}_{\alpha,\lambda}(z) &= -\frac{1}{2}\Big(\frac{1}{2}+\alpha-\lambda\Big)\Big(\frac{1}{2}+\alpha+\lambda\Big)\mathbf{S}_{\alpha+1,\lambda}(z),\\
\big((1-z^2)\partial_z - 2\alpha z\big)\,\mathbf{S}_{\alpha,\lambda}(z) &= -2\mathbf{S}_{\alpha-1,\lambda}(z),
\end{aligned}
$$

$$
\begin{aligned}
\Bigg((1-z^2)\partial_z - \Big(\frac{1}{2}+\alpha+\lambda\Big)z\Bigg)\,\mathbf{S}_{\alpha,\lambda}(z) &= -\Big(\frac{1}{2}+\alpha+\lambda\Big)\mathbf{S}_{\alpha,\lambda+1}(z),\\
\Bigg((1-z^2)\partial_z - \Big(\frac{1}{2}+\alpha-\lambda\Big)z\Bigg)\,\mathbf{S}_{\alpha,\lambda}(z) &= -\Big(\frac{1}{2}+\alpha-\lambda\Big)\mathbf{S}_{\alpha,\lambda-1}(z),
\end{aligned}
$$

$$
\begin{aligned}
\Bigg(z\partial_z + \frac{1}{2}+\alpha-\lambda\Bigg)\,\mathbf{S}_{\alpha,\lambda}(z) &= \frac{1}{2}\Big(\frac{1}{2}+\alpha-\lambda\Big)\Big(\frac{3}{2}+\alpha-\lambda\Big)\mathbf{S}_{\alpha+1,\lambda-1}(z),\\
\Bigg(z(1-z^2)\partial_z+\Big(\frac{1}{2}-\alpha+\lambda\Big)(1-z^2)-2\alpha z^2\Bigg)\,\mathbf{S}_{\alpha,\lambda}(z) &= -2\mathbf{S}_{\alpha-1,\lambda+1}(z),
\end{aligned}
$$

$$
\begin{aligned}
\Bigg(z\partial_z + \frac{1}{2}+\alpha+\lambda\Bigg)\,\mathbf{S}_{\alpha,\lambda}(z) &= \frac{1}{2}\Big(\frac{1}{2}+\alpha+\lambda\Big)\Big(\frac{3}{2}+\alpha+\lambda\Big)\mathbf{S}_{\alpha+1,\lambda+1}(z),\\
\Bigg(z(1-z^2)\partial_z+\Big(\frac{1}{2}-\alpha-\lambda\Big)(1-z^2)-2\alpha z^2\Bigg)\,\mathbf{S}_{\alpha,\lambda}(z) &= -2\mathbf{S}_{\alpha-1,\lambda-1}(z).
\end{aligned}
$$

The function

$$G_{\alpha,\mu}(z_0, z_{-2}, z_2, z_{-3}, z_3)$$
$$:= \int z_2^{\alpha}\left(\sqrt{2}z_0 - \tau^{-1}z_{-3} + \tau z_3\right)^{-\alpha-\frac{1}{2}}\tau^{-\mu-1}\mathrm{d}\tau$$

satisfies

$$\begin{aligned} \Delta_5 G_{\alpha,\mu} &= 0, \\ N_2 G_{\alpha,\mu} &= \alpha G_{\alpha,\mu}, \\ N_3 G_{\alpha,\mu} &= \mu G_{\alpha,\mu}, \end{aligned}$$

and is homogeneous of degree $-\frac{1}{2}$. Therefore, its conformal reduction satisfies the Gegenbauer equation.

Let us express it in the coordinates $w, r, p, u_2, u_3$

$$G_{\alpha,\mu}(w, r, p, u_2, u_3)$$
$$=(\sqrt{2})^{\alpha+\frac{1}{2}} u_2^{\alpha} u_3^{\mu} r^{-\frac{1}{2}} \int \left(2w\sigma + (1+\sigma^2)\frac{p}{r}\right)^{-\alpha-\frac{1}{2}} \sigma^{\alpha-\mu-\frac{1}{2}} \mathrm{d}\sigma.$$

We have two kinds of interal representations of the Gegenbauer equation, described in the next slide. a) is the approach to integral representations inherited from the hypergeometric equation. b) is the approach suggested by the above wave packet in 5 dimensions.

a) Let $[0,1] \ni t \mapsto \gamma(t)$ satisfy

$$(t^2-1)^{\frac{b-a+1}{2}}(t-z)^{-b-1}\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$\mathcal{S}(a,b;z,\partial_z)\int_\gamma (t^2-1)^{\frac{b-a-1}{2}}(t-z)^{-b}\mathrm{d}t = 0.$$

b) Let $[0,1] \ni t \mapsto \gamma(t)$ satisfy

$$(t^2+2tz+1)^{\frac{-b-a}{2}+1}t^{b-2}\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$\mathcal{S}(a,b;z,\partial_z)\int_\gamma (t^2+2tz+1)^{\frac{-b-a}{2}}t^{b-1}\mathrm{d}t = 0.$$

# The Schrödinger Lie algebra

# and Symmetries of the Heat Equation

We consider again the space $\mathbb{R}^{n+2}$ with the split scalar product given by

$$\langle z|z\rangle = \sum_i z_{-i} z_i, \quad z \in \mathbb{R}^{n+2},$$

and the Laplacian

$$\Delta_{n+2} = \sum_i \partial_{z_{-i}} \partial_{z_i}.$$

A special role will be played by the operator

$$B_{m+1,m} = z_{-m-1}\partial_{z_m} - z_{-m}\partial_{z_{m+1}} \in \mathrm{so}(n+2).$$

We define the Schrödinger Lie algebra

$$\mathrm{sch}(n-2) := \{B \in \mathrm{so}(n+2) \ : \ [B, B_{m+1,m}] = 0\}.$$

We also have the Schrödinger group

$$\mathrm{Sch}(n-2) := \{\alpha \in \mathrm{O}(n+2) \ : \ \alpha B_{m+1,m} = B_{m+1,m}\alpha\}.$$

Recall that

$$N_m = -z_{-m}\partial_{z_{-m}} + z_m\partial_{z_m},$$
$$N_{m+1} = -z_{-m-1}\partial_{z_{-m-1}} + z_{m+1}\partial_{z_{m+1}}.$$

Define

$$M := -N_m + N_{m+1}.$$

Note that $M$ belongs to $\mathrm{sch}(n-2)$ and commutes with $\mathrm{so}(n-2)$, which is naturally embedded in $\mathrm{sch}(n-2)$.

$\mathrm{sch}(n-2)$ is spanned by the following operators:

(1) $B_{m+1,m}$, which spans the center of $\mathrm{sch}(n-2)$.

(2) $B_{m,j}$, $B_{m+1,j}$, $j = 1, \ldots, m-1$, which have the following nonzero commutator:

$$[B_{m,j}, B_{m+1,-j}] = B_{m+1,m}.$$

(3) $B_{m+1,-m}$, $B_{-m-1,m}$, $M$, which have the usual commutation relations of $\mathrm{sl}(2) \simeq \mathrm{so}(3)$:

$$\begin{aligned} [B_{m+1,-m}, B_{-m-1,m}] &= M, \\ [M, B_{m+1,-m}] &= -B_{m+1,m}, \\ [M, B_{-m-1,m}] &= B_{-m-1,m}. \end{aligned}$$

(4) $B_{i,j}$, $|i| < |j| \leq m-1$, $N_i$, $i = 1, \ldots, m-1$, with the usual commutation relations of $\mathrm{so}(n-2)$.

The span of (2) can be identified with $\mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2} \simeq \mathbb{R}^2 \otimes \mathbb{R}^{n-2}$, which has a natural structure of a symplectic space. The span of (1) and (2) is the central extension of the abelian algebra $\mathbb{R}^2 \otimes \mathbb{R}^{n-2}$. Such a Lie algebra is usually called the Heisenberg Lie algebra over $\mathbb{R}^2 \otimes \mathbb{R}^{n-2}$ and can be denoted by

$$\mathrm{heis}(2(n-2)) = \mathbb{R} \rtimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2}).$$

$\mathrm{sl}(2)$ acts in the obvious way on $\mathbb{R}^2$ and $\mathrm{so}(n-2)$ acts on $\mathbb{R}^{n-2}$. Thus $\mathrm{sl}(2) \oplus \mathrm{so}(n-2)$ acts on $\mathbb{R}^2 \otimes \mathbb{R}^{n-2}$. Thus

$$\mathrm{sch}(n-2) \simeq \mathbb{R} \rtimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2}) \rtimes (\mathrm{sl}(2) \oplus \mathrm{so}(n-2))\,.$$

Note, in particular, that neither $\mathrm{sch}(n-2)$ nor $\mathrm{SSch}(n-2)$ are semisimple.

The subalgebra spanned by
the usual Cartan algebra of $\mathrm{so}(n-2)$, $M$ and $B_{-m-1,m}$
is a maximal commutative subalgebra of $\mathrm{sch}(n-2)$. It will be called the "Cartan algebra" of $\mathrm{sch}(n-2)$.

Let us introduce $\kappa \in \mathrm{SO}(n+2)$:

$$\kappa(\ldots, z_{-m}, z_m, z_{-m-1}, z_{m+1}) :$$
$$=(\ldots, z_{-m-1}, z_{m+1}, -z_{-m}, -z_m).$$

Note that $\kappa^4 = \iota$ and $\kappa \in \mathrm{SSch}(n-2)$. On the level of functions

$$\kappa K(\ldots, z_{-m}, z_m, z_{-m-1}, z_{m+1}) :$$
$$=K(\ldots, -z_{-m-1}, -z_{m+1}, z_{-m}, z_m).$$

The subgroup of $\mathrm{Sch}(n-2)$ generated by $W(n-2) \subset \mathrm{O}(n-2)$ and $\kappa$ will be called the group of Weyl symmetries of $\mathrm{sch}(n-2)$.

Recall that we have the representations

$$\mathrm{so}(n+2) \ni B \;\mapsto\; B^{\mathrm{fl},\eta},$$
$$\mathrm{O}(n+2) \ni \alpha \;\mapsto\; \alpha^{\mathrm{fl},\eta},$$

and the generalized symmetry

$$B^{\mathrm{fl},\frac{-2-n}{2}}\Delta_n \;=\; \Delta_n B^{\mathrm{fl},\frac{2-n}{2}}, \quad B \in \mathrm{so}(n+2),$$
$$\alpha^{\mathrm{fl},\frac{-2-n}{2}}\Delta_n \;=\; \Delta_n \alpha^{\mathrm{fl},\frac{2-n}{2}}, \quad \alpha \in \mathrm{O}(n+2).$$

We consider now the space $\mathbb{R}^{n-2} \oplus \mathbb{R}$ with the generic variables $(y, t) = (\ldots, y_{m-1}, t)$. Note that $t$ should be understood as a new name for $y_{-m}$, and we keep the old names for the first $n-2$ coordinates.

We define the map $\theta : C^\infty(\mathbb{R}^{n-2} \oplus \mathbb{R}) \to C^\infty(\mathbb{R}^n)$ by setting for $h$

$$(\theta h)(\ldots, y_{m-1}, y_{-m}, y_m) := h(\ldots, y_{m-1}, y_{-m})\mathrm{e}^{y_m}.$$

We also define $\zeta : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-2} \oplus \mathbb{R})$, which to $f$ associates

$$(\zeta f)(\ldots, y_{m-1}, t) := f(\ldots, y_{m-1}, t, 0).$$

Clearly, $\zeta$ is a left inverse of $\theta$:

$$\zeta \circ \theta = \iota.$$

Therefore, $\theta \circ \zeta = \iota$ is true on the range of $\theta$.

The heat operator in $n-2$ spatial dimensions can be obtained from the Laplacian in $n$ dimension:

$$\mathcal{L}_{n-2} := \Delta_{n-2} + 2\partial_t = \zeta \Delta_n \theta.$$

For $B \in \mathrm{sch}(n-2) \subset \mathrm{so}(n+2)$ we define

$$B^{\mathrm{sch},\eta} := \zeta B^{\mathrm{fl},\eta} \theta,$$

and for $\alpha \in \mathrm{Sch}(n-2) \subset \mathrm{O}(n+2)$,

$$\alpha^{\mathrm{sch},\eta} := \zeta \alpha^{\mathrm{fl},\eta} \theta.$$

It is easy to see that $\mathrm{sch}(n-2)$, $\mathrm{Sch}(n-2)$ and $\Delta_n$ preserve the range of $\theta$.

For any $\eta$, we have representations

$$\begin{aligned} \mathrm{sch}(n-2) \ni B &\mapsto B^{\mathrm{sch},\eta}, \\ \mathrm{Sch}(n-2) \ni \alpha &\mapsto \alpha^{\mathrm{sch},\eta} \end{aligned}$$

is a representations.

We also have a generalized symmetry

$$\begin{aligned} B^{\mathrm{sch},\frac{-2-n}{2}}\mathcal{L}_{n-2} &= \mathcal{L}_{n-2}B^{\mathrm{sch},\frac{2-n}{2}}, \qquad B \in \mathrm{sch}(n-2), \\ \alpha^{\mathrm{sch},\frac{-2-n}{2}}\mathcal{L}_{n-2} &= \mathcal{L}_{n-2}\alpha^{\mathrm{sch},\frac{2-n}{2}}, \qquad \alpha \in \mathrm{Sch}(n-2). \end{aligned}$$

Let us sum up information about Schrödinger symmetries on 3 levels:

$$z \in \mathbb{R}^{n+2}, \ \langle z|z\rangle_{n+2} = \sum_j z_{-j} z_j,$$

$$y \in \mathbb{R}^{n}, \quad \langle y|y\rangle_{n} = \sum_j y_{-j} y_j,$$

$$(y,t) \in \mathbb{R}^{n-2} \oplus \mathbb{R}, \ \langle y|y\rangle_{n-2} = \sum_j y_{-j} y_j.$$

**Cartan algebra of** $\mathrm{sch}(n-2)$**.** Central element:

$$\begin{aligned} B_{m+1,m} &= z_{-m-1}\partial_{z_m} - z_{-m}\partial_{z_{m+1}}, \\ B^{\mathrm{fl}}_{m+1,m} &= \partial_{y_m}, \\ B^{\mathrm{sch}}_{m+1,m} &= 1. \end{aligned}$$

Cartan algebra of $\mathrm{so}(n-2)$, $j = 1, \ldots, m-1$:

$$\begin{aligned} N_j &= -z_{-j}\partial_{z_{-j}} + z_j\partial_{z_j}, \\ N^{\mathrm{fl}}_j &= -y_{-j}\partial_{y_{-j}} + y_j\partial_{y_j}, \\ N^{\mathrm{sch}}_j &= -y_{-j}\partial_{y_{-j}} + y_j\partial_{y_j}. \end{aligned}$$

Generator of scaling:

$$\begin{aligned} M &= z_{-m}\partial_{z_{-m}} - z_m\partial_{z_m} - z_{-m-1}\partial_{z_{-m-1}} + z_{m+1}\partial_{z_{m+1}}, \\ M^{\mathrm{fl},\eta} &= \sum_j y_j\partial_{y_j} + 2y_{-m}\partial_{y_{-m}} - \eta, \\ M^{\mathrm{sch},\eta} &= \sum_j y_j\partial_{y_j} + 2t\partial_t - \eta. \end{aligned}$$

**Root operators of $\mathrm{sch}(n-2)$.** Roots of $\mathrm{so}(n-2)$, $|i| < |j|$:

$$\begin{aligned} B_{i,j} &= z_{-i}\partial_{z_j} - z_{-j}\partial_{z_i}, \\ B^{\mathrm{fl}}_{i,j} &= y_{-i}\partial_{y_j} - y_{-j}\partial_{y_i}, \\ B^{\mathrm{sch}}_{i,j} &= y_{-i}\partial_{y_j} - y_{-j}\partial_{y_i}. \end{aligned}$$

Space translations, $|j| \le n-2$:

$$\begin{aligned} B_{m+1,j} &= z_{-m-1}\partial_{z_j} - z_{-j}\partial_{z_{m+1}}, \\ B^{\mathrm{fl}}_{m+1,j} &= \partial_{y_j}, \\ B^{\mathrm{sch}}_{m+1,j} &= \partial_{y_j}. \end{aligned}$$

Time translation:

$$\begin{aligned} B_{m+1,-m} &= z_{-m-1}\partial_{z_{-m}} - z_m\partial_{z_{m+1}}, \\ B^{\mathrm{fl}}_{m+1,-m} &= \partial_{y_{-m}}, \\ B^{\mathrm{sch}}_{m+1,-m} &= \partial_t. \end{aligned}$$

Additional roots, $j \in I_{n-2}$:

$$\begin{aligned} B_{m,j} &= z_{-m}\partial_{z_j} - z_{-j}\partial_{z_m}, \\ B^{\text{fl}}_{m,j} &= y_{-m}\partial_{y_j} - y_{-j}\partial_{y_m}, \\ B^{\text{sch}}_{m,j} &= t\partial_{y_j} - y_{-j}. \end{aligned}$$

$$\begin{aligned} B_{-m-1,m} &= z_{m+1}\partial_{z_m} - z_{-m}\partial_{z_{-m-1}}, \\ B^{\text{fl},\eta}_{-m-1,m} &= y_{-m}\Big(\sum_j y_j\partial_{y_j} + y_{-m}\partial_{y_{-m}} - \eta\Big) - \sum_j \frac{y_{-j}y_j}{2}\partial_{y_m}, \\ B^{\text{sch},\eta}_{-m-1,m} &= t\Big(\sum_j y_j\partial_{y_j} + t\partial_t - \eta\Big) - \sum_j \frac{y_{-j}y_j}{2} \end{aligned}$$

**Weyl symmetries.** Reflection:

$$\tau_0 K(z_0, \dots) = K(-z_0, \dots),$$
$$\tau_0^{\mathrm{fl}} f(y_0, \dots) = f(-y_0, \dots),$$
$$\tau_0^{\mathrm{sch}} h(y_0, \dots) = h(-y_0, \dots).$$

Flips, $j = 1, \dots, m-1$:

$$\tau_j K(\dots, z_{-j}, z_j, \dots) = K(\dots, z_j, z_{-j}, \dots),$$
$$\tau_j^{\mathrm{fl}} f(\dots, y_{-j}, y_j, \dots) = f(\dots, y_j, y_{-j}, \dots),$$
$$\tau_j^{\mathrm{sch}} h(\dots, y_{-j}, y_j, \dots) = h(\dots, y_j, y_{-j}, \dots).$$

Permutations, $\pi \in S_{m-1}$:

$$\sigma_\pi K(\dots, z_{-m+1}, z_{m-1}, \dots) = K(\dots, z_{-\pi_{m-1}}, z_{\pi_{m-1}}, \dots),$$
$$\sigma_\pi^{\mathrm{fl}} f(\dots, y_{-m+1}, y_{m-1}, \dots) = f(\dots, y_{-\pi_{m-1}}, y_{\pi_{m-1}}, \dots),$$
$$\sigma_\pi^{\mathrm{sch}} h(\dots, y_{-m+1}, y_{m-1}, t) = h(\dots, y_{-\pi_{m-1}}, y_{\pi_{m-1}}, t).$$

Special transformation $\kappa$:

$$\begin{aligned}
\kappa K(\ldots, z_{-m}, z_m, z_{-m-1}, z_{m+1}) \\
&= K(\ldots, -z_{-m-1}, -z_{m+1}, z_{-m}, z_m), \\
\kappa^{\mathrm{fl},\eta} f(\ldots, y_{m-1}, y_{-m}, y_m) \\
&= y_{-m}^{\eta} f\Big(\ldots, \frac{y_{m-1}}{y_{-m}}, -\frac{1}{y_{-m}}, \frac{1}{2y_{-m}} \textstyle\sum_j y_{-j} y_j\Big), \\
\kappa^{\mathrm{sch},\eta} h(\ldots, y_{m-1}, t) \\
&= t^{\eta} \exp\Big(\frac{1}{2t} \textstyle\sum_j y_{-j} y_j\Big) h(\ldots, \frac{y_{m-1}}{t}, -\frac{1}{t}).
\end{aligned}$$

## Laplacian/Laplacian/Heat operator

$$\Delta_{n+2} = \sum_{|j| \le n+2} \partial_{z_{-j}} \partial_{z_j},$$

$$\Delta_n = \sum_{|j| \le n} \partial_{y_{-j}} \partial_{y_j},$$

$$\mathcal{L}_{n-2} = \sum_{|j| \le n-2} \partial_{y_{-j}} \partial_{y_j} + 2\partial_t.$$

It is easy to see that

$$
\begin{aligned}
&K(\dots, z_{-m}, z_m, z_{-m-1}, z_{m+1}) \\
:=& z_1^{\alpha_1} \cdots z_{m-1}^{\alpha_{m-1}} \exp\Big( - \frac{z_{m+1}}{z_{-m}} \Big),
\end{aligned}
$$

solves

$$
\begin{aligned}
\Delta_{n+2} K &= 0, \\
B_{m+1,m} K &= K, \\
N_i K &= \alpha_i K, \quad i = 1, \dots, m-1.
\end{aligned}
$$

We descend to $\mathbb{R}^{n-2} \oplus \mathbb{R}$, and we obtain that

$$f(\ldots, y_{m-1}, t)$$
$$:= t^{1-\frac{n}{2}-\alpha_1-\cdots-\alpha_{m-1}} y_1^{\alpha_1} \cdots y_{m-1}^{\alpha_{m-1}} \exp\left(\frac{y_{-1}y_1 + \cdots + y_{-m}y_m}{t}\right)$$

solves

$$\begin{aligned} \mathcal{L}_{n-2} f &= 0, \\ B^{\mathrm{sch}}_{m+1,m} f &= f, \\ N_i^{\mathrm{sch}} f &= \alpha_i f, \quad i = 1, \ldots, m-1. \end{aligned}$$

Suppose that $]0,1[\ni s \mapsto \tau(s)$ is a contour satisfying

$$f(\tau y, \tau^2 t)\tau^{-\nu}\Big|_{\tau(0)}^{\tau(1)} = 0.$$

Set

$$F_\nu(y,t) := \int f(\tau y, \tau^2 t)\tau^{-1-\nu}\mathrm{d}\tau.$$

Then

$$\Big(\sum_i y\partial_y + 2t\partial_t\Big)F_\nu = \nu F_\nu.$$

Recall that

$$M^{\mathrm{sch},\eta} = \sum_i y\partial_y + 2t\partial_t - \eta.$$

# Heat equation in 2 dimensions

# and the confluent equation

We again consider $\mathbb{R}^6$ with the split coordinates and the product

$$\langle z|z\rangle = 2z_{-1}z_1 + 2z_{-2}z_2 + 2z_{-3}z_3.$$

We describe various object related to the Lie algebra $\mathrm{sch}(2)$ treated as a subalgebra of $\mathrm{so}(6)$.

**Lie algebra $\mathrm{sch}(2)$.** Cartan algebra is spanned by

$$\begin{aligned} M &= z_{-2}\partial_{z_{-2}} - z_2\partial_{z_2} - z_{-3}\partial_{z_{-3}} + z_3\partial_{z_3}, \\ N_1 &= -z_{-1}\partial_{z_{-1}} + z_1\partial_{z_1}, \\ B_{3,2} &= z_{-3}\partial_{z_2} - z_{-2}\partial_{z_3}. \end{aligned}$$

Root operators:

$$\begin{aligned} B_{3,-1} &= z_{-3}\partial_{z_{-1}} - z_1\partial_{z_3}, \\ B_{2,1} &= z_{-2}\partial_{z_1} - z_{-1}\partial_{z_2}, \\ B_{3,1} &= z_{-3}\partial_{z_1} - z_{-1}\partial_{z_3}, \\ B_{2,-1} &= z_{-2}\partial_{z_{-1}} - z_1\partial_{z_2}, \\ B_{3,-2} &= z_{-3}\partial_{z_{-2}} - z_2\partial_{z_3}, \\ B_{-3,2} &= z_3\partial_{z_2} - z_{-2}\partial_{z_{-3}}. \end{aligned}$$

**Weyl symmetries.**

$$\begin{aligned}
\iota K(z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3) &= K(z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3), \\
\tau_1 K(z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3) &= K(z_1, z_{-1}, z_{-2}, z_2, z_{-3}, z_3), \\
\kappa K(z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3) &= K(z_{-1}, z_1, -z_{-3}, -z_3, z_{-2}, z_2), \\
\tau_1 \kappa K(z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3) &= K(z_1, z_{-1}, -z_{-3}, -z_3, z_{-2}, z_2).
\end{aligned}$$

**Laplacian.**

$$\Delta_6 = 2\partial_{z_{-1}}\partial_{z_1} + 2\partial_{z_{-2}}\partial_{z_2} + 2\partial_{z_{-3}}\partial_{z_3}.$$

We descend on the level of $\mathbb{R}^4$, with the coordinates $(y_{-1}, y_1, y_{-2}, y_2)$ and the scalar product given by

$$\langle y|y\rangle = 2y_{-1}y_1 + 2y_{-2}y_2.$$

**Lie algebra** $\mathrm{sch}(2)$**.** Cartan algebra:

$$\begin{aligned} M^{\mathrm{fl},\eta} &= y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + 2y_{-2}\partial_{y_{-2}} - \eta, \\ N_1^{\mathrm{fl}} &= -y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1}, \\ B_{3,2}^{\mathrm{fl}} &= \partial_{y_2}. \end{aligned}$$

Root operators:

$$
\begin{aligned}
B^{\text{fl}}_{3,-1} &= \partial_{y_{-1}}, \\
B^{\text{fl}}_{2,1} &= y_{-2}\partial_{y_1} - y_1\partial_{y_2}, \\
B^{\text{fl}}_{3,1} &= \partial_{y_1}, \\
B^{\text{fl}}_{2,-1} &= y_{-2}\partial_{y_{-1}} - y_1\partial_{y_2}, \\
B^{\text{fl}}_{3,-2} &= \partial_{y_{-2}}, \\
B^{\text{fl},\eta}_{-3,2} &= y_{-2}(y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + y_{-2}\partial_{y_{-2}} - \eta) - y_{-1}y_1\partial_{y_2}.
\end{aligned}
$$

**Weyl symmetries.**

$$\begin{aligned}
\iota f(y_{-1},y_1,y_{-2},y_2) &= f(y_{-1},y_1,y_{-2},y_2),\\
\tau_1^{\mathrm{fl}} f(y_{-1},y_1,y_{-2},y_2) &= f(y_1,y_{-1},y_{-2},y_2),\\
\kappa^{\mathrm{fl},\eta} f(y_{-1},y_1,y_{-2},y_2) &= y_{-2}^{\eta} f\left(\frac{y_{-1}}{y_{-2}},\frac{y_1}{y_{-2}},-\frac{1}{y_{-2}},\frac{y_{-1}y_1+y_{-2}y_2}{y_{-2}}\right),\\
\tau_1\kappa^{\mathrm{fl},\eta} f(y_{-1},y_1,y_{-2},y_2) &= y_{-2}^{\eta} f\left(\frac{y_1}{y_{-2}},\frac{y_{-1}}{y_{-2}},-\frac{1}{y_{-2}},\frac{y_{-1}y_1+y_{-2}y_2}{y_{-2}}\right).
\end{aligned}$$

We apply the ansatz involving the exponential $\mathrm{e}^{y_2}$. We rename $y_{-2}$ to $t$.

**Lie algebra** $\mathrm{sch}(2)$**.** Cartan algebra:

$$\begin{aligned} M^{\mathrm{sch},\eta} &= y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + 2t\partial_t - \eta, \\ N_1^{\mathrm{sch}} &= -y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1}, \\ B_{32}^{\mathrm{sch}} &= 1. \end{aligned}$$

Root operators:

$$\begin{aligned}
B^{\mathrm{sch}}_{3,-1} &= \partial_{y_{-1}}, \\
B^{\mathrm{sch}}_{2,1} &= t\partial_{y_1} - y_{-1}, \\
B^{\mathrm{sch}}_{3,1} &= \partial_{y_1}, \\
B^{\mathrm{sch}}_{2,-1} &= t\partial_{y_{-1}} - y_1, \\
B^{\mathrm{sch}}_{3,-2} &= \partial_t, \\
B^{\mathrm{sch},\eta}_{-3,2} &= t(y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + t\partial_t - \eta) - y_{-1}y_1.
\end{aligned}$$

**Weyl symmetries.**

$$
\begin{aligned}
\iota g(y_{-1}, y_1, t) &= g(y_{-1}, y_1, t),\\
\tau_1^{\mathrm{sch}} h(y_{-1}, y_1, t) &= h(y_1, y_{-1}, t),\\
\kappa^{\mathrm{sch},\eta} h(y_{-1}, y_1, t) &= t^\eta \exp\left(\frac{y_{-1}y_1}{t}\right) h\left(\frac{y_{-1}}{t}, \frac{y_1}{t}, -\frac{1}{t}\right),\\
\tau_1 \kappa^{\mathrm{sch},\eta} h(y_{-1}, y_1, t) &= t^\eta \exp\left(\frac{y_{-1}y_1}{t}\right) h\left(\frac{y_1}{t}, \frac{y_{-1}}{t}, -\frac{1}{t}\right).
\end{aligned}
$$

**Heat operator:**

$$\mathcal{L}_2 = 2\partial_{y_{-1}}\partial_{y_1} + 2\partial_t.$$

It satisfies the following generalized symmetries:

$$B^{\mathrm{sch},-3}\mathcal{L}_2 = \mathcal{L}_2 B^{\mathrm{sch},-1}, \quad B \in sch(2),$$
$$\alpha^{\mathrm{sch},-3}\mathcal{L}_2 = \mathcal{L}_2 \alpha^{\mathrm{sch},-1}, \quad \alpha \in Sch(2).$$

We introduce new coordinates $u, w, s$

$$w = \frac{y_{-1}y_1}{t}\,, \quad u = \frac{y_1}{\sqrt{t}}\,, \quad s = \sqrt{t}\,.$$

**Lie algebra** $\mathrm{sch}(2)$**.** Cartan algebra:

$$M^{\mathrm{sch},\eta} = s\,\partial_s - \eta,$$
$$N_1^{\mathrm{sch}} = u\,\partial_{u,}$$
$$B_{32}^{\mathrm{sch}} = 1.$$

Root operators:

$$
\begin{aligned}
B^{\rm sch}_{3,-1} &= \frac{u}{s}\partial_w,\\
B^{\rm sch}_{2,1} &= \frac{s}{u}(w\partial_w + u\partial_u - w),\\
B^{\rm sch}_{3,1} &= \frac{1}{us}(w\partial_w + u\partial_u),\\
B^{\rm sch}_{2,-1} &= su(\partial_w - 1),\\
B^{\rm sch}_{3,-2} &= \frac{1}{s^2}\big( - w\partial_w - \frac{1}{2}u\partial_u + \frac{1}{2}s\partial_s\big),\\
B^{{\rm sch},\eta}_{-3,2} &= s^2\big(w\partial_w + \frac{1}{2}u\partial_u + \frac{1}{2}s\partial_s - w - \eta\big).
\end{aligned}
$$

**Weyl symmetries.**

$$
\begin{aligned}
\iota h(w,u,s) &= h(w,u,s),\\
\tau_1^{\mathrm{sch}} h\Big(w,u,s\Big) &= h\Big(w,\frac{w}{u},s\Big),\\
\kappa^{\mathrm{sch},\eta} h(w,u,s) &= s^{2\eta}\mathrm{e}^{w} h\Big(-w,-\mathrm{i}u,\frac{\mathrm{i}}{s}\Big),\\
\tau_1\kappa^{\mathrm{sch},\eta} h(w,u,s) &= s^{2\eta}\mathrm{e}^{w} h\Big(-w,-\frac{\mathrm{i}w}{u},\frac{\mathrm{i}}{s}\Big).
\end{aligned}
$$

**Heat operator:**

$$
\mathcal{L}_2 = \frac{2}{s^2}\Big(w\partial_w^2 + (u\partial_u + 1 - w)\partial_w + \frac{1}{2}(-u\partial_u + s\partial_s)\Big).
$$

Let us make an ansatz

$$h(w,u,s) = u^{\alpha} s^{-\theta-1} F(w).$$

Clearly,

$$M^{\mathrm{sch},-1} h = -\theta h, \quad N_1^{\mathrm{sch}} h = \alpha h.$$

Therefore on functions of this form

$$\frac{s^2}{2} \mathcal{L}_2 u^{\alpha} s^{-\theta-1} F(w) = u^{\alpha} s^{-\theta-1} \mathcal{F}_{\theta,\alpha} F(w),$$

where $\mathcal{F}_{\theta,\alpha}$ is the confluent operator

$$\mathcal{F}_{\theta,\alpha}(w,\partial_w) \;=\; w\partial_w^2 + (1+\alpha-w)\partial_w - \frac{1}{2}(1+\theta+\alpha).$$

We have a closely related operator

$$\tilde{\mathcal{F}}_{\theta,\alpha}(z,\partial_z) = z^2\partial_z^2 + (-1+(2+\theta)z)\partial_z + \frac{1}{4}(1+\theta)^2 - \frac{1}{4}\alpha^2.$$

If $z = -w^{-1}$, then

$$(-z)^{\frac{3+\alpha+\theta}{2}}\tilde{\mathcal{F}}_{\theta,\alpha}(z,\partial_z)(-z)^{-\frac{1+\alpha+\theta}{2}} = \mathcal{F}_{\theta,\alpha}(w,\partial_w).$$

We will treat $\mathcal{F}_{\theta,\alpha}(w,\partial_w)$ as the principal operator.

We have the following transmutation relations:

$$
\begin{aligned}
\partial_z \ & \mathcal{F}_{\theta,\alpha} \\
= \quad \mathcal{F}_{\theta+1,\alpha+1} \ & \partial_z, \\
(z\partial_z + \alpha - z) \ & \mathcal{F}_{\theta,\alpha} \\
= \quad \mathcal{F}_{\theta-1,\alpha-1} \ & (z\partial_z + \alpha - z), \\
(z\partial_z + \alpha) \ & \mathcal{F}_{\theta,\alpha} \\
= \quad \mathcal{F}_{\theta+1,\alpha-1} \ & (z\partial_z + \alpha), \\
(\partial_z - 1) \ & \mathcal{F}_{\theta,\alpha}, \\
= \quad \mathcal{F}_{\theta-1,\alpha+1} \ & (\partial_z - 1); \\
\big(z\partial_z + \tfrac{1}{2}(\theta + \alpha + 1)\big) \ & z\mathcal{F}_{\theta,\alpha} \\
= \quad z\mathcal{F}_{\theta+2,\alpha} \ & \big(z\partial_z + \tfrac{1}{2}(\theta + \alpha + 1)\big), \\
\big(z\partial_z + \tfrac{1}{2}(-\theta + \alpha + 1) - z\big) \ & z\mathcal{F}_{\theta,\alpha} \\
= \quad z\mathcal{F}_{\theta-2,\alpha} \ & \big(z\partial_z + \tfrac{1}{2}(-\theta + \alpha + 1) - z\big).
\end{aligned}
$$

## Discrete symmetries:

The following operators equal $\mathcal{F}_{\theta,\alpha}(w,\partial_w)$ for the appropriate $w$:

$$
\begin{aligned}
w = z:&\\
&\mathcal{F}_{\theta,\alpha}(z,\partial_z),\\
&z^{-\alpha}\ \mathcal{F}_{\theta,-\alpha}(z,\partial_z)\ z^{\alpha},\\
w = -z:&\\
&-\mathrm{e}^{-z}\ \mathcal{F}_{-\theta,\alpha}(z,\partial_z)\ \mathrm{e}^{z},\\
&-\mathrm{e}^{-z}z^{-\alpha}\ \mathcal{F}_{-\theta,-\alpha}(z,\partial_z)\ \mathrm{e}^{z}z^{\alpha}.
\end{aligned}
$$

Note the commutation relations

$$[B_{2,-1}, B_{3,1}] = [B_{2,1}\,, B_{3,-1}] = B_{3,2}.$$

Therefore, we have two distinguished subalgebras in $\mathrm{sch}(2)$ isomorphic to the Heisenberg algebra over a 2-dimensional symplectic space:

$$\mathrm{heis}_{-}(2) \text{ spanned by } B_{2,-1},\ B_{3,1},\ B_{3,2},$$
$$\mathrm{heis}_{+}(2) \text{ spanned by } B_{2,1},\ B_{3,-1},\ B_{3,2}.$$

Let us define

$$\begin{aligned}\mathcal{C}_- &= 2\,B_{2,-1}B_{3,1} + M - N_1 - B_{3,2}\\ &= 2\,B_{3,1}B_{2,-1} + M - N_1 + B_{3,2},\\ \mathcal{C}_+ &= 2\,B_{2,1}B_{3,-1} + M + N_1 - B_{3,2}\\ &= 2\,B_{3,-1}B_{2,1} + M + N_1 + B_{3,2}.\end{aligned}$$

$\mathcal{C}_+$ and $\mathcal{C}_-$ can be viewed as the Casimir operators for $\mathrm{heis}_+(2)$ and $\mathrm{heis}_-(2)$ respectively. Indeed, $\mathcal{C}_+$, resp. $\mathcal{C}_-$ commute with all operators in $\mathrm{heis}_+(2)$, resp. $\mathrm{heis}_-(2)$.

On the level of $\mathbb{R}^2 \oplus \mathbb{R}$, the two operators $\mathcal{C}_+$ and $\mathcal{C}_-$ coincide. Indeed, a direct calculation yields

$$\mathcal{C}_+^{\mathrm{sch},\eta} = \mathcal{C}_-^{\mathrm{sch},\eta} = 2t(\partial_{y_{-1}}\partial_{y_1} + \partial_t) - \eta - 1.$$

Second, note the commutation relations

$$[B_{-3,2}, B_{3,-2}] = -N_2 + N_3 = M.$$

Therefore, we have a distinguished subalgebra in $\mathrm{sch}(2)$ isomorphic to $\mathrm{so}(3)$

$$\mathrm{so}_{23}(3) \quad \text{spanned by} \quad B_{-3,2}, B_{3,-2}, M.$$

The Casimir operator for $\mathrm{so}_{23}(3)$ is

$$\begin{aligned} \mathcal{C}_{23} &= 4\, B_{3,-2} B_{-3,2} - (M+1)^2 + 1 \\ &= 4\, B_{-3,2} B_{3,-2} - (M-1)^2 + 1. \end{aligned}$$

We have

$$(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_6^{\diamond} = -1 + \mathcal{C}_{23}^{\diamond,-1} + (N_1^{\diamond,-1})^2.$$

Hence,

$$
\begin{aligned}
&(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_6^{\diamond} \\
=&4B_{2,-3}B_{-2,3} - (N_1 + M + 1)(-N_1 + M + 1), \\
=&4B_{-2,3}B_{2,-3} - (N_1 + M - 1)(-N_1 + M - 1),
\end{aligned}
$$

where the $B$, $N_1$ and $M$ operators should be equipped with the superscript $^{\diamond,-1}$.

Let us sum up the factorizations in the variables $y_{-1}y_1, t$ obtained with the help of the three subalgebras:

$$\begin{aligned}
t\mathcal{L}_2 &= 2\,B_{2,-1}B_{3,1} - (-M + N_1 + 1) \\
&= 2\,B_{3,1}B_{2,-1} - (-M + N_1 - 1), \\
&= 2\,B_{2,1}B_{3,-1} - (-M - N_1 + 1) \\
&= 2\,B_{3,-1}B_{2,1} - (-M - N_1 - 1), \\
2y_{-1}y_1\mathcal{L}_2 &= -4B_{2,-3}B_{-2,3} - (N_1 + M + 1)(N_1 - M - 1), \\
&= -4B_{-2,3}B_{2,-3} - (N_1 + M - 1)(N_1 - M + 1),
\end{aligned}$$

where the $B$, $N_1$ and $M$ operators should be equipped with the superscript $^{\mathrm{sch},-1}$.

In the variables $w, u, s$, we need to make the replacements

$$y_{-1}y_1 \quad \to \quad ws^2, \quad t \quad \to \quad s^2.$$

We obtain several ways of factorizing the ${}_1F_1$ operator:

$$
\begin{aligned}
\mathcal{F}_{\theta,\alpha} &= \Big(z\partial_z + 1 + \alpha - z\Big)\partial_z - \frac{1}{2}(\theta + \alpha + 1),\\
&= \partial_z\Big(z\partial_z + \alpha - z\Big) - \frac{1}{2}(\theta + \alpha - 1),\\
&= \Big(z\partial_z + 1 + \alpha\Big)\Big(\partial_z - 1\Big) - \frac{1}{2}(\theta - \alpha - 1),\\
&= \Big(\partial_z - 1\Big)\Big(z\partial_z + \alpha\Big) - \frac{1}{2}(\theta - \alpha + 1);
\end{aligned}
$$

$$
\begin{aligned}
z\mathcal{F}_{\theta,\alpha} &= \Big(z\partial_z + \frac{1}{2}(\theta + \alpha - 1)\Big)\Big(z\partial_z + \frac{1}{2}(-\theta + \alpha + 1) - z\Big)\\
&\quad -\frac{1}{4}(-\theta + \alpha + 1)(\theta + \alpha - 1),\\
&= \Big(z\partial_z + \frac{1}{2}(-\theta + \alpha - 1) - z\Big)\Big(z\partial_z + \frac{1}{2}(\theta + \alpha + 1)\Big)\\
&\quad -\frac{1}{4}(-\theta + \alpha - 1)(\theta + \alpha + 1).
\end{aligned}
$$

Traditionally, the confluent or the ${}_1F_1$ equation is given by the operator

$$\mathcal{F}(a; c; z, \partial_z) := z\partial_z^2 + (c - z)\partial_z - a.$$

Here is the relationship between the classical parameters parameters and the Lie-algebraic parameters $\alpha, \theta$:

$$\alpha := c - 1, \qquad \theta := -c + 2a;$$

$$a = \frac{1+\alpha+\theta}{2} \qquad c = 1 + \alpha.$$

The confluent equation has a regular singular point at $0$. with indices $0$, $1-c$. The unique solution of the confluent equation analytic at $0$ and equal to $1$ at $0$ is called the ${}_1F_1$ hypergeometric function or the confluent function. It is equal to

$$F(a;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}.$$

It is defined for $c \neq 0, -1, -2, \dots$. Sometimes it is more convenient to consider the function

$$\mathbf{F}(a;c;z) := \frac{F(a;c;z)}{\Gamma(c)} = \sum_{n=0}^{\infty} \frac{(a)_n}{\Gamma(c+n)} \frac{z^n}{n!}.$$

Another useful function proportional to ${}_1F_1$ is

$$\mathbf{F}^{\mathrm{I}}(a;c;z) := \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F(a;c;z).$$

In the Lie-algebraic parameters:

$$
\begin{aligned}
F_{\theta,\alpha}(z) &:= F\Big(\frac{1+\alpha+\theta}{2};1+\alpha;z\Big),\\
\mathbf{F}_{\theta,\alpha}(z) &:= \mathbf{F}\Big(\frac{1+\alpha+\theta}{2};1+\alpha;z\Big)\\
&= \frac{1}{\Gamma(\alpha+1)}F_{\theta,\alpha}(z),\\
\mathbf{F}^{\mathrm{I}}_{\theta,\alpha}(z) &:= \mathbf{F}^{\mathrm{I}}\Big(\frac{1+\alpha+\theta}{2};1+\alpha;z\Big)\\
&= \frac{\Gamma(\frac{1+\alpha+\theta}{2})\Gamma(\frac{1+\alpha-\theta}{2})}{\Gamma(\alpha+1)}F_{\theta,\alpha}(z).
\end{aligned}
$$

A traditional form of the the ${}_2\mathcal{F}_0$ equation is

$$\mathcal{F}(a,b;-;z,\partial_z) := z^2\partial_z^2 + (-1+(1+a+b)z)\partial_z + ab,$$

It does not have a regular singular point at zero. One of its solutions is defined, for $z \in \mathbb{C}\backslash[0,+\infty[$,

$$F(a,b;-;z) := \lim_{c\to\infty} F(a,b;c;cz),$$

where $|\arg c - \pi| < \pi - \epsilon$, $\epsilon > 0$. It extends to an analytic function on the universal cover of $\mathbb{C}\backslash\{0\}$ with a branch point of an infinite order at 0. It has the following asymptotic expansion:

$$F(a,b;-;z) \sim \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!} z^n, \ |\arg z - \pi| < \pi - \epsilon.$$

Sometimes instead of ${}_2F_0$ it is useful to consider the function

$$\mathbf{F}^{\mathrm{I}}(a,b;-;z) := \Gamma(a)F(a,b;-;z).$$

When we use the Lie-algebraic parameters, we denote the ${}_2F_0$ function by $\tilde{F}$ and $\tilde{\mathbf{F}}$. The tilde is needed to avoid the confusion with the ${}_1F_1$ function:

$$\begin{aligned}
\tilde{F}_{\theta,\alpha}(z) &:= F\Big(\frac{1+\alpha+\theta}{2}, \frac{1-\alpha+\theta}{2}; -; z\Big), \\
\tilde{\mathbf{F}}^{\mathrm{I}}_{\theta,\alpha}(z) &:= \mathbf{F}^{\mathrm{I}}\Big(\frac{1+\alpha+\theta}{2}, \frac{1-\alpha+\theta}{2}; -; z\Big) \\
&= \Gamma\Big(\frac{1-\alpha+\theta}{2}\Big)\tilde{F}_{\theta,\alpha}(z).
\end{aligned}$$

We have 4 standard solutions of the $_1F_1$ equation:

$$\begin{aligned}
\text{Solution} \sim 1 \text{ at } 0: \quad & F_{\theta,\alpha}(z) \\
&= e^z F_{-\theta,\alpha}(-z); \\
\text{Solution} \sim z^{-\alpha} \text{ at } 0: \quad & z^{-\alpha} F_{\theta,-\alpha}(z) \\
&= z^{-\alpha} e^z F_{-\theta,-\alpha}(-z); \\
\text{Solution} \sim z^{-a} \text{ at } +\infty: \quad & z^{\frac{-1-\theta-\alpha}{2}} \tilde{F}_{\theta,\alpha}(-z^{-1}) \\
&= z^{\frac{-1-\theta-\alpha}{2}} \tilde{F}_{\theta,-\alpha}(-z^{-1}) \\
\text{Solution} \sim (-z)^{b-1} e^z \text{ at } -\infty: \quad & e^z (-z)^{\frac{-1+\theta-\alpha}{2}} \tilde{F}_{-\theta,\alpha}(z^{-1}) \\
&= e^z (-z)^{\frac{-1+\theta-\alpha}{2}} \tilde{F}_{-\theta,-\alpha}(z^{-1}).
\end{aligned}$$

Recurrence relations for the confluent function correspond to roots of the Lie algebra $sch(2)$:

$$\begin{aligned} \partial_z \mathbf{F}_{\theta,\alpha}(z) &= \frac{1+\theta+\alpha}{2}\mathbf{F}_{\theta+1,\alpha+1}(z), \\ (z\partial_z + \alpha - z)\,\mathbf{F}_{\theta,\alpha}(z) &= \mathbf{F}_{\theta-1,\alpha-1}(z), \end{aligned}$$

$$\begin{aligned} (z\partial_z + \alpha)\,\mathbf{F}_{\theta,\alpha}(z) &= \mathbf{F}_{\theta+1,\alpha-1}(z), \\ (\partial_z - 1)\,\mathbf{F}_{\theta,\alpha}(z) &= \frac{-1+\theta-\alpha}{2}\mathbf{F}_{\theta-1,\alpha+1}(z), \end{aligned}$$

$$\begin{aligned} \left(z\partial_z + \frac{1+\theta+\alpha}{2}\right)\mathbf{F}_{\theta,\alpha}(z) &= \frac{1+\theta+\alpha}{2}\mathbf{F}_{\theta+2,\alpha}(z), \\ \left(z\partial_z + \frac{1-\theta+\alpha}{2} - z\right)\mathbf{F}_{\theta,\alpha}(z) &= \frac{1-\theta+\alpha}{2}\mathbf{F}_{\theta-2,\alpha}(z). \end{aligned}$$

The following functions solve the heat equation:

$$t^{-\nu-1}(y_{-1}-1)^{\nu}\exp\left(\frac{(y_{-1}-1)y_1}{t}\right),$$
$$t^{-\nu-1}(y_1-1)^{\nu}\exp\left(\frac{y_{-1}(y_1-1)}{t}\right).$$

In order to make eigenfunctions of $M$ and $N_1$, we smear them and chose an appropriate $\nu$:

Set

$$H_{\theta,\alpha}(y_{-1}, y_1, t)$$
$$:= \int \tau^{-\alpha-1} t^{\frac{\theta+\alpha-1}{2}} (\tau^{-1} y_{-1} - 1)^{\frac{-\theta-\alpha-1}{2}} \exp\left(\frac{(y_{-1} - \tau) y_1}{t}\right) \mathrm{d}\tau,$$
$$G_{\theta,\alpha}(y_{-1}, y_1, t)$$
$$:= \int \tau^{-\alpha-1} t^{\frac{\theta-\alpha-1}{2}} (\tau y_1 - 1)^{\frac{-\theta+\alpha-1}{2}} \exp\left(\frac{y_{-1}(y_1 - \tau^{-1})}{t}\right) \mathrm{d}\tau.$$

Recall that

$$M = y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + 2t\partial_t + 1.$$

We have

$$\mathcal{L}_2 H_{\theta,\alpha} = 0, \qquad \mathcal{L}_2 G_{\theta,\alpha} = 0$$
$$M H_{\theta,\alpha} = \theta H_{\theta,\alpha}, \qquad M G_{\theta,\alpha} = \theta G_{\theta,\alpha};$$
$$N_1 H_{\theta,\alpha} = \alpha H_{\theta,\alpha}, \qquad N_1 G_{\theta,\alpha} = \alpha G_{\theta,\alpha}$$

Now we express the above wave packets in the coordinates $w, u, s$:

$$
\begin{aligned}
&H_{\theta,\alpha}(w,s,u) \\
=&s^{-1-\theta}u^{\alpha}\int \sigma^{\frac{-\alpha+\theta-1}{2}}(w-\sigma)^{\frac{-\alpha-\theta-1}{2}}\mathrm{e}^{\sigma}\mathrm{d}\sigma, \\
&G_{\theta,\alpha}(w,s,u) \\
=&s^{-1-\theta}u^{\alpha}\int \exp\left(\frac{w}{\sigma}\right)\sigma^{-\alpha-1}(\sigma-1)^{\frac{\alpha-\theta-1}{2}}\mathrm{d}\sigma.
\end{aligned}
$$

We thus have two kinds of integral representations of solutions to the ${}_1F_1$ equation:

a) Let $[0,1]\ni t\mapsto \gamma(t)$ satisfy

$$t^{a-c+1}\mathrm{e}^{t}(t-z)^{-a-1}\Big|_{\gamma(0)}^{\gamma(1)}=0.$$

Then

$$\mathcal{F}(a;c;z,\partial_z)\int_\gamma t^{a-c}\mathrm{e}^{t}(t-z)^{-a}\mathrm{d}t=0.$$

b) Let $[0,1]\ni t\mapsto \gamma(t)$ satisfy

$$\mathrm{e}^{\frac{z}{t}}t^{-c}(1-t)^{c-a}\Big|_{\gamma(0)}^{\gamma(1)}=0.$$

Then

$$\mathcal{F}(a;c;z,\partial_z)\int_\gamma \mathrm{e}^{\frac{z}{t}}t^{-c}(1-t)^{c-a-1}\mathrm{d}t=0.$$

Using the integral representations of type a) and attaching contours to $-\infty$, $0$ and $z$. we can obtain all standard solutions. The natural normalization leads to the function $\mathbf{F}$

Similarly, using the integral representations of type b) and attaching contours to $0-0$, $1$ and $\infty$ we can obtain all standard solutions. The natural normalization leads to the function $\mathbf{F}^{\mathrm{I}}$.

$$
\begin{array}{rll}
 & a) & b) \\
\text{Solution} \sim 1 \text{ at } 0: & ]-\infty,(0,z)^{+},-\infty[, & [1,+\infty[; \\
\text{Solution} \sim z^{-\alpha} \text{ at } 0: & [0,z] & (0-0)^{+}; \\
\text{Solution} \sim z^{-a} \text{ at } +\infty: & ]-\infty,0] & ]-\infty,0]; \\
\text{Solution} \sim (-z)^{b-1}\mathrm{e}^{z} \text{ at } -\infty: & [z,-\infty[ & [0,1].
\end{array}
$$

# Heat equation in 1 dimensions

# and the Hermite equation

We again consider $\mathbb{R}^5$ with the coordinates

$$z_0, z_{-2}, z_2, z_{-3}, z_3$$

and the scalar product given by

$$\langle z|z\rangle = z_0^2 + 2z_{-2}z_2 + 2z_{-3}z_3.$$

Remember that $\mathrm{sch}(1)$ is a subalgebra of $\mathrm{so}(5)$ and we keep the notation from $\mathrm{so}(5)$.

**Lie algebra $\mathrm{sch}(1)$.** The Cartan algebra is spanned by

$$\begin{aligned} M &= z_{-2}\partial_{z_{-2}} - z_2\partial_{z_2} - z_{-3}\partial_{z_{-3}} + z_3\partial_{z_3}, \\ B_{3,2} &= z_{-3}\partial_{z_2} - z_{-2}\partial_{z_3}. \end{aligned}$$

Root operators:

$$\begin{aligned} B_{3,0} &= z_{-3}\partial_{z_0} - z_0\partial_{z_3}, \\ B_{2,0} &= z_{-2}\partial_{z_0} - z_0\partial_{z_2}, \\ B_{3,-2} &= z_{-3}\partial_{z_{-2}} - z_2\partial_{z_3}, \\ B_{-3,2} &= z_3\partial_{z_2} - z_{-2}\partial_{z_{-3}}. \end{aligned}$$

**Weyl symmetries:**

$$\begin{aligned} \iota K(z_0, z_{-2}, z_2, z_{-3}, z_3) &= K(z_0, z_{-2}, z_2, z_{-3}, z_3), \\ \kappa K(z_0, z_{-2}, z_2, z_{-3}, z_3) &= K(z_0, -z_{-3}, -z_3, z_{-2}, z_2), \\ \kappa^2 K(z_0, z_{-2}, z_2, z_{-3}, z_3) &= K(z_0, -z_{-2}, -z_2, -z_{-3}, -z_3), \\ \kappa^3 K(z_0, z_{-2}, z_2, z_{-3}, z_3) &= K(z_0, z_{-3}, z_3, -z_{-2}, -z_2). \end{aligned}$$

**Laplacian:**

$$\Delta_5 = \partial_{z_0}^2 + 2\partial_{z_{-2}}\partial_{z_2} + 2\partial_{z_{-3}}\partial_{z_3}.$$

We descend on the level of $\mathbb{R}^3$ with the scalar product given by

$$\langle y|y\rangle = y_0^2 + 2y_{-2}y_2.$$

**Lie algebra** $\mathrm{sch}(1)$**.** Cartan algebra:

$$\begin{aligned} M^{\mathrm{fl},\eta} &= y_0\partial_{y_0} + 2y_{-2}\partial_{y_{-2}} - \eta, \\ B^{\mathrm{fl}}_{3,2} &= \partial_{y_2}. \end{aligned}$$

Root operators:

$$\begin{aligned} B^{\mathrm{fl}}_{3,0} &= \partial_{y_0}, \\ B^{\mathrm{fl}}_{2,0} &= y_{-2}\partial_{y_0} - y_0\partial_{y_2}, \\ B^{\mathrm{fl}}_{3,-2} &= \partial_{y_{-2}}, \\ B^{\mathrm{fl},\eta}_{-3,2} &= y_{-2}\big(y_0\partial_{y_0} + y_{-2}\partial_{y_{-2}} - \eta\big) - \frac{1}{2}y_0^2\partial_{y_2}. \end{aligned}$$

**Weyl symmetries:**

$$\iota^{\mathrm{fl},\eta} f(y_0, y_{-2}, y_2) = f(y_0, y_{-2}, y_2),$$

$$\kappa^{\mathrm{fl},\eta} f(y_0, y_{-2}, y_2) = y_{-2}^{\eta} f\left(\frac{y_0}{y_{-2}}, -\frac{1}{y_{-2}}, \frac{y_0^2 + 2y_{-2}y_2}{2y_{-2}}\right),$$

$$\left(\kappa^{\mathrm{fl},\eta}\right)^2 f(y_0, y_{-2}, y_2) = (-1)^{\eta} f(-y_0, y_{-2}, y_2),$$

$$\left(\kappa^{\mathrm{fl},\eta}\right)^3 f(y_0, y_{-2}, y_2) = (-y_{-2})^{\eta} f\left(-\frac{y_0}{y_{-2}}, -\frac{1}{y_{-2}}, \frac{y_0^2 + 2y_{-2}y_2}{2y_{-2}}\right).$$

**Laplacian:**

$$\Delta_5^{\mathrm{fl}} = \partial_{y_0}^2 + 2\partial_{y_{-2}}\partial_{y_2}.$$

We descend onto the level of $\mathbb{R} \oplus \mathbb{R}$. We rename $y_{-2}$ to $t$ and $y_0$ to $y$.

**Lie algebra** $\mathrm{sch}(1)$**.** Cartan algebra:

$$M^{\mathrm{sch},\eta} = y\partial_y + 2t\partial_t - \eta,$$
$$B_{3,2} = 1.$$

Root operators:

$$\begin{aligned} B^{\mathrm{sch}}_{3,0} &= \partial_y, \\ B^{\mathrm{sch}}_{2,0} &= t\partial_y - y, \\ B^{\mathrm{sch}}_{3,-2} &= \partial_t, \\ B^{\mathrm{sch},\eta}_{-3,2} &= t(y\partial_y + t\partial_t - \eta) - \frac{1}{2}y^2. \end{aligned}$$

**Weyl symmetry:**

$$
\begin{aligned}
\iota^{\mathrm{sch},\eta} h(y,t) &= h(y,t),\\
\kappa^{\mathrm{sch},\eta} h(y,t) &= t^{\eta} \exp(\tfrac{y^2}{2t}) h(\tfrac{y}{t}, -\tfrac{1}{t}),\\
\left(\kappa^{\mathrm{sch},\eta}\right)^2 h(y,t) &= (-1)^{\eta} h(-y,t),\\
\left(\kappa^{\mathrm{sch},\eta}\right)^3 h(y,t) &= (-t)^{\eta} \exp(\tfrac{y^2}{2t}) h(-\tfrac{y}{t}, -\tfrac{1}{t}).
\end{aligned}
$$

**Heat operator:**

$$\mathcal{L}_1 = \partial_y^2 + 2\partial_t.$$

We have the generalized symmetries:

$$\begin{aligned} B^{\mathrm{sch},-\frac{5}{2}}\mathcal{L}_1 &= \mathcal{L}_1 B^{\mathrm{sch},-\frac{1}{2}}, \quad B \in \mathrm{sch}(1); \\ \alpha^{\mathrm{sch},-\frac{5}{2}}\mathcal{L}_1 &= \mathcal{L}_1 \alpha^{\mathrm{sch},-\frac{1}{2}}, \quad \alpha \in \mathrm{Sch}(1). \end{aligned}$$

Let us define new coordinates:

$$w = \frac{y}{\sqrt{2\,t}}\,, \qquad\qquad s = \sqrt{t}\,.$$

**Lie algebra** $\mathrm{sch}(1)$**.** Cartan operators:

$$\begin{aligned} M^{\mathrm{sch},\eta} &= s\,\partial_s - \eta, \\ B_{-3\,2} &= 1. \end{aligned}$$

Root operators:

$$B^{\mathrm{sch}}_{3,0} = \frac{1}{\sqrt{2}\,s}\ \partial_w,$$
$$B^{\mathrm{sch}}_{2,0} = \frac{s}{\sqrt{2}}\ (\,\partial_w - 2w)\,,$$
$$B^{\mathrm{sch}}_{3,-2} = \frac{1}{2\,s^2}\ (-w\,\partial_w + s\partial_s)\,,$$
$$B^{\mathrm{sch},\eta}_{-3,2} = \frac{s^2}{2}\ \Big(w\,\partial_w + s\partial_s - 2\eta - 2w^2\Big)\,.$$

**Weyl symmetries:**

$$\iota^{\mathrm{sch},\eta} h(w,s) = h(w,s),$$
$$\kappa^{\mathrm{sch},\eta} h(w,s) = s^{2\eta}\mathrm{e}^{w^2} h(\mathrm{i}w, -\tfrac{\mathrm{i}}{s}),$$
$$(\kappa^{\mathrm{sch},\eta})^2 h(w,s) = (-1)^\eta h(-w,s),$$
$$(\kappa^{\mathrm{sch},\eta})^3 h(w,s) = (-s^2)^\eta \mathrm{e}^{w^2} h(-\mathrm{i}w, -\tfrac{\mathrm{i}}{s}).$$

**Heat operator:**

$$\mathcal{L}_1 = \frac{1}{2s^2}\left( \partial_w^2 - 2w\,\partial_w + 2s\,\partial_s \right) \ .$$

Let us set $\eta = -\frac{1}{2}$ and use the ansatz

$$h(w, s) = s^{-\lambda - \frac{1}{2}} F(w).$$

Clearly,

$$M^{\mathrm{sch}, -\frac{1}{2}} h \;=\; -\lambda h.$$

On functions of this, $2s^2 \mathcal{L}_1$ coincides with the Hermite operator

$$S_\lambda(w, \partial_w) := \partial_w^2 - 2w\partial_w - 2\lambda - 1.$$

We will also use an alternative notation

$$S(a; w, \partial_w) := \partial_w^2 - 2w\partial_w - 2a,$$

so that

$$\lambda = a - \frac{1}{2}, \quad a = \lambda + \frac{1}{2}.$$

The Hermite equation is reflection invariant. By using the quadratic transformation we can reduce it to a special case of the confluent equation:

$$\begin{aligned} \mathcal{S}_\lambda(z,\partial_z) &= 4\mathcal{F}_{\lambda,-\frac12}(w,\partial_w), \\ z^{-1}\mathcal{S}_\lambda(z,\partial_z)z &= 4\mathcal{F}_{\lambda,\frac12}(w,\partial_w), \end{aligned}$$

where

$$w = z^2, \qquad z = \sqrt{w}.$$

## Transmutation relations

$$\begin{aligned} \partial_z\ \mathcal{S}_\lambda \\ = \ \mathcal{S}_{\lambda+1}\ \partial_z, \end{aligned}$$

$$\begin{aligned} (\partial_z - 2z)\ \mathcal{S}_\lambda \\ = \ \mathcal{S}_{\lambda-1}\ (\partial_z - 2z), \end{aligned}$$

$$\begin{aligned} (z\partial_z + \lambda + \tfrac{1}{2})\ z^2\mathcal{S}_\lambda \\ = \ z^2\mathcal{S}_{\lambda+2}\ (z\partial_z + \lambda + \tfrac{1}{2}), \end{aligned}$$

$$\begin{aligned} (z\partial_z - \lambda + \tfrac{1}{2} - 2z^2)\ z^2\mathcal{S}_\lambda \\ = \ z^2\mathcal{S}_{\lambda-2}\ (z\partial_z - \lambda + \tfrac{1}{2} - 2z^2). \end{aligned}$$

## Discrete symmetries

The following operators equal $\mathcal{S}_\lambda(w, \partial_w)$ for an appropriate $w$:

$w = \pm z$ :

$$\mathcal{S}_\lambda(z, \partial_z),$$

$w = \pm \mathrm{i} z$ :

$$-\exp(-z^2)\ \mathcal{S}_{-\lambda}(z, \partial_z)\ \exp(z^2).$$

First note the commutation relations

$$[B_{2,0}, B_{3,0}] = B_{3,2}.$$

Therefore, we have a distinguished subalgebra in $\mathrm{sch}(1)$ isomorphic to $\mathrm{heis}(2)$

$$\mathrm{heis}_0(2) \text{ spanned by } B_{2,0},\ B_{3,0},\ B_{3,2}.$$

Let us define

$$\begin{aligned}\mathcal{C}_0 &= 2\, B_{2,0}B_{3,0} + 2M - B_{3,2}\\ &= 2\, B_{3,0}B_{2,0} + 2M + B_{3,2}.\end{aligned}$$

$\mathcal{C}_0$ can be treated as the Casimir operator of $\mathrm{heis}_0(2)$: it commutes with all elements of $\mathrm{heis}_0(2)$.

On the level of $\mathbb{R} \oplus \mathbb{R}$, we have the identity

$$2t\mathcal{L}_1 = \mathcal{C}_0^{\mathrm{sch},-\frac{1}{2}}.$$

Second, note that the triple of operators $B_{-3,2}, B_{3,-2},\ M$ is contained both in $\mathrm{sch}(6)$ and in $\mathrm{sch}(5)$. Therefore. in the context of $\mathrm{sch}(6)$, is also contained in $\mathrm{sch}(5)$. Recall that its Casimir operator is

$$\begin{aligned} \mathcal{C}_{23} &= 4\, B_{3,-2}B_{-3,2} - (M+1)^2 + 1 \\ &= 4\, B_{-3,2}B_{3,-2} - (M-1)^2 + 1. \end{aligned}$$

We have

$$(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_5^{\diamond} = \mathcal{C}_{23}^{\diamond,-\frac{1}{2}} - \frac{3}{4}.$$

Hence

$$\begin{aligned}(2z_{-2}z_2 + 2z_{-3}z_3)\Delta_5^{\diamond} =& 4B_{2,-3}B_{-2,3} \\ & - \Big(N_1 + M + \frac{3}{2}\Big)\Big( - N_1 + M + \frac{1}{2}\Big), \\ =& 4B_{-2,3}B_{2,-3} \\ & - \Big(N_1 + M - \frac{3}{2}\Big)\Big( - N_1 + M - \frac{1}{2}\Big),\end{aligned}$$

where the $B$, $N_1$ and $M$ operators should be equipped with the superscript ${}^{\diamond,-\frac{1}{2}}$.

Let us sum up the factorizations in the variables $y_0, t$ obtained with the help of the two subalgebras:

$$\begin{aligned} 2t\mathcal{L}_1 &= 2\, B_{2,0}B_{3,0} - (-2M+1) \\ &= 2\, B_{3,0}B_{2,0} - (-2M-1), \\ -y_0^2\mathcal{L}_1 &= 4B_{2,-3}B_{-2,3} - \Big(N_1 + M + \frac{3}{2}\Big)\Big( - N_1 + M + \frac{1}{2}\Big), \\ &= 4B_{-2,3}B_{2,-3} - \Big(N_1 + M - \frac{3}{2}\Big)\Big( - N_1 + M - \frac{1}{2}\Big), \end{aligned}$$

where the $B$, $N_1$ and $M$ operators should be equipped with the superscript $^{\mathrm{sch},-\frac{1}{2}}$.

In the coordinates $w, s$ we need to make the replacements

$$\begin{aligned} t &\rightarrow s^2, \\ y_0^2 &\rightarrow 2ws^2. \end{aligned}$$

This leads to the following factorizations of the Hermite operator:

$$\begin{aligned}\mathcal{S}_\lambda &= \big(\partial_z - 2z\big)\partial_z - 2\lambda - 1\\ &= \partial_z\big(\partial_z - 2z\big) - 2\lambda + 1,\end{aligned}$$

$$\begin{aligned}z^2\mathcal{S}_\lambda &= \left(z\partial_z + \lambda - \frac{3}{2}\right)\left(z\partial_z - \lambda + \frac{1}{2} - 2z^2\right)\\ &\quad + \left(\lambda - \frac{3}{2}\right)\left(\lambda - \frac{1}{2}\right)\\ &= \left(z\partial_z - \lambda - \frac{3}{2} - 2z^2\right)\left(z\partial_z + \lambda + \frac{1}{2}\right)\\ &\quad + \left(\lambda + \frac{3}{2}\right)\left(\lambda + \frac{1}{2}\right).\end{aligned}$$

The Hermite equation has only one singular point, $\infty$. We will see that one can define two kinds of solutions with a simple asymptotics at $\infty$.

Solution $\sim z^{-a}$ for $z \to +\infty$:

$$\begin{aligned} S_\lambda(z) &:= z^{-\lambda-\frac{1}{2}} \tilde{F}_{-\frac{1}{2},\lambda}(-z^{-2}) \\ &= z^{-a} F\Big(\frac{a}{2}, \frac{a+1}{2}; -; -z^{-2}\Big), \end{aligned}$$

Solution $\sim (-\mathrm{i}z)^{a-1}\mathrm{e}^{z^2}$ for $z \to +\mathrm{i}\infty$:

$$\mathrm{e}^{z^2} S_{-\lambda}(-\mathrm{i}z) = (-\mathrm{i}z)^{\lambda-\frac{1}{2}} \mathrm{e}^{z^2} \tilde{F}_{-\frac{1}{2},-\lambda}(z^{-2}).$$

Set

$$G_\lambda(y,t) := \int (\tau^2 t)^{-\frac{1}{2}} \exp\Big(\frac{(y-\tau^{-1})^2}{2t}\Big)\tau^{-1+\frac{1}{2}+\lambda}\mathrm{d}\tau,$$

$$H_\lambda(y,t) := \mathrm{e}^{-\sqrt{2}y\tau - t\tau^2}\tau^{-1+\frac{1}{2}+\lambda}\mathrm{d}\tau.$$

Recall that

$$M = y\partial_y + 2t\partial_t + \frac{1}{2}.$$

We have

$$\mathcal{L}_1 G_\lambda = 0, \qquad \mathcal{L}_1 H_\lambda = 0;$$

$$MG_\lambda = -\lambda G_\lambda, \qquad MH_\lambda = -\lambda M_\lambda.$$

Let us express these wave packets in the coordinates $w, s$:

$$G_\lambda(w,s) = \int s^{-1} \exp\left(w - \frac{1}{\sqrt{2}\tau s}\right)^2 \tau^{-2+\frac{1}{2}+\lambda} \mathrm{d}\tau.$$

We set $\sigma := w - \frac{1}{\sqrt{2}\tau s}$, so that $\tau = \frac{1}{(w-\sigma)\sqrt{2}s}$, obtaining

$$G_\lambda(w,s) = (\sqrt{2}s)^{-\frac{1}{2}-\lambda} \int \mathrm{e}^{\sigma^2} (w-\sigma)^{-\frac{1}{2}-\lambda} \mathrm{d}\sigma.$$

Similarly,

$$H_\lambda(w,s) = \int \mathrm{e}^{-2sw\tau - s^2\tau^2}\tau^{-1+\frac{1}{2}+\lambda}\mathrm{d}\tau.$$

We set $\sigma := s\tau$, so that $\tau = \frac{\sigma}{s}$, obtaining

$$H_\lambda(w,s) = s^{-\frac{1}{2}-\lambda}\int \mathrm{e}^{-2\sigma w - \sigma^2}\sigma^{-1+\frac{1}{2}+\lambda}\mathrm{d}\sigma.$$

Below we describe two kinds of integral representations of the Hermite equation.

a) Let $[0,1] \ni t \mapsto \gamma(t)$ satisfy

$$\mathrm{e}^{t^2}(t-z)^{-a-1}\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$\mathcal{S}(a;z,\partial_z)\int_\gamma \mathrm{e}^{t^2}(t-z)^{-a}\mathrm{d}t.$$

b) Let $[0,1] \ni t \mapsto \gamma(t)$ satisfy

$$\mathrm{e}^{-t^2-2zt}t^a\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$\mathcal{S}(a;z,\partial_z)\int_\gamma \mathrm{e}^{-t^2-2zt}t^{a-1}\mathrm{d}t = 0.$$

We can also deduce the second representation from the first by the symmetry involving the multiplication by $\mathrm{e}^{z^2}$ and the change of variables $z \mapsto \mathrm{i}z$. $\square$

In the first case the integrand has a singular point at $0$ and goes to zero as $t \to \pm\infty$. We can thus use a contour with such endpoints. We will see that they give all standard solutions.

In the second case the integrand has a singular point at $z$ and goes to zero as $t \to \pm\mathrm{i}\infty$. Using a contour with such endpoints we will also obtain all standard solutions.

| | $a)$ | $b)$ |
|---|---|---|
| Sol. $\sim z^{-a}$, $z \to +\infty$: | $[0, \infty[$ | $]-\mathrm{i}\infty, z^{-}, -\mathrm{i}\infty[$ |
| Sol. $\sim (-\mathrm{i}z)^{a-1}\mathrm{e}^{z^2}$, $z \to +\mathrm{i}\infty$: | $]-\infty, 0^{+}, -\infty[$ | $[z, \mathrm{i}\infty[$ |

It is convenient to introduce alternatively normalized solutions:

$$\begin{aligned}\mathbf{S}^{\mathrm{I}}_{\lambda}(z) &:= 2^{-\lambda-\frac{1}{2}}\Gamma\Big(\lambda+\frac{1}{2}\Big)S_{\lambda}(z)\\ \mathbf{S}^{0}_{\lambda}(z) &:= \sqrt{\pi}S_{\lambda}(z).\end{aligned}$$

(The normalization of $\mathbf{S}^{0}_{\lambda}$ is somewhat trivial – we introduce it to preserve the analogy with the Gegenbauer equation, which had a less trivially normalized solution $\mathbf{S}^{0}_{\alpha,\lambda}$.)

$$-\frac{1}{2} < \mathrm{Re}\lambda :$$

$$\int_0^\infty \mathrm{e}^{-t^2-2tz} t^{\lambda-\frac{1}{2}} \mathrm{d}t = \mathbf{S}_\lambda^{\mathrm{I}}(z), \quad z \notin ]-\infty, 0];$$

for all parameters:

$$-\mathrm{i} \int_{]-\mathrm{i}\infty, z^-, \mathrm{i}\infty[} \mathrm{e}^{t^2} (z-t)^{-\lambda-\frac{1}{2}} \mathrm{d}t = \mathbf{S}_\lambda^0(z), \quad z \notin ]-\infty, 0];$$

for all parameters:

$$\int_{]-\infty, 0^+, \infty[} \mathrm{e}^{-t^2-2tz} (-\mathrm{i}t)^{\lambda-\frac{1}{2}} \mathrm{d}t = \mathrm{e}^{z^2} \mathbf{S}_{-\lambda}^0(-\mathrm{i}z), \quad z \notin [0, \infty[;$$

$$\mathrm{Re}\lambda < \frac{1}{2} :$$

$$-\mathrm{i} \int_{[z, \mathrm{i}\infty[} \mathrm{e}^{t^2} (-\mathrm{i}(t-z))^{-\lambda-\frac{1}{2}} \mathrm{d}t = \mathrm{e}^{z^2} \mathbf{S}_{-\lambda}^{\mathrm{I}}(-\mathrm{i}z), \quad z \notin [0, \infty[.$$

The following recurrence relations correspond to root operra-tors:

$$\begin{aligned}\partial_z S_\lambda(z) &= -\Big(\frac{1}{2}+\lambda\Big)S_{\lambda+1}(z),\\ (\partial_z - 2z)S_\lambda(z) &= -2S_{\lambda-1}(z),\end{aligned}$$

$$\begin{aligned}\Big(z\partial_z+\frac{1}{2}+\lambda\Big)S_\lambda(z) &= -\frac{1}{2}\Big(\frac{1}{2}+\lambda\Big)\Big(\frac{3}{2}+\lambda\Big)S_{\lambda+2}(z),\\ \Big(z\partial_z+\frac{1}{2}-\lambda-2z^2\Big)S_\lambda(z) &= -2S_{\lambda-2}(z).\end{aligned}$$