RENORMALIZATION OF THE FRIEDRICHS HAMILTONIAN

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(*Received June 25, 2002*)

We discuss the construction of the Friedrichs Hamiltonian with singular off-diagonal terms. This construction resembles the renormalization of mass in quantum field theory.

Keywords: self-adjoint operators, renormalization, singular perturbations.

1. Introduction

Let H_0 be a self-adjoint operator on the Hilbert space $\mathcal{H}$. Let $\epsilon \in \mathbb{R}$ and $h \in \mathcal{H}$. The following operator on the Hilbert space $\mathbb{C} \oplus \mathcal{H}$ is often called the Friedrichs Hamiltonian:

$$G := \begin{bmatrix} \epsilon & (h| \\ |h) & H_0 \end{bmatrix}. \tag{1.1}$$

In the above expression the operators $(h|$ and $|h)$ are defined by

$$\begin{aligned} &\mathcal{H} \ni v \mapsto (h|v := (h|v) \in \mathbb{C}, \\ &\mathbb{C} \ni \alpha \mapsto |h)\alpha := \alpha h \in \mathcal{H}, \end{aligned} \tag{1.2}$$

where $(h|v)$ denotes the scalar product in $\mathcal{H}$.

In our note we would like to describe how to define the Friedrichs Hamiltonian if h is not necessarily a bounded functional on $\mathcal{H}$. It will turn out that it is natural to consider 3 cases:

$$(1)\ h \in \mathcal{H}, \qquad (2)\ h \in \mathcal{H}_{-1} \backslash \mathcal{H}, \qquad (3)\ h \in \mathcal{H}_{-2} \backslash \mathcal{H}_{-1}, \tag{1.3}$$

where by $\mathcal{H}_{-n}$ we denote the usual scale of spaces associated to the operator H_0, that is $\mathcal{H}_{-n} := \langle H_0 \rangle^{n/2} \mathcal{H}$, where $\langle H_0 \rangle := (\mathbf{1} + H_0^2)^{1/2}$.

Clearly, in the case (1) G is self-adjoint on $\mathbb{C} \oplus \mathrm{Dom} H_0$. We will see that in the case (2) one can easily define G as a self-adjoint operator, but its domain is no longer equal to $\mathbb{C} \oplus \mathrm{Dom} H_0$. In the case (3), strictly speaking, the formula (1.1) does not make sense. Nevertheless, it is possible to define a *renormalized Friedrichs Hamiltonian*. To do this one needs to *renormalize the parameter* ϵ. This procedure resembles the *renormalization of mass* in quantum field theory.

Friedrichs Hamiltonian appears in various guises in the physics and mathematics literature. For instance, the Wigner–Weisskopf model, the Jaymes–Cummings model and one of the sectors of the Lee model are essentially equivalent to the Friedrichs Hamiltonian [10]. It was studied by Friedrichs in his well-known book [5]. (Note, however, that the name "Friedrichs Hamiltonian" is sometimes used to denote operators different from (1.1)).

Basic idea of the renormalization of the Friedrichs Hamiltonian seems to belong to the folklore of quantum physics. For instance, it was mentioned in [5]. Nevertheless, we are not aware of a rigorous exposition of this instructive toy model in the literature.

A phenomenon similar to the renormalization of the Friedrichs Hamiltonian occurs in the case of rank one perturbations. To be more precise, let $\lambda \in \mathbb{R}$ and consider the operator of the form $H := H_0 + \lambda|h)(h|$. One of the first mathematical treatments of those operators appeared in [2, 4], therefore, to fix terminology, we will call H the Aronszajn–Donoghue Hamiltonian. It turns out that one can make sense of the Aronszajn–Donoghue Hamiltonian even if $h \in \mathcal{H}_{-2}$ by an appropriate *renormalization of the coupling constant* λ. This fact was observed by many authors, among them let us mention Berezin and Faddeev [3] in the case of a delta-like perturbation of the 3-dimensional Laplacian and Kiselev and Simon [7] in the case of an abstract semibounded operator H_0. The renormalization of Aronszajn–Donoghue Hamiltonians is in fact reminiscent of the *charge renormalization* in QFT. For comparison with the theory of Friedrichs Hamiltonians, in Section 4 we briefly sketch the theory of Aronszajn–Donoghue Hamiltonians (without assuming H_0 to be semibounded).

We will see that the renormalization of Friedrichs Hamiltonians resembles that of Aronszajn–Donoghue Hamiltonians. There is however a difference: in the case of Friedrichs Hamiltonians one applies what can be called the *additive renormalization*, whereas in the case of Aronszajn–Donoghue Hamiltonians the *multiplicative renormalization* is performed.

2. Renormalization

Let us first consider the case $h \in \mathcal{H}$. As we have said earlier, the operator G with $\mathrm{Dom}G = \mathbb{C} \oplus \mathrm{Dom}H_0$ is self-adjoint. It is well known that the resolvent of G can be computed exactly. In fact, for $z \notin \mathrm{sp}H_0$ define the analytic function

$$g(z) := \epsilon + (h|(z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1}h), \tag{2.1}$$

($\mathrm{sp}H_0$ denotes the spectrum of H_0). Then for $z \in \Omega := \{z \in \mathbb{C}\backslash\mathrm{sp}H_0 : g(z) - z \neq 0\}$ the resolvent $Q(z) := (z\,\mathbf{1}_{\mathbb{C}\oplus\mathcal{H}} - G)^{-1}$ is given by

$$Q(z) = \begin{bmatrix} 0 & 0 \\ 0 & (z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1} \end{bmatrix}$$
$$+ \Big(z - g(z)\Big)^{-1} \begin{bmatrix} \mathbf{1}_{\mathbb{C}} & (h|(z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1} \\ (z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1}|h) & (z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1}|h)(h|(z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1} \end{bmatrix}. \tag{2.2}$$

In what follows we will usually omit $\mathcal{H}$, $\mathbb{C}\oplus\mathcal{H}$, etc. from $\mathbf{1}_{\mathcal{H}}$, $\mathbf{1}_{\mathbb{C}\oplus\mathcal{H}}$. Now let us describe how to define G for more general h:

THEOREM 2.1. *Assume that:*
(A) *$h \in \mathcal{H}_{-1}$, $\epsilon \in \mathbb{R}$ and let $Q(z)$ be given by* (2.2) *with $g(z)$ defined by* (2.1), *or*
(B) *$h \in \mathcal{H}_{-2}$, $\gamma \in \mathbb{R}$ and let $Q(z)$ be given by* (2.2) *with $g(z)$ defined by*

$$\begin{aligned} g(z) &:= \gamma + \big(h|((z - H_0)^{-1} + H_0(1 + H_0^2)^{-1})h\big) \\ &= \gamma + \left(h|(\frac{i - z}{2(z - H_0)(i - H_0)} - \frac{i + z}{2(z - H_0)(-i - H_0)})h\right). \end{aligned} \tag{2.3}$$

Then for all $z \in \Omega$:
(i) *$Q(z)$ is a bounded operator which fulfills the first resolvent formula (in the terminology of* [6], *$Q(z)$ is a pseudoresolvent).*
(ii) $\mathrm{Ker}\,Q(z) = \{0\}$.
(iii) $\mathrm{Ran}\,Q(z)$ *is dense in* $\mathbb{C}\oplus\mathcal{H}$.
(iv) $Q(z)^* = Q(\bar{z})$.
Therefore, by [6], *there exists a unique densely defined self-adjoint operator G such that $Q(z) = (z - G)^{-1}$. More precisely, for any $z_0 \in \Omega$, $\mathrm{Dom}\,G = \mathrm{Ran}\,Q(z_0)$, and if $\varphi \in \mathrm{Ran}\,Q(z_0)$ and $Q(z_0)\psi = \varphi$, then*

$$G\varphi := -\psi + z_0 Q(z_0)\psi.$$

Proof: Let $z \in \Omega$. It is obvious that $Q(z)$ is bounded and satisfies (iv). We easily see that both in the case (A) and (B) the function $g(z)$ satisfies

$$g(z_1) - g(z_2) = -(z_1 - z_2)(h|(z_1 - H_0)^{-1}(z_2 - H_0)^{-1}|h). \tag{2.4}$$

Direct computations using (2.4) show the first resolvent formula.

Let $(\alpha, f) \in \mathbb{C}\oplus\mathcal{H}$ be such that $(\alpha, f) \in \mathrm{Ker}\,Q(z)$. Then

$$0 = (z - g(z))^{-1}\Big(\alpha + (h|(z - H_0)^{-1} f)\Big), \tag{2.5}$$

$$0 = (z - H_0)^{-1} f + (z - H_0)^{-1} h (z - g(z))^{-1}\Big(\alpha + (h|(z - H_0)^{-1} f)\Big). \tag{2.6}$$

Inserting (2.5) into (2.6) we get $0 = (z - H_0)^{-1} f$ and hence $f = 0$. Now (2.5) implies $\alpha = 0$, so $\mathrm{Ker}\,Q(z) = \{0\}$.

Using (ii) and (iv) we get $(\mathrm{Ran}\,Q(z))^{\perp} = \mathrm{Ker}\,Q(z)^* = \mathrm{Ker}\,Q(\bar{z}) = \{0\}$. Hence (iii) holds. □

3. Cut-off renormalization

Let $h \in \mathcal{H}_{-2}$ and $\gamma \in \mathbb{R}$. Let us impose a cut-off on h. For $k \in \mathbb{N}$ we define

$$h_k := \mathbf{1}_{[-k,k]}(H_0)\, h, \tag{3.1}$$

where $\mathbf{1}_{[-k,k]}(H_0)$ is the spectral projection for H_0 associated with the interval $[-k, k] \subset \mathbb{R}$. Note that $h_k \in \mathcal{H}$ and hence both $(h_k|$ and $|h_k)$ are well-defined bounded operators. Set

$$\epsilon_k := \gamma + (h_k|H_0(1 + H_0^2)^{-1} h_k).$$

For all $k \in \mathbb{N}$, the cut-off Friedrichs Hamiltonian

$$G_k := \begin{bmatrix} \epsilon_k & (h_k| \\ |h_k) & H_0 \end{bmatrix}$$

is well defined and we can compute its resolvent $Q_k(z) := (z\,\mathbf{1}_{\mathbb{C}\oplus\mathcal{H}} - G_k)^{-1}$:

$$\begin{aligned} Q_k(z) &= \begin{bmatrix} 0 & 0 \\ 0 & (z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1} \end{bmatrix} \\ &+ \Big(z - g_k(z)\Big)^{-1} \begin{bmatrix} \mathbf{1}_{\mathbb{C}} & (h_k|(z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1} \\ (z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1}|h_k) & (z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1}|h_k)(h_k|(z\,\mathbf{1}_{\mathcal{H}} - H_0)^{-1} \end{bmatrix}, \end{aligned} \tag{3.2}$$

where

$$g_k(z) := \epsilon_k + (h_k|(z - H_0)^{-1} h_k). \tag{3.3}$$

Note that ϵ_k is chosen in such a way that the following *renormalization condition* is satisfied: $\frac{1}{2}\,(g_k(\mathrm{i}) + g_k(-\mathrm{i})) = \gamma$. Let us also mention that if H_0 is bounded from below, then $\lim_{k\to\infty} \epsilon_k = \infty$.

THEOREM 3.1. *Assume that $h \in \mathcal{H}_{-2}$. Then $\lim\limits_{k\to\infty} Q_k(z) = Q(z)$, where $Q(z)$ is given by* (2.2) *and $g(z)$ is given by* (2.3).

Proof: The proof is obvious if we note that $\lim\limits_{k\to\infty} \|(z-H_0)^{-1}h-(z-H_0)^{-1}h_k\| = 0$ and $\lim\limits_{k\to\infty} g_k(z) = g(z)$. □

Thus the cut-off Friedrichs Hamiltonian is norm resolvent convergent to the renormalized Friedrichs Hamiltonian.

4. Renormalization of the Aronszajn–Donoghue Hamiltonian

In this section we briefly sketch the renormalization procedure for Aronszajn–Donoghue Hamiltonians (see [1, 3, 7]).

Let H_0 be again a self-adjoint operator on $\mathcal{H}$, $h \in \mathcal{H}$ and $\lambda \in \mathbb{R}$. As before we start with the case $h \in \mathcal{H}$. Note that according to (1.2), $|h)(h|$ equals the orthogonal projection onto h times $\|h\|^2$. Thus

$$H := H_0 + \lambda |h)(h| \tag{4.1}$$

is a rank one perturbation of H_0. We will call (4.1) the Aronszajn–Donoghue Hamiltonian. Clearly, H is self-adjoint on $\mathrm{Dom} H_0$. We can compute its resolvent. In fact, for $z \notin \mathrm{sp} H_0$ we define an analytic function

$$g(z) := -\lambda^{-1} + (h|(z - H_0)^{-1} h). \tag{4.2}$$

Then for $z \in \Theta := \{z \in \mathbb{C} \backslash \mathrm{sp} H_0 : g(z) \neq 0\}$ and $\lambda \neq 0$, the resolvent of the operator H is given by Krein's formula

$$R(z) = (z - H_0)^{-1} - g(z)^{-1} (z - H_0)^{-1} |h)(h| (z - H_0)^{-1}. \tag{4.3}$$

For $\lambda = 0$, we set $\Theta = \mathbb{C} \backslash \mathrm{sp} H_0$ and clearly

$$R(z) = (z - H_0)^{-1}. \tag{4.4}$$

THEOREM 4.1. *Assume that:*
(A) $h \in \mathcal{H}_{-1}$, $\lambda \in \mathbb{R} \cup \{\infty\}$ *and let* $R(z)$ *be given by* (4.4) *or* (4.3) *with* $g(z)$ *given by* (4.2),
or
(B) $h \in \mathcal{H}_{-2}$, $\gamma \in \mathbb{R}$ *and let* $R(z)$ *be given by* (4.3) *with* $g(z)$ *given by*

$$g(z) := \gamma + \left(h|((z - H_0)^{-1} + H_0(1 + H_0^2)^{-1})h\right).$$

Then, for all $z \in \Theta$,
(i) $R(z)$ *is a bounded operator which fulfills the first resolvent formula.*
(ii) $\mathrm{Ker} R(z) = \{0\}$, *unless* $h \in \mathcal{H}$ *and* $\lambda = \infty$.
(iii) $\mathrm{Ran} R(z)$ *is dense in* $\mathcal{H}$, *unless* $h \in \mathcal{H}$ *and* $\lambda = \infty$.
(iv) $R(z)^* = R(\bar{z})$.
Hence, except for the case $h \in \mathcal{H}$, $\lambda = \infty$, *there exists a unique densely defined self-adjoint operator* H *such that* $R(z)$ *is the resolvent of* H.

Another way to define H for the case $h \in \mathcal{H}_{-2}$ is the cut-off method. For all $k \in \mathbb{N}$ we define h_k as in (3.1) and fix the *running coupling constant* by

$$-\lambda_k^{-1} := \gamma + (h_k | H_0 (1 + H_0^2)^{-1} h_k),$$

and set the cut-off Hamiltonian to be

$$H_k := H_0 + \lambda_k |h_k)(h_k|. \tag{4.5}$$

Then the resolvent for H_k is given by

$$R_k(z) = (z - H_0)^{-1} + g_k(z)^{-1} (z - H_0)^{-1} |h_k)(h_k| (z - H_0)^{-1}, \tag{4.6}$$

where

$$g_k(z) := -\lambda_k^{-1} + \left(h_k|(z - H_0)^{-1}h_k\right). \tag{4.7}$$

Note that λ_k is chosen in such a way that the renormalization condition $\frac{1}{2}(g_k(\mathrm{i}) + g_k(-\mathrm{i})) = \gamma$ holds. It is easy to see that if H_0 is bounded from below, then $\lim_{k\to\infty} \lambda_k = 0$. Again, the cut-off Hamiltonian converges to the renormalized Hamiltonian.

THEOREM 4.2. *Assume that* $h \in \mathcal{H}_{-2}$. *Then* $\lim_{k\to\infty} R_k(z) = R(z)$.

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