BOGOLIUBOV HAMILTONIANS

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I will speak about mathematical theory of quadratic Hamiltonians on a bosonic Fock space.

First I will consider finite dimensional one-particle space, which have a relatively simple theory. They satisfy, however, quite a number of nontrivial identities. Then I will consider arbitrary dimension, where the theory becomes quite technical and complicated. Finally, I will describe an example: scalar particle interacting with a mass-like position dependent perturbation. This model illustrates the need for infinite renormalization.
FINITE DIMENSIONS

We will first assume that the one-particle space is $\mathbb{C}^m$. Operators on $\mathbb{C}^m$ are identified with $m \times m$ matrices. If $h = [h_{ij}]$ is a matrix, then $\overline{h}$, $h^*$ and $h^\#$ will denote its complex conjugate, hermitian conjugate and transpose.
It is convenient to consider the doubled Hilbert space $\mathbb{C}^m \oplus \mathbb{C}^m$ equipped with the complex conjugation

$$J(z_1, z_2) = (\bar{z}_2, \bar{z}_1)$$

and the charge form

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

Operators that commute with $J$ have the form

$$R = \begin{bmatrix} p & q \\ \bar{q} & \bar{p} \end{bmatrix},$$

and will be called $J$-real.
Consider a self-adjoint $J$-real operator on the doubled space:

$$A = \begin{bmatrix} h & g \\ \bar{g} & \bar{h} \end{bmatrix}.$$ 

Note that $h = h^*$, $g = g^\#$. 

We also introduce

$$B := HS = \begin{bmatrix} h & -g \\ \bar{g} & -\bar{h} \end{bmatrix}.$$
By a quadratic classical Hamiltonian with associated to $A$, we will mean

$$H_A = \sum h_{ij} a_i^* a_j + \frac{1}{2} \sum g_{ij} a_i^* a_j^* + \frac{1}{2} \sum \bar{g}_{ij} a_i a_j,$$

where $a_i, a_j^*$ are classical (commuting) variables such that $a_i^*$ is the complex conjugate of $a_i$ and the following Poisson bracket relations hold:

$$\{a_i, a_j\} = \{a_i^*, a_j^*\} = 0,$$

$$\{a_i, a_j^*\} = -i\delta_{ij}.$$
Our main interest are operators on the bosonic Fock space $\Gamma_s(\mathbb{C}^m)$. $\hat{a}_i, \hat{a}_j^*$ will denote the standard annihilation and creation operators on $\Gamma_s(\mathbb{C}^m)$, where $\hat{a}_i^*$ is the Hermitian conjugate of $\hat{a}_i$,

$$\begin{align*}
[\hat{a}_i, \hat{a}_j] &= [\hat{a}_i^*, \hat{a}_j^*] = 0, \\
[\hat{a}_i, \hat{a}_j^*] &= \delta_{ij}.
\end{align*}$$
By a quantization of \( H_A \) (or, abusing terminology, a quantization of \( A \)) we will mean an operator on the \( \Gamma_s(\mathbb{C}^m) \) of the form

\[
\hat{H}_A^c := \frac{1}{2} \sum g_{ij} \hat{a}_i^* \hat{a}_j^* + \frac{1}{2} \sum \overline{g}_{ij} \hat{a}_i \hat{a}_j + \sum h_{ij} \hat{a}_i^* \hat{a}_j + c,
\]

where \( c \) is an arbitrary real constant. In the sequel, we will often drop \( A \), and especially \( c \), from \( \hat{H}_A^c \).
Two quantizations of $H_A$ are especially useful: the Weyl (or symmetric) quantization $\hat{H}_A^w$ and the normally ordered (or Wick) quantization $\hat{H}_A^n$:

$$\hat{H}_A^w := \frac{1}{2} \sum g_{ij} \hat{a}_i \hat{a}_j^* + \frac{1}{2} \sum \bar{g}_{ij} \hat{a}_i \hat{a}_j + \frac{1}{2} \sum h_{ij} \hat{a}_i^* \hat{a}_j + \frac{1}{2} \sum h_{ij} \hat{a}_j \hat{a}_i^*,$$

$$\hat{H}_A^n := \frac{1}{2} \sum g_{ij} \hat{a}_i^* \hat{a}_j^* + \frac{1}{2} \sum \bar{g}_{ij} \hat{a}_i \hat{a}_j + \sum h_{ij} \hat{a}_i^* \hat{a}_j.$$

The two quantizations differ by a constant:

$$\hat{H}_A^w = \hat{H}_A^n + \frac{1}{2} \text{Tr} h.$$
We say that a $J$-real operator

$$R = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}.$$ 

is symplectic if $R^* SR = S$. Below there are the equivalent conditions

$$p^* p - q^* \overline{q} = 1, \quad p^* q - q^* \overline{p} = 0,$$

$$pp^* - qq^* = 1, \quad pq^* - qp^* = 0.$$

We denote by $Sp(\mathbb{R}^{2m})$ the group of all symplectic transformations.
Note that

\[ pp^* \geq 1, \quad p^* p \geq 1. \]

Hence \( p^{-1} \) and \( p^{*-1} \) are well defined, and we can set

\[ d_1 := q^#(p^#)^{-1}, \]
\[ d_2 := q\bar{p}^{-1}. \]

Note that \( d_1^# = d_1 \), \( d_2 = d_2^# \). One has the following factorization:

\[
R = \begin{bmatrix}
1 & d_2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
(p^*)^{-1} & 0 \\
0 & \bar{p}
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\bar{d}_1 & 1
\end{bmatrix}.
\]
$U$ is a (Bogoliubov) implementer of a symplectic transformation $R$ if

$$U \hat{a}^*_i U^* = p_{ij} \hat{a}^*_j + q_{ij} \hat{a}_j,$$

$$U \hat{a}_i U^* = \bar{q}_{ij} \hat{a}^*_j + \bar{p}_{ij} \hat{a}_j.$$

Every symplectic transformation has an implementer, unique up to a choice of a phase factor.
We have the following canonical choices: the natural implementer $U_R^{\text{nat}}$, and a pair of metaplectic implementers $\pm U_R^{\text{met}}$:

$$
U_R^{\text{nat}} := |\det pp^*|^{-1} e^{-\frac{1}{2} \hat{a}^*(d_2)} \Gamma((p^*)^{-1}) e^{\frac{1}{2} \hat{a}(d_1)},
$$

$$
\pm U_R^{\text{met}} := \pm (\det p^*)^{-\frac{1}{2}} e^{\frac{1}{2} \hat{a}^*(d_2)} \Gamma((p^*)^{-1}) e^{\frac{1}{2} \hat{a}(d_1)}.
$$

Above, we used a compact notation for double annihilators/creators: If $d = [d_{ij}]$ is a symmetric matrix, then

$$
\hat{a}^*(d) = \sum_{ij} d_{ij} \hat{a}_i^* \hat{a}_j^*,
$$

$$
\hat{a}(d) = \sum_{ij} \overline{d_{ij}} \hat{a}_i \hat{a}_j,
$$
The set of Bogoliubov implementers is a group called sometimes the \( c\text{-metaplectic group} \) \( Mp^c(\mathbb{R}^{2m}) \). We have an obvious homomorphism \( Mp^c(\mathbb{R}^{2m}) \ni U \mapsto R \in Sp(\mathbb{R}^{2m}) \).

The set of metaplectic Bogoliubov implementers is a subgroup \( Mp^c(\mathbb{R}^{2m}) \) called the metaplectic group \( Mp(\mathbb{R}^{2m}) \). For any quadratic Hamiltonian \( A \), we have \( e^{it\hat{H}_A^w} \in Mp(\mathbb{R}^{2m}) \).
Various homomorphisms related to the metaplectic group can be described by the following diagram

\[
\begin{array}{ccc}
1 & \rightarrow & \mathbb{Z}_2 \\
\downarrow & & \downarrow \\
1 & \rightarrow & U(1) \\
\downarrow & & \downarrow \\
1 & \rightarrow & Mp(\mathbb{R}^{2m}) \\
\downarrow & & \downarrow \\
1 & \rightarrow & Sp(\mathbb{R}^{2m}) \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
\]
Of special importance are positive symplectic transformations. They satisfy
\[ p = p^*, \quad p > 0, \quad q = q^\#. \]
For such transformations \( d_1 = d_2 \) will be simply denoted by
\[ d := q(p^\#)^{-1}. \]
For positive symplectic transformations the natural implementer coincides with one of the metaplectic implementers:
\[ U^\text{nat}_R := \det p^{-\frac{1}{2}} e^{-\frac{1}{2} \hat{a}^*(d)} \Gamma(p^{-1}) e^{\frac{1}{2} \hat{a}(d)}. \]
Theorem about diagonalization of positive Hamiltonians.

Suppose that $A > 0$. Then,

1. $B$ has real nonzero eigenvalues.

2. $\text{sgn}(B)$ is symplectic.

3. $R_0 := S\text{sgn}B$ is symplectic and has positive eigenvalues.

4. Using the positive square root, define $R := R_0^{\frac{1}{2}}$. Then $R$ is symplectic and diagonalizes $A$. That means, for some $h_{\text{dg}},$

$$R^*^{-1}AR^{-1} = \begin{bmatrix} h_{\text{dg}} & 0 \\ 0 & h_{\text{dg}}^{-1} \end{bmatrix}.$$
Here is an alternative expression for $R_0$:

$$R_0 = A_2^{\frac{1}{2}} (A_2^{\frac{1}{2}} S A S A_2^{\frac{1}{2}})^{-\frac{1}{2}} A_2^{\frac{1}{2}}.$$  

On the quantum level, if $R$ diagonalizes $A$, then the corresponding unitary Bogoliubov implementers $U$ remove double annihilators/creators from $\hat{H}$:

$$U \hat{H}^w U^* = 2h_{dg,ij} \hat{a}_i^* \hat{a}_j + E^w,$$
$$U \hat{H}^n U^* = 2h_{dg,ij} \hat{a}_i^* \hat{a}_j + E^n,$$

where $E^w$, resp. $E^n$ is the infimum of $\hat{H}^w$, resp. of $\hat{H}^n$. 

We can compute the infimum of the Bogoliubov Hamiltonians. The simplest expression is valid for the Weyl quantization, which we present in various equivalent forms:

\[ E^w := \inf \hat{H}^w = \frac{1}{4} \text{Tr} \sqrt{B^2} \]

\[ = \frac{1}{4} \text{Tr} \sqrt{A^\frac{1}{2} S A S A^\frac{1}{2}} \]

\[ = \frac{1}{4} \text{Tr} \int \frac{B^2}{(B^2 + \tau^2)} \frac{d\tau}{2\pi} \]

\[ = \frac{1}{4} \text{Tr} \left[ \begin{array}{cc} h^2 - gg^* & -hg + gh^# \\ g^*h - h^#g^* & h^{#2} - g^*g \end{array} \right]^{\frac{1}{2}} \]
\[ E^n := \inf \hat{H}^n = E^w - \frac{1}{2} \text{Tr}h \]
\[ = \frac{1}{8} \int_0^1 d\sigma \text{Tr} \frac{B_\sigma}{\sqrt{B_\sigma^2}} GS. \]

where

\[ G := A - A_0 = \begin{bmatrix} 0 & g \\ \bar{g} & 0 \end{bmatrix}, \]
\[ B_\sigma = B_0 + \sigma G = \begin{bmatrix} h & -\sigma g \\ \sigma \bar{g} & -\bar{h} \end{bmatrix}. \]
Suppose now that

\[
A_0 = \begin{bmatrix}
h_0 & 0 \\
0 & \bar{h}_0 \\
0 & \bar{h}_0 
\end{bmatrix}
\]  

(1)

is a “free” Hamiltonian. We set

\[
B_0 := A_0 S = \begin{bmatrix}
h_0 & 0 \\
0 & -\bar{h}_0 \\
0 & -\bar{h}_0 
\end{bmatrix}, \quad V = B^2 - B_0^2.
\]  

(2)

We allow \( h_0 \) to be different from \( h \).
The infimum of the Weyl quantization of $H$ can be rewritten as

$$E^w = \sum_{j=0}^{\infty} L_j,$$

where

$$L_0 = \frac{1}{2} \text{Tr} \int \frac{B_0^2}{(B_0^2 + \tau^2)} \frac{d\tau}{2\pi} = \frac{1}{2} \text{Tr}|B_0| = \text{Tr} h,$$

$$L_j = \frac{1}{2} \text{Tr} \int \frac{(-1)^j}{B_0^2 + \tau^2} \left( V \frac{1}{B_0^2 + \tau^2} \right)^j \frac{\tau^2}{2\pi} d\tau$$

$$= \frac{1}{2} \text{Tr} \int \frac{(-1)^j}{2j} \left( V \frac{1}{B_0^2 + \tau^2} \right)^j \frac{d\tau}{2\pi}, \quad j = 1, 2, \ldots.$$
The constant $L_j$ arises in the diagramatic expansions as the evaluation of the loop with $2j$ vertices. To see this, introduce the “propagator”

$$G(t) := \frac{e^{-|B_0|t}}{2|B_0|}.$$ 

Clearly

$$\frac{1}{B_0^2 + \tau^2} = \int G(s)e^{is\tau}ds.$$
Therefore,

\[ L_j = \int dt_{j-1} \cdots \int dt_1 \text{Tr} VG(t_j - t_1) VG(t_1 - t_2) \cdots VG(t_{j-1} - t_j) \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt_j \int_{-T}^{T} dt_{j-1} \cdots \int_{-T}^{T} dt_1 \]

\[ \times \text{Tr} VG(t_j - t_1) VG(t_1 - t_2) \cdots VG(t_{j-1} - t_j). \]
Suppose now that

$$h_1^2 = gg, \quad h_1 g = g h_1. \quad (3)$$

Then $V$ contains only 1st order terms in $g$ and the loop expansion coincides with the expansion into powers of coupling constant. Then the loop expansion for the infimum of the normally ordered Hamiltonian amounts to omitting $L_0$ and $L_1$:

$$\inf E^n = E^w - \frac{1}{2} \text{Tr} h = \sum_{n=2}^{\infty} L_n. \quad (4)$$
$L_1$, and especially $L_0$, are often infinite. Sometimes, $L_2$ is infinite as well. Then we can renormalize even further:

$$E^\text{ren} := E^w - L_0 - L_1 - L_2 = \sum_{n=3}^{\infty} L_n$$

$$= -\frac{1}{4} \int \text{Tr} \frac{1}{B_0^2 + \tau^2} V \frac{1}{B^2 + \tau^2} \left( V \frac{1}{B_0^2 + \tau^2} \right)^2 \tau^2 d\tau.$$

We can also introduce the renormalized Hamiltonian

$$\hat{H}^\text{ren} := \hat{H}^w - L_0 - L_1 - L_2,$$

so that

$$E^\text{ren} = \inf \hat{H}^\text{ren}.$$
ARBITRARY DIMENSIONS

$Sp_{\text{res}}(\mathcal{Y})$ will denote the restricted symplectic group, which consists of $R \in Sp(\mathcal{Y})$ such that $q$ is Hilbert-Schmidt.

**Shale Theorem.** Let $R \in Sp(\mathcal{Y})$. Then $R$ is implementable iff $R \in Sp_{\text{res}}(\mathcal{Y})$. For such $R$, we can define its natural implementer

$$U^\text{nat}_R := |\det pp^*|^{-\frac{1}{4}} e^{-\frac{1}{2} \hat{a}^*(d_2)} \Gamma((p^*)^{-1}) e^{\frac{1}{2} \hat{a}(d_1)}.$$

We have a short exact sequence

$$1 \to U(1) \to Mp^c(\mathcal{Y}) \to Sp_{\text{res}}(\mathcal{Y}) \to 1.$$
$Sp_{af}(\mathcal{Y})$ will denote the anomaly-free symplectic group, which consists of $R \in Sp_{res}(\mathcal{Y})$ such that $\mathbb{1} - p$ is trace class. For $R \in Sp_{af}(\mathcal{Y})$ we can define a pair of metaplectic Bogoliubov implementers

$$\pm U^\text{met}_R := \pm (\det p^*)^{-\frac{1}{2}} e^{-\frac{1}{2} \hat{a}^*(d_2)} \Gamma((p^*)^{-1}) e^{\frac{1}{2} \hat{a}(d_1)}.$$ 

They form a group, which we denote $Sp_{af}(\mathcal{Y})$. We have a short exact sequence

$$\mathbb{1} \rightarrow \mathbb{Z}_2 \rightarrow Mp_{af}(\mathcal{Y}) \rightarrow Sp_{af}(\mathcal{Y}) \rightarrow \mathbb{1}.$$
Theorem. Let $B$ be a closed operator on $\mathcal{W} \oplus \overline{\mathcal{W}}$. The following statements are equivalent:

1. $e^{iBt}$, $t \in \mathbb{R}$, is a strongly continuous 1-parameter group of symplectic transformations.

2. $B = AS$ where $A$ is $J$-real and $A^* \supset A$ (in other words, $A$ is Hermitian), and there exist $c, b$ such that

$$
\|(A + i\tau S)^{-1}\| \leq c(|\tau| - b)^{-1}, \quad |\tau| > b.
$$
**Theorem** Suppose that \( g \subset g^\# \) and \( g = g_1 + g_2 \) such that \( \|g_1\| < \infty \) and \( \|h|^{-\frac{1}{2}}g_2|\bar{h}|^{-\frac{1}{2}}\| =: a < 1 \). Then the form \( A \) defines a classical quadratic Hamiltonian. Besides, \( A \) is self-adjoint.

We say that \( A \) possesses a quantization if there exists a self-adjoint operator \( \hat{H} \) on on \( \Gamma_s(\mathcal{W}) \) such that \( e^{it\hat{H}} \) implements \( e^{itB} \) for any \( t \in \mathbb{R} \). \( \hat{H} \) is uniquely defined up to an additive constant.

If the group \( e^{it\hat{H}} \) implementing \( e^{itB} \) is contained in \( Mp_{af}(\mathcal{Y}) \), then \( \hat{H} \) will be called the Weyl quantization of \( A \). For a given classical \( A \), its Weyl quantization, if it exists, is unique. We will denote it by \( \hat{H}_A^w \).
We say that $\hat{H}$ is the normally ordered quantization of $A$ if $\hat{H}$ implements $e^{itB}$ and
\[
\frac{d}{dt}(\Omega | e^{it\hat{H}} \Omega) \bigg|_{t=0} = 0.
\]
Again, a given classical Hamiltonian $A$ possesses at most one normally ordered quantization. We will denote it by $\hat{H}_A^n$.

If $A$ possess a quantization, which is bounded from below, then all of its quantizations are bounded from below. Then one can introduce the zero-infimum quantization $\hat{H}_A^z$ fixed by the condition
\[
\inf \hat{H}_A^z = 0.
\]
Define
\[ \gamma(g) := (h \otimes 1 + 1 \otimes h)^{-1} g, \]
where we use the tensor interpretation of \( g \) and assume that \( g \in \text{Dom}(h \otimes 1 + 1 \otimes h)^{-1}. \)
Theorem about existence of quantizations.

1. Suppose that $g$ is bounded and $g = g_1 + g_2$, where $\|g_1\|_{HS} < \infty$ and $\|\gamma(g_2)\|_{HS} < \infty$. Then $A$ possesses quantizations.

2. Suppose that $\|g\|_{HS} < \infty$. Then $A$ possesses the normally ordered quantization.

3. Suppose that $\|h\|_1 < \infty$ and $\|g\|_{HS} < \infty$. Then $A$ possesses both the Weyl and the normally ordered quantization. Besides,

$$\hat{H}^w = \hat{H}^n + \text{Tr}h.$$
**Theorem.** Let $h$ be positive and

$$\|h^{-\frac{1}{2}}g\bar{h}^{-\frac{1}{2}}\| =: a < 1.$$  \hspace{1cm} (6)

Then $A$

$$R_0 = SA^{-\frac{1}{2}}(A^{\frac{1}{2}}SASA^{\frac{1}{2}})^{\frac{1}{2}}A^{-\frac{1}{2}}S,$$  \hspace{1cm} (7)

is a bounded invertible positive operator.

$$R = R_0^{\frac{1}{2}}$$  \hspace{1cm} (8)

diagonalizes $A$, that is, for some positive self-adjoint $h_{dg}$

$$R^{-1}A(R^*)^{-1} = \begin{bmatrix} h_{dg} & 0 \\ 0 & \bar{h}_{dg} \end{bmatrix} =: A_{dg}. $$  \hspace{1cm} (9)
Theorem. (Napiórkowski, Nam, Solovej) In addition, suppose that
\[ \| h^{-\frac{1}{2}} g h^{-\frac{1}{2}} \|_{HS} < \infty. \] (10)

Then \( R \in Sp_{\text{res}}(\mathcal{Y}) \) and hence \( R \) is implementable.
Theorem. (Napiórkowski, Nam, Solovej) Assume that $\|h^{-\frac{1}{2}}g\bar{h}^{-\frac{1}{2}}\| < 1$ and $\text{Tr} g^* h^{-1} g < \infty$. Then the form

$$d\Gamma(h) + \frac{1}{2} a^*(g) + \frac{1}{2} a(g)$$

(11)

defined on $\text{Dom} d\Gamma(h)$ is closed and bounded from below. Hence it defines a self-adjoint operator. This operator is the normally ordered quantization of the classical Hamiltonian $A$. 
Set

\[ E^n_A := \frac{1}{4} \int_0^1 d\sigma \text{Tr} A^{\frac{1}{2}}_\sigma (A^{\frac{1}{2}}_\sigma S A^{\frac{1}{2}}_\sigma S A^{\frac{1}{2}}_\sigma)^{-\frac{1}{2}} A^{\frac{1}{2}}_\sigma G, \]

provided that the above integral is well defined.

**Theorem.**

1. Let \( \text{Tr} \sqrt{g} g \) \( < \infty \). Then \( E^n_A \) is well defined.

2. Suppose that \( \text{Tr} g h^{-1} g^* \) \( < \infty \). Then \( E^n_A \) and \( \hat{H}^n_A \) are well defined and

\[ E^n_A = \inf \hat{H}^n_A. \]
EXAMPLE:
SCALAR FIELD WITH
POSITION DEPENDENT MASS

Consider classical variables parametrized by $\vec{x} \in \mathbb{R}^3$ satisfying the Poisson bracket relations

$$\{\phi(\vec{x}), \phi(\vec{y})\} = \{\pi(\vec{x}), \pi(\vec{y})\} = 0,$$
$$\{\phi(\vec{x}), \pi(\vec{y})\} = \delta(\vec{x} - \vec{y}).$$
Consider the classical Hamiltonian of the free scalar field:

\[ H_0 = \int \left( \frac{1}{2} \pi^2(\vec{x}) + \frac{1}{2} (\vec{\partial} \phi(\vec{x}))^2 + \frac{1}{2} m^2 \phi^2(\vec{x}) \right) d\vec{x}, \]

If we assume that the mass squared depends on a position, we obtain a perturbed Hamiltonian:

\[ H = \int \left( \frac{1}{2} \pi^2(\vec{x}) + \frac{1}{2} (\vec{\partial} \phi(\vec{x}))^2 + \frac{1}{2} (m^2 + \kappa(\vec{x})) \phi^2(\vec{x}) \right) d\vec{x}, \]
Let us replace classical variables $\phi, \pi$ with quantum operators $\hat{\phi}, \hat{\pi}$ satisfying the commutation relations

\[
[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0,
\]
\[
[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta(\vec{x} - \vec{y}).
\]
It is well-known how to quantize $H_0$. The one-particle space consists of positive-frequency modes. The normally ordered Hamiltonian

$$\hat{H}_0^n = \int : \left( \frac{1}{2} \hat{\pi}^2(x) + \frac{1}{2} \left( \hat{\partial} \hat{\phi}(x) \right)^2 + \frac{1}{2} m^2 \hat{\phi}^2(x) \right) : dx,$$

acts on the corresponding Fock space. The infimum of $\hat{H}_0$ is zero. (The Weyl prescription $\hat{H}_0^w$ is ill-defined).
In the case of $H$, the normally-ordered prescription does not work. One has to renormalize by subtracting the (infinite) contribution of the loop with 2 vertices $L_2$, which can be formally written as

$$\hat{H}_{\text{ren}} = \int : \left( \frac{1}{2} \hat{\pi}^2(\vec{x}) + \frac{1}{2} (\hat{\partial} \phi(\vec{x}))^2 + \frac{1}{2} (m^2 + \kappa(\vec{x})) \phi^2(\vec{x}) \right) : d\vec{x} - L_2,$$

Let us stress that $\hat{H}_{\text{ren}}$ is a well-defined self-adjoint operator acting on the same space as $\hat{H}_0$. 

The infimum of $\hat{H}^{\text{ren}}$ is the sum of loops

$$\sum_{j=3}^{\infty} L_j$$

with at least 3 vertices. It is called the vacuum energy and is closely related to the so-called effective action.