BOGOLIUBOV HAMILTONIANS

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I will speak about mathematical theory of quadratic Hamiltonians on a bosonic Fock space.

First I will consider finite dimensional one-particle space, which have a relatively simple theory. They satisfy, however, quite a number of nontrivial identities. Then I will consider arbitrary dimension, where the theory becomes quite technical and complicated. Finally, I will describe an example: scalar particle interacting with a mass-like position dependent perturbation. This model illustrates the need for infinite renormalization.

FINITE DIMENSIONS

We will first assume that the one-particle space is $\mathbb{C}^m$. Operators on $\mathbb{C}^m$ are identified with $m \times m$ matrices. If $h = [h_{ij}]$ is a matrix, then $\overline{h}$, h^* and $h^\#$ will denote its complex conjugate, hermitian conjugate and transpose.

It is convenient to consider the doubled Hilbert space $\mathbb{C}^m \oplus \mathbb{C}^m$ equipped with the complex conjugation

$$J(z_1, z_2) = (\overline{z}_2, \overline{z}_1)$$

and the charge form

$$S = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}.$$

Operators that commute with J have the form

$$R = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix},$$

and will be called J-real.

Consider a self-adjoint J-real operator on the doubled space:

$$A = \begin{bmatrix} h & g \\ \overline{g} & \overline{h} \end{bmatrix}.$$

Note that $h = h^*$, $g = g^\#$.

We also introduce

$$B := HS = \begin{bmatrix} h & -g \\ \overline{g} & -\overline{h} \end{bmatrix}.$$

By a quadratic classical Hamiltonian with associated to A, we will mean

$$H_A = \sum h_{ij} a_i^* a_j + \frac{1}{2} \sum g_{ij} a_i^* a_j^* + \frac{1}{2} \sum \overline{g}_{ij} a_i a_j,$$

where a_i, a_j^* are classical (commuting) variables such that a_i^* is the complex conjugate of a_i and the following Poisson bracket relations hold:

$$\begin{aligned} \{a_i, a_j\} = \{a_i^*, a_j^*\} &= 0, \\ \{a_i, a_j^*\} &= -\mathrm{i}\delta_{ij}. \end{aligned}$$

Our main interest are operators on the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathbb{C}^m)$. $\hat{a}_i, \hat{a}_j^*$ will denote the standard annihilation and creation operators on $\Gamma_{\mathrm{s}}(\mathbb{C}^m)$, where $\hat{a}_i^*$ is the Hermitian conjugate of $\hat{a}_i$,

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^*, \hat{a}_j^*] \; = \; 0,$$

$$[\hat{a}_i, \hat{a}_j^*] \; = \; \delta_{ij}.$$

By a quantization of H_A (or, abusing terminology, a quantization of A) we will mean an operator on the $\Gamma_{\mathrm{s}}(\mathbb{C}^m)$ of the form

$$\hat{H}_A^c := \frac{1}{2}\sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \frac{1}{2}\sum \overline{g}_{ij}\hat{a}_i\hat{a}_j + \sum h_{ij}\hat{a}_i^*\hat{a}_j + c,$$

where c is an arbitrary real constant. In the sequel, we will often drop A, and especially c, from $\hat{H}_A^c$.

Two quantizations of H_A are especially useful: the Weyl (or symmetric) quantization $\hat{H}_A^{\mathrm{w}}$ and the normally ordered (or Wick) quantization $\hat{H}_A^{\mathrm{n}}$:

$$\hat{H}_A^{\mathrm{w}} := \frac{1}{2}\sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \frac{1}{2}\sum \overline{g}_{ij}\hat{a}_i\hat{a}_j + \frac{1}{2}\sum h_{ij}\hat{a}_i^*\hat{a}_j + \frac{1}{2}\sum h_{ij}\hat{a}_j\hat{a}_i^*,$$

$$\hat{H}_A^{\mathrm{n}} := \frac{1}{2}\sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \frac{1}{2}\sum \overline{g}_{ij}\hat{a}_i\hat{a}_j + \sum h_{ij}\hat{a}_i^*\hat{a}_j.$$

The two quantizations differ by a constant:

$$\hat{H}_A^{\mathrm{w}} = \hat{H}_A^{\mathrm{n}} + \frac{1}{2}\mathrm{Tr}h.$$

We say that a J-real operator

$$R = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}.$$

is symplectic if $R^*SR = S$. Below there are the equivalent conditions

$$p^*p - q^\#\overline{q} = \mathbb{1}, \quad p^*q - q^\#\overline{p} = 0,$$

$$pp^* - qq^* = \mathbb{1}, \quad pq^\# - qp^\# = 0.$$

We denote by $Sp(\mathbb{R}^{2m})$ the group of all symplectic transformations.

Note that

$$pp^* \geq \mathbb{1}, \qquad p^*p \geq \mathbb{1}.$$

Hence p^{-1} and p^{*-1} are well defined, and we can set

$$\begin{aligned} d_1 &:= q^\#(p^\#)^{-1}, \\ d_2 &:= q\overline{p}^{-1}. \end{aligned}$$

Note that $d_1^\# = d_1$, $d_2 = d_2^\#$. One has the following factorization:

$$R = \begin{bmatrix} \mathbb{1} & d_2 \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} (p^*)^{-1} & 0 \\ 0 & \overline{p} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ \overline{d}_1 & \mathbb{1} \end{bmatrix}.$$

U is a (Bogoliubov) implementer of a symplectic transformation R if

$$
\begin{aligned}
U\hat{a}_i^* U^* &= p_{ij}\hat{a}_j^* + q_{ij}\hat{a}_j, \\
U\hat{a}_i U^* &= \overline{q}_{ij}\hat{a}_j^* + \overline{p}_{ij}\hat{a}_j.
\end{aligned}
$$

Every symplectic transformation has an implementer, unique up to a choice of a phase factor.

We have the following canonical choices: the natural implementer U_R^{nat}, and a pair of metaplectic implementers $\pm U_R^{\mathrm{met}}$:

$$\begin{aligned} U_R^{\mathrm{nat}} &:= |\det pp^*|^{-\frac14} \mathrm{e}^{-\frac12 \hat{a}^*(d_2)} \Gamma\big((p^*)^{-1}\big) \mathrm{e}^{\frac12 \hat{a}(d_1)}, \\ \pm U_R^{\mathrm{met}} &:= \pm(\det p^*)^{-\frac12} \mathrm{e}^{-\frac12 \hat{a}^*(d_2)} \Gamma\big((p^*)^{-1}\big) \mathrm{e}^{\frac12 \hat{a}(d_1)}. \end{aligned}$$

Above, we used a compact notation for double annihilators/creators: If $d = [d_{ij}]$ is a symmetric matrix, then

$$\begin{aligned} \hat{a}^*(d) &= \sum_{ij} d_{ij} \hat{a}_i^* \hat{a}_j^*, \\ \hat{a}(d) &= \sum_{ij} \overline{d}_{ij} \hat{a}_i \hat{a}_j, \end{aligned}$$

The set of Bogoliubov implementers is a group called sometimes the c-metaplectic group $Mp^c(\mathbb{R}^{2m})$. We have an obvious homomorphism $Mp^c(\mathbb{R}^{2m}) \ni U \mapsto R \in Sp(\mathbb{R}^{2m})$.

The set of metaplectic Bogoliubov implementers is a subgroup $Mp^c(\mathbb{R}^{2m})$ called the metaplectic group $Mp(\mathbb{R}^{2m})$. For any quadratic Hamiltonian A, we have $\mathrm{e}^{\mathrm{i}t\hat{H}_A^{\mathrm{w}}} \in Mp(\mathbb{R}^{2m})$.

Various homomorphisms related to the metaplectic group can be described by the following diagram

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathbb{Z}_2 & \to & U(1) & \to & U(1) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Mp(\mathbb{R}^{2m}) & \to & Mp^{\mathrm{c}}(\mathbb{R}^{2m}) & \to & U(1) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Sp(\mathbb{R}^{2m}) & \to & Sp(\mathbb{R}^{2m}) & \to & 1 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

Of special importance are positive symplectic transformations. They satisfy

$$p = p^*, \quad p > 0, \quad q = q^{\#}.$$

For such transformations $d_1 = d_2$ will be simply denoted by

$$d \ := \ q(p^{\#})^{-1}.$$

For positive symplectic transformations the natural implementer coincides with one of the metaplectic implementers:

$$U_R^{\mathrm{nat}} \ := \ \det p^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\hat{a}^*(d)} \Gamma\big(p^{-1}\big) \mathrm{e}^{\frac{1}{2}\hat{a}(d)}.$$

Theorem about diagonalization of positive Hamiltonians.

Suppose that $A > 0$. Then,

1. B has real nonzero eigenvalues.
2. $\mathrm{sgn}(B)$ is symplectic.
3. $R_0 := S\mathrm{sgn}B$ is symplectic and has positive eigenvalues.
4. Using the positive square root, define $R := R_0^{\frac{1}{2}}$. Then R is symplectic and diagonalizes A. That means, for some h_{dg},

$$R^{*-1}AR^{-1} = \begin{bmatrix} h_{\mathrm{dg}} & 0 \\ 0 & h_{\mathrm{dg}}^{\#} \end{bmatrix}.$$

Here is an alternative exppression for R_0:

$$R_0 = A^{\frac{1}{2}}\big(A^{\frac{1}{2}}SASA^{\frac{1}{2}}\big)^{-\frac{1}{2}}A^{\frac{1}{2}}.$$

On the quantum level, if R diagonalizes A, then the corresponding unitary Bogoliubov implementers U remove double annihilators/creators from $\hat{H}$:

$$\begin{aligned} U\hat{H}^{\mathrm{w}}U^* &= 2h_{\mathrm{dg},ij}\hat{a}_i^*\hat{a}_j + E^{\mathrm{w}}, \\ U\hat{H}^{\mathrm{n}}U^* &= 2h_{\mathrm{dg},ij}\hat{a}_i^*\hat{a}_j + E^{\mathrm{n}}, \end{aligned}$$

where E^{w}, resp. E^{n} is the infimum of $\hat{H}^{\mathrm{w}}$, resp. of $\hat{H}^{\mathrm{n}}$.

We can compute the infimum of the Bogoliubov Hamiltonians The simplest expression is valid for the Weyl quantization, which we present in various equivalent forms:

$$
\begin{aligned}
E^{\mathrm{w}} := \inf \hat{H}^{\mathrm{w}} &= \frac{1}{4}\mathrm{Tr}\sqrt{B^2} \\
&= \frac{1}{4}\mathrm{Tr}\sqrt{A^{\frac{1}{2}}SASA^{\frac{1}{2}}} \\
&= \frac{1}{4}\mathrm{Tr}\int \frac{B^2}{(B^2+\tau^2)}\frac{\mathrm{d}\tau}{2\pi} \\
&= \frac{1}{4}\mathrm{Tr}\begin{bmatrix} h^2 - gg^* & -hg + gh^{\#} \\ g^*h - h^{\#}g^* & h^{\#2} - g^*g \end{bmatrix}^{\frac{1}{2}}
\end{aligned}
$$

$$E^{\mathrm{n}} := \inf \hat{H}^{\mathrm{n}} = E^{\mathrm{w}} - \frac{1}{2}\mathrm{Tr}h$$

$$= \frac{1}{8}\int_0^1 \mathrm{d}\sigma \mathrm{Tr}\frac{B_\sigma}{\sqrt{B_\sigma^2}} GS.$$

where

$$G := A - A_0 = \begin{bmatrix} 0 & g \\ \overline{g} & 0 \end{bmatrix},$$

$$B_\sigma = B_0 + \sigma G = \begin{bmatrix} h & -\sigma g \\ \sigma \overline{g} & -\overline{h} \end{bmatrix}.$$

Suppose now that

$$A_0 = \begin{bmatrix} h_0 & 0 \\ 0 & \overline{h}_0 \end{bmatrix} \tag{1}$$

is a "free" Hamiltonian. We set

$$B_0 := A_0 S = \begin{bmatrix} h_0 & 0 \\ 0 & -\overline{h}_0 \end{bmatrix}, \qquad V = B^2 - B_0^2. \tag{2}$$

We allow h_0 to be different from h.

The infimum of the Weyl quantization of H can be rewritten as

$$E^{\mathrm{w}} = \sum_{j=0}^{\infty} L_j,$$

where

$$\begin{aligned} L_0 &= \frac{1}{2}\mathrm{Tr}\int \frac{B_0^2}{(B_0^2+\tau^2)}\frac{\mathrm{d}\tau}{2\pi} = \frac{1}{2}\mathrm{Tr}|B_0| = \mathrm{Tr}h, \\ L_j &= \frac{1}{2}\mathrm{Tr}\int \frac{(-1)^{j+1}}{B_0^2+\tau^2}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^j \tau^2\frac{\mathrm{d}\tau}{2\pi} \\ &= \frac{1}{2}\mathrm{Tr}\int \frac{(-1)^j}{2j}\Big(V\frac{1}{B_0^2+\tau^2}\Big)^j \frac{\mathrm{d}\tau}{2\pi}, \qquad j = 1, 2, \ldots. \end{aligned}$$

The constant L_j arises in the diagramatic expasions as the evaluation of the loop with 2j vertices. To see this, introduce the "propagator"

$$G(t) := \frac{\mathrm{e}^{-|B_0|t}}{2|B_0|}.$$

Clearly

$$\frac{1}{B_0^2 + \tau^2} = \int G(s)\mathrm{e}^{\mathrm{i}s\tau}\mathrm{d}s.$$

Therefore,

$$\begin{aligned} L_j &= \int \mathrm{d}t_{j-1} \cdots \int \mathrm{d}t_1 \mathrm{Tr} VG(t_j - t_1) VG(t_1 - t_2) \cdots VG(t_{j-1} - t_j) \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}t_j \int_{-T}^{T} \mathrm{d}t_{j-1} \cdots \int_{-T}^{T} \mathrm{d}t_1 \\ &\quad \times \mathrm{Tr} VG(t_j - t_1) VG(t_1 - t_2) \cdots VG(t_{j-1} - t_j). \end{aligned}$$

Suppose now that

$$h_1^2 = g\overline{g}, \qquad h_1 g = g\overline{h}_1. \tag{3}$$

Then V contains only 1st order terms in g and the loop expansion coincides with the expansion into powers of coupling constant. Then the loop expansion for the infimum of the normally ordered Hamiltonian amounts to omitting L_0 and L_1:

$$\inf E^{\mathrm{n}} = E^{\mathrm{w}} - \frac{1}{2}\mathrm{Tr} h = \sum_{n=2}^{\infty} L_n. \tag{4}$$

L_1, and especially L_0, are often infinite. Sometimes, L_2 is infinite as well. Then we can renormalize even further:

$$\begin{aligned} E^{\mathrm{ren}} &:= E^{\mathrm{w}} - L_0 - L_1 - L_2 = \sum_{n=3}^{\infty} L_n \\ &= -\frac{1}{4} \int \mathrm{Tr} \frac{1}{B_0^2 + \tau^2} V \frac{1}{B^2 + \tau^2} \Big(V \frac{1}{B_0^2 + \tau^2} \Big)^2 \tau^2 \frac{\mathrm{d}\tau}{2\pi}. \end{aligned}$$

We can also introduce the renormalized Hamiltonian

$$\hat{H}^{\mathrm{ren}} := \hat{H}^{\mathrm{w}} - L_0 - L_1 - L_2, \tag{5}$$

so that

$$E^{\mathrm{ren}} = \inf \hat{H}^{\mathrm{ren}}.$$

ARBITRARY DIMENSIONS

$Sp_{\rm res}(\mathcal{Y})$ will denote the restricted symplectic group, which consists of $R \in Sp(\mathcal{Y})$ such that q is Hilbert-Schmidt.

Shale Theorem. Let $R \in Sp(\mathcal{Y})$. Then R is implementable iff $R \in Sp_{\rm res}(\mathcal{Y})$. For such R, we can define its natural implementer

$$U_R^{\rm nat} := |\det pp^*|^{-\frac{1}{4}} \mathrm{e}^{-\frac{1}{2}\hat{a}^*(d_2)} \Gamma\big((p^*)^{-1}\big) \mathrm{e}^{\frac{1}{2}\hat{a}(d_1)}.$$

We have a short exact sequence

$$\mathbb{1} \to U(1) \to Mp^c(\mathcal{Y}) \to Sp_{\rm res}(\mathcal{Y}) \to \mathbb{1}.$$

$Sp_{\mathrm{af}}(\mathcal{Y})$ will denote the anomaly-free symplectic group, which consists of $R \in Sp_{\mathrm{res}}(\mathcal{Y})$ such that $\mathbb{1} - p$ is trace class. For $R \in Sp_{\mathrm{af}}(\mathcal{Y})$ we can define a pair of metaplectic Bogoliubov implementers

$$\pm U_R^{\mathrm{met}} := \pm(\det p^*)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\hat{a}^*(d_2)} \Gamma\big((p^*)^{-1}\big) \mathrm{e}^{\frac{1}{2}\hat{a}(d_1)}.$$

They form a group, which we denote $Sp_{\mathrm{af}}(\mathcal{Y})$. We have a short exact sequence

$$\mathbb{1} \to \mathbb{Z}_2 \to Mp_{\mathrm{af}}(\mathcal{Y}) \to Sp_{\mathrm{af}}(\mathcal{Y}) \to \mathbb{1}.$$

Theorem. Let B be a closed operator on $\mathcal{W} \oplus \overline{\mathcal{W}}$. The following statements are equivalent:

1. $\mathrm{e}^{\mathrm{i}Bt}$, $t \in \mathbb{R}$, is a strongly continuous 1-parameter group of symplectic transformations.
2. $B = AS$ where A is J-real and $A^* \supset A$ (in other words, A is Hermitian), and there exist c, b such that

$$\|(A + \mathrm{i}\tau S)^{-1}\| \leq c(|\tau| - b)^{-1}, \quad |\tau| > b.$$

Theorem Suppose that $g \subset g^{\#}$ and $g = g_1 + g_2$ such that $\|g_1\| < \infty$ and $\||h|^{-\frac{1}{2}} g_2 |\overline{h}|^{-\frac{1}{2}}\| =: a < 1$. Then the form A defines a classical quadratic Hamiltonian. Besides, A is self-adjoint.

We say that A possesses a quantization if there exists a self-adjoint operator $\hat{H}$ on on $\Gamma_{\mathrm{s}}(\mathcal{W})$ such that $\mathrm{e}^{\mathrm{i}t\hat{H}}$ implements $\mathrm{e}^{\mathrm{i}tB}$ for any $t \in \mathbb{R}$. $\hat{H}$ is uniquely defined up to an additive constant.

If the group $\mathrm{e}^{\mathrm{i}t\hat{H}}$ implementing $\mathrm{e}^{\mathrm{i}tB}$ is contained in $Mp_{\mathrm{af}}(\mathcal{Y})$, then $\hat{H}$ will be called the Weyl quantization of A. For a given classical A, its Weyl quantization, if it exists, is unique. We will denote it by $\hat{H}^{\mathrm{w}}_A$.

We say that $\hat{H}$ is the normally ordered quantization of A if $\hat{H}$ implements $\mathrm{e}^{\mathrm{i}tB}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Omega|\mathrm{e}^{\mathrm{i}t\hat{H}}\Omega)\Big|_{t=0} = 0.$$

Again, a given classical Hamiltonian A possesses at most one normally ordered quantization. We will denote it by $\hat{H}_A^{\mathrm{n}}$.

If A possess a quantization, which is bounded from below, then all of its quantizations are bounded from below. Then one can introduce the zero-infimum quantization $\hat{H}^{\mathrm{z}}$ fixed by the condition

$$\inf \hat{H}_A^{\mathrm{z}} = 0.$$

Define

$$\gamma(g) := (h \otimes \mathbb{1} + \mathbb{1} \otimes h)^{-1} g,$$

where we use the tensor interpretation of g and assume that $g \in \mathrm{Dom}(h \otimes \mathbb{1} + \mathbb{1} \otimes h)^{-1}$.

Theorem about existence of quantizations.

1. Suppose that g is bounded and $g = g_1 + g_2$, where $\|g_1\|_{\mathrm{HS}} < \infty$ and $\|\gamma(g_2)\|_{\mathrm{HS}} < \infty$. Then A possesses quantizations.
2. Suppose that $\|g\|_{\mathrm{HS}} < \infty$. Then A possesses the normally ordered quantization.
3. Suppose that $\|h\|_1 < \infty$ and $\|g\|_{\mathrm{HS}} < \infty$. Then A possesses both the Weyl and the normally ordered quantization. Besides,

$$\hat{H}^{\mathrm{w}} = \hat{H}^{\mathrm{n}} + \mathrm{Tr}h.$$

Theorem. Let h be positive and

$$\|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\| =: a < 1. \tag{6}$$

Then A

$$R_0 = S A^{-\frac{1}{2}} (A^{\frac{1}{2}} S A S A^{\frac{1}{2}})^{\frac{1}{2}} A^{-\frac{1}{2}} S, \tag{7}$$

is a bounded invertible positive operator.

$$R = R_0^{\frac{1}{2}} \tag{8}$$

diagonalizes A, that is, for some positive self-adjoint h_{dg}

$$R^{-1} A (R^*)^{-1} = \begin{bmatrix} h_{\mathrm{dg}} & 0 \\ 0 & \overline{h_{\mathrm{dg}}} \end{bmatrix} =: A_{\mathrm{dg}}. \tag{9}$$

Theorem. (Napiórkowski, Nam, Solovej) In addition, suppose that

$$\|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\|_{\mathrm{HS}} < \infty. \tag{10}$$

Then $R \in Sp_{\mathrm{res}}(\mathcal{Y})$ and hence R is implementable.

Theorem. (Napiórkowski, Nam, Solovej) Assume that $\|h^{-\frac{1}{2}} g \overline{h}^{-\frac{1}{2}}\| < 1$ and $\mathrm{Tr} g^* h^{-1} g < \infty$. Then the form

$$\mathrm{d}\Gamma(h) + \frac{1}{2} a^*(g) + \frac{1}{2} a(g) \tag{11}$$

defined on $\mathrm{Dom}\,\mathrm{d}\Gamma(h)$ is closed and bounded from below. Hence it defines a self-adjoint operator. This operator is the normally ordered quantization of the classical Hamiltonian A.

Set

$$E_A^{\mathrm{n}} := \frac{1}{4}\int_0^1 \mathrm{d}\sigma \mathrm{Tr} A_\sigma^{\frac12}(A_\sigma^{\frac12} S A_\sigma S A_\sigma^{\frac12})^{-\frac12} A_\sigma^{\frac12} G,$$

provided that the above integral is well defined.

Theorem.

1. Let $\mathrm{Tr}\sqrt{\overline{g}g} < \infty$. Then E_A^{n} is well defined.
2. Suppose that $\mathrm{Tr} g h^{-1} g^* < \infty$. Then E_A^{n} and $\hat{H}_A^{\mathrm{n}}$ are well defined and

$$E_A^{\mathrm{n}} = \inf \hat{H}_A^{\mathrm{n}}.$$

EXAMPLE:
SCALAR FIELD WITH
POSITION DEPENDENT MASS

Consider classical variables parametrized by $\vec{x} \in \mathbb{R}^3$ satisfying the Poisson bracket relations

$$\begin{aligned} \{\phi(\vec{x}), \phi(\vec{y})\} = \{\pi(\vec{x}), \pi(\vec{y})\} &= 0, \\ \{\phi(\vec{x}), \pi(\vec{y})\} &= \delta(\vec{x} - \vec{y}). \end{aligned}$$

Consider the classical Hamiltonian of the free scalar field:

$$H_0 = \int \Big(\frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\phi(\vec{x})\big)^2 + \frac{1}{2}m^2\phi^2(\vec{x})\Big)\mathrm{d}\vec{x},$$

If we assume that the mass squared depends on a position, we obtain a perturbed Hamiltonian

$$H = \int \Big(\frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\phi(\vec{x})\big)^2 + \frac{1}{2}(m^2 + \kappa(\vec{x}))\phi^2(\vec{x})\Big)\mathrm{d}\vec{x},$$

Let us replace classical variables ϕ, π with quantum operators $\hat{\phi}, \hat{\pi}$ satisfying the commutation relations

$$[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0,$$

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = \mathrm{i}\delta(\vec{x} - \vec{y}).$$

It is well-known how to quantize H_0. The one-particle space consists of positive-frequency modes. The normally ordered Hamiltonian

$$\hat{H}_0^{\mathrm{n}} = \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\hat{\phi}(\vec{x})\big)^2 + \frac{1}{2}m^2\hat{\phi}^2(\vec{x})\Big):\mathrm{d}\vec{x},$$

acts on the corresponding Fock space. The infimum of $\hat{H}_0$ is zero.

(The Weyl prescription $\hat{H}_0^{\mathrm{w}}$ is ill-defined).

In the case of H, the normally-ordered prescription does not work. One has to renormalize by subtracting the (infinite) contribution of the loop with 2 vertices L_2, which can be formally written as

$$\hat{H}^{\rm ren} = \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\hat{\phi}(\vec{x})\big)^2 + \frac{1}{2}(m^2+\kappa(\vec{x}))\hat{\phi}^2(\vec{x})\Big):\mathrm{d}\vec{x} - L_2,$$

Let us stress that $\hat{H}^{\rm ren}$ is a well-defined self-adjoint operator acting on the same space as $\hat{H}_0^{\rm n}$

The infimum of $\hat{H}^{\mathrm{ren}}$ is the sum of loops

$$\sum_{j=3}^{\infty} L_j$$

with at least 3 vertices. It is called the vacuum energy and is closely related to the so-called effective action.