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Extended weak coupling limit for Friedrichs Hamiltonians

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We study a class of self-adjoint operators defined on the direct sum of two Hilbert spaces: a finite dimensional one called sometimes a "small subsystem" and an infinite dimensional one called a "reservoir." The operator, which we call a "Friedrichs Hamiltonian," has a small coupling constant in front of its off-diagonal term. It is well known that under some conditions in the weak coupling limit the appropriately rescaled evolution in the interaction picture converges to a contractive semigroup when restricted to the subsystem. We show that in this model, the properly renormalized and rescaled evolution converges on the whole space to a new unitary evolution, which is a dilation of the above mentioned semigroup. Similar results have been studied before (Accardi *et al.*, 1990) in more complicated models under the name of "stochastic limit." *© 2007 American Institute of Physics.* [DOI: 10.1063/1.2405402]

I. INTRODUCTION

A. Weak coupling limit

The weak coupling limit is often invoked to justify various approximations in quantum physics, at least since Van Hove (1955). It involves a dynamics depending on a small coupling constant λ. One assumes that

$$\lambda \searrow 0, \quad t \to \infty, \quad \lambda^2 t \text{ fixed}. \tag{1.1}$$

Usually one separates the system into two parts: a "small subsystem" and a "reservoir." The long cumulative effect of the reservoir on the small system can in this limit lead to a Markovian dynamics (i.e., a dynamics given by a semigroup).

There exists a large literature devoted to the weak coupling limit reduced to the small subsystem. It was first put on a rigorous footing by Davies (1974). The setup considered by Davies, in its abstract version, consists of a dynamics generated by $H_\lambda := H_0 + \lambda W$, a projection P commuting with H_0 and such that $PH_0P=0$. Davies proved that under appropriate assumptions there exists the limit of the dynamics in the interaction picture restricted from the left and right by P, and this limit is a semigroup on Ran P. Perhaps, it would be appropriate to call the weak coupling limit reduced to the small subsystem the "Davies limit." Another name which one can use is the "reduced weak coupling limit." (In the literature the names "weak coupling" and "van Hove limit" are used—both are rather imprecise and the latter name is especially ambiguous, since it is also used for a completely different concept in statistical physics.)

Davies and a number of other authors gave applications of the above idea to physically interesting situations describing a dynamics of a composite quantum system, where P is a condi-

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tional expectation onto the small system and the resulting semigroup is completely positive. Note, however, that the reduced weak coupling limit is an interesting mathematical phenomenon also in its more general version.

Some authors point out that it should be possible to use the idea of the weak coupling limit not just for the dynamics restricted to the small system, but for the whole system as well. Accardi *et al.* (1990) argued that the full unreduced dynamics of a quantum system in an appropriate limit can be described by the solution of a quantum Langevin (stochastic) equation. One can express this solution in terms of a 1–parameter group of $*$-automorphisms, which is a dilation of the completely positive semigroup obtained by Davies. They call it the "stochastic limit." We prefer to call it the "extended weak coupling limit," since in itself this concept does not have to involve "stochasticity."

We believe that the above idea is interesting and worth exploring. In our next paper [Dereziński and De Roeck (e-print math-ph/0610054)] we would like to present our version of the extended weak coupling limit applied to quantum systems, with some improvements as compared to Accardi *et al.* (1990). In particular, we believe that the approach of Dereziński and De Roeck (e-print math-ph/0610054) proposes a more satisfactory kind of convergence (strong*) than that of Accardi *et al.* (1990) (convergence of correlation functions) and that the proofs of Dereziński and De Roeck (e-print math-ph/0610054) are considerably simpler than those of Accardi *et al.* (1990).

B. Weak coupling limit for Friedrichs Hamiltonians

In the present paper we present results of the same flavor for a class of simple operators on a Hilbert space, which we call Friedrichs Hamiltonians. We will show that for Friedrichs Hamiltonians the idea of the extended weak coupling limit works very well and yields in a rather natural fashion a unitary dilation of the semigroup Λ_t.

By a "Friedrichs Hamiltonian" we mean a self-adjoint operator H_λ on a Hilbert space $\mathcal{H}=\mathcal{E}\oplus\mathcal{H}_R$ given by the expression

$$H_\lambda := \begin{bmatrix} E & \lambda V \\ \lambda V^* & H_R \end{bmatrix}, \tag{1.2}$$

where E is a self-adjoint operator on the space $\mathcal{E}, V\in\mathcal{B}(\mathcal{E},\mathcal{H}_R)$ and H_R is a self-adjoint operator on $\mathcal{H}_R$. We will assume that $\mathcal{E}$ is finite dimensional. The subscript R stands for the reservoir.

The Friedrichs model, often under other names such as the Wigner-Weisskopf atom, is frequently used as a toy model in mathematical physics. In particular, one often considers its second quantization on the bosonic or fermionic Fock space. (Note that the latter is extensively discussed in Aschbacher *et al.* (2006)).

For a large class of Friedrichs Hamiltonians it is easy to prove that the Davies limit exists. In this case, the Davies limit says that under appropriate assumptions the following limit exists:

$$\lim_{\lambda\searrow 0} e^{itE/\lambda^2} 1_{\mathcal{E}}\, e^{-itH_\lambda/\lambda^2} 1_{\mathcal{E}} =: \Lambda_t, \tag{1.3}$$

where Λ_t is a contractive semigroup on $\mathcal{E}$.

By enlarging the space $\mathcal{E}$ to a larger Hilbert space $\mathcal{Z}=\mathcal{E}\oplus\mathcal{Z}_R$, one can construct a dilation of Λ_t. This means, a unitary group e^{-itZ} such that

$$1_{\mathcal{E}}\, e^{-itZ} 1_{\mathcal{E}} = \Lambda_t.$$

The operator Z is actually another example of a Friedrichs Hamiltonian. We devote Sec. II to the construction of a dilation of a contractive semigroup that is well adapted to the analysis of the weak coupling limit. Note that this construction is quite different from the usual one due to Nagy and Foias (1970). Even though it can be found in many disguises in the literature, we have never seen a systematic description of some of its curious properties. Therefore, in Sec. II we devote some space to study this construction. Note, in particular, that Z is an example of a Friedrichs

Hamiltonian whose definition requires a "renormalization" in the terminology of Dereziński and Früboes (2002).

The main results of our paper are described in Sec. IV. We start from a rather arbitrary Friedrichs Hamiltonian. First we describe its Davies limit. Then we show that for an appropriate "scaling operator" J_λ and a "renormalizing operator" Z_{ren}

$$\lim_{\lambda \searrow 0} e^{it\lambda^{-2}Z_{\mathrm{ren}}} J_\lambda^* e^{-it\lambda^{-2}H_\lambda} J_\lambda = e^{-itZ}.$$

It is this convergence of the dynamics to a dilation of the semigroup Λ_t that we call extended weak coupling limit. Following Dereziński and Früboes (2006) and Dereziński and Früboes (2005), we will give two versions of these results: stationary and time dependent.

Note that the Davies limit follows from the extended weak coupling limit, since

$$1_{\mathcal{E}} e^{i\lambda^{-2}tZ_{\mathrm{ren}}} J_\lambda^* e^{-i\lambda^{-2}tH_\lambda} J_\lambda 1_{\mathcal{E}} = 1_{\mathcal{E}} e^{i\lambda^{-2}tE} e^{-i\lambda^{-2}tH_\lambda} 1_{\mathcal{E}}. \tag{1.4}$$

C. The case of one-dimensional $\mathcal{E}$

The main idea of the extended weak coupling limit can be explained already in the case of a one-dimensional small Hilbert space $\mathcal{E}$. If E has more than one eigenvalue, which is possible if $\dim \mathcal{E} \geqslant 2$, then the extended weak coupling limit is more complicated to formulate and prove, which tends to obscure the whole picture. Therefore, in this subsection we describe the main idea of our result in the case $\dim \mathcal{E}=1$.

Let $\mathcal{E}=\mathbb{C}$ and $\mathcal{H}_R=L^2(\mathbb{R})$. Let $e \in \mathbb{R}$ and let ω be a function on $\mathbb{R}$. Assume that there is a unique $\hat{e} := \omega^{-1}(e)$. Let ω also stand for the corresponding multiplication operator on $\mathcal{H}_R$. Fix a function $v \in L^2(\mathbb{R})$ and denote by $\langle v|$ the operator in $\mathcal{B}(\mathcal{H}_R, \mathcal{E})$ which acts as $\langle v| := \langle v|f\rangle \in \mathcal{E}$ and let $|v\rangle := (\langle v|)^*$. Consider the following Hamiltonian on $\mathcal{E} \oplus \mathcal{H}_R$:

$$H_\lambda := \begin{bmatrix} e & \lambda\langle v| \\ \lambda|v\rangle & \omega \end{bmatrix}. \tag{1.5}$$

(Note that in the literature the name "Friedrichs Hamiltonian" is usually reserved for an operator of the form (1.5). Operators of the form (1.2), should be perhaps called "generalized Friedrichs Hamiltonians.")

The weak coupling limit in this model simply states that, under some mild assumptions,

$$\lim_{\lambda \downarrow 0} 1_{\mathcal{E}} e^{-i\lambda^{-2}t(H_\lambda - e)} 1_{\mathcal{E}} = e^{-i\gamma t}, \tag{1.6}$$

where $1_{\mathcal{E}}$ is the orthogonal projection on $\mathcal{E}$,

$$\gamma := P\int_{\mathbb{R}} \mathrm{d}x \frac{v^*(x)v(x)}{\omega(x) - e} + i\pi v^*(\hat{e})v(\hat{e}), \tag{1.7}$$

and $P1/x$ is the principal value of $1/x$.

$\mathrm{e}^{-i\gamma t}$ is a contractive semigroup on $\mathcal{E}$. It can be dilated to a unitary group e^{-itZ} on the Hilbert space on $\mathcal{E} \oplus \mathcal{H}_R$. The generator of the dilating group can be formally written in the form of a Friedrichs Hamiltonian as

$$Z := \begin{bmatrix} \mathrm{Re}\,\gamma & v(\hat{e})\langle 1| \\ v(\hat{e})|1\rangle & \omega'(\hat{e})x \end{bmatrix}. \tag{1.8}$$

$\omega'(\hat{e})x \in \mathbb{R}$ is the new multiplication operator on $\mathcal{H}_R = L^2(\mathbb{R})$. 1 is the constant function with value 1 which is of course not an element of $L^2(\mathbb{R})$. Because of this Eq. (1.8) does not make sense as an operator. Nevertheless, one can give it a precise meaning, e.g., by constructing its resolvent or its unitary group, or by imposing a cutoff and taking it away [see, e.g., Dereziński and Früboes (2002) and Kümmerer and Schröder (1984)].

To state the stochastic limit, we need the unitary rescaling operator $J_\lambda \in \mathcal{B}(L^2(\mathbb{R}))$ defined as

$$(J_\lambda f)(x) = \frac{1}{\lambda} f\left(\frac{x-\hat{e}}{\lambda^2}\right), \quad f \in L^2(\mathbb{R}). \tag{1.9}$$

If the function v is sufficiently regular in $\hat{e}$, we show the following results:

(1) Theorem 6: The rescaled resolvent $J_\lambda^*(z-\lambda^{-2}(H_\lambda - e))^{-1}J_\lambda$ converges in norm to $(z-Z)^{-1}$.

(2) Theorem 7: The rescaled unitary family $J_\lambda^* \mathrm{e}^{-it\lambda^{-2}t(H_\lambda - e)}J_\lambda$ converges strongly to e^{-itZ}.

D. Notation

We will often make the following abuse of notation. If $\mathcal{H}_0$ is a closed subspace of a Hilbert space $\mathcal{H}$, $A \in \mathcal{B}(\mathcal{H}_0)$, and f is a function on the spectrum of A, then the expression $f(A)$ stands for

$$j_0^* f(A) j_0, \tag{1.10}$$

where j_0 is the embedding of $\mathcal{H}_0$ into $\mathcal{H}$.

We set

$$\mathbb{C}_+ := \{z \in \mathbb{C}, \mathrm{Im}\, z > 0\}, \quad \mathbb{C}_- := \{z \in \mathbb{C}, \mathrm{Im}\, z < 0\}. \tag{1.11}$$

II. DILATIONS

Let $\mathcal{E}$ be a Hilbert space and let the family $\Lambda_{t\in\mathbb{R}^+}$ be a contractive semigroup on $\mathcal{E}$,

$$\Lambda_t \Lambda_s = \Lambda_{t+s}, \quad \|\Lambda_t\| \leq 1, \quad t,s \in \mathbb{R}^+. \tag{2.1}$$

Definition 1:

(1) We say that $(\mathcal{Z}, 1_\mathcal{E}, U_{t\in\mathbb{R}})$ is a unitary dilation of $\Lambda_{t\in\mathbb{R}^+}$ if

- (i) $\mathcal{Z}$ is a Hilbert space and $U_{t\in\mathbb{R}} \in \mathcal{B}(\mathcal{Z})$ is a one-parameter unitary group,
- (ii) $\mathcal{E} \subset \mathcal{Z}$ and $1_\mathcal{E}$ is the orthogonal projection from $\mathcal{Z}$ onto $\mathcal{E}$,
- (iii) for all $t \in \mathbb{R}^+$

$$1_\mathcal{E} U_t 1_\mathcal{E} = \Lambda_t. \tag{2.2}$$

(2) We call a dilation $(\mathcal{Z}, 1_\mathcal{E}, U_{t\in\mathbb{R}})$ minimal iff

$$\{U_t \mathcal{E} | t \in \mathbb{R}\}^{\mathrm{cl}} = \mathcal{Z}. \tag{2.3}$$

We have the following theorem due to Nagy and Foias (1970):

Theorem 1:

(1) *Every contractive semigroup $\Lambda_{t\in R^+}$ has a minimal unitary dilation $(\mathcal{Z}, 1_\mathcal{E}, U_{t\in\mathbb{R}})$, unique up to unitary equivalence.*

(2) $1_\mathcal{E} U_t 1_\mathcal{E} = \Lambda_{-t}^*$ *for* $t<0$.

(3) *If $\Lambda_{t\in R^+}$ is strongly continuous in t, then $U_{t\in R}$ can be chosen to be strongly continuous.*

In the following we present a construction of a unitary dilation, which is well suited for the extended weak coupling limit.

In what follows we assume that the contractive semigroup Λ_t is norm continuous. Hence it has a generator, denoted $-i\Gamma \in \mathcal{B}(\mathcal{E})$, so that $\Lambda_t = \mathrm{e}^{-it\Gamma}$. Since Λ_t is contractive, $-i\Gamma$ is dissipative,

$$\operatorname{Im}\Gamma = \frac{1}{i2}(\Gamma - \Gamma^*) \leqslant 0. \tag{2.4}$$

Let $\mathfrak{h}$ be a Hilbert space. Set $\mathcal{Z}_R = L^2(\mathbb{R}) \otimes \mathfrak{h} = L^2(\mathbb{R}, \mathfrak{h})$ and $\mathcal{Z} = \mathcal{E} \oplus \mathcal{Z}_R$. Let $1_\mathcal{E}$ be the orthogonal projection from $\mathcal{Z}$ onto $\mathcal{E}$.

We define an unbounded linear functional on $L^2(\mathbb{R})$ with the domain $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, denoted $\langle 1|$, given by the obvious prescription

$$\langle 1| f = \int_{\mathbb{R}} f(x) \mathrm{d}x.$$

By $|1\rangle$, we denote the adjoint of $\langle 1|$ in the sense of forms. (Note that the adjoint of $\langle 1|$ in the sense of forms is different from the adjoint in the sense of operators; in particular, the latter has a trivial domain).

Introduce the operator Z_R on $\mathcal{Z}_R$ as the operator of multiplication by the variable x,

$$(Z_R f)(x) = x f(x).$$

Let $\nu \in \mathcal{B}(\mathcal{E}, \mathfrak{h})$ be an operator satisfying the condition

$$\frac{1}{2i}(\Gamma - \Gamma^*) = -\pi \nu^* \nu. \tag{2.5}$$

Put $W = |1\rangle \otimes \nu$ and $W^* = \langle 1| \otimes \nu^*$ and remark that the expressions

$$W, \quad W^*, \quad WSW^*, \quad \text{with } S \in \mathcal{B}(\mathcal{E}), \tag{2.6}$$

are well defined quadratic forms on $\mathcal{D} := \mathcal{E} \oplus ((L^1(\mathbb{R}) \cap L^2(\mathbb{R})) \otimes_{\mathrm{al}} \mathfrak{h})$ (where $\otimes_{\mathrm{al}}$ denotes the algebraic tensor product).

Now we combine these objects into something that is *a priori* a quadratic form on $\mathcal{D}$, but turns out to be a bounded operator. For clarity we will explicitly write the projections $1_\mathcal{E}$ onto $\mathcal{E}$ and 1_R onto $\mathcal{Z}_R$. For $t \geqslant 0$, we define

$$U_t = 1_R\, e^{-itZ_R} 1_R + 1_\mathcal{E}\, e^{-it\Gamma} 1_\mathcal{E} - i 1_\mathcal{E} \int_0^t \mathrm{d}u\, e^{-i(t-u)\Gamma} W^*\, e^{-iuZ_R} 1_R - i 1_R \int_0^t \mathrm{d}u\, e^{-i(t-u)Z_R} W\, e^{-iu\Gamma} 1_\mathcal{E}$$

$$- 1_R \int_{0 \leqslant u_1, u_2, u_1+u_2 \leqslant t} \mathrm{d}u_1 \mathrm{d}u_2\, e^{-iu_2 Z_R} W\, e^{-i(t-u_2-u_1)\Gamma} W^*\, e^{-iu_1 Z_R} 1_R,$$

$$U_{-t} = U_t^*. \tag{2.7}$$

For $z \in \mathbb{C}_+$, we define

$$Q(z) := \begin{bmatrix} 0 & 0 \\ 0 & (z - Z_R)^{-1} \end{bmatrix} + \begin{bmatrix} (z-\Gamma)^{-1} & (z-\Gamma)^{-1} W^* (z - Z_R)^{-1} \\ (z - Z_R)^{-1} W (z-\Gamma)^{-1} & (z - Z_R)^{-1} W (z-\Gamma)^{-1} W^* (z - Z_R)^{-1} \end{bmatrix},$$

$$Q(\bar{z}) := Q(z)^*. \tag{2.8}$$

Next we define the following quadratic form on $\mathcal{D}$, using the matrix notation with respect to the decomposition $\mathcal{Z} = \mathcal{E} \oplus \mathcal{Z}_R$:

$$Z^+ = \begin{bmatrix} \Gamma & W^* \\ W & Z_R \end{bmatrix}, \quad Z^- = \begin{bmatrix} \Gamma^* & W^* \\ W & Z_R \end{bmatrix}. \tag{2.9}$$

Last, for $k \in N$, we define approximants $W_k \in \mathcal{B}(\mathcal{E}, L^2(R, \mathfrak{h}))$ to the form W

$$W_k u := |1_{[-k,k]}\rangle \otimes \nu, \tag{2.10}$$

and approximants $Z_{R,k} \in \mathcal{B}(\mathcal{Z}_R)$ for Z_R

$$Z_{R,k} := 1_{[-k,k]}(Z_R) Z_R, \tag{2.11}$$

where $1_{[-k,k]}$ denotes the characteristic function of $[-k,k]$. We set

$$Z_k = \begin{bmatrix} \mathrm{Re}\,\Gamma & W_k^* \\ W_k & Z_{R,k} \end{bmatrix}. \tag{2.12}$$

We have

Theorem 2: *Let U_t be as in* Eq. (2.7), *$Q(z)$ as in* Eq. (2.8), *$Z^{\pm}$ as in* Eq. (2.9), *and Z_k as in* Eq. (2.12), *with Γ satisfying condition* (2.5).

(1) *The family $Q(z)$ is the resolvent of a self-adjoint operator Z; that is, there exists a unique self-adjoint operator Z such that for all $z \in \mathbb{C} \setminus \mathbb{R}$*

$$Q(z) = (z - Z)^{-1}. \tag{2.13}$$

(2) *U_t extends to a unitary, strongly continuous one-parameter group in $\mathcal{B}(\mathcal{Z})$ and*

$$U_t = e^{-itZ}. \tag{2.14}$$

(3) *Fix* $\mathrm{Im}\, z_0 > 0$. $\mathrm{Dom}\, Z$ *consists of vectors ψ of the following form:*

$$\psi = \begin{bmatrix} u \\ (z_0 - Z_R)^{-1} W u + g \end{bmatrix}, \quad u \in \mathcal{E}, g \in \mathrm{Dom}\, Z_R, \tag{2.15}$$

and Z transforms ψ into

$$Z\psi = \begin{bmatrix} \Gamma u + W^* g \\ z_0 (z_0 - Z_R)^{-1} W u + Z_R g \end{bmatrix}. \tag{2.16}$$

(4) *For $\psi \in \mathrm{Dom}\, Z$, we have*

$$Z\psi = \lim_{k \to \infty} Z_k \psi. \tag{2.17}$$

(5) *For $\psi, \psi' \in \mathcal{D}$, the function $\mathbb{R} \ni t \mapsto \langle \psi | U_t \psi' \rangle$ is differentiable away from $t=0$, its derivative $t \mapsto (\mathrm{d}/\mathrm{d}t)\langle \psi | U_t \psi' \rangle$ is continuous away from 0 and at $t=0$ it has the left and the right limits equal respectively to*

$$-i\langle \psi | Z^+ \psi' \rangle = \lim_{t \downarrow 0} t^{-1} \langle \psi | (U_t - 1)\psi' \rangle, \tag{2.18}$$

$$-i\langle \psi | Z^- \psi' \rangle = \lim_{t \uparrow 0} t^{-1} \langle \psi | (U_t - 1)\psi' \rangle. \tag{2.19}$$

(6) The group *U_t dilates the semigroup generated by $-i\Gamma$, that is, for $t \geq 0$,*

$$1_{\mathcal{E}} U_t 1_{\mathcal{E}} = e^{-it\Gamma}. \tag{2.20}$$

(7) *This dilation is minimal if $\mathfrak{h} = \mathrm{Ran}\, \nu$.*

Remark 1: Naturally, every densely defined operator gives rise to a quadratic form on its domain. However, Z^+ and Z^- are not derived from Z in this way. This is seen from the explicit description of these domains, as well as from the fact that for $\psi \in \mathrm{Dom}\, Z$ we have $\mathrm{d}/\mathrm{d}\, t U(t)\psi|_{t=0} = -iZ\psi$, which should be compared with Eqs. (2.18) and (2.19).

Remark 2: Motivated by Eqs. (2.12) and (2.17), we can say that in some sense the operator Z is given by the matrix

$$Z = \begin{bmatrix} \mathrm{Re}\,\Gamma & W^* \\ W & Z_R \end{bmatrix}. \tag{2.21}$$

One should, however, remember, that strictly speaking the expression (3.20) does not define an operator. To define it an appropriate renormalization is needed: one needs to impose a symmetric cutoff and then remove it. The precise meaning of this renormalization is described by Eqs. (2.15) and (2.16), or by Eq. (2.17). Nevertheless, in the sequel, we will freely use expressions of the form (2.21), remembering that its meaning is given by Theorem 2.

Remark 3: For $\lambda \in \mathbb{R}$, introduce the following unitary operator on $\mathcal{Z}$:

$$j_\lambda u = u, \quad u \in \mathcal{E}; \quad j_\lambda g(y) := \lambda^{-1} g(\lambda^{-2} y), \quad g \in \mathcal{Z}_R.$$

Note that

$$j_\lambda^* Z_R j_\lambda = \lambda^2 Z_R, \quad j_\lambda^* |1\rangle = \lambda |1\rangle.$$

Therefore, the operator Z enjoys the following scaling property, which plays an important role in the extended weak coupling limit:

$$\lambda^{-2} J_\lambda^* \begin{bmatrix} \lambda^2\,\mathrm{Re}\,\Gamma & \lambda W^* \\ \lambda W & Z_R \end{bmatrix} J_\lambda = \begin{bmatrix} \mathrm{Re}\,\Gamma & W^* \\ W & Z_R \end{bmatrix}.$$

III. WEAK COUPLING LIMIT

A. Notation and assumptions

Let $\mathcal{E}$ and $\mathcal{H}_R$ be Hilbert spaces. We assume that $\mathcal{E}$ is finite dimensional. We set $\mathcal{H} = \mathcal{E} \oplus \mathcal{H}_R$.

Fix a self-adjoint operator H_R on $\mathcal{H}_R$ and E on $\mathcal{E}$. Let the free Hamiltonian H_0 on $\mathcal{H}$ be given as

$$H_0 = E \oplus H_R.$$

Let $V \in \mathcal{B}(\mathcal{E}, \mathcal{H}_R)$. By a slight abuse of notation we denote by V the corresponding operator on $\mathcal{H}$. For $\lambda \in \mathbb{R}$, let the interacting Friedrichs Hamiltonian be

$$H_\lambda = H_0 + \lambda(V + V^*). \tag{3.1}$$

We write $E = \sum_{e \in \mathrm{sp}E} e 1_e(E)$ where $e, 1_e(E)$ are the eigenvalues and spectral projections of E. The spectral subspace of E for e is denoted $\mathcal{E}_e$. Let us list the assumptions that we will use in our construction.

A1: *Let $\mathfrak{h}_0, \mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_\infty$ denote the Hilbert spaces of dimension $0, 1, 2, \ldots, \infty$. We assume that there exists a partition of $\mathbb{R}$ into measurable sets $I_0, I_1, I_2, \ldots, I_\infty$ and a unitary identification*

$$\mathcal{H}_R \simeq \int_{\mathbb{R}}^{\oplus} \mathfrak{h}(x)\mathrm{d}x \simeq \bigoplus_{n=0}^{\infty} L^2(I_n) \otimes \mathfrak{h}_n, \tag{3.2}$$

where $\mathfrak{h}(x) := \mathfrak{h}_n$ for $x \in I_n$ and H_R is the operator of the multiplication by the variable x. Thus, if $f = \int_{\mathbb{R}}^{\oplus} f(x)\mathrm{d}x \in \mathcal{H}_R$, then

$$(H_R f)(x) = x f(x)$$

for Lebesgue almost all x. Moreover, there exists a measurable function

$$\mathbb{R} \ni x \mapsto v(x) \in B(\mathcal{E}, \mathfrak{h}(x))$$

such that for Lebesgue a.a. $x \in \mathbb{R}$ and all $u \in \mathcal{E}$

$$(Vu)(x) = v(x)u. \tag{3.3}$$

In what follows, the identification (3.2) is fixed and will be used to define the scaling operator J_λ. Note that if $\mathcal{H}$ is separable, the existence of such an identification is guaranteed by abstract measure-theoretic arguments.

A2: *For any $e \in \mathrm{sp}E$, there exists $n(e) \in \{0,1,2,\ldots\infty\}$ such that e belongs to the interior of $I_{n(e)}$. We will write $\mathfrak{h}_e$ for $\mathfrak{h}_{n(e)}$. Moreover, we assume that v is continuous at $\mathrm{sp}E$, so that for $e \in \mathrm{sp}E$, we can unambiguously define $v(e) \in \mathcal{B}(\mathcal{E},\mathfrak{h}_e)$.*

A3: *There is $\delta > 0$, such that for a certain $c > 0$ and for all $e \in \mathrm{sp}E$,*

$$\|v^*(x)v(x) - v^*(e)v(e)\| \leq c|x-e|^\delta. \tag{3.4}$$

We also assume that $x \mapsto \|v(x)\|$ is bounded.

B. The reduced weak coupling limit

In this subsection we describe the reduced weak coupling limit (or the Davies limit) for Friedrichs Hamiltonians. The Davies limit is usually given in its time dependent version described in Theorem 4. Its stationary form, which comes from Dereziński and Früboes (2006) and Dereziński and Früboes (2005), has some technical advantages over the time dependent version.

In both theorems about the Davies limit we do not suppose Assumptions A1, A2, and A3.

Theorem 3 (Stationary Davies limit): *Suppose that for $e \in \mathrm{sp}E$ and $z \in \mathbb{C}_+$*

$$\lim_{\epsilon \downarrow 0} V^*(e + \epsilon z - H_R)^{-1} V$$

exists and is independent of z. Set

$$\Gamma_e^{\mathrm{st}} := \lim_{\epsilon \downarrow 0} 1_{\mathcal{E}_e} V^*(e + \epsilon z - H_R)^{-1} V 1_{\mathcal{E}_e},$$

$$\Gamma^{\mathrm{st}} := \sum_{e \in \mathrm{sp}E} \Gamma_e^{\mathrm{st}}.$$

Then

(1) *for $z \in \mathbb{C}_+$,*

$$\lim_{\lambda \to 0} 1_{\mathcal{E}}(z - \lambda^{-2}(H_\lambda - e))^{-1} 1_{\mathcal{E}} = (z - \Gamma_e^{\mathrm{st}})^{-1} 1_{\mathcal{E}}; \tag{3.5}$$

(2) *for all continuous functions with compact support $f \in \mathcal{C}_c([0,+\infty])$,*

$$\lim_{\lambda \downarrow 0} \int_{\mathbb{R}^+} \mathrm{d}t f(t) \mathrm{e}^{i\lambda^{-2}tE} 1_{\mathcal{E}} \mathrm{e}^{-i\lambda^{-2}tH_\lambda} 1_{\mathcal{E}} = \int_{\mathbb{R}^+} \mathrm{d}t f(t) \mathrm{e}^{-it\Gamma^{\mathrm{st}}}, \tag{3.6}$$

where all limits are in operator norm.

Theorem 4 (Time dependent Davies limit): *Assume that*

$$\lim_{t \to \infty} \int_0^t \mathrm{e}^{isE} V^* \mathrm{e}^{-isH_R} V \mathrm{d}s$$

exists. Set

$$\Gamma_e^{\mathrm{dyn}} := \lim_{t \to \infty} \int_0^t 1_{\mathcal{E}_e} V^* \mathrm{e}^{-is(H_R - e)} V 1_{\mathcal{E}_e} \mathrm{d}s,$$

$$\Gamma^{\mathrm{dyn}} := \sum_{e \in \mathrm{sp}E} \Gamma_e^{\mathrm{dyn}}.$$

Then

$$\lim_{\lambda\to 0}\sup_{0\leq t\leq T}\|\mathrm{e}^{i\lambda^{-2}tE}1_{\mathcal{E}}\mathrm{e}^{-i\lambda^{-2}tH_\lambda}1_{\mathcal{E}}-\mathrm{e}^{-it\Gamma^{\mathrm{dyn}}}\|=0. \tag{3.7}$$

In practice, Γ^{st} and Γ^{dyn} coincide. They will be denoted simply by Γ and called the Davies generator.

Theorem 5 (Formula for the Davies generator): *Suppose that Assumptions* A1, A2, *and* A3 *are true. Then the assumptions of Theorems 3 and 4 are true. Moreover, for* $e\in \mathrm{sp}E$ *and* $z\in\mathbb{C}^+$,

$$-i\lim_{t\to+\infty}\int_0^t \mathrm{d}sV^*\,\mathrm{e}^{-is(H_R-e)}V=\lim_{\epsilon\downarrow 0}V^*(e+\epsilon z-H_R)^{-1}V=\mathrm{P}\int_{\mathbb{R}}\mathrm{d}x\frac{v^*(x)v(x)}{x-e}+i\pi v^*(e)v(e),$$

where P *denotes the principal value. Consequently, the stationary and time dependent Davies generator coincide*,

$$\Gamma_e := \Gamma_e^{\mathrm{dyn}}=\Gamma_e^{\mathrm{st}}=1_{\mathcal{E}_e}\left(\mathrm{P}\int_{\mathbb{R}}\mathrm{d}x\frac{v^*(x)v(x)}{x-e}+i\pi v^*(e)v(e)\right)1_{\mathcal{E}_e}. \tag{3.8}$$

C. Asymptotic space and dynamics

Let $e\in\mathrm{sp}E$. The asymptotic reservoir space and "total" space corresponding to e is

$$\mathcal{Z}_{R_e} := L^2(\mathbb{R})\otimes\mathfrak{h}_e=L^2(\mathbb{R},\mathfrak{h}_e),$$

$$\mathcal{Z}_e := \mathcal{E}_e\oplus\mathcal{Z}_{R_e}.$$

We have the projections

$$1_{\mathcal{E}_e}:\mathcal{Z}_e\to\mathcal{E}_e,\quad 1_{R_e}:\mathcal{Z}_e\to\mathcal{Z}_{R_e}.$$

Let Z_{R_e} be the operator of multiplication by the variable in $\mathbb{R}$ on $\mathcal{Z}_{R_e}$. We define the map $\nu_e:\mathcal{E}_e\to\mathfrak{h}_e$

$$\nu_e := v(e)1_{\mathcal{E}_e}.$$

Under the assumptions A1, A2, and A3, we define the operator Γ_e on $\mathcal{E}_e$, as in Eq. (3.8).

Note that $-\pi\nu_e^*\nu_e=(1/2i)(\Gamma_e-\Gamma_e^*)$, which is the analog of the condition (2.5) for Z_{R_e}, Γ_e, and ν_e for the space $\mathcal{Z}_e=\mathcal{E}_e\oplus\mathcal{Z}_{R_e}$. One can thus apply the procedure of Sec. II and construct a unitary dilation of the semigroup $\mathrm{e}^{-it\Gamma_e}$, as defined in Eq. (2.7). We will denote this dilation by e^{-itZ_e}.

We construct the full asymptotic space as a direct sum of independent reservoirs, for each eigenvalue of E:

$$\mathfrak{h} := \bigoplus_{e\in\mathrm{sp}E}\mathfrak{h}_e,$$

$$\mathcal{Z}_R := \bigoplus_{e\in\mathrm{sp}E}\mathcal{Z}_{R_e}=L^2(\mathbb{R},\mathfrak{h}),$$

$$\mathcal{Z} := \bigoplus_{e\in\mathrm{sp}E}\mathcal{Z}_e=\mathcal{E}\oplus\mathcal{Z}_R.$$

We have the asymptotic reservoir Hamiltonian

$$Z_R = \underset{e \in \mathrm{sp}E}{\oplus} Z_{R_e}.$$

We define the map $\nu: \mathcal{E} \to \mathfrak{h}$,

$$\nu := \underset{e \in \mathrm{sp}E}{\oplus} \nu_e,$$

where we used the decomposition $\mathcal{E} = \underset{e \in \mathrm{sp}E}{\oplus} \mathcal{E}_e$ and $\mathfrak{h} = \underset{e \in \mathrm{sp}E}{\oplus} \mathfrak{h}_e$. We also have the operator Γ on $\mathcal{E}$ as defined in Sec. II.

Clearly, Z_R, Γ, and ν satisfy the condition (2.5). One can thus apply the procedure of Sec. II and construct a unitary dilation e^{-itZ} of the semigroup $\mathrm{e}^{-it\Gamma}$ on $\mathcal{Z} = \mathcal{E} \oplus \mathcal{Z}_R$.

Obviously, everything we constructed commutes with the orthogonal projections $1_e: \mathcal{Z} \to \mathcal{Z}_e$, and we have

$$Z = \underset{e \in \mathrm{sp}E}{\oplus} Z_e.$$

We define the renormalizing Hamiltonian Z_{ren} on $\mathcal{Z}$,

$$Z_{\mathrm{ren}} := \sum_{e \in \mathrm{sp}E} e 1_e = E + \sum_{e \in \mathrm{sp}E} e 1_{R_e}. \tag{3.9}$$

D. Scaling

For any $e \in \mathrm{sp}E$, we choose an open set $\tilde{I}_e$ such that $e \in \tilde{I}_e \subset I_e$ and $\tilde{I}_e$ are mutually disjoint. For $\lambda \in \mathbb{R}_+$, define the family of contractions $J_{\lambda,e}: \mathcal{E}_e \oplus \mathcal{Z}_{R_e} = \mathcal{Z}_e \to \mathcal{E} \oplus L^2(\tilde{I}_e, \mathfrak{h}_e)$, which on $g_e \in \mathcal{Z}_{R_e}$ act as

$$(J_{\lambda,e} g_e)(y) = \begin{cases} \dfrac{1}{\lambda} g_e\left(\dfrac{y-e}{\lambda^2}\right) & \text{if } y \in \tilde{I}_e \\ 0 & \text{if } y \in \mathbb{R} \setminus \tilde{I}_e, \end{cases} \tag{3.10}$$

and on $\mathcal{E}_e$ equals $1_{\mathcal{E}_e}$. Note that

$$J_{\lambda,e}^* J_{\lambda,e} = 1_{\mathcal{E}_e} \oplus 1_{\lambda^{-2}(\tilde{I}_e - e)}(Z_{R_e}), \quad J_{\lambda,e} J_{\lambda,e}^* = 1_{\mathcal{E}_e} \oplus 1_{\tilde{I}_e}(H_R). \tag{3.11}$$

For $\psi = \oplus_{e \in \mathrm{sp}E} \psi_e$ we set

$$J_\lambda \psi := \underset{e \in \mathrm{sp}(E)}{\oplus} J_{\lambda,e} \psi_e.$$

Note that J_λ is a partial isometry from $\mathcal{Z}$ to $\mathcal{H}$.

Remark 4: The precise form of J_λ only matters in a neighborhood of $\mathrm{sp}E$. For instance, let $\tilde{I}_e \ni y \mapsto \eta_e(y)$ be increasing functions differentiable at e such that $(\mathrm{d}\eta_e/\mathrm{d}y)(e) = 1$ for $e \in \mathrm{sp}(E)$. Set

$$(J_{\lambda,e}^\eta g_e)(y) = \begin{cases} \dfrac{1}{\lambda} g_e\left(\dfrac{\eta_e(y) - \eta_e(e)}{\lambda^2}\right) & \text{if } y \in \tilde{I}_e \\ 0 & \text{if } y \in \mathbb{R} \setminus \tilde{I}_e. \end{cases} \tag{3.12}$$

Then all the statements in this paper remain true if one replaces J_λ by J_λ^η.

E. Main results

In this subsection we state the two main results of our paper. They say that the dynamics generated by H_λ after an appropriate rescaling and renormalization for a small coupling approaches the asymptotic dynamics. Again, we present two versions of the result: stationary and time dependent.

Theorem 6 (Stationary extended weak coupling limit): *Assume* A1, A2, and A3. *Let* Z_e *and* Z *be as defined in Sec. III C and let* J_λ *be as defined in Sec. III D.*

(1) *For any* $e \in \mathrm{sp}E$ *and* $z \in \mathbb{C}^+$,

$$\lim_{\lambda \downarrow 0} J_\lambda^*(z - \lambda^{-2}(H_\lambda - e))^{-1} J_\lambda = (z - Z_e)^{-1} 1_e. \tag{3.13}$$

(2) *For all continuous functions with compact support* $f \in \mathcal{C}_c([0,+\infty])$,

$$\lim_{\lambda \downarrow 0} \int_{\mathbb{R}^+} \mathrm{d}t f(t) \mathrm{e}^{i\lambda^{-2} t Z_{\mathrm{ren}}} J_\lambda^* \mathrm{e}^{-i\lambda^{-2} t H_\lambda} J_\lambda = \int_{\mathbb{R}^+} \mathrm{d}t f(t) \mathrm{e}^{-itZ}, \tag{3.14}$$

where all limits are in operator norm.

Theorem 7 (Time dependent extended weak coupling limit): *Assume* A1, A2, and A3. *Let* Z_e *and* Z *be as defined in Sec. III C and let* J_λ *be as defined in Sec. III D. For all* $\psi \in \mathcal{Z}$ *and* $t \in \mathbb{R}$,

$$\lim_{\lambda \downarrow 0} \mathrm{e}^{i\lambda^{-2} t Z_{\mathrm{ren}}} J_\lambda^* \mathrm{e}^{-i\lambda^{-2} t H_\lambda} J_\lambda \psi = \mathrm{e}^{-itZ} \psi. \tag{3.15}$$

Remark 5: From the proof of Theorem 7, it follows immediately that Eq. (3.15) can be stated uniformly in t on compact intervals, but in weak operator topology. For all $\psi, \psi' \in \mathcal{Z}$ and $0 < T < \infty$,

$$\lim_{\lambda \downarrow 0} \sup_{0 \le |t| \le T} |\langle \psi' | \mathrm{e}^{i\lambda^{-2} t Z_{\mathrm{ren}}} J_\lambda^* \mathrm{e}^{-i\lambda^{-2} t H_\lambda} J_\lambda \psi - \mathrm{e}^{-itZ} \psi \rangle| = 0. \tag{3.16}$$

Remark 6: One can also state Eq. (3.15) in the interaction picture, avoiding the renormalizing Hamiltonian Z_{ren}. For all $t \in \mathbb{R}$ and $\psi \in \mathcal{Z}$,

$$\lim_{\lambda \downarrow 0} J_\lambda^* \mathrm{e}^{i\lambda^{-2} t H_0} \mathrm{e}^{-i\lambda^{-2} t H_\lambda} J_\lambda \psi = \mathrm{e}^{itZ_R} \mathrm{e}^{-itZ} \psi. \tag{3.17}$$

This is seen most easily by remarking that for all $t \in \mathbb{R}$ and $\psi \in \mathcal{Z}$,

$$\lim_{\lambda \downarrow 0} J_\lambda^* \mathrm{e}^{i\lambda^{-2} t H_0} J_\lambda \mathrm{e}^{-i\lambda^{-2} t Z_{\mathrm{ren}}} \psi = \mathrm{e}^{itZ_R} \psi. \tag{3.18}$$

IV. PROOFS

A. Proof of Theorem 2

Statement (1) of Theorem 2 follows by the arguments described in a slightly different context in Theorem 2.1 of Dereziński and Früboes (2002). One can take over the proof of Dereziński and Früboes (2002) almost verbatim. For completeness, we reproduce an adjusted proof.

Let $W_k \in \mathcal{B}(\mathcal{E}, L^2(\mathbb{R}, \mathfrak{h}))$ for $k \in \mathbb{N}$ be defined as in Eq. (2.10)

$$\Gamma_k(z) := \mathrm{Re}\,\Gamma + W_k^*(z - Z_R)^{-1} W_k. \tag{4.1}$$

Obviously, the operator

$$Z_k := \mathrm{Re}\,\Gamma + Z_R + W_k^* + W_k \tag{4.2}$$

is a well defined self-adjoint operator on Dom Z_R (since it is a bounded perturbation of Z_R). By the Feshbach formula [see Eq. (4.36)], one checks that the resolvent $(z-Z_k)^{-1}$ is norm convergent to $Q(z)$: It suffices to remark that for all $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\lim_{k\to\infty} \Gamma_k(z) = \Gamma(z), \tag{4.3}$$

$$\lim_{k\to\infty} W_k^*(z-Z_R)^{-1} = W^*(z-Z_R)^{-1} \tag{4.4}$$

in norm. It follows that $Q(z)$ satisfies the resolvent formula. To obtain that $Q(z)$ is actually the resolvent of a (uniquely defined) self-adjoint operator, it suffices [see Kato (1984)] to establish for all $z \in \mathbb{C}\backslash\mathbb{R}$, (1) Ker $Q(z)=\{0\}$; (2) Ran $Q(z)$ is dense in $\mathcal{Z}$; (3) $Q^*(z)=Q(\bar{z})$.

(3) is obvious. To prove (1), we let $u \oplus g \in \mathcal{E} \oplus \mathcal{Z}_R$ and we assume $Q(z)u \oplus g=0$. Suppose that, e.g., $z \in \mathbb{C}_+$. Then

$$(z-\Gamma)^{-1}(u + W^*(z-Z_R)^{-1}g) = 0, \tag{4.5}$$

$$(z-Z_R)^{-1}W(z-\Gamma)^{-1}(u + W^*(z-Z_R)^{-1}g) + (z-Z_R)^{-1}g = 0. \tag{4.6}$$

Inserting Eqs. (4.5) into (4.6) yields $(z-Z_R)^{-1}g=0$ and hence $g=0$. Combined with Eq. (4.5), the latter implies $u\oplus g=0$.

Using (1) and (3), we get (2), since

$$\mathrm{Ran}\, Q(z)^{\perp} = \mathrm{Ker}\, Q(z)^* = \mathrm{Ker}\, Q(\bar{z}) = \{0\}. \tag{4.7}$$

Hence, statement 1 of Theorem 2 is proven.

To prove statement (2) we take $\psi,\psi' \in \mathcal{D}$ and compute the following Laplace transform:

$$-i\int_0^{+\infty} \mathrm{d}t\, \mathrm{e}^{izt}\langle\psi|U_t\psi'\rangle = \langle\psi|Q(z)\psi'\rangle. \tag{4.8}$$

By functional calculus and the fact that $Q(z)=(z-Z)^{-1}$,

$$-i\int_0^{+\infty} \mathrm{d}t \mathrm{e}^{izt}\langle\psi|\mathrm{e}^{-itZ}\psi'\rangle = \langle\psi|Q(z)\psi'\rangle. \tag{4.9}$$

Both $t\mapsto\langle\psi|U_t\psi'\rangle$ and $t\mapsto\langle\psi|\mathrm{e}^{-itZ}\psi'\rangle$ are continuous functions, and we can apply the inverse Laplace transform to Eqs. (4.8) and (4.9), which yields $\langle\psi|U_t\psi'\rangle=\langle\psi|\mathrm{e}^{-itZ}\psi'\rangle$. By the density of $\mathcal{D}$ we obtain $U_t=\mathrm{e}^{-itZ}$. This, in particular, proves that U_t satisfies the group property.

To prove statement (3) we note that any vector in $\mathcal{Z}_R$ can be written as $(z_0-Z_R)g$ for some $g\in\mathrm{Dom}\,Z_R$. Given such g, any vector in $\mathcal{E}$ can be written as $(z_0-\Gamma)u-W^*g$ (here we use the invertibility of $z_0-\Gamma$). Set

$$\phi := \begin{bmatrix} (z_0-\Gamma)u - W^*g \\ (z_0-Z_R)g \end{bmatrix}.$$

Then $\psi=Q(z_0)\phi$ equals Eq. (2.15) and $Z\psi=-\phi+z_0Q(z_0)\phi$ equals Eq. (2.16).

Statements (4)–(6) follow by straightforward calculations.

To prove statement (7), we observe that

$$\mathrm{Span}\{\mathrm{e}^{-itZ}\mathcal{E}, t\in\mathbb{R}\}^{\mathrm{cl}} = \mathrm{Span}\{(z-Z)^{-1}\mathcal{E}, z\in\mathbb{C}\setminus\mathbb{R}\}^{\mathrm{cl}}. \tag{4.10}$$

Since Span$\{x\mapsto(z-x)^{-1}, z\in\mathbb{C}\backslash\mathbb{R}\}$ is dense in $L^2(\mathbb{R})$, and using the fact that $(z-\Gamma)^{-1}, z\in\mathbb{C}_+$ and $(z-\Gamma^*)^{-1}, z\in\mathbb{C}_-$ are invertible, we have

$$\mathrm{Span}\{(z-Z)^{-1}\mathcal{E}, z \in \mathbb{C} \setminus \mathbb{R}\}^{\mathrm{cl}} = \{u \oplus (L^2(\mathbb{R}) \otimes \nu u), u \in \mathcal{E}\}. \tag{4.11}$$

This easily implies statement (7). □

B. Proof of Theorem 3

Theorem 3 is essentially a special case of Theorem 2 from Dereziński and Früboes (2005) [see also Dereziński and Früboes (2006)]. For the convenience of the reader, and because the case we consider allows for some simplifications, we sketch the proof below.

Let

$$G^{-1}(e,\lambda,z) := 1_{\mathcal{E}}(z-\lambda^{-2}(H_\lambda - e))^{-1}1_{\mathcal{E}}, \tag{4.12}$$

which yields immediately the bound

$$\|G^{-1}(e,\lambda,z)\| \leq |\mathrm{Im}\, z|^{-1}. \tag{4.13}$$

In the following we simplify the notation $G(e,\lambda,z)$ into G (hence, we fix a certain $e \in \mathrm{sp}E$) and we put

$$G_{\mathrm{d}} = \sum_{e' \in \mathrm{sp}E} 1_{\mathcal{E}_{e'}} G 1_{\mathcal{E}_{e'}}, \quad G_{\mathrm{o}} := G - G_{\mathrm{d}}, \tag{4.14}$$

and $1_{\mathcal{E}_{\underline{e}}} := 1_{\mathcal{E}} - 1_{\mathcal{E}_e}$.

By the Feshbach formula [see further Eq. (4.36)], we have

$$G = z - \lambda^{-2}(E-e) - \lambda^{-2} 1_{\mathcal{E}} V^* (z - \lambda^{-2}(H_R - e))^{-1} V 1_{\mathcal{E}}.$$

By the assumption of Theorem 3, it is immediate that

$$\lim_{\lambda \downarrow 0} 1_{\mathcal{E}_e} G_{\mathrm{d}}^{-1} = (z - \Gamma_e^{\mathrm{st}})^{-1}. \tag{4.15}$$

By the Neumann expansion and the assumption of Theorem 3, one has for small enough λ and some $c>0$

$$\|1_{\mathcal{E}_{\underline{e}}} G_{\mathrm{d}}^{-1}\| \leq c\lambda^2, \quad \|G_{\mathrm{o}}\| < c. \tag{4.16}$$

From $G = G_d + G_o$, we deduce

$$G^{-1} = G_{\mathrm{d}}^{-1} - G_{\mathrm{d}}^{-1} G_{\mathrm{o}} G_{\mathrm{d}}^{-1} + G_{\mathrm{d}}^{-1} G_{\mathrm{o}} G_{\mathrm{d}}^{-1} G_{\mathrm{o}} G^{-1}, \tag{4.17}$$

from which

$$1_{\mathcal{E}_e}(G^{-1} - G_{\mathrm{d}}^{-1}) = -1_{\mathcal{E}_e} G_{\mathrm{d}}^{-1} G_{\mathrm{o}} 1_{\mathcal{E}_{\underline{e}}} G_{\mathrm{d}}^{-1}(1 - G_{\mathrm{o}} G^{-1}). \tag{4.18}$$

Using Eqs. (4.13), (4.15), and (4.16), we see that the right hand side of Eq. (4.18) vanishes, yielding

$$\lim_{\lambda \downarrow 0} 1_{\mathcal{E}_e} G^{-1} = \lim_{\lambda \downarrow 0} 1_{\mathcal{E}_e} G_{\mathrm{d}}^{-1} = (z - \Gamma_e^{\mathrm{st}})^{-1}. \tag{4.19}$$

Writing

$$1_{\mathcal{E}_{\underline{e}}} G^{-1} = 1_{\mathcal{E}_{\underline{e}}} G_{\mathrm{d}}^{-1} - 1_{\mathcal{E}_{\underline{e}}} G_{\mathrm{d}}^{-1} G_{\mathrm{o}} G^{-1} \tag{4.20}$$

and using Eq. (4.16), one sees that

$$\lim_{\lambda\downarrow 0} 1_{\mathcal{E}_e} G^{-1} = 0. \tag{4.21}$$

Together, Eqs. (4.19) and (4.21) end the proof of (1).

Statement (2) follows from (1) as in Dereziński and Früboes (2006). □

C. Proof of Theorem 4

Theorem 4 is a special case of a well known result of Davies (1974) reproduced, e.g., in Dereziński and Früboes (2006). For the convenience of the reader, and because some simplifications are possible, we sketch the proof below.

We start from the following representation for $\Lambda_{t,\lambda} := e^{it\lambda^{-2}E} 1_{\mathcal{E}} e^{-it\lambda^{-2}H_\lambda} 1_{\mathcal{E}}$:

$$\Lambda_{t,\lambda} = 1 + \int_0^t D_{\lambda,t}(u)\Lambda_{u,\lambda} \mathrm{d}u, \tag{4.22}$$

with

$$\begin{aligned} D_{\lambda,t}(u) &= \lambda^{-2}\int_u^t e^{i\lambda^{-2}vE} V^* e^{-i\lambda^{-2}(v-u)H_R} V e^{-i\lambda^{-2}uE} \mathrm{d}v \\ &= \sum_{e,e'\in \mathrm{sp}E} \int_0^{\lambda^{-2}(t-u)} 1_{\mathcal{E}_e} V^* e^{-is(H_R-e)} V 1_{\mathcal{E}_{e'}} e^{-i\lambda^{-2}u(e'-e)} \mathrm{d}s. \end{aligned} \tag{4.23}$$

For $T>0$, let $Q := \mathcal{C}_0([0,T])$ be the Banach space of continuous functions, equipped with the supremum norm. Define the operators K_λ and K on Q by (for $0 \leqslant t \leqslant T$)

$$(K_\lambda f)(t) = \int_0^t D_{\lambda,t}(s) f(s)\mathrm{d}s, \quad (Kf)(t) = -i\Gamma^{\mathrm{dyn}} \int_0^t f(s)\mathrm{d}s. \tag{4.24}$$

We will prove that

$$\mathrm{s}-\lim_{\lambda\downarrow 0} K_\lambda = K. \tag{4.25}$$

Let

$$\tilde{\Gamma} := -i \lim_{t\to+\infty} \int_0^t V^* e^{-is(H_R-e)} V \mathrm{d}s, \tag{4.26}$$

whose existence was proven in Theorem 5.

One checks that for all $t\in[0,T]$

$$\lim_{\lambda\downarrow 0} \left| (K_\lambda f)(t) + i\sum_{e,e'} \int_0^t 1_{\mathcal{E}_e} \tilde{\Gamma} 1_{\mathcal{E}_{e'}} e^{-i\lambda^{-2}s(e'-e)} f(s)\mathrm{d}s \right| = 0, \tag{4.27}$$

which follows by the assumption of Theorem 4 and dominated convergence. Since f is (bounded and continuous, hence) integrable, the Riemann-Lesbegue lemma yields, for $e,e' \in \mathrm{sp}E$,

$$\lim_{\lambda\downarrow 0} i\int_0^t 1_{\mathcal{E}_e} \tilde{\Gamma} 1_{\mathcal{E}_{e'}} e^{-i\lambda^{-2}s(e-e')} f(s)\mathrm{d}s = \delta_{e,e'} \int_0^t \Gamma_e^{\mathrm{dyn}} f(s)\mathrm{d}s, \tag{4.28}$$

and hence Eq. (4.27) proves (4.25). Note that $\Lambda_{t,\lambda}$ and $\Lambda_t := e^{-it\Gamma^{\mathrm{dyn}}}$ satisfy the equations

$$\Lambda_\lambda = \Lambda_0 + K_\lambda \Lambda_\lambda, \quad \Lambda = \Lambda_0 + K\Lambda, \tag{4.29}$$

where Λ_0 is the constant function with value $\Lambda_0 = \Lambda_{0,\lambda} = 1$. Remark that by the assumption of Theorem 4, there exists a constant c and a λ_0 such that for all $\lambda \leqslant \lambda_0$ and for all $n \in \mathbb{N}$, we have

$$\|K_\lambda^n\| \leqslant \frac{(ct)^n}{n!}, \quad \|K^n\| \leqslant \frac{(ct)^n}{n!}. \tag{4.30}$$

This means that both

$$(1-K_\lambda)^{-1} = \sum_{n=0}^{+\infty} K_\lambda^n, \quad (1-K)^{-1} = \sum_{n=0}^{+\infty} K^n \tag{4.31}$$

exist and that for each $n \in \mathbb{N}, \mathrm{s}-\lim_{\lambda\downarrow 0} K_\lambda^n = K^n$. By Eq. (4.29), we thus have

$$\Lambda_\lambda - \Lambda = ((1-K_\lambda)^{-1} - (1-K)^{-1})\Lambda_0 = \sum_{n=0}^{+\infty} (K_\lambda^n - K_\lambda^n)\Lambda_0. \tag{4.32}$$

Since each term in the right-hand side vanishes as $\lambda \downarrow 0$ and the sequence is absolutely convergent by Eq. (4.30), Theorem 4 follows.

D. Proof of Theorem 5

Let us first state a general lemma about the principal value.

Lemma 1: *Let f be a bounded function on $\mathbb{R}$ such that $f/1+|x| \in L^1(\mathbb{R}), f$ is continuous at* 0 *and there exist $\delta, C > 0$ such that for $|x| < C \Rightarrow |f(x) - f(0)| \leqslant |x|^\delta$. Then, for $z \in C_+$,*

$$-i \lim_{T\to+\infty} \int_0^{+T} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x f(x) \mathrm{e}^{-itx} = \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}} f(x)(\epsilon z - x)^{-1} \mathrm{d}x = -\mathrm{P}\int_{\mathbb{R}} \frac{f(x)}{x} \mathrm{d}x + i\pi f(0). \tag{4.33}$$

Proof: For the first expression of Eq. (4.33), we write

$$\begin{aligned}
-i \lim_{T\to+\infty} \int_0^{+T} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x f(x) \mathrm{e}^{-itx} &= \int_{\mathbb{R}} \frac{(-1+\mathrm{e}^{-iTx})}{x} f(x) \mathrm{d}x \\
&= f(0) \int_{|x|\leqslant C} \frac{-1+\mathrm{e}^{-iTx}}{x} \mathrm{d}x + \int_{|x|\leqslant C} \frac{-f(x)+f(0)}{x} \mathrm{d}x \\
&\quad + \int_{|x|\leqslant C} \frac{(f(x)-f(0))\mathrm{e}^{-iTx}}{x} \mathrm{d}x - \int_{|x|>C} \frac{f(x)}{x} \mathrm{d}x + \int_{|x|>C} \frac{f(x)\mathrm{e}^{-iTx}}{x} \mathrm{d}x.
\end{aligned} \tag{4.34}$$

The first term, by the residue calculus, goes to $f(0)i\pi$. By the Riemann-Lebesgue Lemma, the third and the fifth terms on the right of Eq. (4.34) go to zero. The second and fourth terms yield $\mathrm{P}\int (f(x)/x)\mathrm{d}x$.

To get the second equality in Eq. (4.33), we write $z = a + ib$ and compute:

$$\begin{aligned}
\int_{\mathbb{R}} f(x)(\epsilon z - x)^{-1} \mathrm{d}x &= \int_{\mathbb{R}} \frac{\epsilon i b f(x)}{(\epsilon a - x)^2 + (\epsilon b)^2} \mathrm{d}x + \int_{|x|<\mu} f(x) \left(\frac{(\epsilon a - x)}{(\epsilon a - x)^2 + (\epsilon b)^2} - \frac{-x}{x^2 + (\epsilon b)^2} \right) \mathrm{d}x \\
&\quad - \int_{|x|<\mu} \frac{x f(x)}{x^2 + (\epsilon b)^2} \mathrm{d}x + \int_{|x|>\mu} \frac{(\epsilon a - x) f(x)}{(\epsilon a - x)^2 + (\epsilon b)^2} \mathrm{d}x,
\end{aligned} \tag{4.35}$$

where $0 < \mu < 1$ is fixed. The sum of the last two terms converges to $-\mathrm{P}\int_{\mathbb{R}} [f(x)/x] \mathrm{d}x$. The second term can be estimated by

$$\sup|f|\int_{|x|<\mu}\left(\frac{|\epsilon a-x|}{(x-\epsilon a)^2+\epsilon^2b^2}-\frac{|x|}{x^2+\epsilon^2b^2}\right)\mathrm{d}x$$
$$=\frac{\sup|f|}{2}\left|\log\frac{((\mu+\epsilon a)^2+\epsilon^2b^2)((\mu-\epsilon a)^2+\epsilon^2b^2)}{(\mu^2+\epsilon^2b^2)^2}\right|\underset{\epsilon\to0}{\to}0.$$

□

To apply this lemma, it suffices to note that $f(x):=v^*(x)v(x)$ is a bounded L^1 function, continuous and Hölder at sp$E$.

E. Proof of Theorem 6

Lemma 2: *Let $e,e'\in \mathrm{sp}E$ and $z\in\mathbb{C}_+$. Then*

$$\lim_{\lambda\downarrow0}\frac{1}{\lambda}V^*(z-\lambda^{-2}(H_R-e))^{-1}J_{\lambda,e'}=\langle1|\otimes v^*(e)(z-Z_{R_e})^{-1}\delta_{e,e'}.$$

Proof: Let $g_{e'}\in\mathcal{Z}_{R_{e'}}$. Then

$$\frac{1}{\lambda}V^*(z-\lambda^{-2}(H_R-e))^{-1}J_{\lambda,e'}g=\frac{1}{\lambda^2}\int_{\tilde{I}_{e'}}v^*(y)\left(z-\frac{y-e}{\lambda^2}\right)^{-1}g_{e'}\left(\frac{y-e'}{\lambda^2}\right)\mathrm{d}y$$
$$=\int_{\lambda^{-2}(\tilde{I}_{e'}-e')}v^*(e'+\lambda^2x)(z-x+\lambda^{-2}(e-e'))^{-1}g_{e'}(x)\mathrm{d}x.$$

For $e\neq e'$, we estimate the square of the norm by

$$\int_{\lambda^{-2}(\tilde{I}_{e'}-e')}\|v^*(e'+\lambda^2x)(z-x+\lambda^{-2}(e-e'))^{-1}\|^2\mathrm{d}x\int_{\mathbb{R}}\|g_{e'}(x)\|^2\mathrm{d}x$$
$$\leqslant\sup_{y\in\mathbb{R}}\|v(y)\|^2\int_{\lambda^{-2}(\tilde{I}_{e'}-e')}|(z-x+\lambda^{-2}(e-e'))^{-1}|^2\mathrm{d}x\int_{\mathbb{R}}\|g_{e'}(x)\|^2\mathrm{d}x\to0.$$

The first integral in the last line vanishes by Lesbegue dominated convergence since $e\notin(I_{e'}-e')$. For $e=e'$,

$$\left\|\int_{\lambda^{-2}(\tilde{I}_e-e)}(v^*(e+\lambda^2x)-v^*(e))(z-x)^{-1}g_e(x)\mathrm{d}x\right\|^2$$
$$\leqslant\int_{\mathbb{R}}\|(v^*(e+\lambda^2x)-v^*(e))(z-x)^{-1}\|^2\mathrm{d}x\int_{\mathbb{R}}\|g_e(x)\|^2\mathrm{d}x\to0,$$

by the Lebesgue dominated convergence theorem, since v is bounded and continuous in e. Since $g_{e'}$ enters the above estimates only via $\|g_{e'}\|^2=\int_{\mathbb{R}}\|g_{e'}(x)\|^2\mathrm{d}x$, the convergence is in norm. □

The proof of Theorem 4 is based on the formula

$$(z-H_\lambda)^{-1}=(z-H_R)^{-1}+(1_{\mathcal{E}}+\lambda(z-H_R)^{-1}V)G_\lambda(z)(1_{\mathcal{E}}+\lambda V^*(z-H_R)^{-1}),\tag{4.36}$$

where $G_\lambda(z):=1_{\mathcal{E}}(z-H_\lambda)^{-1}1_{\mathcal{E}}$. After appropriate rescaling and sandwiching with $J^*_{\lambda,e'}$ and $J_{\lambda,e''}$, Eq. (4.36) becomes

$$J^*_{\lambda,e'}(z-\lambda^{-2}(H_\lambda-e))^{-1}J_{\lambda,e''}=\delta_{e',e''}1_{\lambda^{-2}(\tilde{I}_{e'}-e')}(Z_{R,e'})(z-Z_{R,e'}-\lambda^{-2}(e'-e))^{-1}$$
$$+\left(1_{\mathcal{E}_{e'}}+J^*_{\lambda,e'}\frac{1}{\lambda}(z-\lambda^{-2}(H_\lambda-e))^{-1}V\right)$$
$$\times G_\lambda(z,e)\left(V^*(z-\lambda^{-2}(H_\lambda-e))^{-1}\frac{1}{\lambda}J_{\lambda,e''}+1_{\mathcal{E}_{e''}}\right), \qquad (4.37)$$

where

$$G_\lambda(z,e):=1_{\mathcal{E}}(z-\lambda^{-2}(H_\lambda-e))^{-1}1_{\mathcal{E}}.$$

The first term of Eq. (4.37) has $\delta_{e',e''}$ because $\tilde{I}_{e'}$ and $\tilde{I}_{e''}$ are disjoint. It converges to

$$\delta_{e,e'}\delta_{e,e''}(z-Z_{R_e})^{-1}.$$

By the stationary Davies limit [Theorem 3, statement (1)],

$$G_\lambda(z,e)\to 1_{\mathcal{E}_e}(z-\Gamma_e)^{-1}1_{\mathcal{E}_e}.$$

Finally, by application of Lemma 2, the second term on the right-hand side of Eq. (4.37) converges to

$$\delta_{e,e'}\delta_{e,e''}(1_{\mathcal{E}_e}+(z-Z_{R,e})^{-1}|1\rangle\otimes v(e))1_{\mathcal{E}_e}(z-\Gamma_e)^{-1}1_{\mathcal{E}_e}(\langle 1|\otimes v^*(e)(z-Z_{R,e})^{-1}+1_{\mathcal{E}_e}).$$

□

F. Proof of Theorem 7

We start with the time dependent analog of Lemma 2.

Lemma 3: *Let $g_e\in L^1(\mathbb{R},\mathfrak{h}_e)\cap L^2(\mathbb{R},\mathfrak{h}_e)=\mathcal{D}\cap\mathcal{Z}_{R,e}$. Then, uniformly for $|t|<T$, we have the convergence*

$$\lambda^{-1}V^*\,\mathrm{e}^{it\lambda^{-2}(e-H_R)}J_{\lambda,e}g_e\to\langle 1|\otimes v^*(e)\mathrm{e}^{-itZ}g_e.$$

Proof:

$$\frac{1}{\lambda}V^*\mathrm{e}^{-it\lambda^{-2}(H_R-e)}J_{\lambda,e'}g_e=\frac{1}{\lambda^2}\int_{\tilde{I}_e}v^*(y)\mathrm{e}^{-it\lambda^{-2}(y-e)}g_e\left(\frac{y-e'}{\lambda^2}\right)\mathrm{d}y$$
$$=\int_{\lambda^{-2}(\tilde{I}_e-e)}v^*(e+\lambda^2x)\mathrm{e}^{itx}g_e(x)\mathrm{d}x\to v^*(e)\int\mathrm{e}^{-itx}g(x)\mathrm{d}x.$$

□

The proof of Theorem 7 is based on the time dependent analog of the formula (4.36):

$$\mathrm{e}^{-itH_\lambda}=\mathrm{e}^{-itH_R}+T_\lambda(t)+i\lambda\int_0^t T_\lambda(t-s)V^*\,\mathrm{e}^{-isH_R}\mathrm{d}s+i\lambda\int_0^t\mathrm{e}^{-isH_R}VT_\lambda(t-s)\mathrm{d}s$$
$$-\lambda^2\int\int_{0\leqslant s_1,s_2,s_1+s_2\leqslant t}\mathrm{e}^{-is_1H_R}VT_\lambda(t-s_1-s_2)V^*\,\mathrm{e}^{-is_2H_R}\mathrm{d}s_1\mathrm{d}s_2,$$

where

$$T_\lambda(t):=1_{\mathcal{E}}\,\mathrm{e}^{-itH_\lambda}1_{\mathcal{E}}.$$

Rescaling, multiplying from the left by $e^{it\lambda^{-2}Z_{\mathrm{ren}}}J^*_{\lambda,e}$ and from the right by $J_{\lambda,e'}$, we obtain

$$
\begin{aligned}
&1_e\,\mathrm{e}^{i\lambda^{-2}tZ_{\mathrm{ren}}}J_\lambda^*\,\mathrm{e}^{-i\lambda^{-2}tH_\lambda}J_\lambda 1_{e'} \\
&\quad = J_{\lambda,e}^*\,\mathrm{e}^{-i\lambda^{-2}t(H_R-e)}J_{\lambda,e'} + \mathrm{e}^{it\lambda^{-2}e}1_{\mathcal{E}_e}T_\lambda(t)1_{\mathcal{E}_{e'}} + \frac{\mathrm{i}}{\lambda}\mathrm{e}^{it\lambda^{-2}e}1_{\mathcal{E}_e}\int_0^t T_\lambda(t-s)V^*\,\mathrm{e}^{-i\lambda^{-2}sH_R}J_{\lambda,e'}\mathrm{d}s \\
&\quad + \frac{\mathrm{i}}{\lambda}\mathrm{e}^{it\lambda^{-2}e}\int_0^t J_{\lambda,e}^*\,\mathrm{e}^{-i\lambda^{-2}sH_R}VT_\lambda(t-s)\mathrm{d}s1_{\mathcal{E}_{e'}} - \lambda^{-2}\mathrm{e}^{it\lambda^{-2}e}\int\int_{0\le s_1,s_2,s_1+s_2\le t}J_{\lambda,e}^*\,\mathrm{e}^{-is_1\lambda^{-2}H_R} \\
&\quad \times VT_\lambda(t-s_1-s_2)V^*\,\mathrm{e}^{-i\lambda^{-2}s_2H_R}J_{\lambda,e'}\mathrm{d}s_1\mathrm{d}s_2,
\end{aligned}
\tag{4.38}
$$

where

$$T_\lambda(t) := 1_{\mathcal{E}}\,\mathrm{e}^{-i\lambda^{-2}tH_\lambda}1_{\mathcal{E}}.$$

The first term of Eq. (4.38) converges to

$$\delta_{e,e'}\,\mathrm{e}^{-itZ_R}1_{R_e}. \tag{4.39}$$

To handle the next terms we use repeatedly the fact that

$$\|T_\lambda(s) - \mathrm{e}^{is\lambda^{-2}E}\,\mathrm{e}^{-is\Gamma}\| \underset{\lambda\searrow 0}{\longrightarrow} 0$$

uniformly for $0\le s\le t$. The second term converges to

$$\mathrm{e}^{it\lambda^{-2}e}1_{\mathcal{E}_e}\,\mathrm{e}^{-it\lambda^{-2}E}\,\mathrm{e}^{-it\Gamma}1_{\mathcal{E}_{e'}} = \delta_{e,e'}1_{\mathcal{E}_e}\,\mathrm{e}^{-it\Gamma}.$$

The third term acting on $g_{e'}\in L^1(\mathbb{R},\mathfrak{h}_e)\cap L^2(R,\mathfrak{h}_e)$ converges to

$$
\begin{aligned}
&ie^{it\lambda^{-2}e}1_{\mathcal{E}_e}\int_0^t \mathrm{e}^{-i(t-s)\lambda^{-2}E}\,\mathrm{e}^{-i(t-s)\Gamma}\langle 1|v^*(e')\mathrm{e}^{-is(Z_R+\lambda^{-2}e')}g_{e'}\mathrm{d}s \\
&\quad = i1_{\mathcal{E}_e}\int_0^t \mathrm{e}^{-i(t-s)\Gamma}\langle 1|v^*(e')\mathrm{e}^{-isZ_R}g_{e'}\,\mathrm{e}^{is\lambda^{-2}(e-e')}\mathrm{d}s.
\end{aligned}
\tag{4.40}
$$

If $e-e'\neq 0$, this goes to zero by the Lebesgue-Riemann Lemma. Therefore, Eq. (4.40) equals

$$\delta_{e,e'}1_{\mathcal{E}_e}\int_0^t \mathrm{e}^{-i(t-s)\Gamma}\langle 1|v^*(e')\mathrm{e}^{-isZ_R}g_e\mathrm{d}s. \tag{4.41}$$

The fourth term sandwiched between $g_e\in L^1(\mathbb{R},\mathfrak{h}_e)\cap L^2(\mathbb{R},\mathfrak{h}_e)$ and $u\in\mathcal{E}$ converges to

$$
\begin{aligned}
&i\int_0^t \langle g_e|\mathrm{e}^{-isZ_R}|1\rangle\otimes v(e)\mathrm{e}^{-i(t-s)\Gamma}\mathrm{e}^{-i\lambda^{-2}(t-s)E}1_{\mathcal{E}_{e'}}u\rangle\mathrm{e}^{i(t-s)\lambda^{-2}e}\mathrm{d}s \\
&\quad = i\int_0^t \langle g_e|\mathrm{e}^{-isZ_R}|1\rangle\otimes v(e)\mathrm{e}^{-i(t-s)\Gamma}1_{\mathcal{E}_{e'}}u\rangle\mathrm{e}^{i(t-s)\lambda^{-2}(e-e')}\mathrm{d}s.
\end{aligned}
\tag{4.42}
$$

Again, if $e-e'\neq 0$, this goes to zero by the Lebesgue-Riemann Lemma. Therefore, Eq. (4.42) equals

$$\delta_{e,e'}i\int_0^t \langle g_e|\mathrm{e}^{-isZ_R}|1\rangle\otimes v(e)\mathrm{e}^{-i(t-s)\Gamma}1_{\mathcal{E}_e}u\rangle\mathrm{d}s. \tag{4.43}$$

The fifth term sandwiched between $g_e\in L^1(\mathbb{R},\mathfrak{h}_e)\cap L^2(\mathbb{R},\mathfrak{h}_e)$ and $g_{e'}\in L^1(\mathbb{R},\mathfrak{h}_{e'})\cap L^2(\mathbb{R},\mathfrak{h}_{e'})$ converges to

$$
\begin{aligned}
&-\int\int_{0\leq s_1,s_2,s_1+s_2\leq t} \mathrm{e}^{i(t-s_1)\lambda^{-2}e}\langle g_e|\mathrm{e}^{-is_1Z_R}v(e)\otimes|1\rangle\mathrm{e}^{-i(t-s_1-s_2)\lambda^{-2}E}\,\mathrm{e}^{-i(t-s_1-s_2)\Gamma}v(e')^*\\
&\quad\otimes\langle 1|\mathrm{e}^{-is_2(Z_R+\lambda^{-2}e')}g_{e'}\rangle\mathrm{d}s_1\mathrm{d}s_2\\
&\quad=-\sum_{e''\in\mathrm{sp}E}\int\int_{0\leq s_1,s_2,s_1+s_2\leq t}\mathrm{e}^{i(t-s_1)\lambda^{-2}(e-e'')-s_2\lambda^{-2}(e'-e'')}\langle g_e|\mathrm{e}^{-is_1Z_R}v(e)\otimes|1\rangle\mathrm{e}^{-i(t-s_1-s_2)\Gamma}1_{\mathcal{E}_{e''}}v(e')^*\\
&\quad\otimes\langle 1|\mathrm{e}^{-is_2Z_R}g_{e'}\rangle\mathrm{d}s_1\mathrm{d}s_2.
\end{aligned}
\tag{4.44}
$$

By the Riemann-Lebesgue Lemma the terms with $e-e''\neq 0$ or $e'-e''\neq 0$ go to zero. Thus Eq. (4.44) equals

$$
-\delta_{e,e'}\int\int_{0\leq s_1,s_2,s_1+s_2\leq t}\langle g_e|\mathrm{e}^{-is_1Z_R}v(e)\otimes|1\rangle\mathrm{e}^{-i(t-s_1-s_2)\Gamma}1_{\mathcal{E}_e}v(e)^*\otimes\langle 1|\mathrm{e}^{-is_2Z_R}g_e\rangle\mathrm{d}s_1\mathrm{d}s_2.
$$

Thus we proved that for $\psi,\psi'\in\mathcal{D}$ we have

$$
\sup_{0\leq t\leq T}\left|\langle\psi|\mathrm{e}^{it\lambda^{-2}Z_{\mathrm{ren}}}J_\lambda^*\,\mathrm{e}^{-it\lambda^{-2}H_\lambda}J_\lambda\psi'\rangle-\langle\psi|\mathrm{e}^{-itZ}\psi'\rangle\right|\underset{\lambda\to 0}{\to}0.
$$

By density, this can be extended to the whole $\mathcal{Z}$. Using the fact that e^{-itZ} is unitary and $\mathrm{e}^{it\lambda^{-2}Z_{\mathrm{ren}}}J_\lambda^*\,\mathrm{e}^{-it\lambda^{-2}H_\lambda}J_\lambda$ is contractive we obtain that for $\psi\in\mathcal{Z}$

$$
\lim_{\lambda\downarrow 0}\mathrm{e}^{it\lambda^{-2}Z_{\mathrm{ren}}}J_\lambda^*\,\mathrm{e}^{-it\lambda^{-2}H_\lambda}J_\lambda\psi=\mathrm{e}^{-itZ}\psi.
$$

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