Generalized integrals of Macdonald and Gegenbauer functions

Jan Dereziński, Christian Gaß, and Błażej Ruba

Abstract. We compute bilinear integrals involving Macdonald and Gegenbauer functions. These integrals are convergent only for a limited range of parameters. However, when one uses *generalized integrals* they can be computed essentially without restricting the parameters. The generalized integral is a linear functional extending the standard integral to a certain class of functions involving finitely many homogeneous non-integrable terms at the edpoints of the interval. For generic values of parameters, generalized bilinear integrals of Macdonald and Gegenbauer functions can be obtained by analytic continuation from the region in which the integrals are convergent. In the case of integer parameters we obtain expressions with explicit additional terms related to an *anomaly*, namely the failure of the generalized integral to be scaling invariant.

1. Introduction

Consider a *Sturm-Liouville operator*

$$\mathcal{C} := -\rho(r)^{-1}\big(\partial_r p(r)\partial_r + q(r)\big) \tag{1.1}$$

acting on functions on an interval $]a,b[$. $\mathcal{C}$ is formally symmetric for the bilinear scalar product with the *density* ρ:

$$\langle f|g\rangle := \int_a^b f(r)g(r)\rho(r)\mathrm{d}r. \tag{1.2}$$

Let f be an eigenfunction of $\mathcal{C}$, that is, $\mathcal{C}f = Ef$. In important applications one needs to know the value of the scalar product of f with itself:

$$\langle f|f\rangle = \int_a^b f(r)^2\rho(r)\mathrm{d}r. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 33C05, 33C10, 47A52.

The work of J.D. and C.G. was supported by the National Science Center of Poland under the grant UMO-2019/35/B/ST1/01651. J.D. would like to thank Howard Cohl for an inspiring discussion at an early stage of this work.

There exists a simple method, which often allows us to evaluate (1.3). If f_i are eigenfunctions corresponding to two different eigenvalues E_i, then

$$\int_a^b f_1(r)f_2(r)\rho(r)\mathrm{d}r = \frac{\mathcal{W}(f_1,f_2,b)-\mathcal{W}(f_1,f_2,a)}{E_1-E_2}, \tag{1.4}$$

$$\text{where}\quad \mathcal{W}(f_1,f_2,r) := p(r)\big(f_1(r)f_2'(r)-f_1'(r)f_2(r)\big). \tag{1.5}$$

(1.4) is sometimes called *Green's identity*, or the *integrated Lagrange identity.* The rhs of (1.4) can often be easily evaluated. Typically this is possible if the endpoints a, b are singular points of the corresponding differential equation. We will then say that $[a, b]$ is a *natural interval* for the operator $\mathcal{C}$. (1.4) is undefined for $f = f_1 = f_2$, but one can often use a limiting procedure to derive (1.3) from (1.4). We will call integrals of the form (1.3) or (1.4) *bilinear integrals*.

In this paper we consider two families of Sturm-Liouville operators: the *Bessel operator* and the *Gegenbauer operator*:

$$\mathcal{B}_\alpha := -\partial_r^2 - \frac{1}{r}\partial_r + \frac{\alpha^2}{r^2}, \tag{1.6}$$

$$\mathcal{G}_\alpha := -(1-w^2)\partial_w^2 + 2(1+\alpha)w\partial_w. \tag{1.7}$$

$\mathcal{B}_\alpha$ and $\mathcal{G}_\alpha$ are very common in applications, e.g. in mathematical physics. Separation of variables in the Laplace equation on $\mathbb{R}^d$ leads to the Bessel operator with $\alpha = \frac{d-2}{2}$, while separation of variables in the Laplace equation on the sphere $\mathbb{S}^d$ and on the hyperbolic space $\mathbb{H}^d$ lead to the Gegenbauer operator.

The well-known *modified Bessel equation* is the eigenequation of (1.6) with the eigenvalue -1. The even better-known *(standard) Bessel equation* is its eigenequation for the eigenvalue 1. $[0,\infty[$ is a natural interval for $\mathcal{B}_\alpha$, and the density is $\rho(r) = 2r$. *Macdonald functions* are eigenfunctions of $\mathcal{B}_\alpha$ decaying fast at infinity.

The *Gegenbauer equation* is the eigenequation of (1.7) with an eigenvalue that we parametrize as $\lambda^2 - \left(\alpha + \frac{1}{2}\right)^2$. The natural intervals for $\mathcal{G}_\alpha$ are $[-1, 1]$ with $\rho(w) = (1-w^2)^\alpha$, and $[1,\infty[$ with $\rho(w) = (w^2-1)^\alpha$.

The main goal of our paper is to describe bilinear integrals involving Macdonald and Gegenbauer functions. These integrals are convergent only for a limited range of parameters. Therefore, we introduce a concept of the *generalized integral*, which allows us to compute (1.3) and (1.4) for essentially all parameters.

The generalized integral is a linear functional on a certain class of functions on the halfline. This class consists of functions integrable on $[1,\infty[$ whose restriction to $[0, 1]$ is a linear combination of an integrable function and homogeneous functions. On integrable functions, the generalized and the usual (Lebesgue) integral coincide. The generalized integral of non-integrable homogeneous terms is defined as follows:

$$\mathrm{gen}\int_0^1 r^\lambda \mathrm{d}r = \begin{cases} \frac{1}{\lambda+1} & \text{for } \lambda \neq -1, \\ 0 & \text{for } \lambda = -1. \end{cases} \tag{1.8}$$

Note that the case $\lambda = -1$ is troublesome.

We say that the generalized integral is *anomalous* if at leat one of coefficients at negative integer powers of r is nonzero. In the anomalous case the generalized integral has more complicated properties. This case often appears in applications and is especially interesting.

The behavior of Macdonald and Gegenbauer functions near the endpoints of integration crucially depends on the the parameter α. This parameter determines the index of the singular point of the Bessel/Gegenbauer equation. The bilinear integrals of Macdonald and Gegenbauer functions are convergent in the standard sense if and only if $|\mathrm{Re}(\alpha)| < 1$. For all other α they are well-defined only in the generalized sense. For $\alpha \in \mathbb{Z}\backslash\{0\}$, these integrals are anomalous.

Let us describe an application of bilinear integrals (1.3) of Macdonald and Gegenbauer functions that we have in mind. Consider a Schrödinger operator with a potential confined to a very small region. One often approximates its Green's function (that is, the integral kernel of its resolvent) by an expression derived by assuming that the potential is supported at a point. This expression involves a bilinear integral of Macdonald functions for $\mathbb{R}^d$ and of Gegenbauer functions for $\mathbb{H}^d$ and $\mathbb{S}^d$. This is a usual convergent integral in dimensions $d = 1, 2, 3$, which corresponds to $\alpha = -\frac{1}{2}, 0, \frac{1}{2}$. In these dimensions one obtains a 1-parameter family of self-adjoint realizations of the Laplacian perturbed by a point-like potential. The situation is different in dimensions $d \geq 4$. The usual bilinear integral is divergent and point interactions do not lead to self-adjoint realizations of the Laplacian. However, one can use the generalized integral to define Green's functions describing point interactions. These Green's functions are not kernels of bounded operators. However, we expect them to approximate at large distances Green's functions of Laplace operators with perturbations with shrinking support [**7**].

Note that there is a considerable difference between even and odd dimensions. In even dimension bilinear integrals are anomalous, while in odd dimensions they are not. This corresponds to logarithmic terms in even dimensions, which are absent in odd dimensions.

The case $d = 4$, corresponding to $\alpha = 1$, is of particular interest. It is the borderline case: the bilinear integrals are well defined in the usual sense for smaller α, but for $\alpha = 1$ one has to use their generalized version. Besides, they are anomalous.

The above analysis is closely related to renormalization in Quantum Field Theory. After the Wick rotation our spacetime becomes the Euclidean space $\mathbb{R}^4$, and the d'Alembertian becomes the Laplacian. Renormalization, especially based on the dimensional regularization, can be interpreted as an application of the generalized integral.

1.1. Remarks about literature. The generalized integral is closely related to the standard theory of extensions of *homogeneous distributions*, as described by Hörmander [**15**], and earlier by Hadamard [**13, 14**], Riesz [**22**], and Gelfand [**11**].

Various kinds of generalized integrals appear in the literature. One can divide them into two kinds: those involving non-integrability at a finite point and those for which the problem comes from ∞. In our paper we use only the former setting: we assume that the integrand is integrable close to infinity, however close to the finite endpoint at the left hand side of the interval it may involve non-integrable homogeneous terms. In this setting definitions equivalent to our generalized integrable were given by Hadamard [**13, 14**] and Riesz [**22**]. A recent exposition of this topic can be found in a book by Paycha [**20**], Chapter 1.

Let us mention that various forms of the generalized integrals where the integrand is not integrable near ∞ are probably more common in the literature. Starting from Chapter 2, the book [**20**] is devoted mostly to this setting. The noncommutative version of the generalized integral for polyhomogeneous pseudodifferential operators in the non-anomalous case is sometimes called the *Vishik-Kontsevich trace*. In the anomalous case it is related to the *Wodzicki residue*.

The Gegenbauer equation is essentially a special case of the hypergeometric equation, where we put the finite singular points at $-1, 1$ and we assume the symmetry with respect to $w \to -w$. Its particular solutions for special values of parameters are the well-known *Gegenbauer polynomials*. The Gegenbauer equation is equivalent through a simple transformation to the *associated Legendre equation*. In the literature, *associated Legendre functions* are much more common than Gegenbauer functions. However, the use of Gegenbauer functions with the conventions that we introduce, rather than associated Legendre functions, leads to a simplification of various identities, and therefore seems to be preferable.

The application of bilinear integrals to Green's functions of the Laplacian on with point interactions will be described in a separate paper **[7]**. It is a classic result that Green's functions without point interactions for $\mathbb{R}^d$, $\mathbb{S}^d$ and $\mathbb{H}^d$ can be expressed in terms of Macdonald functions for $\mathbb{R}^d$ and Gegenbauer (or associated Legendre) functions for $\mathbb{H}^d$ and $\mathbb{S}^d$ (cf. e.g. **[4, 23]**). Green's functions with point potentials in dimensions 1,2,3, at least in the flat case, are also well-known **[1, 2, 3, 16, 21]**. Green's functions with point potentials in all dimensions, and also for $\mathbb{S}^d$ and $\mathbb{H}^d$ will be described in our subsequent paper **[7]**.

1.2. Outline of the paper. The main goal of the present article is to pedagogically describe elements of the theory the generalized integral and of Bessel and Gegenbauer equation needed in **[7]**.

In Section 2, we define the generalized integral and study its properties. We describe its behavior under a change of variables. We describe a method of computing generalized integrals which resembles the *dimensional regularization* in Quantum Field Theory, and actually can be traced back to an earlier work of M.Riesz.

In Section 3 we recall elements of the theory of the Bessel equation. The main new result is the computation of generalized bilinear integrals of Macdonald functions using the definitions and methods of Section 2.

Section 4 is devoted to the Gegenbauer equation. We introduce two kinds of Gegenbauer functions: $\mathbf{S}_{\alpha,\lambda}$ with a simple behavior near 1 and $\mathbf{Z}_{\alpha,\lambda}$ with a simple behavior near $+\infty$. We compute bilinear integrals of $\mathbf{S}_{\alpha,\lambda}$ and $\mathbf{Z}_{\alpha,\lambda}$, both usual and generalized. We discuss various kinds of integral representations of Gegenbauer functions. These representations are then used to show that Gegenbauer functions are asymptotic to Macdonald functions. We also prove that if we choose the variables correctly, then bilinear integrals of Gegenbauer functions are asymptotic to bilinear integrals of Macdonald functions.

This paper has two appendices. In Appendix A, we list several useful properties of special functions related to the Gamma function, which are needed in the main part of the paper. Appendix B contains an overview of conventions for functions related to Gegenbauer functions.

2. Generalized integral

2.1. Definition of the generalized integral. In this section we introduce the *generalized integral*, which extends the usual integral to a certain class of not integrable functions. We restrict ourselves to functions which fail to be integrable near the (finite) left endpoint of the integration interval. Our definition essentially coincides with similar concepts introduced in **[20]**, Chapter 1, and goes back to the works of Riesz and Hadamard.

One should mention that it is also natural to define the generalized integral for functions not integrable close to ∞. This is described e.g. in **[20]** starting from Chapter 2, and is actually more common in the literature. In our work this will not be considered.

DEFINITION 2.1. Let $a \in \mathbb{R}$. We say that a function f on $]a,\infty[$ is *integrable in the generalized sense* if it is integrable on $]a+1,\infty[$ and there exists a finite set $\Omega \subset \mathbb{C}$ and complex coefficients $(f_k)_{k\in\Omega}$ such that

$$f - \sum_{k\in\Omega} f_k (r-a)^k \tag{2.1}$$

is integrable on $]a, a+1[$. We define

$$\begin{aligned} &\mathrm{gen}\!\int_a^\infty f(r)\mathrm{d}r \\ &:= \sum_{k\in\Omega\setminus\{-1\}} \frac{f_k}{k+1} + \int_a^{a+1} \Big(f(r) - \sum_{k\in\Omega} f_k (r-a)^k\Big)\mathrm{d}r + \int_{a+1}^\infty f(r)\mathrm{d}r. \end{aligned} \tag{2.2}$$

We note that the set $\{k \in \Omega \,|\, \mathrm{Re}(k) \le -1\}$ and the corresponding f_k are uniquely determined by f. It is convenient to allow $k \in \Omega$ with $\mathrm{Re}(k) > -1$. The generalized integral of f does not depend on the choice of Ω.

The generalized integral extends the standard integral:

$$\mathrm{gen}\!\int_a^\infty f(r)\mathrm{d}r = \int_a^\infty f(r)\mathrm{d}r \quad \text{for } f \in L^1[a,\infty[. \tag{2.3}$$

It is translation invariant in the following sense:

$$\mathrm{gen}\!\int_a^\infty f(r)\mathrm{d}r = \mathrm{gen}\!\int_{a-\alpha}^\infty f(u+\alpha)\mathrm{d}u, \quad \alpha \in \mathbb{R}. \tag{2.4}$$

In particular, we can without loss of generality assume $a = 0$.

If f is defined on an interval $]0,b[$ with $b > 0$, we define $\mathrm{gen}\!\int_0^b f(r)\mathrm{d}r$ as the generalized integral of the extension of f to $[0,\infty[$ by zero.

PROPOSITION 2.2. *If f satisfies Def. 2.1 on $[0,\infty[$, then*

$$\mathrm{gen}\!\int_0^\infty f(r)\mathrm{d}r = \lim_{\delta\searrow 0}\left(\int_\delta^\infty f(r)\mathrm{d}r + \sum_{k\in\Omega\setminus\{-1\}} \frac{f_k}{k+1}\delta^{k+1} + f_{-1}\ln(\delta)\right). \tag{2.5}$$

PROOF. For any $0 < \delta \le 1$ we have

$$\begin{aligned} \mathrm{gen}\!\int_0^\infty f(r)\mathrm{d}r = \int_\delta^\infty f(r)\mathrm{d}r &+ \sum_{k\in\Omega\setminus\{-1\}} \frac{f_k}{k+1}\delta^{k+1} + f_{-1}\ln(\delta) \\ &+ \int_0^\delta \Big(f(r) - \sum_{k\in\Omega} f_k r^k\Big)\mathrm{d}r. \end{aligned} \tag{2.6}$$

The last term of (2.6) converges to 0 as $\delta \to 0$. □

2.2. Change of variables. The standard formula for a change of variables in an integral is not true for the generalized integral, but one may write down the correction terms explicitly.

PROPOSITION 2.3. *Suppose that $g : [0,\infty[\to [0,\infty[$ is a bijection, smooth down to zero, such that $g(0) = 0$ and $g'(0) \neq 0$. If f is integrable in the generalized sense, then the same is true for $(f \circ g)g'$ and we have*

$$\begin{aligned}(2.7)\qquad &\mathrm{gen}\!\int_0^\infty f(g(u))g'(u)\mathrm{d}r - \mathrm{gen}\!\int_0^\infty f(r)\mathrm{d}r \\ &= - f_{-1}\ln g'(0) + \sum_{l=2}^{\infty} \frac{f_{-l}}{(l-1)(l-1)!} \frac{\mathrm{d}^{l-1}}{\mathrm{d}u^{l-1}} \left(\frac{u}{g(u)}\right)^{l-1}\Bigg|_{u=0},\end{aligned}$$

where we put $f_k = 0$ for $k \notin \Omega$. In particular for $g(u) = \alpha u$, $\alpha > 0$:

$$\mathrm{gen}\!\int_0^\infty f(\alpha u)\alpha\mathrm{d}u = \mathrm{gen}\!\int_0^\infty f(r)\mathrm{d}r - f_{-1}\ln\alpha. \tag{2.8}$$

PROOF. The first statement easily follows from the Taylor expansion of g around zero. Let $\Delta_f(g)$ be the left hand side of (2.7). We have

$$\Delta_f(g) = \int_{g(1)}^1 f(r)\mathrm{d}r + \mathrm{gen}\!\int_0^1 \left(f(g(r))g'(r) - f(r)\right)\mathrm{d}r, \tag{2.9}$$

where we used the standard change of variables formula for the integral over $[1,\infty[$ and $\int_{g(1)}^1 - \int_1^\infty = \int_{g(1)}^1$. Now decompose $f(r) = F(r) + \sum_{k\in\Omega} f_k r^k$. It is easy to check that F gives no contribution in (2.9), so

$$\Delta_f(g) = \sum_{k\in\Omega} f_k \left(\int_{g(1)}^1 r^k\mathrm{d}r + \mathrm{gen}\!\int_0^1 \left(g(r)^k g'(r) - r^k\right)\mathrm{d}r\right). \tag{2.10}$$

We will analyze this expression term by term. First consider $k \neq -1$. Then

$$\begin{aligned}(2.11)\qquad &\int_{g(1)}^1 r^k\mathrm{d}r + \mathrm{gen}\!\int_0^1 \left(g(r)^k g'(r) - r^k\right)\mathrm{d}r \\ &= -\frac{g(1)^{k+1}}{k+1} + \frac{1}{k+1}\mathrm{gen}\!\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(r)^{k+1}\mathrm{d}r.\end{aligned}$$

We have a Taylor expansion of the form

$$g(r)^{k+1} = r^{k+1}\left(\sum_{j=0}^N a_j r^j + R(r)\right), \tag{2.12}$$

where $R(r) = O(r^{N+1})$ and N is such that $\mathrm{Re}(N + k + 2) > 0$. Then

$$\begin{aligned}(2.13)\qquad &\mathrm{gen}\!\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(r)^{k+1}\mathrm{d}r = \sum_{j=0}^N (k+j+1)a_j\,\mathrm{gen}\!\int_0^1 r^{k+j}\mathrm{d}r + R(1) \\ &= \sum_{\substack{j=0\\ k+j\neq -1}}^N a_j + R(1) = g(1)^{k+1} - a_{-k-1},\end{aligned}$$

where we put $a_{-k-1} = 0$ if $-k-1 \notin \mathbb{N}$. Therefore, (2.11) vanishes for $-k-1 \notin \mathbb{N}$. If $k = -l$ with $l \in \{2, 3, \dots\}$, then (2.11) is equal to

$$-\frac{1}{-l+1} a_{l-1} = \frac{1}{(l-1)(l-1)!} \frac{\mathrm{d}^{l-1}}{\mathrm{d}r^{l-1}} \left(\frac{r}{g(r)}\right)^{l-1} \Bigg|_{r=0}. \tag{2.14}$$

To handle the case $k = -1$, we write $g(r) = g'(0) r \mathrm{e}^{R(r)}$ with $R(0) = 0$ and compute

$$\mathrm{gen}\int_0^1 \frac{g'(r)}{g(r)} \mathrm{d}r = R(1) = \ln g(1) - \ln g'(0). \tag{2.15}$$

□

Formula (2.8) shows that the generalized integral is invariant under scaling if and only if $f_{-1} = 0$, and invariant under a large class of a change of variables if $f_k = 0$ for every negative integer k.

Definition 2.4. The generalized integral (2.2) is called *anomalous* if there exists $n = 1, 2, \dots$ such that $f_{-n} \neq 0$.

Remarkably, the generalized integral is always invariant under changes of variables given by power functions.

Proposition 2.5. *Let f be integrable in the generalized sense. Then for $\alpha > 0$*

$$\mathrm{gen}\int_0^\infty f(r)\mathrm{d}r = \mathrm{gen}\int_0^\infty f(u^\alpha)\, \alpha u^{\alpha-1} \mathrm{d}u. \tag{2.16}$$

Proof. The formula is true for integrable f, so by linearity it is sufficient to verify it for $f(r) = r^k \mathbb{1}_{[0,1]}(r)$. This is an elementary calculation. □

2.3. Dimensional regularization. In this subsection we describe a method to compute generalized integrals by analytic continuation. It is closely related to dimensional regularization used in QFT.

Let F be a holomorphic function on $U \setminus \{z_0\}$, where $z_0 \in U$ and U is open. Then one has a Laurent expansion

$$F(z) = \sum_{j \in \mathbb{Z}} F_j (z - z_0)^j \tag{2.17}$$

convergent for $z \neq z_0$ sufficiently close to z_0. Coefficients $\operatorname{res} F(z_0) := F_{-1}$ and $\operatorname*{fp}_{z \to z_0} F(z) := F_0$ are called the residue and the finite part of F at z_0.

Let $N \in \mathbb{N}$ and let $f :]0, \infty[\times \{\alpha \in \mathbb{C} \mid \mathrm{Re}(\alpha) > -N-1\} \to \mathbb{C}$ be a function such that $f(r, \cdot)$ is holomorphic for each r, $\|f(\cdot, \alpha)\|_{L^1[1,\infty[}$ is bounded locally uniformly in α, and there exist holomorphic functions $f_0, \dots, f_N$ of α such that the $L^1]0,1]$ norm of $f(r,\alpha) - \sum_{n=0}^N r^{\alpha+n} f_n(\alpha)$ is bounded locally uniformly in α. Then $f(\cdot, \alpha)$ is integrable in the generalized sense, and for $-\alpha \notin \{1, \dots, N\}$ one has

$$\begin{aligned} &\mathrm{gen}\int_0^\infty f(r,\alpha)\mathrm{d}r \\ &= \sum_{n=0}^N \frac{f_n(\alpha)}{\alpha+n+1} + \int_0^1 \Big(f(r,\alpha) - \sum_{n=0}^N r^{\alpha+n} f_n(\alpha)\Big)\mathrm{d}r + \int_1^\infty f(r,\alpha)\mathrm{d}r. \end{aligned} \tag{2.18}$$

By Morera's theorem, the right hand side is, away from the poles at $-1, \dots, -N$, a holomorphic function of α. Therefore, to obtain (2.18) in the non-anomalous case it is enough to compute (2.18) in the region where the usual integral is convergent and continue analytically.

Let $m \in \{1, \dots, N\}$. The right hand side of (2.18) has a simple pole at $\alpha = -m$ with residue $f_{m-1}(-m)$ (possibly zero). Its finite part is

$$\underset{\alpha \to -m}{\mathrm{fp}} \ \mathrm{gen}\int_0^\infty f(r, \alpha)\mathrm{d}r = \lim_{\alpha \to -m} \left(\mathrm{gen}\int_0^\infty f(r, \alpha)\mathrm{d}r - \frac{f_{m-1}(-m)}{\alpha + m} \right). \tag{2.19}$$

One may be tempted to think that the finite part is equal to the generalized integral of $f(\cdot, -m)$, but in fact one has to subtract also a certain finite term.

Proposition 2.6. *The generalized integral at $\alpha = -m$ is given by*

$$\mathrm{gen}\int_0^\infty f(r, -m)\mathrm{d}r = \underset{\alpha \to -m}{\mathrm{fp}} \ \mathrm{gen}\int f(r, \alpha)\mathrm{d}r - f'_{m-1}(-m). \tag{2.20}$$

Proof. By definition, the generalized integral of $f(\cdot, -m)$ is given by (2.18) evaluated at $-m$, with the term $n = m - 1$ in the summation omitted. Therefore,

$$\mathrm{gen}\int_0^\infty f(r, -m)\mathrm{d}r = \lim_{\alpha \to -m} \left(\mathrm{gen}\int_0^\infty f(r, \alpha)\mathrm{d}r - \frac{f_{m-1}(\alpha)}{\alpha + m} \right). \tag{2.21}$$

The formula (2.20) immediately follows. □

The following facts deserve emphasis.

- The term $f'_{m-1}(-m)$ in (2.20) may be nonzero even if the generalized integral of $f(r, \alpha)$ has a removable singularity at $\alpha = -m$.
- The finite part for $\alpha \to -m$ of the analytic continuation of $\int_0^\infty f(r, \alpha)\mathrm{d}r$ is not uniquely determined by the function $f(r, -m)$, in contrast to the generalized integral of $f(r, -m)$.

2.4. Examples.

Let us give a few examples of generalized integrals.

Example 2.7.

$$\mathrm{gen}\int_0^1 \alpha r^{-1+\alpha}\mathrm{d}r = \begin{cases} 1 & \text{for } \alpha \neq 0; \\ 0 & \text{for } \alpha = 0. \end{cases} \tag{2.22}$$

Therefore, the the limit of generalized integrals of $f(r, \alpha) = \alpha r^{-1+\alpha}\mathbb{1}_{[0,1]}(r)$ for $\alpha \to 0$ is not the generalized integral of $f(r, 0)$. The finite part for $\alpha \to 0$ (in this case, the limit) is nonzero even though $f(r, 0) = 0$.

Example 2.8 (Gamma integral).

$$\mathrm{gen}\int_0^\infty \mathrm{e}^{-r} r^{-1+\alpha}\mathrm{d}r = \begin{cases} \Gamma(\alpha), & \alpha \notin -\mathbb{N}_0; \\ \frac{(-1)^m}{m!}\psi(m+1), & \alpha = -m \in -\mathbb{N}_0. \end{cases}$$

Example 2.9 (Beta integrals). Assume that $\mathrm{Re}(v) > 0$. Then

$$\mathrm{gen}\int_0^1 r^{-1+u}(1-r)^{v-1}\mathrm{d}r = \begin{cases} \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, & u \notin -\mathbb{N}_0; \\ \frac{(-1)^m (v-m)_m \left(\psi(m+1) - \psi(v-m)\right)}{m!}, & u = -m \in -\mathbb{N}_0. \end{cases}$$

$$\mathrm{gen}\int_0^\infty r^{-1+u}(1+r)^{v-1}\mathrm{d}r = \begin{cases} \frac{\Gamma(u)\Gamma(1-u-v)}{\Gamma(1-v)}, & u \notin -\mathbb{N}_0; \\ \frac{(-1)^m (1-v)_m \left(\psi(m+1) - \psi(1+m-v)\right)}{m!}, & u = -m \in -\mathbb{N}_0. \end{cases}$$

3. Bessel equation

3.1. Modified Bessel equation. Here is the *modified Bessel equation*:

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\alpha^2}{r^2} - 1\right) v(r) = 0. \tag{3.1}$$

There are two standard solutions of (3.1). The first is the *modified Bessel function*, which can be defined by the power series

$$I_\alpha(r) = \sum_{n=0}^\infty \frac{\left(\frac{r}{2}\right)^{2n+\alpha}}{n!\Gamma(\alpha+n+1)}.$$

The second solution is the *Macdonald function*, which for $\mathrm{Re}(r) > 0$ and all α can be defined by the absolutely convergent integral

$$K_\alpha(r) := \frac{1}{2}\int_0^\infty \exp\left(-\frac{r}{2}(s+s^{-1})\right) s^{\alpha-1}\mathrm{d}s. \tag{3.2}$$

We have the identities

$$K_{-\alpha}(r) = K_\alpha(r) = \frac{\pi}{2\sin\pi\alpha}(I_{-\alpha}(r) - I_\alpha(r)), \tag{3.3}$$

$$I_\alpha(r) = \frac{1}{\mathrm{i}\pi}\big(K_\alpha(\mathrm{e}^{-\mathrm{i}\pi}r) - \mathrm{e}^{\mathrm{i}\pi\alpha}K_\alpha(r)\big), \tag{3.4}$$

the asymptotics for $|\arg r| < \pi - \epsilon$, $\epsilon > 0$,

$$\lim_{|r|\to\infty}\left(\sqrt{\frac{\pi}{2r}}\mathrm{e}^{-r}\right)^{-1} K_\alpha(r) = 1, \tag{3.5}$$

and the recurrence relations:

$$\left(\frac{1}{r}\partial_r\right)^n r^{\pm\alpha} I_\alpha(r) = r^{\pm\alpha-n} I_{\alpha\mp n}(r), \tag{3.6}$$

$$\left(-\frac{1}{r}\partial_r\right)^n r^{\pm\alpha} K_\alpha(r) = r^{\pm\alpha-n} K_{\alpha\mp n}(r). \tag{3.7}$$

We note also the inequality **[19]**

$$\frac{K_\alpha(r)}{K_\alpha(R)} < \mathrm{e}^{R-r}\left(\frac{R}{r}\right)^\alpha, \qquad \alpha > \frac{1}{2},\ 0 < r < R. \tag{3.8}$$

To see (3.8), first note that (3.2) can be transformed into

$$K_\alpha(r) := \frac{1}{2}\int_1^\infty \exp\left(-\frac{r}{2}(s+s^{-1})\right)(s^\alpha + s^{-\alpha})s^{-1}\mathrm{d}s, \tag{3.9}$$

which easily implies

$$K_\alpha(r) < K_{\alpha'}(r), \quad 0 \le \alpha < \alpha'. \tag{3.10}$$

By (3.7)

$$\frac{K_\alpha'(r)}{K_\alpha(r)} = -\frac{K_{\alpha-1}(r)}{K_\alpha(r)} - \frac{\alpha}{r}. \tag{3.11}$$

Hence, for $\alpha \ge \frac{1}{2}$, using (3.10) we obtain

$$\frac{K_\alpha'(r)}{K_\alpha(r)} > -1 - \frac{\alpha}{r}. \tag{3.12}$$

Integrating (3.12) from r to R we obtain (3.8).

3.2. Degenerate case. The case $\alpha \in \mathbb{Z}$ is called degenerate, because then the standard solutions characterized by asymptotics at 1 coincide. Using for (3.14) the de l'Hôpital rule, one finds for $\alpha \in \{0, 1, 2, \dots\}$

$$I_{\pm\alpha}(r) = \sum_{n=0}^{\infty} \frac{\left(\frac{r}{2}\right)^{2n+\alpha}}{n!(n+\alpha)!}, \tag{3.13}$$

$$\begin{aligned} K_{\pm\alpha}(r) = &\frac{1}{2}\sum_{k=0}^{\alpha-1} \frac{(-1)^k(\alpha-k-1)!}{k!}\left(\frac{r}{2}\right)^{2k-\alpha} \\ &+ \frac{(-1)^\alpha}{2}\sum_{j=0}^{\infty} \frac{\left(H_j + H_{\alpha+j} - 2\gamma_{\mathrm{E}} - 2\ln(\frac{r}{2})\right)}{j!(\alpha+j)!}\left(\frac{r}{2}\right)^{2j+\alpha}. \end{aligned} \tag{3.14}$$

3.3. Half-integer case. In the half-integer case reduce to elementary functions. More precisely, for $k = 0, 1, 2, \dots,$ we have

$$I_{\frac12+k}(r) = \left(\frac{2}{\pi}\right)^{\frac12} r^{\frac12+k}\left(\frac{1}{r}\partial_r\right)^k \frac{\sinh r}{r}, \tag{3.15}$$

$$I_{-\frac12-k}(r) = \left(\frac{2}{\pi}\right)^{\frac12} r^{\frac12+k}\left(\frac{1}{r}\partial_r\right)^k \frac{\cosh r}{r}, \tag{3.16}$$

$$K_{\pm(\frac12+k)}(r) = \left(\frac{\pi}{2}\right)^{\frac12} r^{\frac12+k}\left(-\frac{1}{r}\partial_r\right)^k \frac{\mathrm{e}^{-r}}{r}. \tag{3.17}$$

3.4. Bilinear integrals. First let us describe some integral identities satisfied by Bessel functions involving the usual integral. For $\mathrm{Re}(a+b) > 0$ and $|\mathrm{Re}(\alpha)| < 1$ we have

$$\int_0^\infty K_\alpha(ar)K_\alpha(br)2r\mathrm{d}r = \frac{\pi}{\sin(\pi\alpha)}\frac{(a/b)^\alpha - (b/a)^\alpha}{a^2-b^2}, \qquad \alpha \neq 0, \tag{3.18}$$

$$\int_0^\infty K_0(ar)K_0(br)2r\mathrm{d}r = \frac{2\ln\frac{a}{b}}{a^2-b^2}. \tag{3.19}$$

If $b = a$ with $\mathrm{Re}(a) > 0$, the above integrals reduce to

$$\int_0^\infty K_\alpha(ar)^2 2r\mathrm{d}r = \frac{\pi\alpha}{a^2\sin(\pi\alpha)}, \qquad \alpha \neq 0, \tag{3.20}$$

$$\int_0^\infty K_0(ar)^2 2r\mathrm{d}r = \frac{1}{a^2}, \quad \mathrm{Re}(a) > 0. \tag{3.21}$$

Formula (3.18) is well-known [**12**, Chapter 6.521]. It can be obtained from the identity

$$\begin{aligned} &(a^2-b^2)K_\alpha(ar)K_\alpha(br) \\ =&(\partial_r^2 + r^{-1}\partial_r)K_\alpha(ar)K_\alpha(br) - K_\alpha(ar)(\partial_r^2 + r^{-1}\partial_r)K_\alpha(br), \end{aligned}$$

which follows directly from Bessel's equation and integration by parts. The remaining formulas are then obtained by applying the de l'Hôpital rule.

If one replaces the integration in (3.18–3.21) by generalized integration, it makes sense for every α. If we treat r^2 as the variable of integration, these generalized integrals are non-anomalous for $\alpha \in \mathbb{Z} \setminus \{0\}$, so expressions stated in (3.18, 3.20) for $\alpha \neq 0$ are true for all $\alpha \notin \mathbb{Z}$. Below we calculate the anomalous integrals for $\alpha \in \mathbb{Z}$.

PROPOSITION 3.1. *Let $\alpha \in \mathbb{Z}$ and let $\mathrm{Re}(a+b) > 0$. If $a \neq b$, then*

$$\text{gen}\int_0^\infty K_\alpha(ar)K_\alpha(br)2r\mathrm{d}r = (-1)^\alpha 2\frac{\left(\frac{a}{b}\right)^\alpha \ln\left(\frac{a}{2}\right) - \left(\frac{b}{a}\right)^\alpha \ln\left(\frac{b}{2}\right)}{a^2-b^2} - \frac{(-1)^\alpha}{ab}\sum_{k=0}^{|\alpha|-1}\left(\frac{a}{b}\right)^{2k-|\alpha|+1}\big(\psi(1+k)+\psi(|\alpha|-k)\big). \tag{3.22}$$

In the case $a = b$, one has

$$\text{gen}\int_0^\infty K_\alpha(br)^2 2r\mathrm{d}r = \frac{(-1)^\alpha}{b^2}\Big(1+|\alpha|\ln\left(\tfrac{b^2}{4}\right)+2|\alpha|\big(1-\psi(1+|\alpha|)\big)\Big). \tag{3.23}$$

PROOF. Both sides of (3.22) are invariant with respect to the sign flip of α, so it is enough to consider $\alpha = -m$ with $m \in \mathbb{N}$. We will use dimensional regularization.

We change integration variables to r^2 (using Proposition 2.5). Let

$$f(r,\alpha) := \left(\frac{ab}{4}\right)^{-\alpha} K_\alpha(ar)K_\alpha(br). \tag{3.24}$$

The generalized integral of $f(\cdot,\alpha)$ is non-anomalous for non-integer α, so from (3.18)

$$\text{gen}\int_0^\infty f(r,\alpha)\mathrm{d}r^2 = \frac{\pi}{\sin\pi\alpha}\frac{\left(\frac{b}{2}\right)^{-2\alpha} - \left(\frac{a}{2}\right)^{-2\alpha}}{a^2-b^2}. \tag{3.25}$$

(3.25) has a simple pole at $\alpha = -m$, with residue and finite part given by

$$\lim_{\alpha\to-m}(\alpha+m)\text{gen}\int_0^\infty f(r,\alpha)\mathrm{d}r^2 = (-1)^m\frac{\left(\frac{b}{2}\right)^{2m} - \left(\frac{a}{2}\right)^{2m}}{a^2-b^2}, \tag{3.26}$$

$$\underset{\alpha\to-m}{\text{fp}}\ \text{gen}\int_0^\infty f(r,\alpha)\mathrm{d}r^2 = (-1)^m 2\frac{\left(\frac{a}{2}\right)^{2m}\ln\left(\frac{a}{2}\right) - \left(\frac{b}{2}\right)^{2m}\ln\left(\frac{b}{2}\right)}{a^2-b^2}. \tag{3.27}$$

By (3.3), all terms of $f(r,\alpha)$ singular for non-integer α with $\mathrm{Re}(\alpha) < 0$ are in

$$\left(\frac{ab}{4}\right)^{-\alpha}\frac{\pi^2}{4\sin^2\pi\alpha}I_\alpha(ar)I_\alpha(br) = \frac{1}{4}\left(\sum_{k=0}^\infty \frac{\left(\frac{a}{2}\right)^{2k} r^{2k+\alpha}(-1)^k\Gamma(-\alpha-k)}{k!}\right)\left(\sum_{j=0}^\infty \frac{\left(\frac{b}{2}\right)^{2j} r^{2j+\alpha}(-1)^j\Gamma(-\alpha-j)}{j!}\right). \tag{3.28}$$

The coefficient of (3.28) at $(r^2)^{\alpha+m-1}$ is

$$f_{m-1}(\alpha) = \frac{(-1)^{m-1}}{4}\sum_{k=0}^{m-1}\frac{\left(\frac{a}{2}\right)^{2k}\left(\frac{b}{2}\right)^{2m-2-2k}\Gamma(-\alpha-k)\Gamma(-\alpha-m+1+k)}{k!(m-1-k)!}. \tag{3.29}$$

We calculate its derivative

$$f'_{m-1}(-m) = \frac{(-1)^{m-1}}{4}\sum_{k=0}^{m-1}\left(\frac{a}{2}\right)^{2k}\left(\frac{b}{2}\right)^{2m-2-2k}\big(-\psi(1+k)-\psi(m-k)\big). \tag{3.30}$$

Formula (3.22) follows from (2.20). It is easy to see that the generalized integral of $K_\alpha(ar)K_\alpha(br)2r$ is a holomorphic function of b, even in the anomalous case. Therefore, (3.23) may be obtained from (3.22) using the de l'Hôpital rule and identity (A.24). ∎

3.5. Poisson-type integral representations. There exist two basic kinds of integral representations of solutions of the Bessel equation. Representations similar to or derived from (3.2) are sometimes called *Bessel-Schläfli–type*. There exists also another family of integral representations, sometimes called *Poisson-type*:

$$I_\alpha(r) = \frac{1}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(\frac{r}{2}\right)^\alpha \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \mathrm{e}^{\pm rt}\mathrm{d}t, \quad \mathrm{Re}(\alpha) > -\frac{1}{2}, \tag{3.31}$$

$$K_\alpha(r) = \frac{\sqrt{\pi}\left(\frac{r}{2}\right)^\alpha}{\Gamma(\alpha+\frac{1}{2})}\int_1^\infty \mathrm{e}^{-sr}(s^2-1)^{\alpha-\frac{1}{2}}\mathrm{d}s, \quad \mathrm{Re}(\alpha) > -\frac{1}{2},\ \mathrm{Re}(r) > 0, \tag{3.32}$$

$$K_\alpha(r) = \frac{\Gamma(\alpha+\frac{1}{2})}{2\sqrt{\pi}}\left(\frac{r}{2}\right)^{-\alpha}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sr}(s^2+1)^{-\alpha-\frac{1}{2}}\mathrm{d}s, \quad \mathrm{Re}(\alpha) > 0,\ r > 0. \tag{3.33}$$

3.6. Standard Bessel equation. The *(standard) Bessel equation* is obtained by setting $r \to \pm\mathrm{i}r$ in the modified one:

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\alpha^2}{r^2} + 1\right) v(r) = 0. \tag{3.34}$$

We regard the modified Bessel equation as more basic than the standard Bessel equation because its standard solutions holomorphic on $\mathbb{C}\backslash]-\infty,0]$ have simpler properties than those of the standard Bessel equation. The latter is useful mostly with r restricted to $\mathbb{R}_+$.

We have several kinds of standard solutions of (3.34). The most important is the *Bessel function*, defined as

$$J_\alpha(r) = \mathrm{e}^{\pm\mathrm{i}\pi\frac{\alpha}{2}} I_\alpha(\mp\mathrm{i}r). \tag{3.35}$$

The two *Hankel functions* also solve (3.34):

$$H_\alpha^\pm(r) = \frac{2}{\pi}\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}(\alpha+1)} K_\alpha(\mp\mathrm{i}r). \tag{3.36}$$

We have an integral representation

$$H_\alpha^\pm(r) = \pm\frac{1}{\pi\mathrm{i}}\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}\alpha}\int_0^\infty \exp\left(\pm\mathrm{i}\frac{r}{2}(s+s^{-1})\right) s^{\alpha-1}\mathrm{d}s, \quad \pm\mathrm{Im}(r) > 0. \tag{3.37}$$

Remark 3.2. In the literature the usual notation for Hankel functions is

$$H_\alpha^{(1)}(r) = H_\alpha^+(r), \quad H_\alpha^{(2)}(r) = H_\alpha^-(r). \tag{3.38}$$

Note the identities

$$J_\alpha(r) = \frac{1}{2}\left(H_\alpha^+(r) + H_\alpha^-(r)\right), \tag{3.39}$$

$$J_{-\alpha}(r) = \frac{1}{2}\left(\mathrm{e}^{\alpha\pi\mathrm{i}}H_\alpha^+(r) + \mathrm{e}^{-\alpha\pi\mathrm{i}}H_\alpha^-(r)\right), \tag{3.40}$$

$$\mathrm{e}^{\mp\alpha\pi\mathrm{i}}H_{-\alpha}^\pm(r) = H_\alpha^\pm(r) = \pm\frac{\mathrm{i}\mathrm{e}^{\mp\alpha\pi\mathrm{i}}J_\alpha(r) - \mathrm{i}J_{-\alpha}(r)}{\sin\alpha\pi}. \tag{3.41}$$

We have recurrence relations

$$\left(\pm\frac{1}{r}\partial_r\right)^n r^{\pm\alpha}J_\alpha(r) = r^{\pm\alpha-n}J_{\alpha\mp n}(r). \tag{3.42}$$

In the above recurrence relations one may replace $J_\alpha(r)$ with $H_\alpha^\pm(r)$.

3.7. Special values of parameters. For $\alpha = 0, 1, 2, \ldots$, we have

$$(-1)^\alpha J_{-\alpha}(r) = J_\alpha(r) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{r}{2}\right)^{2n+\alpha}}{n!(n+\alpha)!}, \tag{3.43}$$

$$\begin{aligned}(-1)^\alpha H^{\pm}_{-\alpha}(r) = H^{\pm}_\alpha(r) = &\mp \frac{\mathrm{i}}{\pi} \sum_{k=0}^{\alpha-1} \left(\frac{r}{2}\right)^{2k-\alpha} \frac{(\alpha-k-1)!}{k!} \\ &\mp \frac{\mathrm{i}}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j \big(H_j + H_{\alpha+j} - 2\gamma_{\mathrm{E}} \pm \mathrm{i}\pi - 2\ln(\frac{r}{2})\big)}{j!(\alpha+j)!} \left(\frac{r}{2}\right)^{2j+\alpha}.\end{aligned} \tag{3.44}$$

For $k = 0, 1, 2, \ldots$ we have

$$J_{\frac{1}{2}+k}(r) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} r^{\frac{1}{2}+k} \left(-\frac{1}{r}\partial_r\right)^k \frac{\sin r}{r}, \tag{3.45}$$

$$J_{-\frac{1}{2}-k}(r) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} r^{\frac{1}{2}+k} \left(\frac{1}{r}\partial_r\right)^k \frac{\cos r}{r}, \tag{3.46}$$

$$H^{\pm}_{-\frac{1}{2}-k}(r) = \pm\mathrm{i}(-1)^k H^{\pm}_{\frac{1}{2}+k}(r) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} r^{\frac{1}{2}+k} \left(\frac{1}{r}\partial_r\right)^k \frac{\mathrm{e}^{\pm\mathrm{i}r}}{r}. \tag{3.47}$$

4. The Gegenbauer equation

4.1. Gegenbauer functions. Here is the *Gegenbauer equation*:

$$\left((1-w^2)\partial_w^2 - 2(1+\alpha)w\partial_w + \lambda^2 - \left(\alpha + \frac{1}{2}\right)^2\right) f(w) = 0. \tag{4.1}$$

Its solutions can be expressed in terms of the Gauss hypergeometric function $F(a, b; c; z)$. We will often use this function with the so-called *Olver's normalization*

$$\mathbf{F}(a, b; c; z) := \frac{F(a, b; c; z)}{\Gamma(c)} = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{\Gamma(c+n) n!}. \tag{4.2}$$

The defining series converges only in the unit disc, but $\mathbf{F}(a, b; c; z)$ extends to a holomorphic function on $\mathbb{C}\backslash[1, \infty[$ (as well as on a universal cover of $\mathbb{C}\backslash\{0, 1\}$, but we will not use the latter point of view here).

In what follows complex power functions should be interpreted as their principal branches (holomorphic on $\mathbb{C}\backslash]-\infty, 0]$). For example $w \mapsto (1-w)^\alpha$ is holomorphic away from $[1, \infty[$. In addition, we will frequently use the notation

$$(w^2-1)^\alpha_\bullet := (w-1)^\alpha (w+1)^\alpha. \tag{4.3}$$

The function $(w^2-1)^\alpha_\bullet$ is holomorphic on $\mathbb{C}\backslash]-\infty, 1]$, whereas $(w^2-1)^\alpha$ is holomorphic on $\mathbb{C}\backslash\big([-1, 1] \cup \mathrm{i}\mathbb{R}\big)$. One has $(w^2-1)^\alpha_\bullet = (w^2-1)^\alpha$ only for $\mathrm{Re}(w) > 0$. However, $(1-w^2)^\alpha = (1-w)^\alpha(1+w)^\alpha$ for all $w \notin]-\infty, -1] \cup [1, \infty[$.

Standard solutions of the Gegenbauer equations are characterized by simple behaviour at one of the three singular points $1, -1, \infty$. Due to the $w \mapsto -w$ symmetry of the equation (4.1), solutions of the second type are obtained from solutions of the first type by negating the argument. Therefore we consider 4 functions, corresponding to 2 behaviors at 1 and 2 behaviors at ∞. All of them are holomorphic on $\mathbb{C}\backslash]-\infty, 1]$.

- The solution characterized by asymptotics ~ 1 at 1:

$$S_{\alpha,\pm\lambda}(w) := F\Big(\frac{1}{2}+\alpha+\lambda, \frac{1}{2}+\alpha-\lambda; \alpha+1; \frac{1-w}{2}\Big) \tag{4.4}$$

$$= \Big(\frac{2}{w+1}\Big)^{\alpha} F\Big(\frac{1}{2}+\lambda, \frac{1}{2}-\lambda; \alpha+1; \frac{1-w}{2}\Big). \tag{4.5}$$

  $S_{\alpha,\lambda}$ is distinguished among the four solutions introduced here by the fact that it is holomorphic on $\mathbb{C}\backslash]-\infty,-1]$ rather than only on $\mathbb{C}\backslash]-\infty,1]$. On the right half-plane we have an alternative expression:

$$S_{\alpha,\lambda}(w) = F\Big(\frac{1}{4}+\frac{\alpha}{2}+\frac{\lambda}{2}, \frac{1}{4}+\frac{\alpha}{2}-\frac{\lambda}{2}; \alpha+1; 1-w^2\Big), \qquad \mathrm{Re}(w) > 0. \tag{4.6}$$

- The solution $\frac{2^{2\alpha}}{(w^2-1)_\bullet^\alpha} S_{-\alpha,\lambda}(w)$ is characterized by asymptotics $\sim \frac{2^\alpha}{(w-1)^\alpha}$ at 1.
- The solution characterized by asymptotics $\sim w^{-\frac{1}{2}-\alpha-\lambda}$ at ∞:

$$\begin{aligned} Z_{\alpha,\lambda}(w) &= (w\pm 1)^{-\frac{1}{2}-\alpha-\lambda} F\Big(\frac{1}{2}+\lambda, \frac{1}{2}+\lambda+\alpha; 1+2\lambda; \frac{2}{1\pm w}\Big) \\ &= w^{-\frac{1}{2}-\alpha-\lambda} F\Big(\frac{1}{4}+\frac{\alpha}{2}+\frac{\lambda}{2}, \frac{3}{4}+\frac{\alpha}{2}+\frac{\lambda}{2}; 1+\lambda; \frac{1}{w^2}\Big). \end{aligned} \tag{4.7}$$

- The solution $Z_{\alpha,-\lambda}(w)$ is characterized by asymptotics $\sim w^{-\frac{1}{2}-\alpha+\lambda}$ at ∞.

All these 4 functions can be expressed in terms of $S_{\alpha,\lambda}$, but for typographical reasons it is convenient to introduce also $Z_{\alpha,\lambda}$. We will often use Olver's normalization:

$$\mathbf{S}_{\alpha,\lambda}(w) := \frac{1}{\Gamma(\alpha+1)} S_{\alpha,\lambda}(w), \qquad \mathbf{Z}_{\alpha,\lambda}(w) := \frac{1}{\Gamma(\lambda+1)} Z_{\alpha,\lambda}(w). \tag{4.8}$$

We note the identities

$$\mathbf{S}_{\alpha,\lambda}(w) = \mathbf{S}_{\alpha,-\lambda}(w), \quad \mathbf{Z}_{\alpha,\lambda}(w) = \frac{\mathbf{Z}_{-\alpha,\lambda}(w)}{(w^2-1)_\bullet^\alpha} \tag{4.9}$$

as well as the slightly more subtle *Whipple transformations*:

$$\mathbf{Z}_{\alpha,\lambda}(w) := (w^2-1)_\bullet^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}} \mathbf{S}_{\lambda,\alpha}\left(\frac{w}{(w^2-1)_\bullet^{\frac{1}{2}}}\right), \tag{4.10}$$

$$\mathbf{S}_{\alpha,\lambda}(w) := (w^2-1)_\bullet^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}} \mathbf{Z}_{\lambda,\alpha}\left(\frac{w}{(w^2-1)_\bullet^{\frac{1}{2}}}\right), \qquad \mathrm{Re}(w) > 0. \tag{4.11}$$

Note that $w \mapsto \frac{w}{(w^2-1)_\bullet^{\frac{1}{2}}}$, defined on $\mathbb{C} \setminus [-1,1]$, is a holomorphic double cover of $\{z \in \mathbb{C} \setminus [-1,1] \,|\, \mathrm{Re}(z) > 0\}$. One has $f(-w) = f(w)$ for $w \notin [-1,1]$ and $f(f(w)) = \mathrm{sgn}(\mathrm{Re}(w))w$ for $w \notin [-1,1] \cup \mathrm{i}\mathbb{R}$. The reason why (4.11) (but not (4.10)) holds only on the right half-plane is that $\mathbf{Z}_{\lambda,\alpha}(w)$ has a branch cut on $[-1,1]$, implying that the right hand side of (4.11) is discontinuous on $\mathrm{i}\mathbb{R}$.

Here are the connection formulas:

$$\mathbf{S}_{\alpha,\lambda}(-w) = -\frac{\cos(\pi\lambda)}{\sin(\pi\alpha)}\mathbf{S}_{\alpha,\lambda}(w) + \frac{2^{2\alpha}\pi}{\sin(\pi\alpha)\Gamma(\frac12+\alpha+\lambda)\Gamma(\frac12+\alpha-\lambda)}\frac{\mathbf{S}_{-\alpha,-\lambda}(w)}{(1-w^2)^{\alpha}}, \tag{4.12}$$

$$\mathbf{Z}_{\alpha,\lambda}(w) = -\frac{2^{\lambda-\alpha-\frac12}\sqrt{\pi}}{\sin(\pi\alpha)\Gamma(\frac12-\alpha+\lambda)}\mathbf{S}_{\alpha,\lambda}(w) + \frac{2^{\lambda+\alpha-\frac12}\sqrt{\pi}}{\sin(\pi\alpha)\Gamma(\frac12+\alpha+\lambda)}\frac{\mathbf{S}_{-\alpha,-\lambda}(w)}{(w^2-1)^{\alpha}_{\bullet}}. \tag{4.13}$$

Using the connection formulas (4.12) and (4.13) (and for $\alpha \in \mathbb{Z}$ in addition the de l'Hôpital rule, see also (4.23) and (4.24)) we derive the asymptotics of the Gegenbauer functions near $w = 1$ for $\mathrm{Re}(\alpha) > 0$:

$$\mathbf{S}_{\alpha,\lambda}(-w) \simeq \frac{2^{\alpha}\Gamma(\alpha)}{\Gamma(\frac12+\alpha+\lambda)\Gamma(\frac12+\alpha-\lambda)(1-w)^{\alpha}}, \tag{4.14}$$

$$\mathbf{Z}_{\alpha,\lambda}(w) \simeq \frac{2^{-\frac12+\lambda}\Gamma(\alpha)}{\sqrt{\pi}\Gamma(\frac12+\alpha+\lambda)(w-1)^{\alpha}}. \tag{4.15}$$

4.2. Recurrence relations. The Gegenbauer functions satisfy various recurrence relations, whose proof is straightforward:

$$\begin{aligned}
\partial_w \mathbf{S}_{\alpha,\lambda}(w) &= -\frac12\Big(\Big(\frac12+\alpha\Big)^2-\lambda^2\Big)\mathbf{S}_{\alpha+1,\lambda}(w),\\
\big((1-w^2)\partial_w - 2\alpha w\big)\,\mathbf{S}_{\alpha,\lambda}(w) &= -2\mathbf{S}_{\alpha-1,\lambda}(w),\\
\Big((1-w^2)\partial_w - \Big(\frac12+\alpha+\lambda\Big)w\Big)\,\mathbf{S}_{\alpha,\lambda}(w) &= -\Big(\frac12+\alpha+\lambda\Big)\mathbf{S}_{\alpha,\lambda+1}(w),\\
\Big((1-w^2)\partial_w - \Big(\frac12+\alpha-\lambda\Big)w\Big)\,\mathbf{S}_{\alpha,\lambda}(w) &= -\Big(\frac12+\alpha-\lambda\Big)\mathbf{S}_{\alpha,\lambda-1}(w);
\end{aligned}$$

$$\begin{aligned}
\partial_w \mathbf{Z}_{\alpha,\lambda}(w) &= -\Big(\frac12+\alpha+\lambda\Big)\mathbf{Z}_{\alpha+1,\lambda}(w),\\
\big((1-w^2)\partial_w - 2\alpha w\big)\,\mathbf{Z}_{\alpha,\lambda}(w) &= \Big(\frac12-\alpha+\lambda\Big)\mathbf{Z}_{\alpha-1,\lambda}(w),\\
\Big((1-w^2)\partial_w - \Big(\frac12+\alpha+\lambda\Big)w\Big)\,\mathbf{Z}_{\alpha,\lambda}(w) &= -\frac12\Big(\Big(\frac12+\lambda\Big)^2-\alpha^2\Big)\mathbf{Z}_{\alpha,\lambda+1}(w),\\
\Big((1-w^2)\partial_w - \Big(\frac12+\alpha-\lambda\Big)w\Big)\,\mathbf{Z}_{\alpha,\lambda}(w) &= 2\mathbf{Z}_{\alpha,\lambda-1}(w).
\end{aligned}$$

Remark 4.1. It might be interesting to note that the above recurrence relations correspond to short roots of the Lie algebra $so(5)$, as explained in [**5, 6**].

4.3. Gegenbauer polynomials. In the literature one can find two kinds of polynomials related to the functions $S_{\alpha,\lambda}$. The *Jacobi polynomials* with $\alpha = \beta$ are defined as

$$\begin{aligned}
P_n^{\alpha,\alpha}(w) &:= \frac{(\alpha+1)_n}{n!}F\Big(-n, n+2\alpha+1; 1+\alpha; \frac{1-w}{2}\Big)\\
&= \frac{\Gamma(\alpha+1+n)}{n!}\mathbf{S}_{\alpha,\frac12+\alpha+n}(w).
\end{aligned} \tag{4.16}$$

However, one usually prefers the *Gegenbauer polynomials*

$$\begin{aligned} C_n^{\alpha+\frac12}(w) &:= \frac{(2\alpha+1)_n}{(\alpha+1)_n} P_n^{\alpha,\alpha}(w) \\ &= \frac{(2\alpha+1)_n}{n!} F\Big(-n, n+2\alpha+1; 1+\alpha; \frac{1-w}{2}\Big). \end{aligned} \tag{4.17}$$

Note that

$$P_n^{\alpha,\alpha}(-w) = (-1)^n P_n^{\alpha,\alpha}(w), \quad C_n^{\frac12+\alpha}(-w) = (-1)^n C_n^{\frac12+\alpha}(w). \tag{4.18}$$

We have the well-known special cases: the *Legendre polynomials*

$$P_n(w) = P_n^{0,0}(w) = \mathbf{S}_{0,\frac12+n}(w); \tag{4.19}$$

and the *Chebyshev polynomials of the first and second kind*:

$$\begin{aligned} T_n(w) &= \frac{n!}{(\frac12)_n} P_n^{-\frac12,-\frac12}(w) = \sqrt{\pi}\mathbf{S}_{-\frac12,n}(w) \\ &= \frac12\Big(\big(w+\mathrm{i}\sqrt{1-w^2}\big)^n + \big(w-\mathrm{i}\sqrt{1-w^2}\big)^n\Big), \end{aligned} \tag{4.20}$$

$$\begin{aligned} U_n(w) &= \frac{n!}{(\frac32)_n} P_n^{\frac12,\frac12}(w) = C_n^1(w) = \frac{n+1}{2}\sqrt{\pi}\mathbf{S}_{\frac12,n+1}(w) \\ &= \frac{1}{2\mathrm{i}\sqrt{1-w^2}}\Big(\big(w+\mathrm{i}\sqrt{1-w^2}\big)^{n+1} - \big(w-\mathrm{i}\sqrt{1-w^2}\big)^{n+1}\Big). \end{aligned} \tag{4.21}$$

4.4. Degenerate case. The case $\alpha \in \mathbb{Z}$ is called degenerate, because then the standard solutions characterized by asymptotics at 1 become linearly dependent. More precisely, for $\alpha \in \mathbb{N}$ we have

$$= \frac{2^{2\alpha}}{(\frac12+\lambda)_\alpha(\frac12-\lambda)_\alpha(1-w^2)^\alpha}\mathbf{S}_{-\alpha,\pm\lambda}(w) \tag{4.22}$$

The functions $\mathbf{S}_{\alpha,\lambda}(-w)$ and $\mathbf{Z}_{\alpha,\lambda}(w)$ with $\alpha \in \mathbb{N}$ have the following expansions in the disc $|w-1| < 2$, involving a logarithmic singularity near 1:

$$\begin{aligned} &\Gamma(\tfrac12+\lambda+\alpha)\Gamma(\tfrac12-\lambda+\alpha)\mathbf{S}_{\alpha,\lambda}(-w) \\ &= \Big(\frac{1-w}{2}\Big)^{-\alpha}\sum_{k=0}^{\alpha-1}\frac{(\frac12+\lambda-k)_{2k}(\alpha-k-1)!}{k!}\Big(\frac{1-w}{2}\Big)^k \\ &\quad + \sum_{j=0}^{\infty}\frac{(\frac12+\lambda-\alpha-j)_{2\alpha+2j}}{j!(j+\alpha)!}\Big(\frac{w-1}{2}\Big)^j\Big(\psi(1+\alpha+j)+\psi(1+j) \\ &\quad - \psi(\tfrac12+\lambda+\alpha+j) - \psi(\tfrac12-\lambda+\alpha+j) - \ln\Big(\frac{1-w}{2}\Big)\Big). \end{aligned} \tag{4.23}$$

$$\begin{aligned} &\sqrt{2\pi}\,(-1)^\alpha 2^{\alpha-\lambda}\Gamma(\tfrac12+\lambda+\alpha)\mathbf{Z}_{\alpha,\lambda}(w) \\ &= \Big(\frac{1-w}{2}\Big)^{-\alpha}\sum_{k=0}^{\alpha-1}\frac{(\frac12+\lambda-k)_{2k}(\alpha-k-1)!}{k!}\Big(\frac{1-w}{2}\Big)^k \\ &\quad + \sum_{j=0}^{\infty}\frac{(\frac12+\lambda-\alpha-j)_{2\alpha+2j}}{j!(j+\alpha)!}\Big(\frac{w-1}{2}\Big)^j\Big(\psi(1+\alpha+j)+\psi(1+j) \\ &\quad - \psi(\tfrac12+\lambda+\alpha+j) - \psi(\tfrac12+\lambda-\alpha-j) - \ln\Big(\frac{w-1}{2}\Big)\Big). \end{aligned} \tag{4.24}$$

Let us sketch a proof of (4.24). Using the connection formula (4.13) we can write

$$\begin{aligned}
&\sqrt{2\pi}\, 2^{\alpha-\lambda}\Gamma(\tfrac12+\lambda+\alpha)\mathbf{Z}_{\alpha,\lambda}(w)\\
=&\frac{\pi}{\sin(\pi\alpha)}\Bigg(-\frac{\Gamma(\frac12+\alpha+\lambda)}{\Gamma(\frac12-\alpha+\lambda)}\mathbf{F}\Big(\tfrac12+\alpha+\lambda,\tfrac12+\alpha-\lambda;\alpha+1;\frac{1-w}{2}\Big)\\
&+\frac{2^{2\alpha}}{(w^2-1)_\bullet^{\alpha}}\Big(\frac{2}{w+1}\Big)^{-\alpha}\mathbf{F}\Big(\tfrac12+\lambda,\tfrac12-\lambda;-\alpha+1;\frac{1-w}{2}\Big)\Bigg)\\
=&\frac{\pi}{\sin(\pi\alpha)}\Bigg(-\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\frac12+\alpha+\lambda+j)}{\Gamma(\frac12-\alpha+\lambda-j)\Gamma(\alpha+1+j)j!}\Big(\frac{1-w}{2}\Big)^j\\
&+\Big(\frac{2}{w-1}\Big)^{\alpha}\sum_{k=0}^{\infty}\frac{(\frac12+\lambda)_k(\frac12-\lambda)_k}{\Gamma(-\alpha+1+k)k!}\Big(\frac{1-w}{2}\Big)^k\Bigg).
\end{aligned}$$

Then we apply the de l'Hôpital rule. Similar for (4.23).

4.5. Half-integer case. If α is a half-integer, then the Gegenbauer functions can be expressed in terms of elementary functions. They are particularly simple for $\alpha=-\frac12$ and $\alpha=\frac12$:

$$\sqrt{\pi}\mathbf{S}_{-\frac12,\lambda}(w)=S_{-\frac12,\lambda}(w)=\frac{\big(w+\mathrm{i}\sqrt{1-w^2}\big)^{\lambda}+\big(w-\mathrm{i}\sqrt{1-w^2}\big)^{\lambda}}{2},\tag{4.25}$$

$$\frac{\sqrt{\pi}}{2}\mathbf{S}_{\frac12,\lambda}(w)=S_{\frac12,\lambda}(w)=\frac{\big(w+\mathrm{i}\sqrt{1-w^2}\big)^{\lambda}-\big(w-\mathrm{i}\sqrt{1-w^2}\big)^{\lambda}}{2\lambda\mathrm{i}\sqrt{1-w^2}},\tag{4.26}$$

$$\Gamma(1+\lambda)\mathbf{Z}_{-\frac12,\lambda}(w)=Z_{-\frac12,\lambda}(w)=2^{\lambda}\big((w^2-1)_\bullet^{\frac12}+w\big)^{-\lambda},\tag{4.27}$$

$$\Gamma(1+\lambda)\mathbf{Z}_{\frac12,\lambda}(w)=Z_{\frac12,\lambda}(w)=2^{\lambda}\frac{\big(w^2-1)_\bullet^{\frac12}+w\big)^{-\lambda}}{(w^2-1)_\bullet^{\frac12}}.\tag{4.28}$$

Using the recurrence relations we can find expressions for Gegenbauer functions with $\alpha\in\frac12+\mathbb{Z}$. More precisely, for $n=0,1,\dots,$ we have

$$\mathbf{S}_{-\frac12-n,\lambda}(w)=\frac{(1-w^2)^{\frac12+n}}{2\sqrt{\pi}(-2)^n}\partial_w^n\frac{\big(w+\mathrm{i}\sqrt{1-w^2}\big)^{\lambda}+\big(w-\mathrm{i}\sqrt{1-w^2}\big)^{\lambda}}{\sqrt{1-w^2}},\tag{4.29}$$

$$\mathbf{S}_{\frac12+n,\lambda}(w)=\frac{2^n}{\mathrm{i}\sqrt{\pi}(\lambda-n)_{2n+1}}\partial_w^n\frac{\big(w+\mathrm{i}\sqrt{1-w^2}\big)^{\lambda}-\big(w-\mathrm{i}\sqrt{1-w^2}\big)^{\lambda}}{\sqrt{1-w^2}},\tag{4.30}$$

$$\mathbf{Z}_{-\frac12-n,\lambda}(w)=\frac{(-1)^n2^{\lambda}(w^2-1)_\bullet^{\frac12+n}}{\Gamma(1+\lambda+n)}\partial_w^n\frac{\big((w^2-1)_\bullet^{\frac12}+w\big)^{-\lambda}}{(w^2-1)_\bullet^{\frac12}},\tag{4.31}$$

$$\mathbf{Z}_{\frac12+n,\lambda}(w)=\frac{(-1)^n2^{\lambda}}{\Gamma(1+\lambda+n)}\partial_w^n\frac{\big((w^2-1)_\bullet^{\frac12}+w\big)^{-\lambda}}{(w^2-1)_\bullet^{\frac12}}.\tag{4.32}$$

4.6. Integral representations. In this subsection we describe the four basic types of integral representations of the Gegenbauer functions $\mathbf{S}_{\alpha,\lambda}$ and $\mathbf{Z}_{\alpha,\lambda}$. In order to understand their form it is useful to start with recalling some basic facts about the hypergeometric equation.

The hypergeometric equation has 6 standard solutions, which are traditionally presented in *Kummer's table*. To represent these solutions in a form as symmetric as possible, instead of the usual parameters a, b, c it is convenient to use the parameters $\alpha = c-1$, $\beta = a+b-c$, $\mu = a-b$. Various elements of Kummer's table correspond to permutations of α, β, μ and switching their signs, as explained e.g. in [**5, 6**].

The hypergeometric function $\mathbf{F}(a, b; c; z)$ is of course one of standard solutions of the hypergeometric equation. For $\mathrm{Re}(1+\alpha) > |\mathrm{Re}(\beta \mp \mu)|$ it satisfies the following pair of identities

$$\mathbf{F}\Big(\frac{1+\alpha+\beta+\mu}{2}, \frac{1+\alpha+\beta-\mu}{2}; \alpha+1; z\Big) \tag{4.33}$$
$$= \frac{1}{\Gamma(\frac{1+\alpha+\beta\mp\mu}{2})\Gamma(\frac{1+\alpha-\beta\pm\mu}{2})} \int_1^\infty t^{\frac{-1-\alpha+\beta\pm\mu}{2}} (t-1)^{\frac{-1+\alpha-\beta\pm\mu}{2}} (t-z)^{\frac{-1-\alpha-\beta\mp\mu}{2}} \mathrm{d}t.$$

(see e.g. [**5, 6**] and [**17**, Eq. (15.6.1)]). Note that the change of the sign of μ does not affect the left hand side, but yields two distinct integral representations on the right hand side.

By choosing intervals that join pairs of singular points $0, 1, \infty, z$ of the integrand of (4.33) one obtains integral representations of all 6 standard solutions. Let us quote the pair of such representations of another standard solution, the first for $\mathrm{Re}(1+\mu) > |\mathrm{Re}(\alpha-\beta)|$, the second for $\mathrm{Re}(1+\mu) > |\mathrm{Re}(\alpha+\beta)|$:

$$z^{\frac{-1-\alpha-\beta-\mu}{2}} \mathbf{F}\Big(\frac{1+\mu+\beta+\alpha}{2}, \frac{1+\mu+\beta-\alpha}{2}; \mu+1; z^{-1}\Big)$$
$$= \frac{1}{\Gamma(\frac{1+\mu+\beta-\alpha}{2})\Gamma(\frac{1+\mu-\beta+\alpha}{2})} \int_0^1 t^{\frac{-1-\alpha+\beta+\mu}{2}} (1-t)^{\frac{-1+\alpha-\beta+\mu}{2}} (z-t)^{\frac{-1-\alpha-\beta-\mu}{2}} \mathrm{d}t, \tag{4.34}$$
$$= \frac{1}{\Gamma(\frac{1+\mu+\beta+\alpha}{2})\Gamma(\frac{1+\mu-\beta-\alpha}{2})} \int_z^\infty t^{\frac{-1-\alpha+\beta-\mu}{2}} (t-1)^{\frac{-1+\alpha-\beta-\mu}{2}} (t-z)^{\frac{-1-\alpha-\beta+\mu}{2}} \mathrm{d}t. \tag{4.35}$$

To obtain representations of the Gegenbauer functions we need to set $\alpha = \beta$, $\mu = 2\lambda$, $t = \frac{s+1}{2}$ in the above formulas. By setting $z = \frac{1-w}{2}$ in (4.33), we obtain the following pair of representations valid for $\frac{1}{2} > \mp\mathrm{Re}(\lambda) > -\frac{1}{2} - \mathrm{Re}(\alpha)$:

$$\mathbf{S}_{\alpha,\lambda}(w) = \frac{2^{\frac{1}{2}+\alpha\mp\lambda}}{\Gamma(\frac{1}{2}+\alpha\mp\lambda)\Gamma(\frac{1}{2}\pm\lambda)} \int_1^\infty (s^2-1)^{-\frac{1}{2}\pm\lambda} (s+w)^{-\frac{1}{2}-\alpha\mp\lambda} \mathrm{d}s. \tag{4.36}$$

Setting $z = \frac{1+w}{2}$ in (4.34) we obtain the identity valid for $\mathrm{Re}(\lambda) + \frac{1}{2} > 0$:

$$\mathbf{Z}_{\alpha,\lambda}(w) = \frac{1}{\sqrt{\pi}\Gamma(\frac{1}{2}+\lambda)} \int_{-1}^1 (1-s^2)^{-\frac{1}{2}+\lambda} (w-s)^{-\frac{1}{2}-\alpha-\lambda} \mathrm{d}s, \tag{4.37}$$

Setting $z = \frac{1+w}{2}$ in (4.35) yields the formula valid for $\mathrm{Re}(\lambda) + \frac{1}{2} > |\mathrm{Re}(\alpha)|$:

$$\mathbf{Z}_{\alpha,\lambda}(w) = \frac{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}+\lambda-\alpha)\Gamma(\frac{1}{2}+\lambda+\alpha)} \int_w^\infty (s^2-1)_\bullet^{-\frac{1}{2}-\lambda} (s-w)^{-\frac{1}{2}-\alpha+\lambda} \mathrm{d}s. \tag{4.38}$$

Derivation of (4.37) and (4.38) uses the duplication formula (A.8).

Inserting $w = \frac{z}{(z^2-1)_\bullet^{\frac{1}{2}}}$ and using the Whipple transformations (4.10) and (4.11) we derive another family of integral representations. The first follows by inserting

$s=\frac{t-z}{(z^2-1)_\bullet^{\frac12}}$ into (4.36) and is valid for $\mathrm{Re}(\lambda)+\frac12>-\mathrm{Re}(\alpha)>-\frac12$:

$$\mathbf{Z}_{\alpha,\lambda}(z)=\frac{2^{\frac12+\alpha+\lambda}}{\Gamma(\frac12+\alpha+\lambda)\Gamma(\frac12-\alpha)}\int_{(z^2-1)_\bullet^{\frac12}+z}^{\infty}(t^2-2tz+1)^{-\alpha-\frac12}t^{-\frac12+\alpha-\lambda}\mathrm{d}t. \tag{4.39}$$

This one follows by inserting $s=\frac{-t+z}{(z^2-1)_\bullet^{\frac12}}$ into (4.37) and holds for $\mathrm{Re}(\alpha)+\frac12>0$:

$$\mathbf{S}_{\alpha,\lambda}(z)=-\frac{\mathrm{i}(1-z^2)^{-\alpha}}{\sqrt{\pi}\Gamma(\frac12+\alpha)}\int_{z-\mathrm{i}\sqrt{1-z^2}}^{z+\mathrm{i}\sqrt{1-z^2}}(t^2-2tz+1)_\bullet^{\alpha-\frac12}t^{-\frac12-\alpha\pm\lambda}\mathrm{d}t, \tag{4.40}$$

where $(t^2-2tz+1)_\bullet^{\mu}=(t-z-\mathrm{i}\sqrt{1-z^2})^{\mu}(t-z+\mathrm{i}\sqrt{1-z^2})^{\mu}$, and the integration contour is chosen so that it passes zero from the right without encircling it and does not cross the horizontal half-lines $z\pm\mathrm{i}\sqrt{1-z^2}-\mathbb{R}_+$. Such contours varying smoothly with z on $\mathbb{C}\setminus(]-\infty,-1]\cup[1,\infty[)$ exist because on this region $\mathrm{Im}(z-\mathrm{i}\sqrt{1-z^2})<0$ and $\mathrm{Im}(z+\mathrm{i}\sqrt{1-z^2})>0$. Since $\mathbf{S}_{\alpha,\lambda}(z)$ is continuous on $[1,\infty[$, the right hand side of (4.40) has the same limits as z approaches an element of $[1,\infty[$ from above or below. This is not obvious directly from (4.40). The next representation follows by inserting $s=\frac{t+z}{(z^2-1)_\bullet^{\frac12}}$ into (4.38), and is valid for $\mathrm{Re}(\alpha)+\frac12>|\mathrm{Re}(\lambda)|$:

$$\mathbf{S}_{\alpha,\lambda}(z)=\frac{2^{2\alpha}\Gamma(\alpha+\frac12)}{\sqrt{\pi}\Gamma(\frac12+\alpha-\lambda)\Gamma(\frac12+\alpha+\lambda)}\int_0^{\infty}(t^2+2tz+1)^{-\alpha-\frac12}t^{-\frac12+\alpha-\lambda}\mathrm{d}t. \tag{4.41}$$

It is convenient to make an exponential change of variables in the representations above. Substituting $z=\cosh\theta$, with $\mathrm{Re}(\theta)>0$, $|\mathrm{Im}(\theta)|<\pi$, and $t=\mathrm{e}^{\phi}$ in (4.39), we obtain (with the same restrictions on α,λ):

$$\mathbf{Z}_{\alpha,\lambda}(\cosh\theta)=\frac{2^{\lambda}}{\Gamma(\frac12+\alpha+\lambda)\Gamma(\frac12-\alpha)}\int_{\theta}^{\infty}\big(\cosh\phi-\cosh\theta\big)^{-\alpha-\frac12}\mathrm{e}^{-\lambda\phi}\mathrm{d}\phi. \tag{4.42}$$

Under the same conditions on θ, combining (4.9) and (4.42) gives

$$\mathbf{Z}_{\alpha,\lambda}(\cosh\theta)=\frac{2^{\lambda}}{\Gamma(\frac12-\alpha+\lambda)\Gamma(\frac12+\alpha)(\sinh\theta)^{2\alpha}}\int_{\theta}^{\infty}(\cosh\phi-\cosh\theta)^{\alpha-\frac12}\mathrm{e}^{-\lambda\phi}\mathrm{d}\phi. \tag{4.43}$$

Substituting $t=\mathrm{e}^{\mathrm{i}\phi}$ in (4.40) we obtain for $0<\mathrm{Re}(\theta)<\pi$

$$\mathbf{S}_{\alpha,\lambda}(\cos\theta)=\frac{2^{\alpha-\frac12}}{\sqrt{\pi}\Gamma(\frac12+\alpha)(\sin\theta)^{2\alpha}}\int_{-\theta}^{\theta}(\cos\phi-\cos\theta)^{\alpha-\frac12}\mathrm{e}^{\pm\mathrm{i}\lambda\phi}\mathrm{d}\phi. \tag{4.44}$$

Substituting $t=\mathrm{e}^{\phi}$, in (4.41) we get for $|\mathrm{Re}(\theta)|<\pi$

$$\mathbf{S}_{\alpha,\lambda}(\cos\theta)=\frac{2^{\alpha-\frac12}\Gamma(\alpha+\frac12)}{\sqrt{\pi}\Gamma(\frac12+\alpha-\lambda)\Gamma(\frac12+\alpha+\lambda)}\int_{-\infty}^{\infty}(\cosh\phi+\cos\theta)^{-\alpha-\frac12}\mathrm{e}^{-\lambda\phi}\mathrm{d}\phi. \tag{4.45}$$

4.7. Asymptotics. From the integral representations of the Gegenbauer functions, we can deduce the following asymptotics:

THEOREM 4.2. *Let $\alpha \geq -\frac{1}{2}$ and $\pi > \delta > 0$ be fixed. Then we have the following estimates:*

(1) *Uniformly for $\theta \in [0, \pi - \delta]$ and $\beta \to \infty$,*

$$\frac{(\sin\theta)^{\alpha+\frac{1}{2}}}{2^\alpha \theta^{\alpha+\frac{1}{2}}} \mathbf{S}_{\alpha,\pm \mathrm{i}\beta}(\cos\theta) = (\theta\beta)^{-\alpha} I_\alpha(\theta\beta)\Big(1 + \theta O\big(\beta^{-1}\big)\Big). \tag{4.46}$$

(2) *Uniformly for $\theta \in [0, \pi - \delta]$ and $\beta \to \infty$,*

$$\frac{\pi \mathrm{e}^{-\pi\beta}(\sin\theta)^{\alpha+\frac{1}{2}}}{2^\alpha \theta^{\alpha+\frac{1}{2}}} \mathbf{S}_{\alpha,\pm \mathrm{i}\beta}(-\cos\theta) = (\theta\beta)^{-\alpha} K_\alpha(\beta\theta)\big(1 + O(\beta^{-1})\big). \tag{4.47}$$

(3) *Uniformly for $\theta \geq 0$ and $\lambda \to \infty$,*

$$\frac{\sqrt{\pi}\Gamma(\frac{1}{2} - \alpha + \lambda)(\sinh\theta)^{\alpha+\frac{1}{2}}}{2^{\lambda+\frac{1}{2}} \theta^{\alpha+\frac{1}{2}}} \mathbf{Z}_{\alpha,\lambda}(\cosh\theta) = (\lambda\theta)^{-\alpha} K_\alpha(\lambda\theta)\big(1 + O(\lambda^{-1})\big). \tag{4.48}$$

REMARK 4.3. Similar statements about associated Legendre functions can for example be found in [**4, 10**]. See also [**18**, Chapter 12, §§12,13] for detailed asymptotics of Legendre functions including also non-leading terms.

In the proof of Theorem 4.2, we will use the functions

$$\operatorname{sinc}\theta := \frac{\sin\theta}{\theta}, \quad \operatorname{sinhc}\theta := \frac{\sinh\theta}{\theta}. \tag{4.49}$$

We note that $\operatorname{sinc} 0 = \operatorname{sinhc} 0 = 1$, sinc is decreasing on $[0, \pi]$, sinhc and all its derivatives are increasing on $[0, \infty[$, and for $\theta \in [0, \pi]$ we have

$$0 \leq \operatorname{sinc}\theta \leq 1 - \frac{\theta}{\pi}, \qquad -\frac{1}{3}\theta \leq \operatorname{sinc}'\theta \leq 0. \tag{4.50}$$

PROOF. We note first that the limiting case $\alpha = -\frac{1}{2}$ can be proved explicitly by using the representations of modified Bessel functions and Gegenbauer functions in the half-integer case, given in Sections 3.3 and 4.5, respectively. For the following, let us thus assume $\alpha > -\frac{1}{2}$. In the proof, c will be the notation for various constants independent of θ, s.

Let us prove (4.46). The integral representation (3.31) implies

$$(\theta\beta)^{-\alpha} I_\alpha(\theta\beta) = \frac{1}{\sqrt{2\pi}\Gamma\big(\alpha + \frac{1}{2}\big)} \int_{-1}^{1} \Big(\frac{1 - s^2}{2}\Big)^{\alpha - \frac{1}{2}} \mathrm{e}^{-\theta\beta s} \mathrm{d}s. \tag{4.51}$$

We use the integral representation (4.44), the substitution $\phi = \theta s$ and the identity $\cos(s\theta) - \cos\theta = 2\sin\big(\theta\frac{1-s}{2}\big)\sin\big(\theta\frac{1+s}{2}\big)$ to find

$$\begin{aligned} &\frac{(\sin\theta)^{\alpha+\frac{1}{2}}}{2^\alpha \theta^{\alpha+\frac{1}{2}}} \mathbf{S}_{\alpha,\pm \mathrm{i}\beta}(\cos\theta) \\ =&\frac{1}{\sqrt{2\pi}\Gamma(\frac{1}{2} + \alpha)} \int_{-1}^{1} \left(\frac{2\sin\big(\theta\frac{1-s}{2}\big)\sin\big(\theta\frac{1+s}{2}\big)}{\theta\sin\theta}\right)^{\alpha - \frac{1}{2}} \mathrm{e}^{-\theta\beta s} \mathrm{d}s. \end{aligned} \tag{4.52}$$

Subtracting (4.51) from (4.52), we obtain

$$\begin{aligned}(4.53)\qquad &\frac{(\sin\theta)^{\alpha+\frac12}}{2^\alpha\theta^{\alpha+\frac12}}\mathbf{S}_{\alpha,\pm\mathrm{i}\beta}(\cos\theta)-(\theta\beta)^{-\alpha}I_\alpha(\theta\beta)\\ &=\frac{1}{\sqrt{2\pi}\Gamma(\alpha+\frac12)}\int_{-1}^1\Big(\frac{1-s^2}{2}\Big)^{\alpha-\frac12}\bigg(\Big(\frac{\operatorname{sinc}\big(\theta\frac{1-s}{2}\big)\operatorname{sinc}\big(\theta\frac{1+s}{2}\big)}{\operatorname{sinc}\theta}\Big)^{\alpha-\frac12}-1\bigg)\mathrm{e}^{-s\beta\theta}\mathrm{d}s.\end{aligned}$$

Next,

$$\begin{aligned}(4.54)\qquad &\frac{\operatorname{sinc}\big(\theta\frac{1-s}{2}\big)\operatorname{sinc}\big(\theta\frac{1+s}{2}\big)}{\operatorname{sinc}\theta}-1\\ &=\Big(\operatorname{sinc}\frac{(1-|s|)\theta}{2}-1\Big)\frac{\operatorname{sinc}\frac{(1+|s|)\theta}{2}}{\operatorname{sinc}\theta}+\frac{\operatorname{sinc}\frac{(1+|s|)\theta}{2}-\operatorname{sinc}\theta}{\operatorname{sinc}\theta}=:I+II.\end{aligned}$$

Now

$$|I|\le c\frac{\theta^2(1-|s|)^2}{\operatorname{sinc}\theta},\quad |II|\le c\frac{\theta^2(1-|s|)}{\operatorname{sinc}\theta},\quad\text{hence}\quad |(4.54)|\le c\frac{\theta^2(1-|s|)}{\operatorname{sinc}\theta}.$$

II is estimated as follows:

$$(4.55)\qquad \Big|\operatorname{sinc}\frac{(1+|s|)\theta}{2}-\operatorname{sinc}\theta\Big|\le\Big|\frac{(1+|s|)}{2}\theta-\theta\Big|\sup_{\phi\in[0,\theta]}|\operatorname{sinc}'\phi|\le c\theta^2(1-|s|).$$

Hence, using

$$(4.56)\qquad |x^\kappa-1|=\Big|\kappa\int_1^x y^{\kappa-1}\mathrm{d}y\Big|\le|\kappa||x-1|\max(x^{\kappa-1},1)$$

and $\frac{1}{\operatorname{sinc}(c)}\le c$, $\frac{\operatorname{sinc}\left(\theta\frac{1-s}{2}\right)\operatorname{sinc}\left(\theta\frac{s+1}{2}\right)}{\operatorname{sinc}\theta}\ge c>0$ on $[0,\pi-\delta]$, we obtain

$$(4.57)\qquad \bigg|\Big(\frac{\operatorname{sinc}\big(\theta\frac{1-s}{2}\big)\operatorname{sinc}\big(\theta\frac{s+1}{2}\big)}{\operatorname{sinc}\theta}\Big)^{\alpha-\frac12}-1\bigg|\le c\theta^2(1-|s|).$$

Thus, applying (4.57) and then integrating by parts, we obtain

$$\begin{aligned}(4.58)\qquad |(4.53)|&\le c\theta^2\int_{-1}^1\Big(\frac{1-s^2}{2}\Big)^{\alpha-\frac12}(1-|s|)\mathrm{e}^{s\beta\theta}\mathrm{d}s\\ &=\frac{c\theta}{\beta}\int_{-1}^1\Big(\frac{1-s^2}{2}\Big)^{\alpha-\frac12}\frac{2\alpha s+\operatorname{sgn}(s)}{1+|s|}\mathrm{e}^{s\beta\theta}\mathrm{d}s\\ &\le\frac{c\theta}{\beta}(\beta\theta)^{-\alpha}I_\alpha(\beta\theta).\end{aligned}$$

This proves (4.46).

We next prove (4.47). The integral representation (3.32) gives

$$(4.59)\qquad (\theta\beta)^{-\alpha}K_\alpha(\theta\beta)=\frac{\sqrt{\pi}}{\sqrt{2}\Gamma(\frac12+\alpha)}\int_1^\infty\Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac12}\mathrm{e}^{-\theta\beta s}\mathrm{d}s.$$

The integral representation (4.44), the substitution $\phi = \theta s - \pi$ and the identity $\cos\theta - \cos(s\theta) = 2\sin\left(\theta\frac{s-1}{2}\right)\sin\left(\theta\frac{s+1}{2}\right)$ yield

$$\begin{aligned}
&\frac{\pi \mathrm{e}^{-\pi\beta}(\sin\theta)^{\alpha+\frac12}}{2^\alpha\theta^{\alpha+\frac12}}\mathbf{S}_{\alpha,\pm\mathrm{i}\beta}(-\cos\theta) \\
=&\frac{\sqrt{\pi}}{\sqrt{2}\Gamma(\frac12+\alpha)}\int_1^{\frac{2\pi}{\theta}-1}\left(\frac{2\sin\left(\theta\frac{s-1}{2}\right)\sin\left(\theta\frac{1+s}{2}\right)}{\theta\sin\theta}\right)^{\alpha-\frac12}\mathrm{e}^{-\theta\beta s}\mathrm{d}s.
\end{aligned} \tag{4.60}$$

Subtracting (4.59) from (4.60) we obtain

$$\begin{aligned}
&\frac{\pi(\sin\theta)^{\alpha+\frac12}\mathrm{e}^{-\beta\pi}}{2^\alpha\theta^{\alpha+\frac12}}\mathbf{S}_{\alpha,\pm\mathrm{i}\beta}(-\cos\theta)-(\beta\theta)^{-\alpha}K_\alpha(\beta\theta) \\
&=\frac{\sqrt{\pi}}{\sqrt{2}\Gamma(\alpha+\frac12)}\int_1^{\frac{2\pi}{\theta}-1}\Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac12}\left(\Big(\frac{\operatorname{sinc}\theta\frac{s-1}{2}\operatorname{sinc}\theta\frac{s+1}{2}}{\operatorname{sinc}\theta}\Big)^{\alpha-\frac12}-1\right)\mathrm{e}^{-s\beta\theta}\mathrm{d}s \\
&-\frac{\sqrt{\pi}}{\sqrt{2}\Gamma(\alpha+\frac12)}\int_{\frac{2\pi}{\theta}-1}^\infty\Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac12}\mathrm{e}^{-s\beta\theta}\mathrm{d}s \;=:\; \frac{\sqrt{\pi}}{\sqrt{2}\Gamma(\alpha+\frac12)}\int_1^\infty F(s)\mathrm{d}s.
\end{aligned} \tag{4.61}$$

Arguing as in the proof of (4.55), we obtain

$$\left|\frac{\operatorname{sinc}\left(\theta\frac{s-1}{2}\right)\operatorname{sinc}\left(\theta\frac{s+1}{2}\right)}{\operatorname{sinc}\theta}-1\right|\le c\theta(s-1). \tag{4.62}$$

The power of θ is worse than in (4.57), because for $|s|>1$ we can no longer bound II by a term proportional to θ^2.

Unfortunately, due to the zero of $\operatorname{sinc}\theta\frac{s+1}{2}$ at $s=\frac{2\pi}{\theta}-1$, the remaining part of the argument has to be split into 3 cases.

First we consider $\alpha\ge\frac32$. Using (4.56) we obtain

$$\left|\Big(\frac{\operatorname{sinc}\left(\theta\frac{s-1}{2}\right)\operatorname{sinc}\left(\theta\frac{s+1}{2}\right)}{\operatorname{sinc}\theta}\Big)^{\alpha-\frac12}-1\right|\le c\theta(s-1). \tag{4.63}$$

Therefore,

$$\Big|\int_1^{\frac{2\pi}{\theta}-1}F(s)\mathrm{d}s\Big|\le c\theta\int_1^{\frac{2\pi}{\theta}-1}\Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac12}(s-1)\mathrm{e}^{-s\beta\theta}\mathrm{d}s. \tag{4.64}$$

Next, using $\theta(s-1)\ge 2\pi-2\theta$ we obtain

$$\Big|\int_{\frac{2\pi}{\theta}-1}^\infty F(s)\mathrm{d}s\Big|\le\theta(2\pi-2\theta)^{-1}\int_{\frac{2\pi}{\theta}-1}^\infty\Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac12}(s-1)\mathrm{e}^{-s\beta\theta}\mathrm{d}s \tag{4.65}$$

Then we sum up (4.64) and (4.65). We choose the new c to be the maximum of the old c and $(2\delta)^{-1}$. Next we integrate by parts:

$$\begin{aligned}
\Big|\int_1^\infty F(s)\mathrm{d}s\Big| &\le c\theta\int_1^\infty\Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac12}(s-1)\mathrm{e}^{-s\beta\theta}\mathrm{d}s \\
&=\frac{c}{\beta}\int_1^\infty\Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac12}\frac{2\alpha s+1}{1+s}\mathrm{e}^{-s\beta\theta}\mathrm{d}s \\
&\le\frac{c}{\beta}(\beta\theta)^{-\alpha}K_\alpha(\beta\theta).
\end{aligned} \tag{4.66}$$

Next we consider $\frac{1}{2} \le \alpha \le \frac{3}{2}$. Choose $\sigma \in]\pi, \pi+\delta[$. For $1 \le s \le \frac{\sigma}{\theta}$, we have

$$\pi - \theta\frac{s+1}{2} \ge \frac{\pi-\sigma+\delta}{2} > 0, \tag{4.67}$$

so for such s the argument of $\operatorname{sinc}\big(\theta\frac{s+1}{2}\big)$ is separated from the location of the zero by a constant independent of θ and s. Hence we have a bound

$$\frac{\operatorname{sinc}\big(\theta\frac{s-1}{2}\big)\operatorname{sinc}\big(\theta\frac{s+1}{2}\big)}{\operatorname{sinc}\theta} \ge c > 0. \tag{4.68}$$

Therefore, we can apply (4.56) and obtain (4.63) on this interval. Thus

$$\Big|\int_1^{\frac{\sigma}{\theta}} F(s)\mathrm{d}s\Big| \le c\theta\int_1^{\frac{\sigma}{\theta}} \Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac{1}{2}}(s-1)\mathrm{e}^{-s\beta\theta}\mathrm{d}s. \tag{4.69}$$

On $\frac{\sigma}{\theta} \le s \le \frac{2\pi}{\theta}-1$ we have $\delta < \sigma-\theta \le \theta(s-1)$. We estimate the left hand side of (4.63) by a constant and get

$$\Big|\int_{\frac{\sigma}{\theta}}^{\frac{2\pi}{\theta}-1} F(s)\mathrm{d}s\Big| \le c\frac{\theta}{\delta}\int_{\frac{\sigma}{\theta}}^{\frac{2\pi}{\theta}-1} \Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac{1}{2}}(s-1)\mathrm{e}^{-s\beta\theta}\mathrm{d}s. \tag{4.70}$$

Now combine (4.69), (4.70), and (4.65) and argue as in (4.66).

Finally, consider $-\frac{1}{2} < \alpha \le \frac{1}{2}$. We need to consider only the interval $\frac{\sigma}{\theta} \le s \le \frac{2\pi}{\theta}-1$. Contributions of the remaining part of the integration region are $O(\beta^{-1})$ by previous arguments.

First,

$$s^2-1 \ge \frac{\sigma^2-\theta^2}{\theta^2} \ge \frac{c}{\theta^2} \quad\Rightarrow\quad \Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac{1}{2}} \le c\theta^{1-2\alpha}. \tag{4.71}$$

Moreover, using (4.50) we have

$$\frac{1}{\sin\theta} \ge c, \quad \operatorname{sinc}\big(\theta\tfrac{s\pm1}{2}\big) \ge c\theta\big(\tfrac{2\pi}{\theta}-s\mp1\big), \tag{4.72}$$

which implies

$$\begin{aligned}\Big(\frac{\operatorname{sinc}\big(\theta\frac{s-1}{2}\big)\operatorname{sinc}\big(\theta\frac{s+1}{2}\big)}{\operatorname{sinc}\theta}\Big)^{\alpha-\frac{1}{2}} - 1 &\le \Big(\frac{\operatorname{sinc}\big(\theta\frac{s-1}{2}\big)\operatorname{sinc}\big(\theta\frac{s+1}{2}\big)}{\operatorname{sinc}\theta}\Big)^{\alpha-\frac{1}{2}} \\ \le c\theta^{2\alpha-1}\Big(\big(\tfrac{2\pi}{\theta}-s\big)^2-1\Big)^{\alpha-\frac{1}{2}} &= c\theta^{2\alpha-1}(t^2-1)^{\alpha-\frac{1}{2}},\end{aligned} \tag{4.73}$$

where we set $t = \frac{2\pi}{\theta}-s$. Then we estimate (very roughly)

$$\Big|\int_{\frac{\sigma}{\theta}}^{\frac{2\pi}{\theta}-1} F(s)\mathrm{d}s\Big| \le c\int_1^{\frac{2\pi-\sigma}{\theta}} (t^2-1)^{\alpha-\frac{1}{2}}\mathrm{e}^{-\beta2\pi+\beta\theta t}\mathrm{d}t \tag{4.74}$$

$$\le c\int_1^{\frac{2\pi-\sigma}{\theta}} (t^2-1)^{\alpha-\frac{1}{2}}\mathrm{e}^{\beta2(\pi-\sigma)-\beta\theta t}\mathrm{d}t \tag{4.75}$$

$$\le\ c\mathrm{e}^{\beta2(\pi-\sigma)}(\beta\theta)^{-\alpha}K_\alpha(\beta\theta) = (\beta\theta)^{-\alpha}K_\alpha(\beta\theta)O(\beta^{-\infty}). \tag{4.76}$$

Finally, let us prove (4.48). Substituting $\phi = s\theta$ in (4.43) and using the identity $\cosh(s\theta) - \cosh(\theta) = 2\sinh\left(\theta\frac{s-1}{2}\right)\sinh\left(\theta\frac{s+1}{2}\right)$, we find

$$\frac{\sqrt{\pi}\Gamma(\frac{1}{2}-\alpha+\lambda)(\sinh\theta)^{\alpha+\frac{1}{2}}}{2^{\lambda+\frac{1}{2}}\theta^{\alpha+\frac{1}{2}}}\mathbf{Z}_{\alpha,\lambda}(\cosh\theta) \tag{4.77}$$

$$=\frac{\sqrt{\pi}}{\sqrt{2}\Gamma(\frac{1}{2}+\alpha)}\int_1^\infty\left(\frac{2\sinh\left(\theta\frac{s-1}{2}\right)\sinh\left(\theta\frac{s+1}{2}\right)}{\theta\sinh\theta}\right)^{\alpha-\frac{1}{2}}\mathrm{e}^{-\lambda\theta s}\mathrm{d}s. \tag{4.78}$$

Subtracting (4.59) with β replaced with λ from (4.77), we obtain

$$\begin{aligned}&\frac{\sqrt{\pi}\Gamma(-\alpha+\lambda+\frac{1}{2})(\sinh\theta)^{\alpha+\frac{1}{2}}}{2^{\lambda+\frac{1}{2}}\theta^{\alpha+\frac{1}{2}}}\mathbf{Z}_{\alpha,\lambda}(\cosh\theta)-(\lambda\theta)^{-\alpha}K_\alpha(\lambda\theta)\\ =&c\int_1^\infty\Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac{1}{2}}\left(\Big(\frac{\operatorname{sinhc}\left(\theta\frac{s-1}{2}\right)\operatorname{sinhc}\left(\theta\frac{s+1}{2}\right)}{\operatorname{sinhc}\theta}\Big)^{\alpha-\frac{1}{2}}-1\right)\mathrm{e}^{-s\lambda\theta}\mathrm{d}s.\end{aligned} \tag{4.79}$$

Function $s\mapsto\frac{\operatorname{sinhc}\left(\theta\frac{s-1}{2}\right)\operatorname{sinhc}\left(\theta\frac{s+1}{2}\right)}{\operatorname{sinhc}(\theta)}$ is monotonically increasing and equals 1 at $s=1$. Arguing as in the proof of (4.57) we obtain

$$\left|\frac{\operatorname{sinhc}\left(\theta\frac{s-1}{2}\right)\operatorname{sinhc}\left(\theta\frac{s+1}{2}\right)}{\operatorname{sinhc}\theta}-1\right|\leq c\theta(s-1)\mathrm{e}^{\theta(s-1)}. \tag{4.80}$$

Next we note that

$$\begin{aligned}&\frac{\operatorname{sinhc}\left(\theta\frac{s-1}{2}\right)\operatorname{sinhc}\left(\theta\frac{s+1}{2}\right)}{\operatorname{sinhc}\theta}=\frac{2}{\theta(s^2-1)}(\mathrm{e}^{\theta(s-1)}-1)\frac{1-\mathrm{e}^{-\theta(s+1)}}{1-\mathrm{e}^{-2\theta}}\\ \leq&\frac{2\mathrm{e}^{\theta(s-1)}}{s+1}\frac{1-\mathrm{e}^{-\theta(s+1)}}{1-\mathrm{e}^{-2\theta}}\leq\mathrm{e}^{\theta(s-1)},\end{aligned} \tag{4.81}$$

where in the first inequality we used $\mathrm{e}^x-1\leq x\mathrm{e}^x$, and in the second one we optimized $\frac{1-\mathrm{e}^{-\theta(s+1)}}{1-\mathrm{e}^{-2\theta}}$ with respect to θ. Invoking (4.56) again, we get

$$\left|\Big(\frac{\operatorname{sinhc}\theta\frac{s-1}{2}\operatorname{sinhc}\theta\frac{s+1}{2}}{\operatorname{sinhc}\theta}\Big)^{\alpha-\frac{1}{2}}-1\right|\leq c\theta(s-1)\mathrm{e}^{\theta(s-1)\rho(\alpha)}, \tag{4.82}$$

where $\rho(\alpha)=\max\{1,\alpha-\frac{1}{2}\}$. Then, integrating by parts, we obtain

$$\begin{aligned}|(4.79)|\leq&c\theta\int_1^\infty\Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac{1}{2}}(s-1)\mathrm{e}^{-s\lambda\theta+(s-1)\theta\rho(\alpha)}\mathrm{d}s\\ =&\frac{c}{\lambda-\rho(\alpha)}\mathrm{e}^{-\theta\rho(\alpha)}\int_1^\infty\Big(\frac{s^2-1}{2}\Big)^{\alpha-\frac{1}{2}}\frac{2\alpha s+1}{1+s}\mathrm{e}^{-s\theta(\lambda-\rho(\alpha))}\mathrm{d}s\\ \leq&\frac{c\mathrm{e}^{-\theta\rho(\alpha)}}{\lambda-\rho(\alpha)}\big(\theta(\lambda-\rho(\alpha))\big)^{-\alpha}K_\alpha\big(\theta(\lambda-\rho(\alpha))\big)\\ \leq&\frac{c}{\lambda-\rho(\alpha)}\Big(\frac{\lambda}{\lambda-\rho(\alpha)}\Big)^{2\alpha}(\lambda\theta)^{-\alpha}K_\alpha(\lambda\theta),\end{aligned} \tag{4.83}$$

where in the last step we used (3.8). □

4.8. Bilinear integrals. In this subsection we compute certain integrals involving products of two Gegenbauer functions. They are analogs of the integrals for products of Macdonald functions considered in Subsect. 3.4. We will use the integration variable $2w$, because this will facilitate comparison of the integrals for Gegenbauer functions with those for Macdonald functions. Indeed, if we consider the spherical, resp. hyperbolic interpretation of these identities, for small r, we have $2(w \mp 1) \approx r^2$. Therefore, $\mathrm{d}2w$ corresponds to $2r\mathrm{d}r = \mathrm{d}r^2$. This is especially important in the case of anomalous generalized integrals in Theorem 4.10, where without a good choice of variables we would not have obtained a correct limit.

THEOREM 4.4. *For $|\mathrm{Re}(\alpha)| < 1$, the following identities hold:*

$$\begin{aligned}(4.84)\qquad &\int_{-1}^{1} \mathbf{S}_{\alpha,\mathrm{i}\beta_1}(w)\mathbf{S}_{\alpha,\mathrm{i}\beta_2}(w)(1-w^2)^{\alpha}\mathrm{d}2w \\ =&\frac{2^{2\alpha+2}}{(\beta_1^2-\beta_2^2)\sin\pi\alpha}\Big(\frac{\cosh(\pi\beta_1)}{\Gamma(\frac12+\alpha-\mathrm{i}\beta_2)\Gamma(\frac12+\alpha+\mathrm{i}\beta_2)} - (\beta_1 \leftrightarrow \beta_2)\Big)\end{aligned}$$

$$\begin{aligned}(4.85)\qquad &\int_{-1}^{1} \mathbf{S}_{0,\mathrm{i}\beta_1}(w)\mathbf{S}_{0,\mathrm{i}\beta_2}(w)\mathrm{d}2w \\ &= \frac{4\cosh(\pi\beta_1)\cosh(\pi\beta_2)\Big(\psi\big(\frac12-\mathrm{i}\beta_1\big)+\psi\big(\frac12+\mathrm{i}\beta_1\big)-(\beta_1\leftrightarrow\beta_2)\Big)}{\pi^2(\beta_1^2-\beta_2^2)},\end{aligned}$$

$$\begin{aligned}(4.86)\qquad &\int_{-1}^{1} \mathbf{S}_{\alpha,\mathrm{i}\beta}(w)^2(1-w^2)^{\alpha}\mathrm{d}2w = \frac{2^{2\alpha+1}\mathrm{i}\cosh(\pi\beta)}{\beta\sin\pi\alpha\,\Gamma(\frac12+\alpha-\mathrm{i}\beta)\Gamma(\frac12+\alpha+\mathrm{i}\beta)} \\ &\times\Big(\psi\big(\tfrac12+\alpha+\mathrm{i}\beta\big)-\psi\big(\tfrac12+\alpha-\mathrm{i}\beta\big)+\psi\big(\tfrac12-\mathrm{i}\beta\big)-\psi\big(\tfrac12+\mathrm{i}\beta\big)\Big),\end{aligned}$$

$$(4.87)\qquad \int_{-1}^{1} \mathbf{S}_{0,\mathrm{i}\beta}(w)^2\mathrm{d}2w = \frac{2\mathrm{i}\cosh^2(\pi\beta)\Big(\psi'(\frac12+\mathrm{i}\beta)-\psi'(\frac12-\mathrm{i}\beta)\Big)}{\beta\pi^2},$$

$$(4.88)\qquad \int_{-1}^{1} \mathbf{S}_{\alpha,0}(w)^2(1-w^2)^{\alpha}\mathrm{d}2w = \frac{2^{2\alpha+1}\big(\pi^2-2\psi'\big(\frac12+\alpha\big)\big)}{\sin(\pi\alpha)\Gamma(\frac12+\alpha)^2},$$

$$(4.89)\qquad \int_{-1}^{1} \mathbf{S}_{0,0}(w)^2\mathrm{d}2w = -\frac{4\psi''\big(\frac12\big)}{\pi^2}.$$

PROOF. The Gegenbauer equation implies

$$\begin{aligned}&(\beta_1^2-\beta_2^2)\int_{-1}^{1} \mathbf{S}_{\alpha,\mathrm{i}\beta_1}(w)\mathbf{S}_{\alpha,\mathrm{i}\beta_2}(w)(1-w^2)^{\alpha}\mathrm{d}w \\ =&\int_{-1}^{1} \mathbf{S}_{\alpha,\mathrm{i}\beta_1}(w)\partial_w(1-w^2)^{\alpha+1}\partial_w\mathbf{S}_{\alpha,\mathrm{i}\beta_2}(w)\mathrm{d}w \\ &-\int_{-1}^{1}\Big(\partial_w(1-w^2)^{\alpha+1}\partial_w\mathbf{S}_{\alpha,\mathrm{i}\beta_1}(w)\Big)\mathbf{S}_{\alpha,\mathrm{i}\beta_2}(w)\mathrm{d}w \\ =&\lim_{w\searrow-1}\Big((1-w^2)^{\alpha+1}\Big(\mathbf{S}_{\alpha,\mathrm{i}\beta_1}(w)\partial_w\mathbf{S}_{\alpha,\mathrm{i}\beta_2}(w)-\big(\partial_w\mathbf{S}_{\alpha,\mathrm{i}\beta_1}(w)\big)\mathbf{S}_{\alpha,\mathrm{i}\beta_2}(w)\Big)\Big).\end{aligned}$$

Then we use the connection formula (4.12) and obtain (4.84). The remaining identities follow by the de l'Hôpital rule. □

Integrals in Theorem 4.4 are defined as generalized integrals for $\mathrm{Re}(\alpha) > -1$ (beyond this region one would need to use a generalized integral which also takes care of non-integrability at the right endpoint). If $\alpha \notin \mathbb{N}$, these generalized integrals are non-anomalous and hence formulas from Theorem 4.4 remain true. Below we compute anomalous generalized integrals for $\alpha \in \mathbb{N}$.

THEOREM 4.5. *Let* $\alpha \in \mathbb{N}$. *Then*

$$
\begin{aligned}
&(-\tfrac14)^{\alpha+1}\Big(\prod_{i=1}^{2}\prod_{\pm}\Gamma(\tfrac12+\alpha\pm \mathrm{i}\beta_i)\Big)\mathrm{gen}\!\int_{-1}^{1}\mathbf{S}_{\alpha,\mathrm{i}\beta_1}(w)\mathbf{S}_{\alpha,\mathrm{i}\beta_2}(w)(1-w^2)^{\alpha}\mathrm{d}2w \\
&=\frac{1}{\beta_1^2-\beta_2^2}\Bigg(\big(\tfrac12+\mathrm{i}\beta_1\big)_\alpha\big(\tfrac12-\mathrm{i}\beta_1\big)_\alpha\Big(\ln 4-\sum_{\pm}\psi\big(\tfrac12+\alpha\pm\mathrm{i}\beta_1\big)\Big)-(\beta_1\leftrightarrow\beta_2)\Bigg) \\
&+\sum_{k=0}^{\alpha-1}\Bigg(\prod_{\pm}\frac{(\frac12\pm\mathrm{i}\beta_2)_\alpha(\frac12\pm\mathrm{i}\beta_1)_k}{(\frac12\pm\mathrm{i}\beta_2)_{k+1}}\Bigg) \\
&\times\Big(\psi(\alpha-k)+\psi(1+k)-\sum_{\pm}H_{\alpha-1-k}\big(\tfrac12-\alpha\pm\mathrm{i}\beta_2\big)\Big).
\end{aligned}
\tag{4.90}
$$

$$
\begin{aligned}
&(-\tfrac14)^{\alpha+1}\frac{\pi}{\cosh\pi\beta}\Big(\prod_{\pm}\Gamma(\tfrac12+\alpha\pm\mathrm{i}\beta)\Big)\mathrm{gen}\!\int_{-1}^{1}\mathbf{S}_{\alpha,\mathrm{i}\beta}(w)^2(1-w^2)^{\alpha}\mathrm{d}2w \\
&=\sum_{k=0}^{\alpha-1}\frac{1}{(k+\frac12)^2+\beta^2}\Big(\ln 4+\psi(\alpha-k)+\psi(k+1)-\sum_{\pm}\psi(-\tfrac12-k\pm\mathrm{i}\beta)\Big) \\
&-\frac{\mathrm{i}}{2\beta}\big(\psi'(\tfrac12+\alpha+\mathrm{i}\beta)-\psi'(\tfrac12+\alpha-\mathrm{i}\beta)\big).
\end{aligned}
\tag{4.91}
$$

$$
\begin{aligned}
&(-\tfrac14)^{\alpha+1}\pi\Gamma(\tfrac12+\alpha)^2\ \mathrm{gen}\!\int_{-1}^{1}\mathbf{S}_{\alpha,0}(w)^2(1-w^2)^{\alpha}\mathrm{d}2w \\
&=\psi''(\tfrac12+\alpha)+\sum_{k=0}^{\alpha-1}\frac{\ln 4+\psi(\alpha-k)+\psi(k+1)-2\psi(-\frac12-k)}{(k+\frac12)^2}
\end{aligned}
\tag{4.92}
$$

REMARK 4.6. The right-hand side of (4.90) is not manifestly symmetric in β_1, β_2, although the left-hand side is. This is a nontrivial sum identity which follows from the equality of (4.4) and (4.5). The same is true for (4.112) below.

PROOF. Let us first compute (4.90). We will use the dimensional regularization. Thus α at first takes arbitrary complex values with $\mathrm{Re}(\alpha) > -1$. We define

$$
f(\alpha,w) := 2^{-4\alpha}\left[\prod_{i=1}^{2}\prod_{\pm}\Gamma(\tfrac12+\alpha\pm\mathrm{i}\beta_i)\right]\mathbf{S}_{\alpha,\mathrm{i}\beta_1}(w)\mathbf{S}_{\alpha,\mathrm{i}\beta_2}(w)(1-w^2)^{\alpha}.
\tag{4.93}
$$

In terms of $f(\alpha, w)$, (4.84) becomes (for $|\mathrm{Re}(\alpha)| \le 1$)

$$
\int_{-1}^{1} f(\alpha,w)\mathrm{d}2w=\frac{2^{-2\alpha+2}\pi}{(\beta_1^2-\beta_2^2)\sin\pi\alpha}\Bigg(\prod_{\pm}\frac{\Gamma(\frac12+\alpha\pm\mathrm{i}\beta_1)}{\Gamma(\frac12\pm\mathrm{i}\beta_1)}-(\beta_1\leftrightarrow\beta_2)\Bigg).
\tag{4.94}
$$

Let $m \in \mathbb{N}_0$. The residue and the finite part of (4.94) at $\alpha = m$ are

$$
\mathrm{res}(m)=\frac{(-1)^m2^{-2m+2}}{\beta_1^2-\beta_2^2}\Big(\big(\tfrac12-\mathrm{i}\beta_1\big)_m\big(\tfrac12+\mathrm{i}\beta_1\big)_m-(\beta_1\leftrightarrow\beta_2)\Big),
\tag{4.95}
$$

$$(4.96)\qquad (-1)^m 2^{2m-2}(\beta_1^2-\beta_2^2)\ \underset{\alpha\to m}{\mathrm{fp}}\int_{-1}^{1} f(\alpha,w)\mathrm{d}2w$$
$$=\big(\tfrac12-\mathrm{i}\beta_1\big)_m\big(\tfrac12+\mathrm{i}\beta_1\big)_m\Big(-\ln(4)+\sum_{\pm}\psi\big(\tfrac12+m\pm\mathrm{i}\beta_1\big)\Big)-(\beta_1\leftrightarrow\beta_2).$$

All negative powers of $w+1$ in $f(\alpha,w)$ are contained in

$$f^{\mathrm{sing}}(\alpha,w):=\frac{2^{-2\alpha}\pi^2\big(\frac{1-w}{2}\big)^{-\alpha}\big(\frac{1+w}{2}\big)^{-\alpha}\mathbf{S}_{-\alpha,\mathrm{i}\beta_1}(-w)\mathbf{S}_{-\alpha,\mathrm{i}\beta_2}(-w)}{\sin^2\pi\alpha}.$$

In order to avoid having to expand $(1-w)^\alpha$ in terms of $1+w$, we choose to expand one Gegenbauer function using (4.4) and the other one using (4.5):

$$(4.97)\qquad (2(1+w))^\alpha f^{\mathrm{sing}}(\alpha,w)=\frac{\pi^2}{\sin^2\pi\alpha}\mathbf{F}\Big(\frac12+\mathrm{i}\beta_1,\frac12-\mathrm{i}\beta_1,-\alpha+1,\frac{1+w}{2}\Big)$$
$$\times\mathbf{F}\Big(\frac12-\alpha+\mathrm{i}\beta_2,\frac12-\alpha-\mathrm{i}\beta_2,-\alpha+1,\frac{1+w}{2}\Big)$$
$$=\Bigg(\sum_{k=0}^{\infty}\frac{(-1)^k(\frac12+\mathrm{i}\beta_1)_k(\frac12-\mathrm{i}\beta_1)_k\Gamma(\alpha-k)}{k!}\Big(\frac{1+w}{2}\Big)^k\Bigg)$$
$$\times\Bigg(\sum_{j=0}^{\infty}\frac{(-1)^j(\frac12-\alpha+\mathrm{i}\beta_2)_j(\frac12-\alpha-\mathrm{i}\beta_2)_j\Gamma(\alpha-j)}{j!}\Big(\frac{1+w}{2}\Big)^j\Bigg).$$

The coefficient at $\big(2(1+w)\big)^{-\alpha+m-1}$ in $f^{\mathrm{sing}}(\alpha,w)$ is

$$f_{m-1}(\alpha)=\sum_{k=0}^{m-1}\frac{\Gamma(\alpha-k)\Gamma(\alpha-m+1+k)}{k!(m-1-k)!(-4)^{m-1}}\prod_{\pm}\big(\tfrac12\pm\mathrm{i}\beta_1\big)_k\big(\tfrac12-\alpha\pm\mathrm{i}\beta_2\big)_{m-1-k}.$$

We evaluate the derivative

$$(4.98)\qquad f'_{m-1}(m)=(-\tfrac14)^{m-1}\sum_{k=0}^{m-1}\prod_{\pm}\big(\tfrac12\pm\mathrm{i}\beta_1\big)_k\big(\tfrac12-m\pm\mathrm{i}\beta_2\big)_{m-1-k}$$
$$\times\Bigg(\psi(m-k)+\psi(1+k)-\sum_{\pm}H_{m-1-k}\Big(\frac12-m\pm\mathrm{i}\beta_2\Big)\Bigg).$$

Then we use

$$(4.99)\qquad \big(\tfrac12-m+\mathrm{i}\beta_2\big)_{m-1-k}\big(\tfrac12-m-\mathrm{i}\beta_2\big)_{m-1-k}=\frac{(\frac12+\mathrm{i}\beta_2)_m(\frac12-\mathrm{i}\beta_2)_m}{(\frac12+\mathrm{i}\beta_2)_{k+1}(\frac12-\mathrm{i}\beta_2)_{k+1}}.$$

Since the singular exponent is $-\alpha+m-1$, we have to *add* $f'_{m-1}(m)$ to the finite part of the bilinear integral.[1] Thus, we obtain the generalized integral (4.90) for positive integers by multiplying

$$(4.100)\qquad \mathrm{gen}\int_{-1}^{1}f(m,w)\mathrm{d}2w=\underset{\alpha\to m}{\mathrm{fp}}\int_{-1}^{1}f(\alpha,w)\mathrm{d}2w+f'_{m-1}(m)$$

[1]For $\mathrm{Re}(\alpha)>0$, the derivative term has to be added to the finite part, not subtracted:

$$-\frac{f_{m-1}(\alpha)}{m-\alpha}=-\frac{f_{m-1}(m)+(\alpha-m)f'_{m-1}(m)}{m-\alpha}+O(\alpha-m)=\frac{\mathrm{res}(m)}{\alpha-m}+f'_{m-1}(m)+O(\alpha-m).$$

with

$$\frac{2^{4m}}{\Gamma(\frac12+m+\mathrm{i}\beta_1)\Gamma(\frac12+m-\mathrm{i}\beta_1)\Gamma(\frac12+m+\mathrm{i}\beta_2)\Gamma(\frac12+m-\mathrm{i}\beta_2)}. \tag{4.101}$$

Let us now prove (4.91). Since the exponents of the non-integrable terms do not depend on β, it suffices to take the limit $\beta_1,\beta_2\to\beta$ of (4.90). The multiplicative factor in front of the left-hand side becomes

$$(-\tfrac14)^{\alpha+1}\prod_{\pm}\Gamma(\tfrac12+\alpha\pm\mathrm{i}\beta)^2=(-\tfrac14)^{\alpha+1}\frac{\pi}{\cosh\pi\beta}\prod_{\pm}\Gamma(\tfrac12+\alpha\pm\mathrm{i}\beta)(\tfrac12\pm\mathrm{i}\beta)_\alpha. \tag{4.102}$$

Applying the de l'Hôpital rule to the second line of (4.90) then yields

$$\begin{aligned}\frac{\mathrm{i}(\frac12+\mathrm{i}\beta)_\alpha(\frac12-\mathrm{i}\beta)_\alpha}{2\beta}\Big(\big(\ln 4-\sum_{\pm}\psi(\tfrac12+\alpha\pm\mathrm{i}\beta)\big)\big(\sum_{\pm}\pm H_\alpha(\tfrac12\pm\mathrm{i}\beta)\big)\\ -\sum_{\pm}\pm\psi'(\tfrac12+\alpha\pm\mathrm{i}\beta)\Big)\end{aligned} \tag{4.103}$$

The third and fourth line of (4.90) can be converted to the following form

$$\begin{aligned}&(\tfrac12+\mathrm{i}\beta)_\alpha(\tfrac12-\mathrm{i}\beta)_\alpha\sum_{k=0}^{\alpha-1}\frac{1}{\big(\frac12+k\big)^2+\beta^2}\\ &\times\Big(\psi(\alpha-k)+\psi(k+1)-\sum_{\pm}H_{\alpha-1-k}\big(\tfrac12-\alpha\pm\mathrm{i}\beta\big)\Big).\end{aligned} \tag{4.104}$$

Next we use the identities

$$\sum_{k=0}^{\alpha-1}\frac{1}{(\frac12+k)^2+\beta^2}=\frac{\mathrm{i}}{2\beta}\Big(H_\alpha\big(\tfrac12+\mathrm{i}\beta\big)-H_\alpha\big(\tfrac12-\mathrm{i}\beta\big)\Big), \tag{4.105}$$

$$\psi\big(\tfrac12+\alpha+\mathrm{i}\beta\big)+\psi\big(\tfrac12+\alpha-\mathrm{i}\beta\big)=\psi\big(\tfrac12-\alpha+\mathrm{i}\beta\big)+\psi\big(\tfrac12-\alpha-\mathrm{i}\beta\big), \tag{4.106}$$

$$H_{\alpha-1-k}(\tfrac12-\alpha\pm\mathrm{i}\beta)=\psi(-\tfrac12-k\pm\mathrm{i}\beta)-\psi(\tfrac12-\alpha\pm\mathrm{i}\beta), \tag{4.107}$$

to combine (4.104) and (4.103). This proves (4.91).

Similarly, applying de l'Hôpital to (4.91) for $\beta\to0$ yields (4.92). □

THEOREM 4.7. *Let* $|\mathrm{Re}(\alpha)|<1$ *and* $\mathrm{Re}(\lambda)>0$. *Then*

$$\begin{aligned}&\int_1^\infty\mathbf{Z}_{\alpha,\lambda_1}(w)\mathbf{Z}_{\alpha,\lambda_2}(w)(w^2-1)^\alpha\mathrm{d}2w\\ &=\frac{2^{\lambda_1+\lambda_2+1}}{(\lambda_1^2-\lambda_2^2)\sin\pi\alpha}\Big(\frac{1}{\Gamma(\frac12-\alpha+\lambda_1)\Gamma(\frac12+\alpha+\lambda_2)}-(\lambda_1\leftrightarrow\lambda_2)\Big),\end{aligned} \tag{4.108}$$

$$\int_1^\infty\mathbf{Z}_{0,\lambda_1}(w)\mathbf{Z}_{0,\lambda_2}(w)\mathrm{d}2w=\frac{2^{\lambda_1+\lambda_2+2}(\psi(\frac12+\lambda_1)-\psi(\frac12+\lambda_2))}{\pi(\lambda_1^2-\lambda_2^2)\Gamma(\frac12+\lambda_1)\Gamma(\frac12+\lambda_2)}, \tag{4.109}$$

$$\int_1^\infty\mathbf{Z}_{\alpha,\lambda}(w)^2(w^2-1)^\alpha\mathrm{d}2w=\frac{2^{2\lambda}(\psi(\frac12+\alpha+\lambda)-\psi(\frac12-\alpha+\lambda))}{\lambda\sin\pi\alpha\Gamma(\frac12-\alpha+\lambda)\Gamma(\frac12+\alpha+\lambda)}, \tag{4.110}$$

$$\int_1^\infty\mathbf{Z}_{0,\lambda}(w)^2\mathrm{d}2w=\frac{2^{2\lambda+1}\psi'(\frac12+\lambda)}{\pi\lambda\Gamma(\frac12+\lambda)^2}. \tag{4.111}$$

PROOF.

$$
\begin{aligned}
&(\lambda_1^2-\lambda_2^2)\int_1^\infty \mathbf{Z}_{\alpha,\lambda_1}(w)\mathbf{Z}_{\alpha,\lambda_2}(w)(w^2-1)^\alpha \mathrm{d}w\\
=&\int_1^\infty \mathbf{Z}_{\alpha,\lambda_1}(w)\partial_w(w^2-1)^{\alpha+1}\partial_w\mathbf{Z}_{\alpha,\lambda_2}(w)\mathrm{d}w\\
&-\int_1^\infty \Big(\partial_w(w^2-1)^{\alpha+1}\partial_w\mathbf{Z}_{\alpha,\lambda_1}(w)\Big)\mathbf{Z}_{\alpha,\lambda_2}(w)\mathrm{d}w\\
=&\lim_{w\searrow 1}\Big(\mathbf{Z}_{\alpha,\lambda_1}(w)(w^2-1)^{\alpha+1}\partial_w\mathbf{Z}_{\alpha,\lambda_2}(w)-(w^2-1)^{\alpha+1}\big(\partial_w\mathbf{Z}_{\alpha,\lambda_1}(w)\big)\mathbf{Z}_{\alpha,\lambda_2}(w)\Big).
\end{aligned}
$$

Then we use the connection formula (4.13) to decompose the **Z** functions into the **S** functions. The remaining formulas follow by the rule of de l'Hôpital. □

REMARK 4.8. The integrand $\mathbf{Z}_{\alpha,\lambda}(w)^2(w^2-1)^\alpha$ has asymptotics $\sim w^{-1-2\lambda}$ for $w\to\infty$ and therefore the case $\mathrm{Re}(\lambda)\le 0$ escapes our considerations.

As before, the above integral formulas remain valid in the non-anomalous case $\alpha\in\mathbb{C}\setminus\mathbb{Z}$ if the integral is replaced by the generalized integral. We compute the anomalous generalized integrals for $\alpha\in\mathbb{Z}$:

THEOREM 4.9. *Let* $\alpha\in\mathbb{Z}$. *For* $\mathrm{Re}(\lambda_i)>0$, *we have*

$$
\begin{aligned}
&\frac{\pi}{2^{\lambda_1+\lambda_2+1}}\Big(\prod_{i=1}^{2}\Gamma\big(\tfrac12+|\alpha|+\lambda_i\big)\Big)\mathrm{gen}\!\int_1^\infty \mathbf{Z}_{\alpha,\lambda_1}(w)\mathbf{Z}_{\alpha,\lambda_2}(w)(w^2-1)^\alpha \mathrm{d}2w \tag{4.112}\\
=&\Bigg(\frac{(\frac12+\lambda_1)_{|\alpha|}(\frac12-\lambda_1)_{|\alpha|}}{\lambda_1^2-\lambda_2^2}\Big(-\ln(4)+\sum_{\pm}\psi\big(\tfrac12\pm\alpha+\lambda_1\big)\Big)+(\lambda_1\leftrightarrow\lambda_2)\Bigg)\\
&-(\tfrac12+\lambda_2)_{|\alpha|}(\tfrac12-\lambda_2)_{|\alpha|}\sum_{k=0}^{|\alpha|-1}\frac{(\frac12+\lambda_1)_k(\frac12-\lambda_1)_k}{(\frac12+\lambda_2)_{k+1}(\frac12-\lambda_2)_{k+1}}\\
&\times\Big(-\psi(|\alpha|-k)-\psi(1+k)+\sum_{\pm}H_{|\alpha|-1-k}\big(\tfrac12-|\alpha|\pm\lambda_2\big)\Big).
\end{aligned}
$$

Moreover, if $\mathrm{Re}(\lambda)>0$,

$$
\begin{aligned}
&\frac{(-1)^\alpha\pi}{2^{2\lambda+1}}\Big(\prod_{\pm}\Gamma\big(\tfrac12\pm\alpha+\lambda\big)\Big)\mathrm{gen}\!\int_1^\infty \mathbf{Z}_{\alpha,\lambda}(w)^2(w^2-1)^\alpha \mathrm{d}2w \tag{4.113}\\
=&\frac{1}{2\lambda}\sum_{\pm}\Big(\psi'(\tfrac12\pm\alpha+\lambda)\mp H_{|\alpha|}(\tfrac12\pm\lambda)\ln(4)\Big)\\
&+\sum_{k=0}^{|\alpha|-1}\frac{\psi(\frac32+k+\lambda)+\psi(-\frac12-k+\lambda)-\psi(|\alpha|-k)-\psi(1+k)}{\lambda^2-\big(\frac12+k\big)^2}.
\end{aligned}
$$

PROOF. The proof works similar to the proof with the anomalous generalized **S**-integrals. For $\alpha\in\mathbb{C}$ we set

$$
f(w,\alpha)=\frac{\Gamma(\frac12-\alpha+\lambda_1)\Gamma(\frac12-\alpha+\lambda_2)}{2^{\lambda_1+\lambda_2-2\alpha-1}}\mathbf{Z}_{\alpha,\lambda_1}(w)\mathbf{Z}_{\alpha,\lambda_2}(w)(w^2-1)^\alpha. \tag{4.114}
$$

Then for $\alpha\notin\mathbb{Z}$,

$$
\mathrm{gen}\!\int_1^\infty f(w,\alpha)\mathrm{d}2w=\frac{2^{2\alpha+2}}{(\lambda_1^2-\lambda_2^2)\sin\pi\alpha}\Bigg(\frac{\Gamma(\frac12-\alpha+\lambda_2)}{\Gamma(\frac12+\alpha+\lambda_2)}-(\lambda_1\leftrightarrow\lambda_2)\Bigg). \tag{4.115}
$$

Let $m \in \mathbb{Z}$. The residue and the finite part of (4.115) at $\alpha = m$ are

$$\operatorname{res}(m) = \frac{(-1)^m 2^{2m+2}}{\pi(\lambda_1^2 - \lambda_2^2)} \left(\frac{\Gamma(\frac12 - m + \lambda_2)}{\Gamma(\frac12 + m + \lambda_2)} - (\lambda_1 \leftrightarrow \lambda_2) \right), \tag{4.116}$$

$$\operatorname{fp} \int_1^\infty f(w,\alpha) \mathrm{d}2w = \frac{(-1)^m 2^{2m+2}}{\pi(\lambda_1^2 - \lambda_2^2)} \tag{4.117}$$
$$\times \left(\frac{\Gamma(\frac12 - m + \lambda_2)}{\Gamma(\frac12 + m + \lambda_2)} \big(\ln(4) - \psi(\tfrac12 - m + \lambda_2) - \psi(\tfrac12 + m + \lambda_2) \big) - (\lambda_1 \leftrightarrow \lambda_2) \right).$$

For $\operatorname{Re}(\alpha) \le 0$, the singular part of $f(\alpha, w)$ is contained in

$$\begin{aligned}
f^{\mathrm{sing}}(\alpha, w) &= \frac{\pi}{\sin^2 \pi\alpha} \mathbf{S}_{\alpha,\lambda_1}(w) \mathbf{S}_{\alpha,\lambda_2}(w) (w^2-1)^\alpha \\
=&\frac{2^\alpha \pi (w-1)^\alpha}{\sin^2 \pi\alpha} \mathbf{F}\Big(\frac12 + \lambda_1, \frac12 - \lambda_1, \alpha+1, \frac{1-w}{2} \Big) \\
&\times \mathbf{F}\Big(\frac12 + \alpha + \lambda_2, \frac12 + \alpha - \lambda_2, \alpha+1, \frac{1-w}{2} \Big) \\
=&\frac{2^\alpha (w-1)^\alpha}{\pi} \left(\sum_{k=0}^\infty \frac{(\frac12+\lambda_1)_k (\frac12 - \lambda_1)_k \Gamma(-\alpha-k)}{k!} \Big(\frac{w-1}{2} \Big)^k \right) \\
&\times \left(\sum_{j=0}^\infty \frac{(\frac12 + \alpha + \lambda_2)_j (\frac12 + \alpha - \lambda_2)_j \Gamma(-\alpha - j)}{j!} \Big(\frac{w-1}{2} \Big)^j \right).
\end{aligned} \tag{4.118}$$

Let m be a positive integer. The coefficient of (4.118) at $\big(2(w-1)\big)^{\alpha+m-1}$ is

$$\begin{aligned}
&f_{m-1}(\alpha) \\
=&\frac{2^{-2m+2}}{\pi} \sum_{k=0}^{m-1} \frac{(\frac12+\lambda_1)_k (\frac12-\lambda_1)_k (\frac12+\alpha+\lambda_2)_{m-1-k} (\frac12+\alpha-\lambda_2)_{m-1-k}}{k!(m-1-k)!} \\
&\times \Gamma(-\alpha-k)\Gamma(-\alpha-m+1+k).
\end{aligned} \tag{4.119}$$

Now

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\alpha} f_{m-1}(\alpha) \Big|_{\alpha=-m} = \frac{2^{-2m+2}}{\pi} \\
&\times \sum_{k=0}^{m-1} (\tfrac12+\lambda_1)_k (\tfrac12-\lambda_1)_k (\tfrac12 - m + \lambda_2)_{m-1-k} (\tfrac12 - m - \lambda_2)_{m-1-k} \\
&\times \Big(H_{m-1-k}(\tfrac12 - m + \lambda_2) + H_{m-1-k}(\tfrac12 - m - \lambda_2) - \psi(m-k) - \psi(1+k) \Big)
\end{aligned}$$

Then we use

$$\big(\tfrac12 - m + \lambda_2\big)_{m-1-k} \big(\tfrac12 - m - \lambda_2\big)_{m-1-k} = \frac{(\frac12+\lambda_2)_m (\frac12 - \lambda_2)_m}{(\frac12 + \lambda_2)_{k+1} (\frac12 - \lambda_2)_{k+1}}. \tag{4.120}$$

Thus we find the integral at negative integers:

$$\begin{aligned}&\mathrm{gen}\int_1^\infty \mathbf{Z}_{-m,\lambda_1}(w)\mathbf{Z}_{-m,\lambda_2}(w)(w^2-1)^{-m}\mathrm{d}2w\\ =&\frac{2^{\lambda_1+\lambda_2-2m+1}}{\Gamma(\frac12+m+\lambda_1)\Gamma(\frac12+m+\lambda_2)}\mathrm{gen}\int_{-1}^1 f(-m,w)\mathrm{d}2w\\ =&\frac{2^{\lambda_1+\lambda_2-2m+1}}{\Gamma(\frac12+m+\lambda_1)\Gamma(\frac12+m+\lambda_2)}\bigg(\mathrm{fp}\int_{-1}^1 f(-m,w)\mathrm{d}2w-\frac{\mathrm{d}}{\mathrm{d}\alpha}f_{m-1}(\alpha)\Big|_{\alpha=-m}\bigg).\end{aligned}$$

This proves (4.112) for negative integers. But (4.112) is invariant with respect to the flip of the sign of α, because

$$\mathbf{Z}_{-m,\lambda_1}(w)\mathbf{Z}_{-m,\lambda_2}(w)(w^2-1)^{-m}=\mathbf{Z}_{m,\lambda_1}(w)\mathbf{Z}_{m,\lambda_2}(w)(w^2-1)^{m}. \tag{4.121}$$

Thus (4.112) has been proven.

To prove (4.113), we set $\lambda=\lambda_1=\lambda_2$ in (4.112), using the de l'Hôpital rule where needed:

(4.122)

$$\begin{aligned}&\mathrm{gen}\int_1^\infty \mathbf{Z}_{\alpha,\lambda}(w)\mathbf{Z}_{\alpha,\lambda}(w)(w^2-1)^{\alpha}\mathrm{d}2w=\frac{(-1)^\alpha 2^{2\lambda+1}}{\pi\Gamma\big(\frac12-\alpha+\lambda\big)\Gamma\big(\frac12+\alpha+\lambda\big)}\\ &\times\Bigg(\frac{\psi'\big(\frac12-\alpha+\lambda\big)+\psi'\big(\frac12+\alpha+\lambda\big)}{2\lambda}\\ &+\frac{\big(\psi(\frac12-\alpha+\lambda)+\psi(\frac12+\alpha+\lambda)-\ln(4)\big)\big(-H_{|\alpha|}(\frac12-\lambda)+H_{|\alpha|}(\frac12+\lambda)\big)}{2\lambda}\\ &+\sum_{k=0}^{|\alpha|-1}\frac{H_{|\alpha|-1-k}(\frac12-|\alpha|-\lambda)+H_{|\alpha|-1-k}\big(\frac12-|\alpha|+\lambda\big)-\psi(|\alpha|-k)-\psi(1+k)}{\lambda^2-\big(\frac12+k\big)^2}\Bigg).\end{aligned}$$

Then we use

$$\sum_{k=0}^{|\alpha|-1}\frac{1}{\lambda^2-\big(\frac12+k\big)^2}=\frac{H_{|\alpha|}(\frac12+\lambda)-H_{|\alpha|}(\frac12-\lambda)}{2\lambda}, \tag{4.123}$$

$$\psi(\tfrac12-|\alpha|+\lambda)+H_{|\alpha|-1-k}(\tfrac12-|\alpha|+\lambda)=\psi(-\tfrac12-k+\lambda), \tag{4.124}$$

$$\begin{aligned}&\psi(\tfrac12+|\alpha|+\lambda)+H_{|\alpha|-1-k}(\tfrac12-|\alpha|-\lambda)\\ =&\psi(\tfrac12+|\alpha|+\lambda)-H_{|\alpha|-1-k}(\tfrac32+k+\lambda)=\psi(\tfrac32+k+\lambda).\end{aligned} \tag{4.125}$$

□

4.9. Asymptotics of bilinear integrals. The bilinear (generalized) integrals for Gegenbauer functions converge to the corresponding integrals for Macdonald functions consistently with (4.47) and (4.48). This is relatively straightforward in the non-anomalous case. It is also true in the anomalous case, because we have chosen the right variables. Otherwise there would be a discrepancy.

THEOREM 4.10. *For $\beta\to\infty$ with fixed α satisfying* $\mathrm{Re}(\alpha)>-1$ *we have*

$$\begin{aligned}\frac{\pi^2\mathrm{e}^{-2\pi\beta}\beta^{2\alpha}}{2^{2\alpha}}\ \mathrm{gen}\!\int_{-1}^{1}\mathbf{S}_{\alpha,\mathrm{i}\beta}(w)^2(1-w^2)^{\alpha}\mathrm{d}2w \\ =\Big(1+\mathcal{O}\big(\tfrac{1}{\beta}\big)\Big)\ \mathrm{gen}\!\int_{0}^{\infty}K_{\alpha}(\beta r)^2 2r\mathrm{d}r,\end{aligned}\tag{4.126}$$

For $\lambda\to\infty$ and all $\alpha\in\mathbb{C}$ we have

$$\begin{aligned}\frac{\pi\Gamma\big(\frac12+\alpha+\lambda\big)^2}{2^{2\lambda+1}\lambda^{2\alpha}}\ \mathrm{gen}\!\int_{1}^{\infty}\mathbf{Z}_{\alpha,\lambda}(w)^2(w^2-1)^{\alpha}\mathrm{d}2w \\ =\Big(1+\mathcal{O}\big(\tfrac{1}{\lambda}\big)\Big)\ \mathrm{gen}\!\int_{0}^{\infty}K_{\alpha}(\lambda r)^2 2r\mathrm{d}r.\end{aligned}\tag{4.127}$$

PROOF. Let us first consider the non-anomalous case, that is $\alpha\in\mathbb{C}\setminus\mathbb{Z}$. For the $\mathbf{S}$-functions, we have

$$\begin{aligned}&\mathrm{LHS}(4.126)\\ =&\frac{2\pi^2\beta^{2\alpha}}{\mathrm{e}^{2\pi\beta}}\frac{\mathrm{i}\cosh(\pi\beta)\big(\psi\big(\frac12+\alpha+\mathrm{i}\beta\big)-\psi\big(\frac12+\alpha-\mathrm{i}\beta\big)+\psi\big(\frac12-\mathrm{i}\beta\big)-\psi\big(\frac12+\mathrm{i}\beta\big)\big)}{\beta\sin\pi\alpha\,\Gamma(\frac12+\alpha-\mathrm{i}\beta)\Gamma(\frac12+\alpha+\mathrm{i}\beta)}\\ =&\frac{\pi\alpha}{\beta^2\sin\pi\alpha}\Big(1+\mathcal{O}\big(\tfrac{1}{\beta}\big)\Big)\ =\ \mathrm{RHS}(4.126),\end{aligned}$$

where we used

$$\begin{aligned}2\mathrm{e}^{-\pi\beta}\cosh(\pi\beta)&=1+\mathcal{O}\big(\tfrac{1}{\beta}\big),\\ \Gamma(\tfrac12+\alpha-\mathrm{i}\beta)\Gamma(\tfrac12+\alpha+\mathrm{i}\beta)&=2\pi\beta^{2\alpha}\mathrm{e}^{-\beta\pi}(1+O(\tfrac{1}{\beta})),\\ \psi(\tfrac12+\alpha+\mathrm{i}\beta)-\psi(\tfrac12+\alpha-\mathrm{i}\beta)+\psi(\tfrac12-\mathrm{i}\beta)-\psi(\tfrac12+\mathrm{i}\beta)&=-\frac{2\mathrm{i}\alpha}{\beta}+O(\tfrac{1}{\beta^2}).\end{aligned}$$

For the $\mathbf{Z}$-functions, we have

$$\begin{aligned}\mathrm{LHS}(4.127)=&\frac{\pi\Gamma\big(\frac12+\alpha+\lambda\big)\big(\psi\big(\frac12+\alpha+\lambda\big)-\psi\big(\frac12-\alpha+\lambda\big)\big)}{2\lambda^{2\alpha+1}\sin\pi\alpha\,\Gamma\big(\frac12-\alpha+\lambda\big)}\\ =&\frac{\pi\alpha}{\lambda^2\sin\pi\alpha}\Big(1+\mathcal{O}\big(\tfrac{1}{\lambda}\big)\Big)\ =\ \mathrm{RHS}(4.127)\end{aligned}$$

where we used

$$\begin{aligned}\frac{\Gamma(\frac12+\alpha+\lambda)}{\Gamma(\frac12-\alpha+\lambda)}&=\lambda^{2\alpha}(1+O(\tfrac{1}{\lambda})),\\ \psi(\tfrac12+\alpha+\lambda)-\psi(\tfrac12-\alpha+\lambda)&=\frac{2\alpha}{\lambda}+O(\tfrac{1}{\lambda^2}).\end{aligned}$$

In the anomalous case, that is $\alpha \in \mathbb{N}$, we have

$$
\begin{aligned}
&\mathrm{LHS}(4.126)\\
=&\frac{(-1)^\alpha 4\cosh(\pi\beta)\mathrm{e}^{-2\pi\beta}\beta^{2\alpha}}{\pi\Gamma(\frac12+\alpha+\mathrm{i}\beta)\Gamma(\frac12+\alpha-\mathrm{i}\beta)}\Bigg(\frac{\mathrm{i}}{2\beta}\Big(\psi'\big(\tfrac12+\alpha+\mathrm{i}\beta\big)-\psi'\big(\tfrac12+\alpha-\mathrm{i}\beta\big)\Big)\\
&+\sum_{k=0}^{|\alpha|-1}\frac{\psi\big(-\frac12-k+\mathrm{i}\beta\big)+\psi\big(-\frac12-k-\mathrm{i}\beta\big)-\psi(|\alpha|-k)-\psi(1+k)-\ln 4}{\big(\frac12+k\big)^2+\beta^2}\Bigg)\\
=&(-1)^\alpha\Big(\frac{1}{\beta^2}-\frac{|\alpha|}{\beta^2}\ln(4)+\sum_{k=0}^{|\alpha|-1}\frac{\big(2\ln(\beta)-\psi(|\alpha|-k)-\psi(1+k)\big)}{\beta^2}\Big)\Big(1+O\big(\tfrac{1}{\beta}\big)\Big)\\
=&\mathrm{RHS}(4.126),
\end{aligned}
$$

where in the last equality we used (A.24). Similarly, for $\mathbf{Z}$ functions

$$
\begin{aligned}
&\mathrm{LHS}(4.127)\\
=&\frac{(-1)^\alpha\Gamma(\frac12+\alpha+\lambda)}{\Gamma\big(\frac12-\alpha+\lambda\big)\lambda^{2\alpha}}\Bigg(\frac{\psi'\big(\frac12-\alpha+\lambda\big)+\psi'\big(\frac12+\alpha+\lambda\big)}{2\lambda}\\
&+\frac{H_{|\alpha|}(\frac12-\lambda)-H_{|\alpha|}(\frac12+\lambda)}{2\lambda}\ln(4)\\
&+\sum_{k=0}^{|\alpha|-1}\frac{\psi(\frac32+k+\lambda)+\psi\big(-\frac12-k+\lambda\big)-\psi(|\alpha|-k)-\psi(1+k)}{\lambda^2-\big(\frac12+k\big)^2}\Bigg)\\
=&(-1)^\alpha\Big(\frac{1}{\lambda^2}-\frac{|\alpha|}{\lambda^2}\ln(4)+\sum_{k=0}^{|\alpha|-1}\frac{\big(2\ln(\lambda)-\psi(|\alpha|-k)-\psi(1+k)\big)}{\lambda^2}\Big)\Big(1+O\big(\tfrac{1}{\lambda}\big)\Big)\\
=&\mathrm{RHS}(4.127).
\end{aligned}
$$

□

Appendix A. Around the Gamma function

The *Gamma function* and its logarithmic derivative, the *digamma function*, play a central role in the theory of special functions. In this section we collect properties of these functions, which we will often use in our paper. We will also describe the properties of the *Pochhammer symbol* and *harmonic numbers*, which are also closely related to the Gamma and digamma functions.

A.1. Definitions. Recall the definition of the digamma function,

$$\psi(z) := \partial_z \ln\Gamma(z) = \frac{\partial_z\Gamma(z)}{\Gamma(z)}. \tag{A.1}$$

It is also useful to introduce the Pochhammer symbol,

$$(z)_k := \frac{\Gamma(z+k)}{\Gamma(z)} = \begin{cases}(z)(z+1)\cdots(z+k-1), & k\geq 0,\\ \frac{1}{(z+k)(z+k+1)\cdots(z-1)}, & k\leq 0;\end{cases} \tag{A.2}$$

$$\text{the shifted } k\text{th harmonic number} \qquad H_k(z) := \frac{1}{z} + \cdots + \frac{1}{z+k-1}, \tag{A.3}$$

$$\text{and the } k\text{th harmonic number} \qquad H_k := H_k(1) = \frac{1}{1} + \cdots + \frac{1}{k}. \tag{A.4}$$

A.2. Basic properties.

$$\Gamma(z+1) = z\Gamma(z), \tag{A.5}$$

$$\frac{1}{\Gamma(1-z)\Gamma(z)} = \frac{\sin \pi z}{\pi}, \tag{A.6}$$

$$\frac{1}{\Gamma\big(\frac12 + z\big)\Gamma\big(\frac12 - z\big)} = \frac{\cos \pi z}{\pi}, \tag{A.7}$$

$$\frac{\Gamma(2\lambda+1)}{\Gamma(\frac12 + \lambda)\Gamma(1+\lambda)} = \frac{2^{2\lambda}}{\sqrt{\pi}}. \tag{A.8}$$

$\frac{1}{\Gamma(z)}$ is an entire function with zeros at $0, -1, -2, \dots$ and derivative

$$\partial_z \frac{1}{\Gamma(z)} = -\frac{\psi(z)}{\Gamma(z)}. \tag{A.9}$$

The digamma function satisfies the following functional relations [**12**, Chapter 8.365]:

$$\psi(1+z) = \psi(z) + \frac{1}{z}, \tag{A.10}$$

$$\psi(z) - \psi(1-z) = -\pi \cot(\pi z), \tag{A.11}$$

$$\psi\big(\tfrac12 + z\big) - \psi\big(\tfrac12 - z\big) = \pi \tan(\pi z), \tag{A.12}$$

$$2\psi(2z) = 2\ln 2 + \psi(z) + \psi\big(z + \tfrac12\big), \tag{A.13}$$

$$\psi(z+k) = \psi(z) + H_k(z), \tag{A.14}$$

$$\psi(1+k) = -\gamma_{\mathrm{E}} + H_k. \tag{A.15}$$

The following relations among the harmonic numbers and the Pochhammer symbol are immediate from there definitions:

$$H_{k+n}(z) = H_n(z) + H_k(z+n), \tag{A.16}$$

$$H_k(z) = -H_k(1-z-k), \tag{A.17}$$

$$(z)_k (z+k)_n = (z)_{k+n}, \tag{A.18}$$

$$(z)_k = (-1)^k (1-k-z)_k, \tag{A.19}$$

$$(z)_{-k} = \frac{1}{(z-k)_k} = \frac{(-1)^k}{(1-z)_k}, \tag{A.20}$$

$$(\tfrac12 + z)_k (\tfrac12 - z)_k = (-1)^k (\tfrac12 + z - k)_{2k}, \tag{A.21}$$

$$\partial_z (z)_k = H_k(z)(z)_k. \tag{A.22}$$

The Pochammer symbol is also useful to expand powers around a different center:

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \quad |z| < 1. \tag{A.23}$$

The following identities follow by induction:

$$\sum_{k=0}^{m-1} \psi(1+k) = m\big(\psi(1+m)-1\big), \tag{A.24}$$

$$\sum_{k=0}^{m-1} H_k = m\big(H_m - 1\big). \tag{A.25}$$

A.3. Special values. We have [**12**, Chapter 8.366]

$$\psi(1) = -\gamma_{\mathrm{E}}, \qquad \psi\big(\tfrac{1}{2}\big) = -\gamma_{\mathrm{E}} - 2\ln 2, \qquad \psi'\big(\tfrac{1}{2}\big) = \frac{\pi^2}{2}, \tag{A.26}$$

with the Euler-Mascheroni constant γ_{E}. Moreover, we have

$$(1/2)_k = \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}} = \frac{(2k)!}{2^{2k}k!} = \frac{(2k-1)!}{2^{2k-1}(k-1)!} \tag{A.27}$$

and

$$\partial_z \frac{1}{\Gamma(z)}\Big|_{z=-n} = (-1)^n n!, \quad n = 0, 1, 2, \dots \tag{A.28}$$

A.4. Asymptotics for large arguments. Let $\epsilon > 0$. Then for $|\arg(z)| < \pi - \epsilon$, as $|z| \to \infty$, we have

$$\Gamma(z) = z^{z-\frac{1}{2}} \mathrm{e}^{-z}\sqrt{2\pi}\Big(1 + \mathcal{O}\big(\tfrac{1}{z}\big)\Big), \tag{A.29}$$

$$\ln\Gamma(z) = z\ln z - z + \frac{1}{2}\ln\frac{2\pi}{z} + \mathcal{O}\big(\tfrac{1}{z}\big), \tag{A.30}$$

$$\psi(z) = \ln(z) - \frac{1}{2z} + \mathcal{O}\big(\tfrac{1}{z^2}\big), \tag{A.31}$$

$$\psi'(z) = \frac{1}{z} + \frac{1}{2z^2} + \mathcal{O}\big(\tfrac{1}{z^3}\big). \tag{A.32}$$

(A.29) is the famous Stirling formula. (A.30) is obviously equivalent to (A.29). Formally, (A.31) and (A.32) follow from (A.30) by differentiation. However, rigorously, we are not allowed to differentiate asymptotic series, and therefore (A.31) and (A.32) require an independent proof. We can use for this the so-called 2nd Binet's identity [**24**] and its derivatives:

$$\ln\Gamma(z) = \Big(z - \frac{1}{2}\Big)\ln z - z + \frac{1}{2}\ln 2\pi + 2\int_0^\infty \frac{\arctan\frac{t}{z}}{\mathrm{e}^{2\pi t}-1}\mathrm{d}t, \tag{A.33}$$

$$\psi(z) = \ln z - \frac{1}{2z} - 2\int_0^\infty \frac{t\mathrm{d}t}{(z^2+t^2)(\mathrm{e}^{2\pi t}-1)}, \tag{A.34}$$

$$\psi'(z) = \frac{1}{z} + \frac{1}{2z^2} + 4\int_0^\infty \frac{zt\mathrm{d}t}{(z^2+t^2)^2(\mathrm{e}^{2\pi t}-1)}. \tag{A.35}$$

Let us show e.g. that (A.33) implies (A.30). Let $0 < 2\theta < \pi$. We have $|\arctan u| \leq c|u|$ for $|\arg u| < \theta$. We set $t = \mathrm{e}^{\mathrm{i}\theta}s$. Then we use

$$\left|\frac{\arctan\frac{t}{z}}{\mathrm{e}^{2\pi t}-1}\right| \leq \frac{c}{|z|}\frac{1}{\mathrm{e}^{2\pi s\cos\theta}} \tag{A.36}$$

One can present the asymptotics (A.30), (A.31) and (A.32) in a different equivalent form. Let us fix α and let $|\arg\lambda| < \pi - \epsilon$. Then as $|\lambda| \to \infty$, we have

$$\ln\Gamma(\tfrac12+\alpha+\lambda) = (\alpha+\lambda)\ln\lambda - \lambda + \frac12\ln(2\pi) + \mathcal{O}\big(\tfrac{1}{\lambda}\big), \tag{A.37}$$

$$\psi(\tfrac12+\alpha+\lambda) = \ln(\lambda) + \frac{\alpha}{\lambda} + \mathcal{O}\big(\tfrac{1}{\lambda^2}\big), \tag{A.38}$$

$$\psi'(\tfrac12+\alpha+\lambda) = \frac{1}{\lambda} - \frac{\alpha}{\lambda^2} + \mathcal{O}\big(\tfrac{1}{\lambda^3}\big). \tag{A.39}$$

Here is a consequence of the Stirling formula [**4**, Lemma 2.1]:

$$\frac{\Gamma(a+\lambda)}{\Gamma(b+\lambda)} = \lambda^{a-b}\Big(1+\mathcal{O}\big(\tfrac{1}{\lambda}\big)\Big). \tag{A.40}$$

Here is another consequence, where $\beta > 0$ and $\beta \to \infty$:

$$\Gamma(\alpha+\mathrm{i}\beta)\Gamma(\alpha-\mathrm{i}\beta) = 2\pi\beta^{2\alpha-1}\mathrm{e}^{-\pi\beta}\Big(1+\mathcal{O}\big(\tfrac{1}{\beta}\big)\Big). \tag{A.41}$$

Appendix B. Other conventions

This appendix is included to facilitate the comparison of our formulas with those contained in a part of the literature.

B.1. Associated Legendre equation. In the literature (see for example [**4, 17, 18**]), instead of the Gegenbauer equation (4.1), one often considers the *associated Legendre equation* given by the operator

$$\begin{aligned}(z^2-1)^{\frac{\alpha}{2}}\Big((1-z^2)\partial_z^2 - 2(1+\alpha)z\partial_z + \lambda^2 - \Big(\alpha+\frac12\Big)^2\Big)(z^2-1)^{-\frac{\alpha}{2}} \\ =\partial_z(1-z^2)\partial_z - \frac{\alpha^2}{1-z^2} + \lambda^2 - \frac14.\end{aligned} \tag{B.1}$$

In the standard literature the parameter λ is shifted by $\frac12$. Thus, with $\mu = \alpha$ and $\nu = \lambda - \frac12$, (B.1) becomes

$$(1-z^2)\partial_z^2 - 2z\partial_z - \frac{\mu^2}{1-z^2} + \nu(\nu+1). \tag{B.2}$$

Certain functions annihilated by (B.2) are called *associated Legendre functions.* In this subsection we show how to convert the functions $\mathbf{S}$ and $\mathbf{Z}$ into associated Legendre functions. The material of this subsection will not be used elsewhere in this paper.

The associated Legendre function of the 1st kind are defined as

$$\begin{aligned}\mathbf{P}_\nu^\mu(z) &= \Big(\frac{z+1}{z-1}\Big)^{\frac{\mu}{2}}\mathbf{F}\Big(-\nu,\nu+1;1-\mu;\frac{1-z}{2}\Big) \\ &= \frac{2^\mu}{(z^2-1)^{\frac{\mu}{2}}}\mathbf{F}\Big(1-\mu+\nu,-\mu-\nu;1-\mu;\frac{1-z}{2}\Big) \\ &= \frac{2^\mu}{(z^2-1)^{\frac{\mu}{2}}}\mathbf{S}_{-\mu,\nu+\frac12}(z).\end{aligned} \tag{B.3}$$

The functions $\mathbf{P}_\nu^\mu$ are well adapted to the halfline $[0,\infty[$. On the interval $]-1,1[$ one prefers to use *Ferrer's associated Legendre function of the 1st kind*

$$\begin{aligned}\mathrm{P}_\nu^\mu(z) &= \Big(\frac{z+1}{1-z}\Big)^{\frac{\mu}{2}}\mathbf{F}\Big(-\nu,\nu+1;1-\mu;\frac{1-z}{2}\Big)\\ &= \frac{2^\mu}{(1-z^2)^{\frac{\mu}{2}}}\mathbf{F}\Big(1-\mu+\nu,-\mu-\nu;1-\mu;\frac{1-z}{2}\Big)\\ &= \frac{2^\mu}{(1-z^2)^{\frac{\mu}{2}}}\mathbf{S}_{-\mu,\nu+\frac12}(z). \qquad \text{(B.4)}\end{aligned}$$

The associated Legendre function of the 2nd kind are given by

(B.5)

$$\begin{aligned}\mathbf{Q}_\nu^\mu(z) &= \mathrm{e}^{\mu\mathrm{i}\pi}\sqrt{\pi}\Gamma(\mu+\nu+1)\frac{(z^2-1)^{\frac{\mu}{2}}}{2^{\nu+1}z^{\nu+\mu+1}}\mathbf{F}\Big(\frac{\nu+\mu+2}{2},\frac{\nu+\mu+1}{2};\nu+\frac32;z^{-2}\Big)\\ &= \frac{\mathrm{e}^{\mu\mathrm{i}\pi}\sqrt{\pi}\Gamma(\mu+\nu+1)(z^2-1)^{\frac{\mu}{2}}}{2^{\nu+1}(1+z)^{\nu+\mu+1}\Gamma(\nu+\frac32)}F\Big(\nu+1,\nu+\mu+1;2\nu+2;\frac{2}{z+1}\Big)\\ &= \mathrm{e}^{\mu\mathrm{i}\pi}\sqrt{\pi}\Gamma(\mu+\nu+1)\frac{(z^2-1)^{\frac{\mu}{2}}}{2^{\nu+1}}\mathbf{Z}_{\mu,\nu+\frac12}(z).\end{aligned}$$

B.2. Other conventions for Gegenbauer functions. One sometimes finds other conventions for the Gegenbauer functions in the literature [**8, 9, 17**]:

$$C_\lambda^\alpha(z) = \frac{\Gamma(\lambda+2\alpha)}{\Gamma(1+\lambda)\Gamma(2\alpha)}F\big(-\lambda,\lambda+2\alpha;\alpha+\tfrac12;\tfrac{1-z}{2}\big), \qquad \text{(B.6)}$$

$$D_\lambda^\alpha(z) = \frac{\mathrm{e}^{\mathrm{i}\pi\alpha}\Gamma(\lambda+2\alpha)(2z)^{-\lambda-2\alpha}}{\Gamma(\alpha)\Gamma(1+\alpha+\lambda)}F\big(\tfrac12\lambda+\alpha,\tfrac12+\tfrac12\lambda+\alpha;1+\alpha+\lambda;z^{-2}\big), \qquad \text{(B.7)}$$

where the second identity is valid for $|z|>1$. Comparing these definitions with (4.4), (4.7), (B.3) and (B.5), we find

$$\begin{aligned}\frac{\Gamma(\alpha)\Gamma(1+\lambda)}{\sqrt{\pi}\Gamma(\lambda+2\alpha)}C_\lambda^\alpha(z) &= 2^{\frac12-\alpha}(z^2-1)^{\frac14-\frac{\alpha}{2}}\mathbf{P}^{\frac12-\alpha}_{-\frac12+\alpha+\lambda}(z)\\ &= 2^{1-2\alpha}\mathbf{S}_{\alpha-\frac12,\alpha+\lambda}(z), \qquad \text{(B.8)}\end{aligned}$$

$$\begin{aligned}\frac{\Gamma(\alpha)\Gamma(1+\alpha+\lambda)}{\Gamma(\lambda+2\alpha)}D_\lambda^\alpha(z) &= \frac{2^{\frac12-\alpha}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}(z^2-1)^{\frac14-\frac{\alpha}{2}}}{\sqrt{\pi}\Gamma\big(\frac12+2\alpha+\lambda\big)}\mathbf{Q}^{\alpha-\frac12}_{-\frac12+\alpha+\lambda}(z)\\ &= \mathrm{e}^{\mathrm{i}\pi\alpha}2^{-\lambda-2\alpha}Z_{\alpha-\frac12,\alpha+\lambda}(z). \qquad \text{(B.9)}\end{aligned}$$

References

[1] Albeverio, S., Gesztesy, F., Høegh-Krohn, R. and Holden, H.: *Solvable Models in Quantum Mechanics*, AMS Chelsea Publishing, second edition, 2004, (with an appendix by P. Exner).

[2] Albeverio, S. and Kurasov, P.: *Singular Perturbations of Differential Operators: Solvable Schrödinger-type Operators*, Cambridge Univ. Press, Cambridge, 2000.

[3] Berezin, F. A. and Faddeev, L. D.: *Remark on the Schrödinger equation with singular potential*, Dokl. Akad. Nauk. SSSR 137 (1961), 1011–1014.

[4] Cohl, H. S., Dang, T. H., and Dunster, T. M.: *Fundamental Solutions and Gegenbauer Expansions of Helmholtz Operators in Riemannian Spaces of Constant Curvature.* SIGMA 14 (2018), 136, 45 pages.

[5] Dereziński, J.: *Hypergeometric type functions and their symmetries.* Annales Henri Poincaré 2014, 15, 1569-1653.

[6] Dereziński, J.: *Group-theoretical origin of symmetries of hypergeometric class equations and functions*, in Complex differential and difference equations. Proceedings of the school and conference held at Będlewo, Poland, September 2-15, 2018, Filipuk, G, Lastra, A; Michalik, S; Takei, Y, Żołądek, H, eds; De Gruyter Proceedings in Mathematics, Berlin, 2020.

[7] Dereziński, J., Ruba, Błażej, Gaß, C. *Point potentials on the hyperbolic space and sphere in any dimension*, in preparation

[8] Durand, L.: *Expansion formulas and addition theorems for Gegenbauer functions*, J. Math. Phys. 17, 1933-1948, 1976.

[9] Durand, L.: *Complex Asymptotics in* λ *for the Gegenbauer Functions* $C_\lambda^\alpha(z)$ *and* $D_\lambda^\alpha(z)$ *with* $z \in (-1, 1)$, Symmetry 2019, 11, 1465.

[10] Durand, L.: *Asymptotic Bessel-function expansions for Legendre and Jacobi functions*, J. Math. Phys. 60, 013501, 2019.

[11] Gelfand, I. M. and Shilov, G. E.: *Generalized functions I: Properties and operations* Academic Press, New York-London, 1964.

[12] Gradshteyn, I.S., Ryzhik, I.M.: *Table of integrals, series, and products*, translated by Scripta Technica, Inc. (7 ed), Academic Press, 2007.

[13] Hadamard, J.: *Lectures on Cauchy's problem in linear partial differential equations*, Dover Phoenix editions, Dover Publications, New York, 1923.

[14] Hadamard, J.: *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques* (in French), Paris, Hermann & Cie, 1932.

[15] Hörmander., L.: *The Analysis of Linear Partial Differential Operators I.* Springer, Berlin, 2nd edition, 1990.

[16] Lieb, E., Seiringer, R., Solovej, J.P., Yngvason, J.: *The Mathematics of the Bose Gas and its Condensation*, Birkhäuser Verlag, Basel, Boston, Berlin 2005.

[17] Olver, F. W. J., Olde Daalhuis, A. B., Lozier, D. W., Schneider, B. I., Boisvert, R. F., Clark, C. W., Miller, B. R., Saunders, B. V., Cohl, H. S., and McClain, M. A. (Editors): *NIST Digital Library of Mathematical Functions*, Release 1.1.6 of 2022-06-30, available at `http://dlmf.nist.gov/`.

[18] Olver, F.W.J.: *Asymptotics and special functions*, Academic Press, New York, 1974.

[19] Laforgia, A.: *Bounds for modified Bessel functions*, J. Comput. Appl. Math. 34, 263-267, 1991.

[20] Paycha, S.: *Regularised Integrals, Sums and Traces: An Analytic Point of View*, University Lecture Series, volume 59, 2012.

[21] Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, London, 1975.

[22] Riesz, M.: *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Mathematica, 81:1–223, 1949.

[23] Szmytkowski R.: *Closed forms of the Green's function and the generalized Green's function for the Helmholtz operator on the N-dimensional unit sphere*, J. Phys. A: Math. Theor. 40, 995-1009, 2007.

[24] Whittaker, E. T., and Watson, G. N.: *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions*, Cambridge Mathematical Library Series, 1996. (First edn. Cambridge University Press, Cambridge, 1902).

Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warszawa, Poland
Email address: `jan.derezinski@fuw.edu.pl`

Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warszawa, Poland
Email address: `christian.gass@fuw.edu.pl`

Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark
Email address: `btr@math.ku.dk`