# Pauli-Fierz Hamiltonians defined as quadratic forms

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**Abstract**

We study Pauli-Fierz Hamiltonians—self-adjoint operators describing a small quantum system interacting with a bosonic field. Using quadratic form techniques, we extend the results of Dereziński-Gérard and Gérard about the self-adjointness, the location of the essential spectrum and the existence of a ground state to a large class of Pauli-Fierz Hamiltonians.

Keywords: Pauli-Fierz Hamiltonians, Fock spaces, quadratic form, spectral theory.

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## 1 Introduction

Our paper is devoted to the study of spectral properties of self-adjoint operators of the following form:

$$H = K \otimes 1 + 1 \otimes \int h(\xi) a^*(\xi) a(\xi) \mathrm{d}\xi + \int v(\xi) \otimes a^*(\xi) \mathrm{d}\xi + \int v(\xi)^* \otimes a(\xi) \mathrm{d}\xi. \tag{1.1}$$

Above, $K$ denotes a self-adjoint operator on a Hilbert space $\mathcal{K}$. $a^*(\xi)/a(\xi)$ are creation/annihilation operators acting on the bosonic Fock space $\Gamma_s(\mathcal{Z})$. $H$ is understood as a self-adjoint operator on the tensor product $\mathcal{K} \otimes \Gamma_s(\mathcal{Z})$.

The one-particle space $\mathcal{Z}$ will be assumed to be $L^2(\mathbf{R}^d) \otimes \mathbf{C}^n$. $h(\xi)$ describes the dispersion relation of the bosons. We will always assume that $h(\xi)$ is nonnegative. Abusing terminology, $\inf h$ is sometimes called the mass of the bosons. We are particularly interested in the massless case, that is $\inf h = 0$. A typical form of $h(\xi)$ considered in physical applications is $h(\xi) = (m^2 + \xi^2)^{1/2}$, where $m$ is a non-negative number that we call the mass.

$\mathbf{R}^d \ni \xi \mapsto v(\xi)$ is a function with values in operators on $\mathcal{K}$. It is responsible for the interaction between the small system and the bosons.

There is no universally accepted name for operators of the form (1.1). In [5, 8, 11, 12] they are called Pauli-Fierz operators or Pauli-Fierz Hamiltonians, and we will use this name. Note, however, that sometimes the name "the Pauli-Fierz Hamiltonian" is used to denote slightly different objects [15, 17].

Operators similar to (1.1) arise in quantum physics as simplified Hamiltonians describing a small system described by the Hilbert space $\mathcal{K}$ interacting with a bosonic field. For instance, the dipole approximation to nonrelativistic QED is of this form. From recent rigorous work it became apparent that Pauli-Fierz Hamiltonians are not only physically relevant, but also they are interesting mathematical objects.

One of the results about Pauli-Fierz Hamiltonians that can be found in the literature says that the essential spectrum of $H$ is shifted to the right from the bottom of the spectrum of $H$ by the "mass" $\inf h$. This theorem to our knowledge was first proven in [5]. It resembles the Hunziker–van-Winter–Zhislin Theorem about many-body Schrödinger operators [19]. Therefore, we call it the HVZ-type theorem about Pauli-Fierz Hamiltonians.

It is obvious that if the "mass" is positive, then the HVZ-type theorem implies the existence of a ground state. It turns out that even in the massless case, under some additional assumptions, one can show that there exists a ground state of $H$ that "sits" at the tip of the continuous spectrum. This result was first proven in [2, 3] for a small coupling constant. In [21] this result was extended to an arbitrary coupling constant for a Hamiltonian satisfying an appropriate condition that allows to use the Perron-Frobenius method. In the work of Gérard [12] the existence of a ground state was proven for a large class of Pauli-Fierz Hamiltonians without using the Perron-Frobenius method. See also later work [1, 14].

In our paper we extend the HVZ-type theorem of [5] and the theorem about the existence of a ground state from [12] to a larger class of Pauli-Fierz Hamiltonians. The main motivation of our paper is to give an analysis of mathematical tools used in the context of second quantization and of tensor products of Hilbert spaces. Let us make some comments about these tools.

In Equation (1.1) we used the formalism of "operator valued distributions" $a^*(\xi)$, $a(\xi)$, which is a common approach to creation/annihilation operators. In the following sections of our paper we will not use this formalism. Instead, we will write $a^*(v)$ for $\int v(\xi) \otimes a^*(\xi)\mathrm{d}\xi$ and $a(v)$ for $\int v(\xi)^* \otimes a(\xi)\mathrm{d}\xi$, where $v$ is a quadratic form from $\mathcal{K}$ to $\mathcal{K} \otimes \mathcal{Z}$. This clearly leads to a more compact notation (used before in particular in [5, 11]). Note, however, that the advantage of working with the form $v$ instead of the function $\xi \mapsto v(\xi)$ is not just a matter of notation. It also helps to obtain stronger results. This is one of the reasons why the results of this paper are stronger than those of [12].

To define the operator $H$ we use the form boundedness technique based on the KLMN theorem [18], instead of the operator boundedness technique based on the Kato-Rellich theorem, employed commonly in the literature [5, 12]. This allows us to give rather weak and simple conditions for our results, as compared with the literature. (Note that [11] also uses the form boundedness technique in a similar context, see Subsect. 5.3).

In the proof of the HVZ-type theorem we use the so-called extended space. This technique was introduced in [5, 6], and then used e.g. in [10]. In our paper we have to adapt it to the case of a Hamiltonian defined using the quadratic form method.

One of the techniques that proved powerful in the study of 2nd quantized Hamiltonians is the so-called pullthrough formula. It was used in the early works of Glimm, Jaffe and Rosen on constructive quantum field theory, for instance in the work of Rosen on higher order estimates [20]. It was also applied in the work of Fröhlich on the massless translation invariant Nelson model [9]. In [2, 3, 12] it was used as an important step in the proof of the existence of a ground state for Pauli-Fierz operators. The version of the pullthrough formula for Pauli-Fierz operators employed in [2, 3, 12] has the following form:

$$1{\otimes}a(\xi)\ H = (H + h(\xi))\ 1{\otimes}a(\xi) + v(\xi){\otimes}1. \tag{1.2}$$

Note that in the above formula the annihilation operator $a(\xi)$ is not even closable. It can be interpreted as an operator valued distribution, which is a little awkward mathematically. Besides, (1.2) depends explicitly on the identification of the one-particle space with the space $L^2(\mathrm{d}\xi)$, which should not play a role in the arguments.

In our paper we propose a reformulation of the pullthrough formula that is more satisfactory mathematically. To this end, we introduce the so-called Pullthrough Annihilation Operator $A$. It acts from the Fock space $\Gamma_s(\mathcal{Z})$ to the tensor product $\Gamma_s(\mathcal{Z}) \otimes \mathcal{Z}$ . Using the Pullthrough Annihilation Operator, the equation (1.2) can be rewritten as

$$A\ H = (H \otimes 1_{\mathcal{Z}} + 1_{\mathcal{H}} \otimes h)\ A + v, \tag{1.3}$$

We show how to use (1.3) to obtain the existence of a ground state under weaker assumptions than those in the literature.

The identity (1.3) is clearly equivalent to (1.2) but is written in a "more canonical way". Note, however, that it is often not easy to deal with tensor products of vector spaces in a transparent way. Strictly speaking we should have written $1_{\mathcal{K}} \otimes A$ instead of $A$, whereas $v$ should be tensored with $1_{\Gamma_s(\mathcal{Z})}$ "in the middle".

Our paper is organized as follows. In Section 2 we introduce notation and give a precise description of the model. We try to be quite pedantic, since there seems to be no standard terminology concerning some of the constructions that we need. In particular, we introduce *creation and annihilation forms* on a Fock space. Equipped with this terminology it is easy to introduce a natural class of Pauli-Fierz Hamiltonians that are defined as form perturbation of free Pauli-Fierz Hamiltonians using the KLMN theorem. The assumptions on the coupling function are considerably weaker and simpler than those considered in the literature [5, 11, 12]. In this section we also state our main assumptions and formulate the main results of the paper.

In Section 3 we prove a HVZ-type theorem which says that the infimum of the essential spectrum of $H$ equals the infimum of the spectrum of $h$ plus the infimum of the spectrum of $H$. In particular, this result ensures that $H$ admits a ground state in the massive case. The proof is based on the ideas from [5], but applies to a much larger class of Pauli-Fierz Hamiltonians.

In Section 4, we study the question of existence of ground states. Our arguments are based on the ideas of [12] and [7], but again we treat a much larger class of Pauli-Fierz Hamiltonians. In this section we introduce the Pullthrough Operators, study their properties and apply them to Pauli-Fierz operators.

Finally, in Section 5, we compare the results of our paper with the analoguous results from the literature. We describe and correct a minor error contained in [5, 12]. We show that the assumptions of our paper are weaker than those of [5, 11, 12].

# 2 Notation and main results

In this section, we describe basic terminology and notation that we will use, then we define the Pauli-Fierz Hamiltonians as quadratic forms and finally we state our main results.

## 2.1 Basic notation

Let $\mathcal{H}$ be a Hilbert space. The scalar product of two vectors $\Phi, \Psi \in \mathcal{H}$ is denoted by $(\Phi|\Psi)$, (not by $(\Phi, \Psi)$, which denotes an ordered pair).

If $\Phi \in \mathcal{H}$, then we introduce the operators $(\Phi|$ and $|\Phi)$ as follows:

$$\mathcal{H} \ni \Psi \quad \mapsto \quad (\Phi|\Psi := (\Phi|\Psi) \in \mathbf{C}, \tag{2.4}$$

$$\mathbf{C} \ni \lambda \quad \mapsto \quad |\Phi)\lambda := \lambda\Phi \in \mathcal{H}. \tag{2.5}$$

If $A$ is a self-adjoint operator, then $\mathrm{sp}A$, $\mathrm{sp}_{\mathrm{ess}}A$ and $\mathrm{sp}_{\mathrm{pp}}A$ denote its spectrum, essential spectrum and pure point spectrum. $\inf A$ denotes the infimum of its spectrum and $1_\Theta(A)$ denotes its spectral projection onto a Borel set $\Theta \subset \mathbf{R}$.

By saying that $A$ is an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ we mean that it is a linear map

$$\mathrm{Dom}\, A \ni \Phi \mapsto A\Phi \in \mathcal{H}_2,$$

where Dom $A$ is a linear subspace of $\mathcal{H}_1$ called the domain of $A$. We define the adjoint of $A$, denoted by $A^*$, in the usual way. Clearly, $A^*$ is an operator from $\mathcal{H}_2$ to $\mathcal{H}_1$.

$B(\mathcal{H}_1, \mathcal{H}_2)$ will denote bounded everywhere defined operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Dom $A$ will be sometimes treated as a Hilbert space equipped with the graph norm. For instance, if $A$ is positive, this graph norm can be taken to be $\|\Phi\|_A := \|(A+1)\Phi\|$. The space dual to Dom $A$ will be denoted $(A+1)\mathcal{H}$. Note that $A$ extends to a bounded operator from Dom $(A+1)^{1/2}$ to $(A+1)^{1/2}\mathcal{H}$.

## 2.2 Unbounded forms

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We say that the map

$$\mathrm{Dom}_{\mathrm{l}} h \times \mathrm{Dom}_{\mathrm{r}} h \ni (\Phi, \Psi) \mapsto (\Phi|h\Psi) \in \mathbf{C} \tag{2.6}$$

is a form $h$ from $\mathcal{H}_1$ to $\mathcal{H}_2$ iff

(1) $\mathrm{Dom}_{\mathrm{r}} h$ is a subspace of $\mathcal{H}_1$;

(2) $\mathrm{Dom}_{\mathrm{l}} h$ is a subspace of $\mathcal{H}_2$;

(3) (2.6) is linear wrt the second argument;

(4) (2.6) is antilinear wrt the first argument.

$\mathrm{Dom}_{\mathrm{r}} h$ is called the right domain of $h$ and $\mathrm{Dom}_{\mathrm{l}} h$ is called the left domain of $h$. If $\mathcal{H}_1 = \mathcal{H}_2$, they often coincide, and we then write $\mathrm{Dom}_{\mathrm{f}} h$ for $\mathrm{Dom}_{\mathrm{l}} h = \mathrm{Dom}_{\mathrm{r}} h$ and call it the domain of $h$.

The notation (2.6) we use is different from the commonly used $h(\Phi, \Psi)$. It suggests the identification of the quadratic form with a linear map from $\mathrm{Dom}_{\mathrm{r}} h$ to the algebraic dual of $\mathrm{Dom}_{\mathrm{l}} h$.

We define the adjoint of the form $h$, denoted by $h^{*\mathrm{f}}$ as follows: $\mathrm{Dom}_{\mathrm{l}} h^{*\mathrm{f}} := \mathrm{Dom}_{\mathrm{r}} h$, $\mathrm{Dom}_{\mathrm{r}} h^{*\mathrm{f}} := \mathrm{Dom}_{\mathrm{l}} h$ and

$$\mathrm{Dom}_{\mathrm{r}} h \times \mathrm{Dom}_{\mathrm{l}} h \ni (\Psi, \Phi) \mapsto (\Psi|h^{*\mathrm{f}}\Phi) := \overline{(\Phi|h\Psi)}.$$

Note that if $h$ is an operator with domain Dom $h \subset \mathcal{H}_1$, then it generates a form, denoted also $h$, such that $\mathrm{Dom}_{\mathrm{r}} h = \mathrm{Dom}\ h$, $\mathrm{Dom}_{\mathrm{l}} h = \mathcal{H}_2$, which is given by

$$\mathcal{H}_2 \times \mathrm{Dom}\ h \ni (\Phi, \Psi) \mapsto (\Phi|h\Psi).$$

Then we can define the adjoint of $h$ in the sense of operators, denoted $h^*$, and its adjoint in the sense of forms, denoted $h^{*\mathrm{f}}$. If $h$ is bounded and everywhere defined, then these two adjoints coincide. If $h$ is unbounded, then they are different.

If $h$ is a self-adjoint operator on $\mathcal{H}$, then there exists a different form often associated with $h$. Its domain, often called the form domain of $h$, equals Dom $|h|^{1/2}$ and it is given by

$$\mathrm{Dom}\ |h|^{1/2} \times \mathrm{Dom}\ |h|^{1/2} \ni (\Phi, \Psi) \mapsto (|h|^{1/2}\Phi|\ \mathrm{sgn} h\ |h|^{1/2}\Psi).$$

If $\Psi \in \mathrm{Dom}_{\mathrm{r}} v$, then we set

$$\|v\Psi\| := \sup\{|(\Phi|v\Psi)|\ :\ \Phi \in \mathrm{Dom}_{\mathrm{l}} v,\ \ \|\Phi\| = 1\}.$$

Note that the above notation agrees with the usual notation if $v$ is given by an operator and $\Psi \in \mathrm{Dom}\, v$.

We will also write

$$\|v\| := \sup\{|(\Phi|v\Psi)|\ :\ \Phi \in \mathrm{Dom}_{\mathrm{l}} v,\ \ \|\Phi\| = 1,\ \Psi \in \mathrm{Dom}_{\mathrm{r}} v,\ \ \|\Psi\| = 1\}.$$

Again, this notation agrees with the usual operator norm if $v$ is an operator.

Suppose now we are given a form $v$ from $\mathcal{H}_1$ to $\mathcal{H}_2$, an operator $h_1$ on $\mathcal{H}_1$ and an operator $h_2$ on $\mathcal{H}_2$. Then $h_2^* v h_1$ denotes the form $(\Phi, \Psi) \mapsto (h_2\Phi|vh_1\Psi)$, with $\mathrm{Dom}_{\mathrm{r}} h_2^* v h_1 := \{\Psi \in \mathrm{Dom}\, h_1 | h_1\Psi \in \mathrm{Dom}_{\mathrm{r}} v\}$ and $\mathrm{Dom}_{\mathrm{l}} h_2^* v h_1 := \{\Phi \in \mathrm{Dom}\, h_2 | h_2\Phi \in \mathrm{Dom}_{\mathrm{l}} v\}$.

### 2.3 Tensor products and Fock spaces

If $\mathcal{K}_0$ and $\mathcal{Z}_0$ are vector spaces, then $\mathcal{K}_0 \overset{\circ}{\otimes} \mathcal{Z}_0$ will denote their algebraic tensor product. $\overset{\circ}{\Gamma}_{\rm s}(\mathcal{Z}_0)$ will denote the algebraic symmetric Fock space over $\mathcal{Z}_0$.

If $\mathcal{K}$ and $\mathcal{Z}$ are Hilbert spaces, then $\mathcal{K} \otimes \mathcal{Z}$ will denote their Hilbert space tensor product (the completion of $\mathcal{K} \overset{\circ}{\otimes} \mathcal{Z}$). $\Gamma_{\rm s}(\mathcal{Z})$ will denote the symmetric Fock space over $\mathcal{Z}$ (the completion of $\overset{\circ}{\Gamma}_{\rm s}(\mathcal{Z})$). $\Gamma_{\rm s}^n(\mathcal{Z})$ will stand for its $n$ particle subspace (the completion of the algebraic symmetric tensor power of $\mathcal{Z}_0$, denoted by $\overset{\circ}{\Gamma}{}_{\rm s}^{n}(\mathcal{Z}_0)$).

If $v$ is a form from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $w$ is another form from $\mathcal{K}_1$ to $\mathcal{K}_2$, then the tensor product of the forms $v$ and $w$ is defined as the form $v \otimes^{\rm f} w$ from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$, such that ${\rm Dom}_r(v \otimes^{\rm f} w) = {\rm Dom}_{\rm r}(v) \overset{\circ}{\otimes} {\rm Dom}_{\rm r}(w)$, ${\rm Dom}_l(v \otimes^{\rm f} w) = {\rm Dom}_{\rm l}(v) \overset{\circ}{\otimes} {\rm Dom}_{\rm l}(w)$ and

$$\big(\Psi_2 \otimes \Xi_2 | (v \otimes^{\rm f} w) \Psi_1 \otimes \Xi_1\big) := (\Psi_2 | v \Psi_1)(\Xi_2 | w \Xi_1).$$

If $v$ is an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $w$ is an operator from $\mathcal{K}_1$ to $\mathcal{K}_2$, then the tensor product of the operators $v$ and $w$ is defined as the operator $v \otimes w$ from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$, such that ${\rm Dom}\,(v \otimes w) = {\rm Dom}\,(v) \overset{\circ}{\otimes} {\rm Dom}\,(w)$ and

$$(v \otimes w) \Psi \otimes \Xi := (v\Psi) \otimes (w\Xi).$$

Thus if $v$ and $w$ are operators, then we have two slightly different tensor products of $v$ and $w$. In the case of the form tensor product ${\rm Dom}_{\rm l}(v \otimes^{\rm f} w) = \mathcal{H}_2 \overset{\circ}{\otimes} \mathcal{K}_2$ and in the case of the operator tensor product ${\rm Dom}_{\rm l}(v \otimes w) = \mathcal{H}_2 \otimes \mathcal{K}_2$ .

If $v$ and $w$ are closed operators, then the above defined $v \otimes w$ is a closable operator. We will denote by the same symbol its closure. If they are densely defined, then so is $v \otimes w$. If $v$ and $w$ are bounded, then $v \otimes w$ is bounded as well.

If $h$ is a closed operator on $\mathcal{Z}$, then ${\rm d}\Gamma(h)$ denotes the closed operator on $\Gamma_{\rm s}(\mathcal{Z})$ defined in the usual way. If $q$ is a contraction from $\mathcal{Z}_1$ to $\mathcal{Z}_2$, then $\Gamma(q)$ denotes the contraction from $\Gamma_{\rm s}(\mathcal{Z}_1)$ to $\Gamma_{\rm s}(\mathcal{Z}_2)$ defined in the usual way (see e.g. [5, 18]).

Finally, $N$ will denote the number operator, *i.e.* $N = {\rm d}\Gamma(1)$.

### 2.4 Creation and annihilation forms

Let $\mathcal{K}_0$ and $\mathcal{Z}_0$ be subspaces of Hilbert spaces $\mathcal{K}$ and $\mathcal{Z}$, respectively. Let $v$ be a form from $\mathcal{K}$ to $\mathcal{K} \otimes \mathcal{Z}$ with the right domain $\mathcal{K}_0$ and the left domain $\mathcal{K}_0 \overset{\circ}{\otimes} \mathcal{Z}_0$.

We define the annihilation form $a^{\rm f}(v)$ as a form on $\mathcal{K} \otimes \Gamma_{\rm s}(\mathcal{Z})$ with the (left and right) domain $\mathcal{K}_0 \overset{\circ}{\otimes} \overset{\circ}{\Gamma}_{\rm s}(\mathcal{Z}_0)$. It is defined for $\Phi \in \mathcal{K}_0 \overset{\circ}{\otimes} \overset{\circ}{\Gamma}{}_{\rm s}^{m}(\mathcal{Z}_0)$, $\Psi \in \mathcal{K}_0 \overset{\circ}{\otimes} \overset{\circ}{\Gamma}{}_{\rm s}^{n}(\mathcal{Z}_0)$ as

$$(\Phi | a^{\rm f}(v) \Psi) := \begin{cases} 0, & m \neq n-1, \\ \sqrt{n}(\Phi | v^{*{\rm f}} \otimes 1^{(n-1)\otimes} \Psi), & m = n-1. \end{cases}$$

The creation form $a^{*{\rm f}}(v)$ is defined as

$$a^{*{\rm f}}(v) := \big(a^{\rm f}(v)\big)^{*{\rm f}}.$$

Note that if $v$ is bounded with the right domain equal to $\mathcal{K}$, then the form $a^{\rm f}(v)$ is associated with a densely defined closable operator. Then we use the symbol $a(v)$ for the closure of this operator. We can then introduce the creation operator

$$a^*(v) := (a(v))^* .$$

The following lemma is essentially proven in [8].

**Lemma 2.1.** *Let $h$ be a positive operator on $\mathcal{Z}$. Suppose that $\mathcal{Z}_0 \subset \mathrm{Dom}\, h^{1/2}$ and $\Phi \in \mathcal{K}_0 \mathring{\otimes} \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}_0)$. Then*

(1) *$\Phi \in \mathrm{Dom}\,(1 \otimes \mathrm{d}\Gamma(h)^{1/2})$ and*

$$\|a^{\mathrm{f}}(v)\Phi\|^2 \le \|h^{-1/2}v\|^2 (\Phi|1 \otimes \mathrm{d}\Gamma(h)\Phi).$$

(2) *If, moreover, $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$, then*

$$\|a^*(v)\Phi\|^2 \le (\Phi|v^* v \otimes 1\Phi) + \|h^{-1/2}v\|^2 (\Phi|1 \otimes \mathrm{d}\Gamma(h)\Phi).$$

**Remark 2.1.** *If $B \in \mathcal{B}(\mathcal{K})$ and $z \in \mathcal{Z}$, then*

$$a^*\left(B \otimes |z)\right) = B \otimes a^*(z), \qquad a\left(B \otimes |z)\right) = B^* \otimes a(z),$$

*where $a^*(z)$ and $a(z)$ are the usual creation and annihilation operators and $|z)$ is defined in (2.5).*

## 2.5 Pauli-Fierz Hamiltonians defined as forms

Let $\mathcal{K}$ and $\mathcal{Z}$ be Hilbert spaces. The main space used in our paper will be $\mathcal{H} := \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$.

Let $K$ be a positive operator on $\mathcal{K}$ and $h$ be a positive operator on $\mathcal{Z}$. $m := \inf h$ will be sometimes called the mass. (Recall that $\inf h$ denotes the infimum of the spectrum of the self-adjoint operator $h$). The free Pauli-Fierz operator is defined as the self-adjoint operator on $\mathcal{H}$ given by

$$H_{\mathrm{fr}} := K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h).$$

Let $\mathcal{K}_0$ be a dense subspace of $\mathcal{K}$ contained in Dom $K^{1/2}$ and $\mathcal{Z}_0$ a dense subspace of $\mathcal{Z}$ contained in Dom $h^{1/2}$. Let $v$ be such as in the previous subsection. We will refer to $v$ as a coupling form. The Pauli-Fierz interaction is defined as a form on $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$, with the domain $\mathcal{K}_0 \mathring{\otimes} \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}_0)$, equal to

$$V := a^{\mathrm{f}*}(v) + a^{\mathrm{f}}(v).$$

In order to abbreviate the notation, in what follows we will omit the superscripts f in the annihilation and creation forms. We will also often omit the factors of 1.

**Theorem 2.1.** *Suppose that*

$$\alpha := \limsup_{t \to \infty} \|h^{-1/2} v (t+K)^{-1/2}\| < \infty. \tag{2.7}$$

*Then, for any $t > 0$, $h^{-1/2}v(t+K)^{-1/2}$ is bounded and $t \mapsto \|h^{-1/2}v(t+K)^{-1/2}\|$ is a decreasing function. Therefore,* lim sup *in (2.7) can be replaced with* lim. *Moreover, the form $V$ is form bounded wrt $H_{\mathrm{fr}}$ with the $H_{\mathrm{fr}}$-form bound $\le \alpha$.*

**Proof.** Let $t > 0$

$$\begin{aligned} |(\Phi|V\Phi)| \quad = 2|(\Phi|a(v)\Phi)| \quad &\le 2\|(t+K)^{1/2}\Phi\| \|(t+K)^{-1/2}a(v)\Phi\| \\ &\le \epsilon(\Phi|(t+K)\Phi) + \epsilon^{-1}\|(t+K)^{-1/2}a(v)\Phi\|^2 \\ &\le \epsilon(\Phi|(t+K)\Phi) + \epsilon^{-1}\|h^{-1/2}v(t+K)^{-1/2}\|^2 (\Phi|\mathrm{d}\Gamma(h)\Phi). \end{aligned} \tag{2.8}$$

In the last step we used the identity $(t+K)^{-1/2}a(v) = a\left(v(t+K)^{-1/2}\right)$ and Lemma 2.1 (1).

Let $\epsilon > \alpha$. We choose $t$ such that $\|h^{-1/2}v(t+K)^{-1/2}\| \le \epsilon$. Then the right hand side of (2.8) is less than or equal to

$$\le t\epsilon \|\Phi\|^2 + \epsilon(\Phi|H_{\mathrm{fr}}\Phi).$$

$\square$

Throughout the paper we will make the assumption

**Assumption A.** $\lim_{t\to\infty} \|h^{-1/2}v(t+K)^{-1/2}\| < 1$.

The KLMN theorem [18] implies the following result.

**Theorem 2.2.** *Suppose that Assumption A holds. Then the operator*

$$H := H_{\rm fr} + V$$

*is well defined as a form sum. The form domains of $H_{\rm fr}$ and $H$ coincide, that is* $\mathrm{Dom}\,|H|^{\frac{1}{2}} = \mathrm{Dom}\,|H_{\rm fr}|^{\frac{1}{2}}$.

The operator $H$ defined in Theorem 2.2 will be called the Pauli-Fierz Hamiltonian.

We will need also the following assumption:

**Assumption B.** $h^{-1/2}v(1+K)^{-1/2}$ *is compact.*

Note that Assumption B implies A. In fact, Assumption B implies $\lim_{t\to\infty} \|h^{-1/2}v(t+K)^{-1/2}\| = 0$. Hence, Assumption B can be used in Theorem 2.2.

In some of our arguments we will need to use a whole family of Pauli-Fierz operators. Each time we will keep $K$ and $h$ fixed, and we will vary the coupling form $v$. In such situations the following assumption will be often helpful:

**Assumption C.**

$$\|h^{-1/2}v(t+K)^{-1/2}\| < \epsilon.$$

We will refer to the above assumption as Assumption C$(\epsilon, t)$.

The proof of the KLMN theorem and of Theorem 2.2 yield to the following.

**Proposition 2.1.** *Fix $t > 0$ and $\epsilon < 1$. Then there exist $c > 0$ and $c_1$ with the following property: if the coupling operator $v$ satisfies Assumption C($\epsilon, t$), then $H + c > 1$ and*

$$\|(c+H)^{-1/2}(c+H_{\rm fr})^{1/2}\| \le c_1, \quad \|(c+H_{\rm fr})^{-1/2}(c+H)^{1/2}\| \le c_1. \tag{2.9}$$

## 2.6 Main results

We now state our remaining assumptions and describe the main results of our paper.

The first assumption describes the confinement of the small system.

**Assumption D.** $(1+K)^{-1}$ *is compact.*

The one-particle space $\mathcal{Z}$ will be of the form $L^2(\mathbf{R}^d) \otimes \mathbf{C}^n$. To state our assumption on the operator $h$, we use the natural isomorphism between $L^2(\mathbf{R}^d) \otimes \mathbf{C}^n$ and $L^2(\mathbf{R}^d, \mathbf{C}^n)$.

**Assumption E.** *$h$ is the multiplication operator by a continuous function: $\mathbf{R}^d \ni \xi \mapsto h(\xi) \in \mathcal{B}(\mathbf{C}^n)$ such that $h(\xi)$ is selfadjoint positive for all $\xi$, $\nabla h \in L^\infty(\mathbf{R}^d, \mathcal{B}(\mathbf{C}^n))^d$ and $\lim_{|\xi|\to\infty} (\inf h(\xi)) = +\infty$.*

Let us denote by $x$ the operator on $\mathcal{Z}$ equal to $x = -i\nabla_\xi$. Then, using Assumption E, one easily sees that for any $r > 0$, $q > 0$, the operator $1_{[-r,r]}(|x|)1_{[0,q]}(h)$ is compact. Moreover, let $f, g$ be bounded measurable functions on $\mathbf{R}$ such that $\lim_{|t|\to\infty} f(t) = 0$, $\lim_{t\to+\infty} g(t) = 0$, then $f(|x|)g(h)$ is compact.

Our first result concerns the essential spectrum of $H$.

**Theorem 2.3.** *Suppose Assumptions B,D,E are true. Then*

$$\mathrm{sp}_{\rm ess} H \ = [\inf H + \inf h, +\infty[.$$

This theorem says that the essential spectrum of $H$ starts at the distance $m$ (the mass of the bosons) to the right of the ground state energy. This theorem resembles the well known HVZ theorem describing the essential spectrum of $N$-body Schrödinger operators [19]. It is a generalization of a result of [5].

Theorem 2.3 implies the existence of a ground state if $\inf h > 0$. In our analysis of the case $\inf h = 0$, we will use the following assumptions:

**Assumption F.** *$h^{-1}v(1+K)^{-1/2}$ is compact.*

**Assumption G.** *$v$ can be split as*

$$v = 1_{\mathcal{K}} \otimes |z) + v_{\mathrm{ren}},$$

*where $z \in \mathrm{Dom}(h^{-1/2})$ and $h^{-1}v_{\mathrm{ren}}(1+K)^{-1/2}$ is bounded.*

Clearly, in the case $\inf h > 0$, Assumption B implies Assumption F.

Our next result concerns the existence of a ground state, it is a generalization of a result of [12].

**Theorem 2.4.** *Suppose that Assumptions B, D, E, F, are satisfied. Then $\inf H \in \mathrm{sp}_{\mathrm{pp}}(H)$. In other words, $H$ has a ground state.*

This theorem gives sufficient conditions for the existence of a ground state. Finally, we also prove the following result which, in the particular case where $v$ can be split as in Assumption G, gives a necessary condition for the existence of a ground state. This is a generalization of a result of [7].

**Theorem 2.5.** *Assume $v$ satisfies Assumption A and G. If $H$ has a ground state, then $z \in \mathrm{Dom}(h^{-1})$.*

One can see this result as a sort of reciprocal of Theorem 2.4. Indeed, we have the following corollary.

**Corollary 2.1.** *Suppose Assumptions D and E are satisfied. Let $v = 1_{\mathcal{K}} \otimes |z) + v_{\mathrm{ren}}$ such that $v_{\mathrm{ren}}$ satisfies Assumptions B and F and $z \in \mathrm{Dom}(h^{-1/2})$. Then, $H$ has a ground state if and only if $z \in \mathrm{Dom}(h^{-1})$.*

**Proof.** If $H$ has a ground state, the result follows from Theorem 2.5. Assume now $z \in \mathrm{Dom}(h^{-1})$. It is then easy to see that $v$ satisfies Assumptions B and F. Therefore, using Theorem 2.4, $H$ has a ground state. $\square$

# 3 Proof of HVZ-type theorem

Our goal in this section is to prove Theorem 2.3.

## 3.1 Operators $\mathrm{d}\Gamma(\cdot;\cdot)$

In this section, we recall some notations from [5].

Let $q$ be a contraction from $\mathcal{Z}_1$ to $\mathcal{Z}_2$, $v$ a form from $\mathcal{K}$ to $\mathcal{K} \otimes \mathcal{Z}_1$. We assume that $\mathcal{K}_0$ is a subspace of $\mathcal{K}$, $\mathcal{Z}_{i0}$ are subspaces of $\mathcal{Z}_i$, $\mathrm{Dom}_{\mathrm{r}}(v) = \mathcal{K}_0$ and $\mathrm{Dom}_{\mathrm{l}}(v) = \mathcal{K}_0 \overset{\circ}{\otimes} \mathcal{Z}_{10}$. We also assume that $q$ maps $\mathcal{Z}_{10}$ into $\mathcal{Z}_{20}$. Let us note the identity in the sense of forms with the right domain $\mathcal{K}_0 \overset{\circ}{\otimes} \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z}_{10})$ and the left domain $\mathcal{K}_0 \overset{\circ}{\otimes} \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z}_{20})$:

$$\Gamma(q) a^{*\mathrm{f}}(v) = a^{*\mathrm{f}}(qv)\Gamma(q). \tag{3.10}$$

A similar identity, obtained from (3.10) by the hermitian conjugation, is also true:

$$\Gamma(q^*) a^{\mathrm{f}}(qv) = a^{\mathrm{f}}(v)\Gamma(q^*).$$

(We will drop the superscript f from $a^{\mathrm{f}}$ and $a^{*\mathrm{f}}$ in what follows).

If $b$ is an operator from $\mathcal{Z}_1$ to $\mathcal{Z}_2$, then we define $\mathrm{d}\Gamma(q;b)$ as an operator from $\Gamma_{\mathrm{s}}(\mathcal{Z}_1)$ to $\Gamma_{\mathrm{s}}(\mathcal{Z}_2)$, with the domain $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\, b)$, which on the $n$-particle sector equals

$$\mathrm{d}\Gamma_n(q;b) = \sum_{k=0}^{n-1} q^{\otimes k} \otimes b \otimes q^{\otimes(n-k-1)}.$$

If $b$ is closed, then the operator $\mathrm{d}\Gamma(q,b)$ is closable, and we will use the same symbol to denote its closure.

If $c$ is an operator on $\mathcal{Z}_1$ and $d$ on $\mathcal{Z}_2$, then we have the operator identities

$$\Gamma(q)\mathrm{d}\Gamma(c) = \mathrm{d}\Gamma(q;qc) \quad \text{on}\, \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\, c),$$

$$\mathrm{d}\Gamma(d)\Gamma(q) = \mathrm{d}\Gamma(q;dq) \quad \text{on}\, \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\, dq).$$

**Proposition 3.1.** *Assume that $\|q\| \leq 1$, $c$, $d$ are closed and $d^{-1}bc^{-1}$ is bounded. Then, for all $\Phi \in \mathrm{Dom}\,(\mathrm{d}\Gamma(dd^*)^{1/2})$ and $\Psi \in \mathrm{Dom}\,(\mathrm{d}\Gamma(c^*c)^{1/2})$,*

$$|(\Phi|\mathrm{d}\Gamma(q;b)\Psi)| \leq \|d^{-1}bc^{-1}\|\|\mathrm{d}\Gamma(dd^*)^{1/2}\Phi\|\|\mathrm{d}\Gamma(c^*c)^{1/2}\Psi\|.$$

**Proof.** Let $\Phi \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\,(d^*))$ and $\Psi \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\,(c))$. Then

$$\begin{aligned}
& |(\Phi|q^{\otimes(k-1)} \otimes b \otimes q^{\otimes(n-k)}\Psi)| \\
\leq\; & \|1^{\otimes(k-1)} \otimes d^* \otimes 1^{\otimes(n-k)}\Phi\|\|q^{\otimes(k-1)} \otimes d^{-1}bc^{-1} \otimes q^{\otimes(n-k)}\|\|1^{\otimes(k-1)} \otimes c \otimes 1^{\otimes(n-k)}\Psi\| \\
\leq\; & (\Phi|1^{\otimes(k-1)} \otimes dd^* \otimes 1^{\otimes(n-k)}\Phi)^{1/2}\|d^{-1}bc^{-1}\|(\Psi|1^{\otimes(k-1)} \otimes c^*c \otimes 1^{\otimes(n-k)}\Psi)^{1/2}.
\end{aligned}$$

Then we sum up over $k = 1, \dots, n$ and apply the Schwarz ineqality.

We easily show that $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\,(d^*))$ and $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\,(c))$ are dense in $\mathrm{Dom}\,(\mathrm{d}\Gamma(dd^*)^{1/2})$ and $\mathrm{Dom}\,(\mathrm{d}\Gamma(c^*c)^{1/2})$ in the graph norm. Hence, using the closedness of $\mathrm{d}\Gamma(q;b)$ we see that we can extend the inequality to $\Phi \in \mathrm{Dom}\,(\mathrm{d}\Gamma(dd^*)^{1/2})$ and $\Psi \in \mathrm{Dom}\,(\mathrm{d}\Gamma(c^*c)^{1/2})$. □

## 3.2 Extended Hilbert space

First let us fix more notation. If $\mathcal{K}, \mathcal{G}_1, \mathcal{G}_2$ are Hilbert spaces and $B_1 \in B(\mathcal{K}, \mathcal{G}_1)$, $B_2 \in B(\mathcal{K}, \mathcal{G}_2)$, then $(B_1, B_2)$ will denote the operator from $\mathcal{K}$ to $\mathcal{G}_1 \oplus \mathcal{G}_2$ defined by

$$\mathcal{K} \ni \Phi \mapsto (B_1, B_2)\Phi := (B_1\Phi, B_2\Phi) \in \mathcal{G}_1 \oplus \mathcal{G}_2.$$

Note that

$$\|(B_1, B_2)\| \leq (\|B_1\|^2 + \|B_2\|^2)^{1/2}.$$

Apart from the space

$$\mathcal{H} = \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}),$$

we will use the *extended space*

$$\mathcal{H}^{\mathrm{ext}} := \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \mathcal{Z}).$$

Note that we have the well known unitary identification (see e.g. [5])

$$U : \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \mathcal{Z}) \to \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}).$$

Thus we have a natural unitary identification

$$1 \otimes U \;:\; \mathcal{H}^{\mathrm{ext}} \to \mathcal{H} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}).$$

We introduce the extended free Pauli-Fierz Hamiltonian

$$H_{\mathrm{fr}}^{\mathrm{ext}} := K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h \oplus h).$$

We then introduce the extended coupling form $(v, 0)$, which is a form from $\mathcal{K}$ to $\mathcal{K}\otimes\mathcal{Z} \oplus \mathcal{K}\otimes\mathcal{Z} = \mathcal{K}\otimes(\mathcal{Z} \oplus \mathcal{Z})$. More precisely, $\mathrm{Dom}_{\mathrm{l}}(v,0) := \mathrm{Dom}_{\mathrm{l}} v \oplus \mathcal{K}\otimes\mathcal{Z}$, $\mathrm{Dom}_{\mathrm{r}}(v,0) = \mathrm{Dom}_{\mathrm{r}} v$, and $((\Psi,\Phi)|(v,0)\Xi) := (\Psi|v\Xi)$. Then, the extended interaction is defined as

$$V^{\mathrm{ext}} = a^*(v,0) + a(v,0).$$

The extended Pauli-Fierz Hamiltonian equals

$$H^{\mathrm{ext}} := H_{\mathrm{fr}}^{\mathrm{ext}} + V^{\mathrm{ext}}. \tag{3.11}$$

Note that

$$1\otimes U \; H^{\mathrm{ext}} \; 1\otimes U^* = H \otimes 1 + 1\otimes\mathrm{d}\Gamma(h).$$

Clearly, under Assumption A, $H^{\mathrm{ext}}$ is defined by the KLMN theorem and (3.11) and its form domain coincides with that of $H_{\mathrm{fr}}^{\mathrm{ext}}$.

## 3.3 Comparing the Hamiltonian with the extended Hamiltonian

Let $j_0, j_\infty$ be operators on $\mathcal{Z}$ such that $j_0^* j_0 + j_\infty^* j_\infty = 1$. Then the operator $(j_0, j_\infty) \in B(\mathcal{Z}, \mathcal{Z} \oplus \mathcal{Z})$ is isometric.

For simplicity, let us assume that $j_0$ and $j_\infty$ preserve $\mathcal{Z}_0$. Let us note the identities

$$\begin{aligned}
&\Gamma(j_0, j_\infty) a^*(v) = a^*(j_0 v, j_\infty v)\Gamma(j_0, j_\infty),\\
&\Gamma(j_0, j_\infty) a(j_0^* v) = a(v, 0)\Gamma(j_0, j_\infty),\\
&\Gamma(j_0, j_\infty)\mathrm{d}\Gamma(h) = \mathrm{d}\Gamma\left((j_0, j_\infty); (j_0 h, j_\infty h)\right),\\
&\mathrm{d}\Gamma(h \oplus h)\Gamma(j_0, j_\infty) = \mathrm{d}\Gamma\left((j_0, j_\infty); (h j_0, h j_\infty)\right).
\end{aligned}$$

The first two identities should be understood as forms identities with the right domain $\mathcal{K}_0 \mathring{\otimes} \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}_0)$, and the left domain domain $\mathcal{K}_0 \mathring{\otimes} \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}_0 \oplus \mathcal{Z}_0)$. The third and fourth are interpreted as operator identities on $\mathring{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\, h)$ and $\mathring{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\,(h j_0) \oplus \mathrm{Dom}\,(h j_\infty))$ respectively.

**Lemma 3.1.** *Let $f \in C_0^\infty(\mathbf{R})$. Then there exists $c$, which does not depend on $j_0, j_\infty$, such that*

$$\begin{aligned}
\|f(H^{\mathrm{ext}})\Gamma(j_0, j_\infty) - \Gamma(j_0, j_\infty) f(H)\| \leq \; & c\Big(\|h^{-1/2}(j_0^* - 1)v(1+K)^{-1/2}\| + \|h^{-1/2}(j_0 - 1)v(1+K)^{-1/2}\|\\
& + \|h^{-1/2} j_\infty v(1+K)^{-1/2}\|\\
& + \|h^{-1/2}[j_0, h]h^{-1/2}\| + \|h^{-1/2}[j_\infty, h]h^{-1/2}\|\Big).
\end{aligned}$$

**Proof.** The extended Hamiltonian satisfies

$$\begin{aligned}
&\Gamma(j_0, j_\infty) H - H^{\mathrm{ext}}\Gamma(j_0, j_\infty) = R_1 + R_2 + R_3,\\
&(z - H^{\mathrm{ext}})^{-1}\Gamma(j_0, j_\infty) - \Gamma(j_0, j_\infty)(z - H)^{-1} = -(z - H^{\mathrm{ext}})^{-1}(R_1 + R_2 + R_3)(z - H)^{-1},
\end{aligned}$$

where

$$R_1 := \Gamma(j_0, j_\infty) a((1 - j_0^*)v),$$

$$R_2 := a^*\left((j_0 - 1)v, j_\infty v\right) \Gamma(j_0, j_\infty),$$

$$R_3 := \mathrm{d}\Gamma\left((j_0, j_\infty); ([j_0, h], [j_\infty, h])\right).$$

For $\Phi \in \mathrm{Dom}\, |H_{\mathrm{fr}}^{\mathrm{ext}}|^{\frac{1}{2}}$ and $\Psi \in \mathrm{Dom}\, |H_{\mathrm{fr}}|^{\frac{1}{2}}$, we have the estimates

$$\begin{aligned}
|(\Phi|R_1\Psi)| \leq\ & \|(1+K)^{1/2}\Phi\| \|\mathrm{d}\Gamma(h)^{1/2}\Psi\| \|h^{-1/2}(j_0^* - 1)v(1+K)^{-1/2}\|, \\
|(\Phi|R_2\Psi)| \leq\ & \|\mathrm{d}\Gamma(h \oplus h)^{1/2}\Phi\| \|(1+K)^{1/2}\Psi\| \\
& \times \left(\|h^{-1/2}(j_0 - 1)v(1+K)^{-1/2}\|^2 + \|h^{-1/2} j_\infty v(1+K)^{-1/2}\|^2\right)^{1/2}, \\
|(\Phi|R_3\Psi)| \leq\ & \|\mathrm{d}\Gamma(h \oplus h)^{1/2}\Phi\| \|\mathrm{d}\Gamma(h)^{1/2}\Psi\| \\
& \times \left(\|h^{-1/2}[j_0, h]h^{-1/2}\|^2 + \|h^{-1/2}[j_\infty, h]h^{-1/2}\|^2\right)^{1/2}.
\end{aligned}$$

To convert the estimate on the resolvent of $H$ to an estimate on $f(H)$ we can use e.g. the well-known method of almost analytic extensions [16] (see also [4]). Let us take an almost analytic extension $\tilde{f} \in C_0^\infty(\mathbf{C})$ of $f$. Then we can write

$$f(H) = (2\pi)^{-1} \int \partial_{\overline{z}} \tilde{f}(z)(z - H)^{-1} \mathrm{d}z \mathrm{d}\overline{z},$$

and similarly for $H^{\mathrm{ext}}$. Thus

$$f(H^{\mathrm{ext}})\Gamma(j_0, j_\infty) - \Gamma(j_0, j_\infty) f(H) = -(2\pi)^{-1} \int \partial_{\overline{z}} \tilde{f}(z)(z - H^{\mathrm{ext}})^{-1}(R_1 + R_2 + R_3)(z - H)^{-1} \mathrm{d}z \mathrm{d}\overline{z}.$$

Then we use $|\partial_{\overline{z}} \tilde{f}(z)| \leq C_N |\mathrm{Im} z|^N$,

$$\|(z - H)^{-1}(c + H)^{1/2}\| \leq c_1(|\mathrm{Im} z|^{-1} + |\mathrm{Im} z|^{-1/2}),$$

and a similar estimate for $H^{\mathrm{ext}}$. □

### 3.4 Localizing in the configuration space

**Lemma 3.2.** *Suppose Assumption E holds and* $\inf h > 0$. *Then, there exists* $C > 0$ *such that, for all* $\xi$,

$$\|h^{1/2}(\xi - \eta) - h^{1/2}(\xi)\| \leq C \|\nabla h\|_\infty |\eta|,$$

*where* $|\eta|$ *denotes the Euclidean norm of* $\eta$ *and*

$$\|\nabla_\infty h\| := \sup\{\|\nabla h(\xi)\|_{\mathcal{B}(\mathbf{R}^d \otimes \mathbf{C}^n, \mathbf{C}^n)} \ :\ \xi \in \mathbf{R}^d\}.$$

**Proof.** If $A$ is a positive self-adjoint operator, then we have

$$A^{1/2} = \frac{1}{\pi} \int_0^{+\infty} \frac{A}{t + A} \frac{\mathrm{d}t}{\sqrt{t}}.$$

Thus,

$$\begin{aligned}
\|h^{1/2}(\xi-\eta)-h^{1/2}(\xi)\| &\le \frac{1}{\pi}\int_0^{+\infty}\frac{\mathrm{d}t}{\sqrt{t}}\left\|\frac{h(\xi-\eta)}{t+h(\xi-\eta)}-\frac{h(\xi)}{t+h(\xi)}\right\| \\
&\le \frac{1}{\pi}\int_0^{+\infty}\mathrm{d}t\sqrt{t}\left\|\frac{1}{t+h(\xi-\eta)}\right\|\|h(\xi-\eta)-h(\xi)\|\left\|\frac{1}{t+h(\xi)}\right\| \\
&\le \frac{1}{\pi}\int_0^{+\infty}\mathrm{d}t\frac{\sqrt{t}}{(t+m)^2}\|\nabla h\|_\infty|\eta|.
\end{aligned}$$

□

Recall that $x$ is an auxiliary operator that appears in Assumption E.

**Proposition 3.2.** *Suppose Assumption E holds and* $\inf h>0$. *Let $g$ be a measurable function on* $\mathbf{R}^d$ *with* $\int|\eta\hat{g}(\eta)|\mathrm{d}\eta<\infty$. *Then*

(1) $\|h^{1/2}g(x)h^{-1/2}-g(x)\|\le c\int|\eta\hat{g}(\eta)|\mathrm{d}\eta$.

(2) $\|h^{-1/2}[g(x),h]h^{-1/2}\|\le 2c\int|\eta\hat{g}(\eta)|\mathrm{d}\eta$.

**Proof.** If we write $g(x)=(2\pi)^{-\frac{d}{2}}\int \mathrm{e}^{\mathrm{i}\eta\cdot x}\hat{g}(\eta)\mathrm{d}\eta$, then

$$\begin{aligned}
h^{1/2}g(x)h^{-1/2}-g(x) &= (2\pi)^{-\frac{d}{2}}\int \mathrm{e}^{\mathrm{i}\eta\cdot x}\big(\mathrm{e}^{-\mathrm{i}\eta\cdot x}h^{1/2}\mathrm{e}^{\mathrm{i}\eta\cdot x}h^{-1/2}-1\big)\hat{g}(\eta)\mathrm{d}\eta \\
&= (2\pi)^{-\frac{d}{2}}\int \mathrm{e}^{\mathrm{i}\eta\cdot x}\big(h^{1/2}(\xi-\eta)h^{-1/2}(\xi)-1\big)\hat{g}(\eta)\mathrm{d}\eta \\
&= (2\pi)^{-\frac{d}{2}}\int \mathrm{e}^{\mathrm{i}\eta\cdot x}\big(h^{1/2}(\xi-\eta)-h^{1/2}(\xi)\big)h^{-1/2}(\xi)\hat{g}(\eta)\mathrm{d}\eta
\end{aligned}$$

Using Lemma 3.2, the norm of this can be bounded by

$$(2\pi)^{-\frac{d}{2}}\int \mathrm{d}\eta\|\nabla h\|_\infty\|h^{-1/2}\|_\infty|\eta\hat{g}(\eta)|.$$

Using Assumption E and $\|h^{-1/2}\|_\infty<\infty$, this proves (1).
Now, (2) follows from (1) and the following identity

$$h^{-1/2}[g(x),h]h^{-1/2}=h^{-1/2}g(x)h^{1/2}-g(x)-h^{1/2}g(x)h^{-1/2}+g(x).$$

□

Assume now that $j_0\in C_0^\infty(\mathbf{R})$, $j_\infty\in C^\infty(\mathbf{R})$ are positive functions satisfying $j_0^2+j_\infty^2=1$ and $j_0=1$ on a neighborhood of 0. For $r>0$ we define the operators on $\mathcal{Z}$

$$j_0^r:=j_0(|x|/r),\quad j_\infty^r:=j_\infty(|x|/r).$$

**Lemma 3.3.** *Suppose Assumptions B, E, are satisfied, and* $\inf h>0$. *Let* $f\in C_0^\infty(\mathbf{R})$. *Then as* $r\to\infty$

$$f(H^{\mathrm{ext}})\Gamma(j_0^r,j_\infty^r)-\Gamma(j_0^r,j_\infty^r)f(H)=o(r^0).$$

**Proof.** Note that $\widehat{j_0^r}(t)=r\hat{j}_0(rt)$, $\widehat{j_\infty^r}(t)=r\hat{j}_\infty(rt)$. Therefore,

$$\int|\widehat{j_0^r}(t)t|\mathrm{d}t=c_0/r,\quad \int|\widehat{j_\infty^r}(t)t|\mathrm{d}t=c_\infty/r.$$

Thus, using Proposition 3.2 (2)

$$h^{-1/2}[j_\infty^r, h]h^{-1/2} = O(r^{-1}), \quad h^{-1/2}[j_0^r, h]h^{-1/2} = O(r^{-1}).$$

Next,

$$\begin{aligned} h^{-1/2}(j_0^r - 1)v(1+K)^{-1/2} = \; & (h^{-1/2}(j_0^r - 1)h^{1/2} - j_0^r + 1)h^{-1/2}v(1+K)^{-1/2} \\ & + (j_0^r - 1)h^{-1/2}v(1+K)^{-1/2}. \end{aligned} \tag{3.12}$$

Using Proposition 3.2 (1), one sees that the first term on the right is $O(r^{-1})$, and by Assumption B the second is $o(r^0)$. Therefore (3.12) is $o(r^0)$. A similar argument shows that $h^{-1/2}((j_0^r)^* - 1)v(1+K)^{-1/2}$ and $h^{-1/2}j_\infty^r v(1+K)^{-1/2}$ are $o(r^0)$. □

**Lemma 3.4.** *Supppose Assumptions D and E hold. Suppose* $\inf h > 0$. *Let* $g \in C_0^\infty(\mathbf{R})$ *such that* $|g| \leq 1$. *Then* $\Gamma(g(|x|))(1+H_{\rm fr})^{-1/2}$ *is compact.*

**Proof.** We know that $(1+K)^{-1/2}$ is compact. Hence for any $\epsilon > 0$, we can find a finite dimensional projection $P$ commuting with $K$ such that $\|(1+K)^{-1/2}(1-P)\| \leq \epsilon$. Now

$$\begin{aligned} \Gamma(g(|x|))(1+H_{\rm fr})^{-1/2} = \; & \left(P \otimes \Gamma(g(|x|))(1+\mathrm{d}\Gamma(h))^{-1/2}\right) \left(1 \otimes (1+\mathrm{d}\Gamma(h))^{1/2}\right) (1+H_{\rm fr})^{-1/2} \\ & + (1 \otimes \Gamma(g(|x|))) \; (1+H_{\rm fr})^{-1/2} \; \left((1+K)^{1/2} \; (1+K)^{-1/2}(1-P) \otimes 1\right). \end{aligned}$$

By Assumption E and because $\inf h > 0$, $\Gamma(g(|x|))(1+\mathrm{d}\Gamma(h))^{-1/2}$ is a compact operator on $\Gamma_{\rm s}(\mathcal{Z})$. Moreover $P$ is a compact operator on $\mathcal{K}$. Hence the first term on the right is compact. The second term is less than $c\epsilon$. □

### 3.5 Proof of theorem 2.3

**Proof of Theorem 2.3.** We first prove that ${\rm sp}_{\rm ess} H \subset [\inf H + \inf h, +\infty[$. It is enough to assume that $\inf h = \inf h(\xi) > 0$. Let $f \in C_0^\infty(\mathbf{R})$, ${\rm supp} f \subset ]-\infty, \inf H + \inf h[$. We prove that $f(H)$ is compact. Note that because of the support of $f$, $f(H^{\rm ext}) = \Gamma(1 \oplus 0)f(H^{\rm ext})$. Moreover, $\Gamma(j_0^r, j_\infty^r)^*\Gamma(1\oplus 0) = \Gamma(j_0^r, 0)^*$. Now, since $\inf h > 0$, we can apply Lemma 3.3. Therefore,

$$\begin{aligned} f(H) \quad & = \quad \Gamma(j_0^r, j_\infty^r)^*\Gamma(j_0^r, j_\infty^r)f(H) \\ & \stackrel{o(r^0)}{=} \quad \Gamma(j_0^r, j_\infty^r)^* f(H^{\rm ext})\Gamma(j_0^r, j_\infty^r) \\ & = \quad \Gamma(j_0^r, 0)^* f(H^{\rm ext})\Gamma(j_0^r, j_\infty^r) \\ & \stackrel{o(r^0)}{=} \quad \Gamma\left((j_0^r)^2\right) f(H), \end{aligned}$$

where $\stackrel{o(r^0)}{=}$ means that the equality holds up to a $o(r^0)$ term. Finally, by Lemma 3.4, the right hand side is compact.

We now prove that $[\inf H + \inf h, +\infty[ \subset {\rm sp}_{\rm ess} H$. Let $E = \inf H$ and $\lambda > \inf h$. It is enough to prove that $E + \lambda \in {\rm sp} H$.

Let $0 < \epsilon < \lambda - \inf h$. Using Assumption E, one sees that $\lambda \in {\rm sp}_{\rm ess} h$. Therefore, we can find a sequence $z_n \in \mathcal{Z}$ such that $\|z_n\| = 1$, $z_n \to 0$ weakly and $1_{[\lambda-\epsilon, \lambda+\epsilon]}(h)z_n = z_n$. We can also find $\Phi \in \mathcal{H}$ such that $1_{[E, E+\epsilon]}(H)\Phi = \Phi$ and $\|\Phi\| = 1$

Using Lemma 2.1 with $v = 1 \otimes |z)$ we see that there exists $C > 0$ such that, for any $z \in \mathcal{Z}$,

$$\|(H+c)^{-1/2}a^*(z)\| \le C\|h^{-1/2}z\|.$$

We consider vectors of the form $a^*(z_n)\Phi$. One has

$$\begin{aligned}\left((H+c)^{-1} - (E+\lambda+c)^{-1}\right)a^*(z_n)\Phi &= (E+\lambda+c)^{-1}(H+c)^{-1}(H-E-\lambda)a^*(z_n)\Phi \\ &= (E+\lambda+c)^{-1}(H+c)^{-1}a^*(hz_n - \lambda z_n)\Phi \\ &\quad +(E+\lambda+c)^{-1}(H+c)^{-1}v^*1_{\mathcal{K}}\otimes|z_n)\Phi \\ &\quad +(E+\lambda+c)^{-1}(H+c)^{-1}a^*(z_n)(H-E)\Phi \\ &= I + II + III.\end{aligned}$$

From now on, we will denote by the letter $C$ any constant which does not depend on $n$ and $\epsilon$. Now

$$\begin{aligned}\|I\| &\le (E+\lambda+c)^{-1}\|(H+c)^{-1}a^*((h-\lambda)z_n)\Phi\| \\ &\le C\|h^{-1/2}(h-\lambda)z_n\| \le C\epsilon.\end{aligned}$$

$$\begin{aligned}\|III\| &\le (E+\lambda+c)^{-1}\|(H+c)^{-1}a^*(z_n)(H-E)\Phi\| \\ &\le C\|h^{-1/2}z_n\|\epsilon \le C\epsilon.\end{aligned}$$

$$\begin{aligned}\|II\| &\le (E+\lambda+c)^{-1}\|(H+c)^{-1}v^*1\otimes|z_n)\| \\ &\le C\|(1+K)^{-1/2}v^*z_n\| = C\|(1+K)^{-1/2}v^*h^{-1/2}h^{1/2}1_{[\lambda-\epsilon,\lambda+\epsilon]}(h)z_n\| \to 0,\end{aligned}$$

where we used Assumption B and the weak convergence of $h^{1/2}1_{[\lambda-\epsilon,\lambda+\epsilon]}(h)z_n$ to 0. Thus

$$\limsup_{n\to\infty}\left\|\left((H+c)^{-1} - (E+\lambda+c)^{-1}\right)a^*(z_n)\Phi\right\| \le C\epsilon.$$

Now, using that $\Phi \in \mathrm{Dom}\,(H_{\mathrm{fr}}+c)^{1/2}$ one has

$$\|a^*(z_n)\Phi\|^2 = \|z_n\|^2\|\Phi\|^2 + \|a(z_n)\Phi\|^2 \to 1$$

Thus choosing $n$ large enough and setting $\Phi_\epsilon := a^*(z_n)\Phi/\|a^*(z_n)\Phi\|$ we obtain a family of vectors satisfying

$$\left\|\left((H+c)^{-1} - (E+\lambda+c)^{-1}\right)\Phi_\epsilon\right\| \le C\epsilon, \quad \|\Phi_\epsilon\| = 1.$$

for $C$ independent of $\epsilon$. This implies that $E+\lambda \in \mathrm{sp}H$. □

# 4 Existence of ground states

In this section we will prove Theorem 2.4 and 2.5. For that purpose, we first introduce the Pullthrough Operators and study some of their properties.

## 4.1 Pullthrough Operators

Let $\mathcal{Z}$ be a Hilbert space. Note that vectors of the form $z^{\otimes n}$ span $\overset{\circ}{\Gamma}{}_{\mathrm{s}}^{n}(\mathcal{Z})$. Using this it is easy to see that there exists a unique linear operator

$$A : \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z}) \to \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z})\overset{\circ}{\otimes}\mathcal{Z}$$

satisfying for $z \in \mathcal{Z}$ the following condition

$$Az^{\otimes n} = \sqrt{n} z^{\otimes (n-1)} \otimes z.$$

The operator $A$ extends to a unique closed operator from $\Gamma_{\mathrm{s}}(\mathcal{Z})$ to $\Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \mathcal{Z}$ also denoted by $A$. The operator $A$ will be called the Annihilation Pullthrough Operator. It is easy to see that, for $z_1, \dots, z_n \in \mathcal{Z}$,

$$A \; z_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} z_n = n^{-1/2} \sum_{j=1}^{n} (z_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} z_{j-1} \otimes_{\mathrm{s}} z_{j+1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} z_n) \otimes z_j. \tag{4.13}$$

$A^*$, called the Creation Pullthrough Operator, satisfies

$$A^* \;\; (z_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} z_n) \otimes z = a^*(z) \; z_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} z_n = \sqrt{n+1} z \otimes_{\mathrm{s}} z_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} z_n.$$

Let us list basic properties of Pullthrough Operators.

**Lemma 4.1.** *Let $z \in \mathcal{Z}$, and let $b$ be an operator on $\mathcal{Z}$. We have the following identities on $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z})$ or on $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\; b)$:*

(1) $A \; a^*(z) - a^*(z) \otimes 1 \; A = 1 \otimes |z)$,

(2) $A \; a(z) - a(z) \otimes 1 \; A = 0$,

(3) $A \; \mathrm{d}\Gamma(b) - \mathrm{d}\Gamma(b) \otimes 1 \; A = 1 \otimes b \; A$,

(4) $A^* \;\; 1 \otimes b \; A = \mathrm{d}\Gamma(b)$.

(5) $A^* A = N$.

It is easy to prove the above lemma directly. It will also follow from the identities that we give further on in Lemma 4.2.

It is useful to note the relationship between the Pullthrough Annihlation Operator and the Scattering Identification Operator introduced in [5, 6]. Recall that the Scattering Identification Operator $I : \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}) \mapsto \Gamma_{\mathrm{s}}(\mathcal{Z})$ is defined as follows:

$$I \; \Phi \otimes \Psi = \sqrt{\frac{(p+q)!}{p!q!}} \Phi \otimes_{\mathrm{s}} \Psi, \quad \Phi \in \Gamma_{\mathrm{s}}^{p}(\mathcal{Z}), \, \Psi \in \Gamma_{\mathrm{s}}^{q}(\mathcal{Z}).$$

Another formula defining $I$ is

$$I := \Gamma(i) U^{-1},$$

where $U : \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \mathcal{Z}) \mapsto \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$ is the unitary identification introduced in Section 3.2 and

$$\begin{aligned} i : \quad & \mathcal{Z} \oplus \mathcal{Z} \to \mathcal{Z} \\ & (z_1, z_2) \mapsto z_1 + z_2. \end{aligned}$$

Note that $\|i\| = \sqrt{2}$, therefore $I$ is unbounded.

For all $n \in \mathbf{N}$, we define $A^{(n)} : \Gamma_{\mathrm{s}}(\mathcal{Z}) \to \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s}}^{n}(\mathcal{Z})$, by

$$A^{(n)} := (1 \otimes P_n) \, I^*, \tag{4.14}$$

where $P_n$ is the orthogonal projection of $\Gamma_{\mathrm{s}}(\mathcal{Z})$ onto $\Gamma_{\mathrm{s}}^{n}(\mathcal{Z})$. We call $A^{(n)}$ the $n^{\mathrm{th}}$ Annihilation Pullthrough Operator. Note that $\mathrm{Dom}\, A^{(n)} = \mathrm{Dom}\, N^{n/2}$ and $A = A^{(1)}$.

The following identities about the scattering identification operator $I$ follow from standard properties of $\mathrm{d}\Gamma$ and $U$ [5]. They easily imply Lemma 4.1.

**Lemma 4.2.** (1) *Let $b$ be an operator on $\mathcal{Z}$, then on $\mathring{\Gamma}_{\rm s}(\mathrm{Dom}\,(b))\mathring{\otimes}\mathring{\Gamma}_{\rm s}(\mathrm{Dom}\,(b))$*

$$\mathrm{d}\Gamma(b)I = I\big(\mathrm{d}\Gamma(b)\otimes 1 + 1\otimes \mathrm{d}\Gamma(b)\big).$$

(2) *For $z\in\mathcal{Z}$, on $\mathring{\Gamma}_{\rm s}(\mathcal{Z})\mathring{\otimes}\mathring{\Gamma}_{\rm s}(\mathcal{Z})$ we have*

$$\begin{array}{rcl} a(z)I & = & I\big(a(z)\otimes 1 + 1\otimes a(z)\big), \\ a^*(z)I & = & I\big(a^*(z)\otimes 1\big). \end{array}$$

The following proposition describes the relation between the different Pullthrough Operators.

**Proposition 4.1.**

$$A^{(n)} = \frac{1}{\sqrt{n!}}\big(A\otimes 1_{\Gamma_{\rm s}^{n-1}(\mathcal{Z})}\big)\cdots(A\otimes 1_{\mathcal{Z}})\ A.$$

**Proof.** On $\Gamma_s^m(\mathcal{Z})$, we have

$$\begin{array}{rcl} A^{(n)}z_1\otimes_{\rm s}\cdots\otimes_{\rm s}z_m & = & (1\otimes P_n)\,I^*z_1\otimes_{\rm s}\cdots\otimes_{\rm s}z_m \\ & = & (1\otimes P_n)\displaystyle\sum_{k=0}^{m}\sqrt{\frac{(m-k)!k!}{m!}}\sum\big(z_{i_1}\otimes_{\rm s}\cdots\otimes_{\rm s}z_{i_{m-k}}\big)\otimes\big(z_{i_{m-k+1}}\otimes_{\rm s}\cdots\otimes_{\rm s}z_{i_m}\big) \\ & = & \displaystyle\sqrt{\frac{(m-n)!n!}{m!}}\sum\big(z_{i_1}\otimes_{\rm s}\cdots\otimes_{\rm s}z_{i_{m-n}}\big)\otimes\big(z_{i_{m-n+1}}\otimes_{\rm s}\cdots\otimes_{\rm s}z_{i_m}\big)\,, \end{array}$$

where the sum is over the set of indices $\{i_1,\ldots,i_m\}$ such that $i_1<\cdots<i_{m-k}$, $i_{m-k+1}<\cdots<i_m$ and $\{i_1,\ldots,i_{m-k}\}\cap\{i_{m-k+1},\ldots,i_m\}=\emptyset$ in the second line, and over the same set with $k=n$ in the last line.

On the other hand, using (4.13), one has

$$\begin{array}{rl} & \big(A\otimes 1_{\Gamma_{\rm s}^{n-1}(\mathcal{Z})}\big)\cdots(A\otimes 1_{\mathcal{Z}})Az_1\otimes_{\rm s}\cdots\otimes_{\rm s}z_m \\ = & \displaystyle\sqrt{\frac{(m-n)!}{m!}}\sum_{\substack{i_1<\cdots<i_{m-n};\ i_{m-n+1}\neq\cdots\neq i_m \\ \{i_1,\ldots,i_{m-n}\}\cap\{i_{m-n+1},\ldots,i_m\}=\emptyset}}\big(z_{i_1}\otimes_{\rm s}\cdots\otimes_{\rm s}z_{i_{m-n}}\big)\otimes z_{i_{m-n+1}}\otimes\cdots\otimes z_{i_m} \\ = & \displaystyle n!\sqrt{\frac{(m-n)!}{m!}}\sum_{\substack{i_1<\cdots<i_{m-n};\ i_{m-n+1}<\cdots<i_m \\ \{i_1,\ldots,i_{m-n}\}\cap\{i_{m-n+1},\ldots,i_m\}=\emptyset}}\big(z_{i_1}\otimes_{\rm s}\cdots\otimes_{\rm s}z_{i_{m-n}}\big)\otimes\big(z_{i_{m-n+1}}\otimes_{\rm s}\cdots\otimes_{\rm s}z_{i_m}\big)\,. \end{array}$$

□

## 4.2 Pullthrough formula and its consequences

Let $v$ be a form from $\mathcal{K}$ to $\mathcal{K}\otimes\mathcal{Z}$ with the right domain $\mathcal{K}_0$ and the left domain $\mathcal{K}_0\mathring{\otimes}\mathcal{Z}_0$. We will identify it with a form from $\mathcal{H}=\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})$ to $\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})\otimes\mathcal{Z}$ in the obvious way (by tensoring it with $1_{\Gamma_{\rm s}(\mathcal{Z})}$ immediately to the right of $\mathcal{K}$). We will also write $A$ for the operator $1_{\mathcal{K}}\otimes A$, which is an operator from $\mathcal{H}$ to $\mathcal{H}\otimes\mathcal{Z}$. Let $H$ be defined as a quadratic form on $\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})$.

Clearly, as quadratic forms with the right domain $\mathcal{K}_0 \overset{\circ}{\otimes} \overset{\circ}{\Gamma}_{\rm s}(\mathcal{Z}_0)$ and the left domain $\mathcal{K}_0 \overset{\circ}{\otimes} \overset{\circ}{\Gamma}_{\rm s}(\mathcal{Z}_0) \overset{\circ}{\otimes} \mathcal{Z}_0$ we have the identities

$$A\ a^*(v) - a^*(v)\otimes 1\ A = v,$$

$$A\ a(v) - a(v)\otimes 1\ A = 0.$$

$$A\ H - (H\otimes 1 + 1\otimes h)A = v. \tag{4.15}$$

Under Assumption A, the operators $AH$, $(H\otimes 1+1\otimes h)A$ and $v$, that is all terms of the equation (4.15), can be extended to bounded operators from Dom $(H_{\rm fr}+c)^{1/2}$ to $(H_{\rm fr}+c)^{1/2}(1+N)^{1/2}\mathcal{H}\otimes(1+h)^{1/2}\mathcal{Z}$.

**Proposition 4.2.** *Let $H$ satisfy Assumption A. Let $H\Psi = E\Psi$. Then*

(1) *We have*

$$(H\otimes 1 + 1\otimes h - E)A\Psi = -v\Psi,$$

*as an identity in $(H_{\rm fr}+c)^{1/2}(1+N)^{1/2}\mathcal{H}\otimes(1+h)^{1/2}\mathcal{Z}$.*

(2) *Let $E = \inf H$ and let $\|h^{-1}v(K+1)^{-1/2}\| < \infty$. Then $\Psi \in$ Dom $A$ and*

$$A\Psi = -(H\otimes 1 + 1\otimes h - E)^{-1}v\Psi. \tag{4.16}$$

(3) *Fix $t$ and $\epsilon < 1$. Then there exists $c$ such that if Assumption $C(\epsilon, t)$ is satisfied, then*

$$\|N^{1/2}\Psi\| \leq c\|h^{-1}v(1+K)^{-1/2}\|\|\Psi\|. \tag{4.17}$$

**Proof.** Clearly, $\Psi \in$ Dom $(H_{\rm fr}+c)^{1/2}$. Therefore, we can apply (4.15) to $\Psi$. This yields (1).

Using the boundedness of $(H\otimes 1 + 1\otimes h - E)^{-1}1\otimes h$ we see that (1) implies (2).

Under Assumption C$(\epsilon, t)$ there exists $c_1$ such that $\|(1+K)^{1/2}\Psi\| \leq c_1\|\Psi\|$. Therefore, $A^*A = N$ and (2) imply (3). □

**Proposition 4.3.** *Fix $\epsilon$ and $t < 1$. Then there exists $c$ with the following properties. Let $v_1$, $v_2$ be coupling forms satisfying Assumption $C(\epsilon, t)$. Let $H_1$, $H_2$ be the corresponding Pauli-Fierz operators. and* $\inf H_1 = E_1$, $\inf H_2 = E_2$. *Suppose that $H_1$, $H_2$ have ground states. Then*

$$|E_1 - E_2| \leq c\|(v_1 - v_2)(1+K)^{-1/2}\| \max(\|h^{-1}v_1(1+K)^{-1/2}\|, \|h^{-1}v_2(1+K)^{-1/2}\|).$$

**Proof.** We suppose that $\Psi_1, \Psi_2 \in \mathcal{H}$ are normalized and satisfy $H_1\Psi_1 = E_1$, $H_2\Psi_2 = E_2\Psi_2$. It is enough to assume that $E_2 \geq E_1$. Then

$$\begin{aligned}
E_2 - E_1 &\leq (\Psi_1|(H_2 - H_1)\Psi_1) \\
&= 2\mathrm{Re}\,(\Psi_1|a(v_2 - v_1)\Psi_1) \\
&\leq 2\|(c_0 + H_1)^{1/2}\Psi_1\|\|(c_0 + H_1)^{-1/2}(1+K)^{1/2}\|\|(1+K)^{-1/2}a(v_2 - v_1)N^{-1/2}\|\|N^{1/2}\Psi_1\| \\
&\leq c\|(v_1 - v_2)(1+K)^{-1/2}\|\|h^{-1}v_1(1+K)^{-1/2}\|,
\end{aligned}$$

where at the last step we used Proposition 4.2. □

### 4.3 Double pullthrough formula

We will also need some identities related to the 2nd Annihilation Pullthrough Operator $A^{(2)} = \frac{1}{\sqrt{2}}(A\otimes 1)A$.

The Double Pullthrough Formula will involve operators that first act from $\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z})$ to $\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z})\otimes\mathcal{Z}$ and then to $\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z})\otimes\Gamma_{\mathrm{s}}^2(\mathcal{Z})\subset\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z})\otimes\mathcal{Z}\otimes\mathcal{Z}$. Let $\mathcal{Z}_{(1)}$ and $\mathcal{Z}_{(2)}$ denote the first and second copy of $\mathcal{Z}$ in the above tensor product. We denote by $v_{(1)}$ the form $v$ acting from $\mathcal{K}$ to $\mathcal{K}\otimes\mathcal{Z}_{(1)}$ tensored by $1_{\Gamma_{\mathrm{s}}(\mathcal{Z})}$ and $1_{\mathcal{Z}_{(2)}}$. Likewise, we denote by $v_{(2)}$ the form $v$ acting from $\mathcal{K}$ to $\mathcal{K}\otimes\mathcal{Z}_{(2)}$ tensored by $1_{\Gamma_{\mathrm{s}}(\mathcal{Z})}$ and $1_{\mathcal{Z}_{(1)}}$. $\mathrm{d}\Gamma^2(h)$ denotes the operator $\mathrm{d}\Gamma(h)$ restricted to the 2-particle space, that is $(h\otimes 1_{\mathcal{Z}_{(2)}}+1_{\mathcal{Z}_{(1)}}\otimes h)\big|_{\Gamma_{\mathrm{s}}^2(\mathcal{Z})}$.

After these lengthy preparations we can write the Double Pullthrough Formula for Pauli-Fierz Hamiltonians

$$(A\otimes 1)\ A\ H = (H\otimes 1+1\otimes\mathrm{d}\Gamma^2(h))\ (A\otimes 1)\ A + v_{(1)}A + v_{(2)}A.$$

This formula can be understood as a quadratic form with the right domain $\mathcal{K}_0\mathring{\otimes}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}_0)$ and the left domain $\mathcal{K}_0\mathring{\otimes}\mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z}_0)\mathring{\otimes}\mathring{\Gamma}_{\mathrm{s}}^2(\mathcal{Z}_0)$. Under Assumption A we can extend all terms of this formula by continuity to bounded operators from $\mathrm{Dom}\,(H_{\mathrm{fr}}+c)^{1/2}$ to $(H_{\mathrm{fr}}+c)^{1/2}(1+N)\mathcal{H}\otimes(1+\mathrm{d}\Gamma^2(h))^{1/2}\Gamma_{\mathrm{s}}^2(\mathcal{Z})$.

**Proposition 4.4.** *Let $H$ satisfy Assumption A and $H\Psi = E\Psi$.*

(1) *We have*

$$(H\otimes 1+1\otimes\mathrm{d}\Gamma^2(h)-E)\ (A\otimes 1)\ A\Psi = -(v_{(1)}A+v_{(2)}A)\Psi,$$

*understood as an identity in $(H_{\mathrm{fr}}+c)^{1/2}(1+N)\mathcal{H}\otimes(1+\mathrm{d}\Gamma^2(h))^{1/2}\Gamma_{\mathrm{s}}^2(\mathcal{Z})$.*

(2) *Let $E=\inf H$ and $\|h^{-1}v(K+1)^{-1/2}\|<\infty$. Then $(H\otimes 1+1\otimes h-E)^{-1}v\Psi\in\mathrm{Dom}\,A\otimes 1$ and*

$$\begin{aligned}(A\otimes 1)(H\otimes 1+1\otimes h-E)^{-1}v\Psi \;&=\; -(H\otimes 1+1\otimes\mathrm{d}\Gamma^2(h)-E)^{-1}v_{(1)}(H\otimes 1+1\otimes h-E)^{-1}v\Psi\\ &\quad -(H\otimes 1+1\otimes\mathrm{d}\Gamma^2(h)-E)^{-1}v_{(2)}(H\otimes 1+1\otimes h-E)^{-1}v\Psi.\end{aligned}$$

(3) *Fix $t$ and $\epsilon<1$. There exists $c$ such that if Assumption $C(\epsilon,t)$ is satisfied, then*

$$\|N^{1/2}\otimes 1\ (H\otimes 1+1\otimes h-E)^{-1}v\Psi\|\le c\|\max(h^{-1},h^{-1/2})v(1+K)^{-1/2}\|\|h^{-1}v(1+K)^{-1/2}\|\|\Psi\|,$$

$$\|(1+K)^{1/2}\otimes N^{1/2}\otimes 1\ (H\otimes 1+1\otimes h-E)^{-1}v\Psi\|\le c\|\max(h^{-1},h^{-1/2})v(1+K)^{-1/2}\|^2\|\Psi\|.$$

**Proof.** (1) and (2) are proven similarly as the corresponding statements of Proposition 4.2.

Let us prove the first estimate of (3). We will use the identity from (2). It clearly suffices to consider the first term of the right-hand side.

$$\begin{aligned}&(H\otimes 1+1\otimes\mathrm{d}\Gamma^2(h)-E)^{-1}v_{(1)}(H\otimes 1+1\otimes h-E)^{-1}v\Psi\\ &=(H\otimes 1+1\otimes\mathrm{d}\Gamma^2(h)-E)^{-1}\ 1\otimes h\otimes 1\\ &\quad\times 1\otimes h^{-1}\otimes 1\ v_{(1)}\ (1+K)^{-1/2}\otimes 1\\ &\quad\times(1+K)^{1/2}\otimes 1\ (H\otimes 1+1\otimes h-E)^{-1}\ 1\otimes\min(h,h^{1/2})\\ &\quad\times 1\otimes\max(h^{-1},h^{-1/2})\ v\ (1+K)^{-1/2}\\ &\quad\times(1+K)^{1/2}\Psi.\end{aligned}$$

Using Lemma 4.3 below, we see that the first four terms on the right are bounded. Besides, $\|(1+K)^{1/2}\Psi\|\le c\|\Psi\|$. This gives the first estimate of (3).

Similarly, to prove the second estimate we write

$$
\begin{aligned}
&(1+K)^{1/2}\otimes 1\ (H\otimes 1+1\otimes \mathrm{d}\Gamma^2(h)-E)^{-1}v_{(1)}(H\otimes 1+1\otimes h-E)^{-1}v\Psi\\
&=(1+K)^{1/2}\otimes 1\ (H\otimes 1+1\otimes \mathrm{d}\Gamma^2(h)-E)^{-1}\ 1\otimes \min(h,h^{1/2})\otimes 1\\
&\quad \times 1\otimes h\otimes 1\ v_{(1)}\ (1+K)^{-1/2}\otimes 1\\
&\quad \times (1+K)^{1/2}\otimes 1\ (H\otimes 1+1\otimes h-E)^{-1}\ 1\otimes \min(h,h^{1/2})\\
&\quad \times 1\otimes \max(h^{-1},h^{-1/2})\ v\ (1+K)^{-1/2}\\
&\quad \times (1+K)^{1/2}\Psi.
\end{aligned}
$$

□

The following easy lemma follows from the spectral theorem.

**Lemma 4.3.** *Let $\epsilon$, $t<1$ be fixed. Let $E:=\inf H$. Then there exists $c$ such that if Assumption C($\epsilon,t$) is satisfied, then*

$$\|(1+K)^{1/2}\otimes 1\ (H\otimes 1+1\otimes h-E)^{-1}\ 1\otimes \min(h,h^{1/2})\|\leq c.$$

**Proof.** Using $\min(h,h^{1/2})\leq h$ we get

$$
\begin{aligned}
&\|(1+K)^{1/2}\otimes 1\ (H\otimes 1+1\otimes h-E)^{-1}\ 1_{[E,E+1]}(H)\otimes \min(h,h^{1/2})\|\\
\leq\ &\|(1+K)^{1/2}1_{[E,E+1]}(H)\|\times \|(H\otimes 1+1\otimes h-E)^{-1}\ 1\otimes h\|.
\end{aligned}
$$

Using $\min(h,h^{1/2})\leq h^{1/2}$ we get

$$
\begin{aligned}
&\|(1+K)^{1/2}\otimes 1\ (H\otimes 1+1\otimes h-E)^{-1}\ 1_{[E+1,\infty[}(H)\otimes \min(h,h^{1/2})\|\\
\leq\ &\|(1+K)^{1/2}\otimes 1\ (H\otimes 1+1-E)^{-1/2}\|\\
&\times\|(H\otimes 1+1-E)^{1/2}(H\otimes 1+1\otimes h-E)^{-1/2}\ 1_{[E+1,\infty[}(H)\otimes 1\|\\
&\times\|(H\otimes 1+1\otimes h-E)^{-1/2}\ 1_{[E+1,\infty[}(H)\otimes h^{1/2}\|.
\end{aligned}
$$

□

## 4.4 Infrared cutoff Hamiltonian

Let us fix $v$ and the corresponding Pauli-Fierz operator $H$. Set $E:=\inf H$.

Let $f\in C^\infty(\mathbf{R}_+)$ be an increasing function such that $0\leq f\leq 1$, $f=0$ on $[0,1]$, and $f=1$ on $[2,\infty[$. For $\sigma\geq 0$, set

$$
\begin{aligned}
v_\sigma &:= f(h/\sigma)v,\\
V_\sigma &:= a^*(v_\sigma)+a(v_\sigma),\\
H_\sigma &:= H_{\mathrm{fr}}+V_\sigma,\\
E_\sigma &:= \inf H_\sigma.
\end{aligned}
$$

**Proposition 4.5.** (1) *Let $v$ satisfy Assumption A. There exist $\epsilon<1$ and $t$ such that $v_\sigma$ satisfies Assumption C($\epsilon,t$) and hence $H_\sigma$ is well defined for any $\sigma$.*

(2) *If Assumption B is true, then $H_\sigma$ converges to $H$ in the norm resolvent sense when $\sigma$ goes to zero and $\lim_{\sigma\to 0}\inf H_\sigma = \inf H$.*

(3) *Suppose Assumptions B,D,E are true. Then, for any $\sigma > 0$, $H_\sigma$ has a ground state.*

**Proof.** (1) is straightforward.

Let us prove (2). By (1) and Proposition 2.1, we can find $c$ such that $(H_\sigma + c)^{-1/2}(H+c)^{1/2}$ is bounded uniformly in $\sigma$. Now, we can write

$$\begin{aligned}(H_\sigma + c)^{-1} - (H+c)^{-1} &= (H_\sigma + c)^{-1}(H - H_\sigma)(H+c)^{-1} \\ &= (H_\sigma + c)^{-1}(H+c)^{1/2}(H+c)^{-1/2}(H-H_\sigma)(H+c)^{-1/2}(H+c)^{-1/2}.\end{aligned}$$

Thus, it suffices to show that $(H+c)^{-1/2}(H-H_\sigma)(H+c)^{-1/2}$ goes to zero in norm as $\sigma$ goes to zero.

Let $\Phi, \Psi \in \mathcal{H}$, then

$$\begin{aligned}&\left|(\Phi|(H+c)^{-1/2}(H_\sigma - H)(H+c)^{-1/2}\Psi)\right| \\ = \ &\left|((H+c)^{-1/2}\Phi|a^{\mathrm{f}}(v-v_\sigma)(H+c)^{-1/2}\Psi) + (a^{\mathrm{f}}(v-v_\sigma)(H+c)^{-1/2}\Phi|(H+c)^{-1/2}\Psi)\right| \\ \leq \ &\|(1+K)^{1/2}(H+c)^{-1/2}\Phi\|\|(1+K)^{-1/2}a(v-v_\sigma)(H+c)^{-1/2}\Psi\| \\ &+\|(1+K)^{1/2}(H+c)^{-1/2}\Psi\|\|(1+K)^{-1/2}a(v-v_\sigma)(H+c)^{-1/2}\Phi\| \\ \leq \ &\|(1+H_{\mathrm{fr}})^{1/2}(H+c)^{-1/2}\Phi\|\,\|h^{-1/2}(v-v_\sigma)(1+K)^{-1/2}\|\,\|\mathrm{d}\Gamma(h)^{1/2}(H+c)^{-1/2}\Psi\| \\ &+\|(1+H_{\mathrm{fr}})^{1/2}(H+c)^{-1/2}\Psi\|\,\|h^{-1/2}(v-v_\sigma)(1+K)^{-1/2}\|\,\|\mathrm{d}\Gamma(h)^{1/2}(H+c)^{-1/2}\Phi\| \\ \leq \ &C\|\Phi\|\,\|\Psi\|\,\|(1-f(h/\sigma))h^{-1/2}v(1+K)^{-1/2}\| \to 0.\end{aligned}$$

To prove (3) we use a well known trick, applied e.g. in [10], of replacing soft photons by massive photons. We introduce

$$h_\sigma(\xi) := h(\xi)f(2h(\xi)/\sigma) + \frac{\sigma}{2}\left(1 - f(2h(\xi)/\sigma)\right).$$

Then $v_\sigma$ and $h_\sigma$ satisfy Assumptions $\mathrm{C}(\epsilon, t)$, D and E. Moreover, $h_\sigma \geq \sigma/2$. Set

$$\tilde{H}_\sigma := K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h_\sigma) + V_\sigma.$$

By Theorem 2.3, $\tilde{H}_\sigma$ has a ground state.

We define $\mathcal{Z}_\sigma := 1_{[0,\sigma[}(h)\mathcal{Z}$ and $\mathcal{Z}^\sigma := 1_{[\sigma,+\infty[}(h)\mathcal{Z}$. Using the so-called exponential law of Fock spaces, one has that $\Gamma_{\mathrm{s}}(\mathcal{Z})$ is isomorphic to $\Gamma_{\mathrm{s}}(\mathcal{Z}^\sigma) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_\sigma)$. Using this identification, one can write

$$H_\sigma = 1 \otimes \mathrm{d}\Gamma(h 1_{[0,\sigma[}(h)) + H^\sigma \otimes 1$$

$$\tilde{H}_\sigma = 1 \otimes \mathrm{d}\Gamma(h_\sigma 1_{[0,\sigma[}(h)) + H^\sigma \otimes 1,$$

where $H^\sigma = K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h 1_{[\sigma,+\infty[}(h)) + V_\sigma$. Clearly, the ground state of $\tilde{H}_\sigma$ is of the form $\tilde{\Psi}_\sigma = \tilde{\Psi}^\sigma \otimes \Omega_\sigma$, where $\Omega_\sigma$ is the vacuum of $\Gamma_{\mathrm{s}}(\mathcal{Z}_\sigma)$. Hence, it is also a ground state of $H_\sigma$. □

Let us now extend Proposition 4.3 to the case where we do not assume the existence of a ground state.

**Proposition 4.6.** *Fix $\epsilon$ and $t < 1$. Suppose Assumptions B, D and E hold. Then there exists $c$ with the following properties. Let $v_1$, $v_2$ be coupling forms satisfying Assumption $C(\epsilon, t)$. Let $H_1$, $H_2$ be the corresponding Pauli-Fierz operators and* $\inf H_1 = E_1$, $\inf H_2 = E_2$. *Then*

$$|E_1 - E_2| \leq c\|(v_1 - v_2)(1+K)^{-1/2}\| \max(\|h^{-1}v_1(1+K)^{-1/2}\|, \|h^{-1}v_2(1+K)^{-1/2}\|).$$

**Proof.** By applying the cutoff procedure to $H_1$ and $H_2$ we can approximate them by $H_{1,\sigma}$ and $H_{2,\sigma}$, which possess ground states and ground state energies $E_{1,\sigma}$, $E_{2,\sigma}$. Then we use Proposition 4.3,

$$\lim_{\sigma \to 0} |E_{1,\sigma} - E_{2,\sigma}| = |E_1 - E_2|,$$

which follows from Proposition 4.5, and

$$\begin{aligned} \|(v_{1,\sigma} - v_{2,\sigma})(1+K)^{-1/2}\| &= \|f(h/\sigma)(v_1 - v_2)(1+K)^{-1/2}\| \\ &\nearrow \|(v_1 - v_2)(1+K)^{-1/2}\|, \end{aligned}$$

$$\begin{aligned} \|h^{-1}v_{i,\sigma}(1+K)^{-1/2}\| &= \|f(h/\sigma)h^{-1}v_i(1+K)^{-1/2}\| \\ &\nearrow \|h^{-1}v_i(1+K)^{-1/2}\|, \quad i = 1, 2. \end{aligned}$$

□

Take now, for any $\sigma > 0$, a normalized ground state $\Psi_\sigma$ of $H_\sigma$ , that means

$$H_\sigma \Psi_\sigma = E_\sigma \Psi_\sigma, \quad \|\Psi_\sigma\| = 1.$$

**Proposition 4.7.** *Suppose Assumptions B, D, E and F are true. Then*

(1) $E - E_\sigma = o(\sigma)$.

(2) $\lim\limits_{\sigma \to 0} \left(A\Psi_\sigma + (H \otimes 1 + 1 \otimes h - E)^{-1} v \Psi_\sigma\right) = 0$.

(3) *Let $b$ be a bounded positive operator on $\mathcal{Z}$. Then there exists a constant $c$ such that*

$$\limsup_{\sigma \to 0} (\Psi_\sigma | \mathrm{d}\Gamma(b)\Psi_\sigma) \leq c\|b^{1/2}h^{-1}v(1+K)^{-1/2}\|^2.$$

(4) $\lim\limits_{r \to \infty} \limsup\limits_{\sigma \to 0} (\Psi_\sigma | \mathrm{d}\Gamma(1_{[r,\infty[}(|x|))\Psi_\sigma) = 0$.

**Proof.** (1) By Proposition 4.6

$$\begin{aligned} |E - E_\sigma| &\leq c\|(1 - f(h/\sigma))v(1+K)^{-1/2}\| \|h^{-1}v(1+K)^{-1/2}\| \\ &\leq c\|(1 - f(h/\sigma))h\| \|1_{[0,2\sigma]}(h)h^{-1}v(1+K)^{-1/2}\| \|h^{-1}v(1+K)^{-1/2}\|. \end{aligned} \tag{4.18}$$

The first factor of (4.18) is $O(\sigma)$. By Assumption F, the second factor is $o(\sigma^0)$.

(2) Using (4.16), one has

$$\begin{aligned} &A\Psi_\sigma + (H \otimes 1 + 1 \otimes h - E)^{-1} v \Psi_\sigma \\ &= -(H_\sigma \otimes 1 + 1 \otimes h - E_\sigma)^{-1} v_\sigma \Psi_\sigma + (H \otimes 1 + 1 \otimes h - E)^{-1} v \Psi_\sigma \\ &= R_1 + R_2 + R_3 + R_4, \end{aligned}$$

where

$$
\begin{aligned}
R_1 &:= (H\otimes 1 + 1\otimes h - E)^{-1}(v - v_\sigma)\Psi_\sigma,\\
R_2 &:= (H\otimes 1 + 1\otimes h - E)^{-1}a^*(v - v_\sigma)(H_\sigma\otimes 1 + 1\otimes h - E_\sigma)^{-1}v_\sigma\Psi_\sigma,\\
R_3 &:= (H\otimes 1 + 1\otimes h - E)^{-1}a(v - v_\sigma)(H_\sigma\otimes 1 + 1\otimes h - E_\sigma)^{-1}v_\sigma\Psi_\sigma,\\
R_4 &:= (E_\sigma - E)(H\otimes 1 + 1\otimes h - E)^{-1}(H_\sigma\otimes 1 + 1\otimes h - E_\sigma)^{-1}v_\sigma\Psi_\sigma.
\end{aligned}
$$

Now, by Assumption F,

$$\|R_1\| \leq c\|h^{-1}(v - v_\sigma)(1+K)^{-1/2}\| \to 0.$$

Using Proposition 4.3,

$$\|R_4\| \leq |E - E_\sigma|\|h^{-2}v_\sigma(1+K)^{-1/2}\|\|(1+K)^{1/2}\Psi_\sigma\|.$$

Thus, $R_4 \to 0$ because $|E - E_\sigma| = o(\sigma)$, $\|h^{-2}v_\sigma(1+K)^{-1/2}\| = O(\sigma^{-1})$, by Assumption F.

Next,

$$
\begin{aligned}
R_2 &= (H\otimes 1 + 1\otimes h - E)^{-1}\ 1\otimes h\\
&\times 1\otimes 1_{[\sigma,\infty[}(h)h^{-1}\\
&\times a^*(v - v_\sigma)\ (1+K)^{-1/2}\otimes(1+N)^{-1/2}\otimes 1\\
&\times (1+K)^{1/2}\otimes(1+N)^{1/2}\otimes 1\ (H_\sigma\otimes 1 + 1\otimes h - E_\sigma)^{-1}v_\sigma\Psi_\sigma.
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
R_3 &= (H\otimes 1 + 1\otimes h - E)^{-1}\ (1+K)^{1/2}\otimes \min(h^{1/2}, h)\\
&\times 1\otimes 1_{[\sigma,\infty[}(h)\max(h^{-1/2}, h^{-1})\\
&\times (1+K)^{-1/2}a(v - v_\sigma)(1+N)^{-1/2}\ \otimes\ 1\\
&\times (1+N)^{1/2}\otimes 1\ (H_\sigma\otimes 1 + 1\otimes h - E_\sigma)^{-1}v_\sigma\Psi_\sigma.
\end{aligned}
\tag{4.20}
$$

The first terms of (4.19) and (4.20) are uniformly bounded, and using Proposition 4.4, so are their last terms. The second terms are bounded by $O(\sigma^{-1})$. The third terms are bounded by $\|(v - v_\sigma)(1+K)^{-1/2}\| = o(\sigma)$. Hence both (4.19) and (4.20) are $o(\sigma^0)$.

(3) Using first Lemma 4.1 and then (2), we obtain

$$
\begin{aligned}
(\Psi_\sigma|\mathrm{d}\Gamma(b)\Psi_\sigma) &= (A\Psi_\sigma|1\otimes b\ A\Psi_\sigma)\\
&\overset{o(\sigma^0)}{=} \big((H\otimes 1 + 1\otimes h - E)^{-1}v\Psi_\sigma|1\otimes b\ (H\otimes 1 + 1\otimes h - E)^{-1}v\Psi_\sigma\big)\\
&\leq (v\Psi_\sigma|1\otimes h^{-1}bh^{-1}\ v\Psi_\sigma)\\
&= \|b^{1/2}h^{-1}v\Psi_\sigma\|^2\\
&\leq c\|b^{1/2}h^{-1}v(1+K)^{-1/2}\|^2.
\end{aligned}
$$

Using Assumption F, we obtain

$$\lim_{r\to\infty}\|1_{[r,+\infty[}(|x|)h^{-1}v(1+K)^{-1/2}\| = 0.$$

Thus (4) follows from (3) by setting $b = 1_{[r,+\infty[}(|x|)$. □

The following easy lemma follows by the spectral theorem.

**Lemma 4.4.** *Let $B$ be a positive operator and $(\Psi|B\Psi) \leq c$. Then $\|1_{[r,\infty[}(B)\Psi\|^2 \leq c/r$.*

**Proof of Theorem 2.4.** Since the unit ball in any Hilbert space is weakly sequentially compact, we can find a sequence $\sigma_n \to 0$ such that $\Psi_n := \Psi_{\sigma_n}$ converges weakly to some $\Psi \in \mathcal{H}$. It is easy to check that $H\Psi = E\Psi$. It remains to prove that $\Psi \neq 0$.

Assume that $\Psi = 0$. We have $(\Psi_\sigma|H_{\rm fr}\Psi_\sigma) \leq c$, and, using (4.17), $(\Psi_\sigma|1\otimes N\ \Psi_\sigma) \leq c$. Hence, using Lemma 4.4, we see that for any $\epsilon > 0$, we can find $r$ such that

$$\|1_{[r,\infty[}(1\otimes N\ + H_{\rm fr})\Psi_n\| \leq \epsilon. \tag{4.21}$$

Using Proposition 4.7 we see that we can find $n_0$ and $r$ such that for $n > n_0$

$$(\Psi_n|{\rm d}\Gamma(1_{[r,+\infty[}(|x|))\Psi_n) < \epsilon.$$

Hence, using Lemma 4.4,

$$\|1_{[1/2,\infty[}\left({\rm d}\Gamma(1_{[r,\infty[}(|x|))\right)\Psi_n\| < 2\epsilon. \tag{4.22}$$

Thus, by (4.21) and (4.22),

$$\limsup_{n\to\infty}\|C\Psi_n - \Psi_n\| \leq 3\epsilon, \tag{4.23}$$

where

$$C := 1_{[0,1/2[}\left({\rm d}\Gamma(1_{[r,\infty[}(|x|))\right)1_{[0,r[}(1\otimes N\ + H_{\rm fr}).$$

Now, using Assumption D and E, we see that the operator $C$ is compact. Hence, using $\text{w}-\lim\limits_{n\to\infty}\Psi_n = 0$ we get

$$\lim_{n\to\infty}\|C\Psi_n\| = 0.$$

If we choose $\epsilon < 1/3$ in (4.23), this contradicts $\|\Psi_n\| = 1$. □

### 4.5 Proof of Theorem 2.5

Let $\Psi$ be a ground state. Using (4.15), we have

$$A\Psi = -(H\otimes 1 + 1\otimes h - E)^{-1}v\Psi$$

as an identity in $(N+1)^{1/2}\mathcal{H}\otimes h^{-1/2}\mathcal{Z}$. Using Assumption G, we thus have

$$A\Psi + \Psi\otimes h^{-1}z = -(H\otimes 1 + 1\otimes h - E)^{-1}v_{\rm ren}\Psi \in \mathcal{H}\otimes\mathcal{Z} \subset (N+1)^{1/2}\mathcal{H}\otimes\mathcal{Z}.$$

But $A\Psi \in (N+1)^{1/2}\mathcal{H}\otimes\mathcal{Z}$, thus the same is true for $\Psi\otimes h^{-1}z$, which proves that $z \in {\rm Dom}(h^{-1})$.

## 5 Comparison with the literature

In this section, we would like to compare our results with the analogous ones of the literature.

### 5.1 HVZ-type theorem: comparison with [5]

In this paper, the authors consider the following assumptions

$(H0)$ $\quad (K+1)^{-1}$ is compact,

$(H1)$ $\quad \mathcal{Z} = L^2(\mathbf{R}^d)$ and $h \in C(\mathbf{R}^d, \mathbf{R})$ satisfies $\begin{cases} \nabla h \in L^\infty, \\ \nabla h(\xi) \neq 0 \text{ for } \xi \neq 0, \\ \lim_{\xi\to\infty} h(\xi) = +\infty, \\ \inf h(\xi) = h(0) =: m > 0, \end{cases}$

$(I1)$ $\quad v \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathcal{Z})$,

and prove the following HVZ-type theorem.

**Theorem 5.1.** *Suppose* $(H0), (H1), (I1)$ *are satisfied, then*

$$\mathrm{sp}_{\mathrm{ess}} H = [\inf H + m, +\infty[.$$

The assumption on the dispersion relation $h$ is similar to ours except that we use $\mathcal{Z} = L^2(\mathbf{R}^d) \otimes \mathbf{C}^n$. This is the reason why we need Lemma 3.2.

The coupling function $v$ is assumed to be a bounded operator from $\mathcal{K}$ to $\mathcal{K} \otimes \mathcal{Z}$. In our paper, $v$ is a quadratic form, and it is not necessarily bounded. Moreover, in [5] it is considered only the case where $\inf h > 0$. In particular, if $v$ is a bounded operator and $\inf h > 0$, then our Assumption B follows from Assumptions D and E. Hence, we get a similar result but with weaker assumptions.

We also would like to note that Lemmas 3.3 and 3.4 of [5] (the analog of our Lemma 3.3) are not correct, and that one needs to put some additional assumption. Indeed, with the notation of our Lemma 3.3, it is used implicitly the fact that $(1 - j_0^r)v$ goes to zero in norm in $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$ which is not necessarily true under their assumptions. In order to make it ocrrect, either one has to assume

$$\lim_{R \to +\infty} \|1_{[R,+\infty[}(|x|)v\| = 0,$$

or one assumes already there that $(1 + K)^{-1}$ is compact, a condition which is needed anyway to prove the HVZ-type theorem, and then use the compactness of $v(1 + K)^{-1/2}$ [13].

## 5.2 Existence of a ground state: comparison with [12]

In this paper, the author consider the question of existence of a ground state in the massless case. We first give the assumptions used in [12].

$$(H0) \qquad (K+1)^{-1} \text{ is compact,}$$

$$(H1) \qquad \mathcal{Z} = L^2(\mathbf{R}^d) \text{ and } h \in C(\mathbf{R}^d, \mathbf{R}) \text{ satisfies } \begin{cases} \nabla h \in L^\infty, \\ \lim_{\xi \to \infty} h(\xi) = +\infty, \\ \inf h(\xi) = 0, \end{cases}$$

$$(I0) \qquad \mathbf{R}^d \ni k \mapsto v(k) \in \mathcal{B}(\mathrm{Dom}(K^{1/2}), \mathcal{K}) \cap \mathcal{B}(\mathcal{K}, \mathrm{Dom}(K^{1/2})^*) \text{ for a.e. } k,$$

$$(I1) \qquad \begin{cases} \text{For a.e.} k,\ v(k)(K+1)^{-1/2} \text{ and } (K+1)^{-1/2}v(k) \in \mathcal{B}(\mathcal{K}), \\ C(R) := \int \frac{1}{h(k)} (\|v(k)(K+R)^{-1/2}\|^2 + \|(K+R)^{-1/2}v(k)\|^2) < +\infty, \\ \lim_{R \to +\infty} C(R) = 0, \end{cases}$$

$$(I2) \qquad \int \frac{1}{h(k)^2} \|v(k)(K+1)^{-1/2}\|^2 < +\infty.$$

Under these assumptions, the author proves that the corresponding Pauli-Fierz Hamiltonian $H$ is well defined as a quadratic form and has a ground state.

First, the assumptions on the dispersion relation $h$ are the same as those of [5] (except that $\inf h = 0$). The difference with our paper is thus the same as the one we mentioned in the previous section.

Concerning the coupling function $v$, we would first like to stress that all the assumptions in [12] are "fibered" with respect to $k \in \mathbf{R}^d$. They are therefore stronger than similar assumptions made without such a fibering. For instance, Assumption $(I1)$ implies

$$C'(R) := \|h^{-1/2}v(K+R)^{-1/2}\| < +\infty \text{ and } \lim C'(R) = 0.$$

Note in particular that it implies our Assumption A.

Finally, it seems that one needs to put an additional assumption in [12]. This is a consequence of the mistake in [5] we mentioned in the previous section. Since in [12] $v$ is not necessarily bounded, one can not use the compactness of $v(1 + K)^{-1/2}$. One can make the argument correct assuming *e.g.*

$$(I1^*) \quad h^{-1/2}v(K+1)^{-1/2} \text{ is compact,}$$

instead of $(I1)$.

### 5.3 Self-adjointness: comparison with [11]

Consider the following assumptions

$$
\begin{array}{ll}
(H1) & \mathcal{Z} = L^2(\mathbf{R}^d) \text{ and } h(\xi) = |\xi|, \\
(I1a) & v \in \mathcal{B}(\mathrm{Dom}(K^{1/2}), \mathcal{K} \otimes \mathcal{Z}) \text{ and } \lim\limits_{R \to +\infty} \|h^{-1/2} v (K+R)^{-1/2}\| = 0.
\end{array}
$$

In [11], the authors prove the following fact.

**Proposition 5.1.** *Assume that $K$ is bounded from below and $(H1), (I1a)$ are satisfied. Then the corresponding Pauli-Fierz Hamiltonian is well defined as a quadratic form and is self-adjoint.*

This result is essentially the same as ours. The main difference is that their coupling function $v$ is defined as an operator and we consider also unbounded quadratic forms.

### 5.4 Pullthrough formula: comparison with [11]

We explained in the introduction that, in the literature, the usual Pullthrough Formula was usually written in terms of the operator valued distribution $a(\xi)$ (see (1.2)), which is not closed, and therefore not so easy to treat carefuly. In [11], one can find a version of the Pullthrough Formula which is different from ours and avoids the use of mathematically awkward $a(\xi)$. Below, we state the version of the Pullthrough Formula of [11].

**Proposition 5.2.** *Suppose that $K$ is bounded from below and $(H1), (I1a)$ are fulfilled. Let $z \in \mathrm{Dom}(h) \cap \mathrm{Dom}(h^{-1/2})$ and $\lambda \in \mathbf{C} \setminus \mathrm{sp}H$. Then the closure $[a^*(z), (H-\lambda)^{-1}]^{\mathrm{cl}}$ of the form $[a^*(z), (H-\lambda)^{-1}]$ is a bounded operator and we have*

$$
[a^*(z), (H-\lambda)^{-1}]^{\mathrm{cl}} = (H-\lambda)^{-1} \left(a^*(hz) + (v^* \ 1_{\mathcal{K}} \otimes |z)) \otimes 1\right) (H-\lambda)^{-1}.
$$

(The assumptions $(H1), (I1a)$ are the ones given in the previous section.)

The version of the Pullthrough Formula of [11] seems to be a little more complicated and less canonical than ours because it uses an additional vector $z$ in the one particle space.

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