FROM HEUN CLASS TO PAINLEVÉ

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Introduction.

Heun class equations is an important class of linear differential equations.

Painlevé equations is a celebrated class of well-behaved non-linear differential equations.

I will discuss a unified derivation of various types of Painlevé quations from various types of Heun class equations. My talk is based on

J. Dereziński, A. Ishkhanyan, A. Latosiński "From Heun class equations to Painlevé equations", SIGMA 17 (2021), 056.

Inspired by

S. Y. Slavyanov, W. Lay, "Special Functions. A Unified Theory Based on Singularities"; Oxford, 2000,

Y. Ohyama, S. Okumura, "A coalescent diagram of the Painlevé equations from the viewpoint of isomonodromic deformations", J. Phys. A 39 (2006).

The Heun class and Painlevé equations, just as many other classes of equations are divided into several types. Among these types one is generic (the standard Heun and the VI Painlevé), and other types are obtained by confluence.

It is known that each type of Painlevé can be derived from one of the types of Heun class. In the literature, this derivation is described separately for each type. I will present a derivation which to a large extent is unified.

Such a unified derivation has obvious advantages from the pedagogical point of view. It also automatically describes the coalescence of various types.

The idea to present classes of special functions in a unified way is borrowed from a well-known book by Nikiforov–Uvarov on hypergeometric class equations. It is especially successful in the description of hypergeometric polynomials, which comprise all classical orthogonal polynomials (Hermite, Laguerre and Jacobi). In the next two slides I sketch the theory of hypergeometric polynomials, to illustrate how a unified treatment of a class consisting of several types can work. (These slides are beside the main topic of my talk).

Unified theory of hypergeometric polynomials (following Nikiforov-Uvarov).

Fix polynomials σ, κ such that $\deg \sigma \leq 2$, $\deg \kappa \leq 1$. Let ρ be the weight solving

$$\sigma(z)\partial_z \rho(z) = \kappa(z)\rho(z). \qquad (*)$$

More generally, for $\delta \in \mathbb{C}$ set $\kappa_\delta := \kappa + \delta\sigma'(z)$ and $\rho_\delta(z) = \rho(z)\sigma(z)^\delta$, which also solves $(*)$. Then we have

Rodriguez formula
$$P_n^\delta(z) := \frac{1}{n!}\rho_\delta(z)^{-1}\partial_z^n \sigma^n(z)\rho_\delta(z);$$

generating function
$$\frac{\rho_\delta(z + t\sigma(z))}{\rho_\delta(z)} = \sum_{n=0}^{\infty} t^n P_n^{\delta-n}(z);$$

differential eq.
$$\Big(\sigma(z)\partial_z^2 + (\sigma'(z) + \kappa_\delta(z))\partial_z - n(n+1)\frac{\sigma''}{2} - n\kappa_\delta'\Big) P_n^\delta(z) = 0;$$

lowering
$$\partial_z P_{n+1}^{\delta-1}(z) = \Big(n\frac{\sigma''}{2} + \kappa_\delta'\Big) P_n^\delta(z);$$

raising
$$\big(\sigma(z)\partial_z + \kappa_\delta(z)\big) P_n^\delta(z) = (n+1) P_{n+1}^{\delta-1}(z);$$

orthogonality
$$\int_a^b P_m^\delta(x)P_k^\delta(x)\rho_\delta(x)\mathrm{d}x$$
$$= \delta_{mk}\frac{\big(-\kappa_\delta' - (m+1)\frac{\sigma''}{2}\big)\dots\big(-\kappa_\delta' - 2m\frac{\sigma''}{2}\big)}{m!}\int_a^b \sigma^m(x)\rho_\delta(z)\mathrm{d}x,$$

where $\sigma(a)\rho_\delta(a) = 0,$ for $-\infty < a;$

$\lim\limits_{x\to\infty} \sigma(x)\rho_\delta(x)x^n = 0,$ for $-\infty = a;$ similarly for b.

Hermite: $\sigma(z) = 1, \quad \kappa(z) = -2z, \quad \rho(z) = \mathrm{e}^{-z^2}, \quad]a,b[=]-\infty,\infty[;$

Laguerre: $\sigma(z) = z, \quad \kappa(z) = \alpha - z, \quad \rho(z) = \mathrm{e}^{-z}z^\alpha, \quad]a,b[=]0,\infty[;$

Bessel: $\sigma(z) = z^2, \quad \kappa(z) = -1 + \theta z, \quad \rho(z) = \mathrm{e}^{-z^{-1}}z^\theta,$ no orthogonality;

Jacobi: $\sigma(z) = 1 - z^2, \quad \kappa(z) = \alpha(1-z) + \beta(1+z),$

$$\rho(z) = (1-z)^\alpha(1+z)^\beta, \quad]a,b[=]-1,1[.$$

Sigularities of 2nd order differential equations (following Slavyanov-Lay).

Consider $$\left(\partial_z^2 + p(z)\partial_z + q(z)\right) u(z) = 0. \qquad (**)$$

Suppose that $z_0 \in \mathbb{C}$ is a singularity of p or q:

$$p(z) = \sum_{k=-m}^{\infty} p_k (z-z_0)^k, \qquad p_{-m} \neq 0;$$
$$q(z) = \sum_{k=-\ell}^{\infty} q_k (z-z_0)^k, \qquad q_{-\ell} \neq 0.$$

We say that z_0 is Fuchsian or regular-singular if $m \leq 1$ and $\ell \leq 2$. Otherwise, we define the rank of the singularity z_0

$$\mathrm{rk}(z_0) := \max\left\{m,\ \frac{\ell}{2}\right\},$$

which is an integer or half-integer. Trasforming $z \to \frac{1}{z}$ we obtain the corresponding definitions for a singularity at ∞.

Consider a singularity at $z_0 = 0$. As we learn in standard courses, if 0 is Fuchsian, generically solutions of $(**)$ can be written as a convergent series

$$z^\rho \sum_{j=0}^{\infty} u_j z^j.$$

Let the rank at 0 be m. If m is an integer > 1, we can find formal solutions in the form of a (usually divergent) series

$$\exp\left(\frac{w_{-m+1}}{m-1} z^{-m+1} + \cdots + w_{-1} z^{-1}\right) z^{w_0} \sum_{j=0}^{\infty} u_j z^j.$$

If m is a half-integer $> \frac{1}{2}$, we also have similar formal solutions, where the sums go in steps of $\frac{1}{2}$:

$$\exp\left(\frac{w_{-m+1}}{m-1} z^{-m+1} + \cdots + w_{-\frac{1}{2}} z^{-\frac{1}{2}}\right) z^{w_0} \sum_{j=0}^{\infty} u_j z^j.$$

These series are often asymptotic to true solutions.

$(**)$ has type $(m_1 \dots m_n; m_{n+1})$ if its finite singular points have rank $m_1, \dots, m_n$ and ∞ has rank m_{n+1}. $\underline{1}$ will denote a Fuchsian singularity.

Example: The Riemann equations is the 2nd order equation with 3 Fuchsian singular points. One of them can be put at ∞:

$$\Big(\partial_z^2 + \sum_{j=1}^{2} \frac{a_j}{z - z_j}\partial_z + \sum_{j=1}^{2} \frac{b_j}{z - z_j} + \sum_{j=1}^{2} \frac{c_j}{(z - z_j)^2}\Big)u(z) = 0, \qquad (***)$$

where $b_1 + b_2 = 0$ and z_1, z_2 are distinct points in $\mathbb{C}$. Its symbol is $(\underline{11}; \underline{1})$.
We multiply $(***)$ by $\sigma(z) := (z - z_1)(z - z_2)$ obtaining

$$\Big(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta(z)\Big)u(z) = 0, \qquad (\triangle)$$

where $\tau(z)$ is a polynomial of degree ≤ 1 and $\sigma(z)\eta(z)$ of degree ≤ 2.

More generally, Riemann class equations is the class containing the Riemann equations and its confluent cases. They have the form $(\triangle)$ with $\deg \sigma \leq 2$, $\deg \tau \leq 1$ and $\deg \sigma\eta \leq 2$.

In the following table we give the classification of Riemann class equations according to the ranks of their singularities:

${}_2F_1$ hypergeometric	$(\underline{11};\underline{1})$	$z(1-z)\partial_z^2+\big(c-(a+b+1)z\big)\partial_z-ab;$
${}_2F_0$ confluent	$(2;\underline{1})$	$z^2\partial_z^2+\big(-1+(a+b+1)z\big)\partial_z+ab;$
${}_1F_1$ confluent	$(\underline{1};2)$	$z\partial_z^2+(c-z)\partial_z-a;$
${}_0F_1$ Bessel	$(\underline{1};\frac{3}{2})$	$z\partial_z^2+c\partial_z-1;$
Hermite	$(;3)$	$\partial_z^2-2z\partial_z-2a;$
Airy	$(;\frac{5}{2})$	$\partial_z^2+z;$
Euler	$(\underline{1};\underline{1})$	$z\partial_z^2+c\partial_z;$
1d Helmholtz	$(;2)$	$\partial_z^2+1;$
1d Laplace	$(;\underline{1})$	$\partial_z^2.$

As we see, this table consists of the most useful equations of mathematical physics.

The Heun equation is the 2nd order equation with 4 Fuchsian singular points, one of them put at ∞:

$$\left(\partial_z^2 + \sum_{j=1}^{3} \frac{a_j}{z - z_j}\partial_z + \sum_{j=1}^{3} \frac{b_j}{z - z_j} + \sum_{j=1}^{n} \frac{c_j}{(z - z_j)^2}\right)u(z) = 0,$$

where $\sum_{j=1}^{3} b_j = 0$ and z_1, z_2, z_3 are distinct points in $\mathbb{C}$. We multiply this by $\sigma(z) := (z - z_1)(z - z_2)(z - z_3)$ obtaining

$$\Big(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta(z)\Big)u(z) = 0,$$

where $\tau(z)$ is a polynomial of degree ≤ 2

and $\sigma(z)\eta(z)$ of degree ≤ 4.

Equations are Heun class if they are obtained by confluence from the Heun equation. They have the form

$$\big(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta(z)\big)u(z) = 0,$$

where $\deg \sigma \leq 3$, $\deg \tau \leq 2$ and $\deg \sigma\eta \leq 4$.

If σ has three distinct roots at z_1, z_2, z_3, they are standard Heun equations from the previous slide.

Heun class equations have many applications in physics: anharmonic oscillator, Laplace equation in various coordinates, wave equation on the Kerr black hole, etc.

From 10 types of Heun class to 10 types of Painlevé.

(following Ohyama, Okumura)

The Painlevé equations is a famous class of nonlinear Hamiltonian differential equations with the so called Painlevé property–the absence of moving essential and branch singularities in its solutions. Traditionally, Painlevé equations are divided into 6 types, called Painlevé I, II, III, IV, V and VI. It is actually natural to subdivide some of them into smaller types, obtaining altogether 10 types.

The method of isomonodromic deformations allows us to derive Painlevé equations from linear equations. There are several approaches. The approach that we use starts from 2nd order scalar equations. There is an alternative approach involving 1st order systems of equations.

The first step of the derivation that we describe is a choice of a family of Heun class equations

$$\big(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta(z)\big)u(z) = 0,$$

where σ, τ, η depend on a parameter denoted t and called the time. Then we consider the corresponding deformed Heun class equation which depends on two additional variables, λ, μ:

$$\begin{aligned}&\bigg(\sigma(z)\partial_z^2 + \Big(\tau(z) - \frac{\sigma(z)}{z-\lambda}\Big)\partial_z \\ &+ \eta(z) - \eta(\lambda) - \mu^2\sigma(\lambda) - \mu\big(\tau(\lambda) - \sigma'(\lambda)\big) + \frac{\mu\sigma(\lambda)}{z-\lambda}\bigg)v(z) = 0.\end{aligned}$$

All finite singularities of the deformed equation are the same as in the original equation except for one additional non-logarithmic singularity (also called an apparent singularity). λ is the position of this singularity and $\mu = \frac{v'(\lambda)}{v(\lambda)}$.

We assume that there exists a family of solutions $v(z,t)$ of the deformed equation satisfying the conditions of constant monodromy

$$\partial_t v(z,t) = a(z,t)\partial_z v(z,t) + b(z,t)v(z,t),$$

for some a ,b. These conditions lead to a set of nonlinear differential equation for λ, μ in terms of t, which can be interpreted as Hamilton equations generated by certain Painlevé Hamiltonians $H(t,\lambda,\mu)$, that is

$$\partial_t \lambda = \partial_\mu H(t,\lambda,\mu),$$
$$\partial_t \mu = -\partial_\lambda H(t,\lambda,\mu).$$

$H(t,\lambda,\mu)$ has always the form

$$H = a(t,\lambda)\mu^2 + b(t,\lambda)\mu + c(t,\lambda).$$

The above Hamilton equations transformed to 2nd order equations for λ are the usual presentations of Painlevé equations.

Here is the list of notrivial types of Heun class equations and the corresponding types of Painleveé equations:

$(\underline{1111})$	(standard) Heun	Painlevé VI
$(\underline{11}2)$	confluent Heun	Painlevé nondegenerate V
$(\underline{11}\frac{3}{2})$	degenerate confluent Heun	Painlevé degenerate V ($\simeq$ ndeg III')
(22)	doubly confluent Heun	Painlevé nondegenerate III'
$(\frac{3}{2}2)$	degenerate doubly confluent Heun	Painlevé degenerate III'
$(\frac{3}{2}\frac{3}{2})$	doubly degenerate doubly confluent Heun	Painlevé doubly degenerate III'
$(\underline{1}3)$	bi-confluent Heun	Painlevé IV
$(\underline{1}\frac{5}{2})$	degenerate bi-confluent Heun	Painlevé 34 ($\simeq$ II)
(4)	tri-confluent Heun	Painlevé II
$(\frac{7}{2})$	degenerate tri-confluent Heun	Painlevé I .

The rank can be an integer or a half-integer. We also introduce the rounded rank: if the rank is m or $m - \frac{1}{2}$, where m is an integer, then we say that its rounded rank is m. We denote this by the symbol $\underline{m}$.

Using the rounded rank we can obtain a coarser classification:

$(\underline{1111})$	(standard) Heun	Painlevé VI
$(\underline{112})$	confluent Heun	Painlevé V
$(\underline{22})$	doubly confluent Heun	Painlevé III'
$(\underline{13})$	bi-confluent Heun	Painlevé IV-34
$(\underline{4})$	tri-confluent Heun	Painlevé II-I

The derivation of Painlevé VI from Heun $(\underline{111};\underline{1})$ by this method can be traced back to a paper by Fuchs from the early 20th century. This approach was later generalized to other Painlevé equations by Okamoto and refined by Ohyama-Okumura and Slavyanov-Lay.

From Heun ($\underline{111};\underline{1}$) to Painlevé VI

$$
\begin{aligned}
&z(z-1)(z-t)\partial_z^2 \\
&+\Big((1-\kappa_0)(z-1)(z-t)+(1-\kappa_1)z(z-t)+(1-\kappa_t)z(z-1)\Big)\partial_z \\
&+\frac{\Big((\kappa_0+\kappa_1+\kappa_t-1)^2-\kappa_\infty^2\Big)z}{4}
\end{aligned}
$$

$$
a(z)=\frac{(\lambda-t)z(z-1)}{t(t-1)(z-\lambda)}, \quad b(z)=-\frac{\lambda(\lambda-1)(\lambda-t)\mu}{t(t-1)(z-\lambda)}.
$$

$$
\begin{aligned}
t(t-1)H =&\lambda(\lambda-1)(\lambda-t)\mu^2 \\
&-\Big(\kappa_0(\lambda-1)(\lambda-t)+\kappa_1\lambda(\lambda-t)+(\kappa_t-1)\lambda(\lambda-1)\Big)\mu \\
&+\frac{\Big((\kappa_0+\kappa_1+\kappa_t-1)^2-\kappa_\infty^2\Big)(\lambda-t)}{4}.
\end{aligned}
$$

From Heun ($2\underline{1};\underline{1}$) to Painlevé V

$$z(z-1)^2\partial_z^2 + \big((2-\chi_1)z(z-1) + (1-\kappa_0)(z-1)^2 + tz\big)\,\partial_z$$
$$+ \frac{\big((\kappa_0+\chi_1-1)^2 - \kappa_\infty^2\big)}{4}(z-1).$$

$$a(z) = \frac{(\lambda-1)z(z-1)}{t(z-\lambda)}, \quad b(z) = -\frac{(\lambda-1)^2\lambda\mu}{t(z-\lambda)}.$$

$$tH = (\lambda-1)^2\lambda\mu^2 - \big(\kappa_0(\lambda-1)^2 + (\chi_1-1)\lambda(\lambda-1) - t\lambda\big)\mu$$
$$+ \frac{\big((\kappa_0+\chi_1-1)^2 - \kappa_\infty^2\big)(\lambda-1)}{4}.$$

From Heun ($\frac{3}{2}\underline{1};\underline{1}$) to degenerate Painlevé V

$$(z-1)^2 z\partial_z^2 + \left((z-1)z + (1-\kappa_0)(z-1)^2\right)\partial_z \\ - \frac{t}{(z-1)} + \frac{(\kappa_0^2-\kappa_\infty^2)}{4}(z-1)$$

$$a(z) = \frac{(\lambda-1)z(z-1)}{t(z-\lambda)}, \quad b(z) = -\frac{\lambda(\lambda-1)^2\mu}{t(z-\lambda)}.$$

$$tH = \lambda(\lambda-1)^2\mu^2 - \kappa_0(\lambda-1)^2\mu + \frac{(\kappa_0^2-\kappa_\infty^2)(\lambda-1)}{4} - \frac{t\lambda}{(\lambda-1)}.$$

From Heun (2;2) to non-degenerate Painlevé III'

$$z^2\partial_z^2 + \left(t + (2-\chi_0)z - z^2\right)\partial_z + \frac{(\chi_0+\chi_\infty-1)z}{2}$$

$$a(z) := \frac{\lambda z}{t(z-\lambda)}, \quad b(z) = -\frac{\lambda^2\mu}{t(z-\lambda)}.$$

$$tH := \lambda^2\mu^2 - \left(\lambda^2 + (\chi_0-1)\lambda - t\right)\mu + \frac{1}{2}(\chi_0+\chi_\infty-1)\lambda$$

From Heun $(2; \frac{3}{2})$ to degenerate Painlevé III'

$$z^2\partial_z^2 + (t + (2-\chi_0)z)\,\partial_z + \frac{1}{2}z.$$

$$a(z) := \frac{\lambda z}{t(z-\lambda)}, \quad b(z) = -\frac{\lambda^2\mu}{t(z-\lambda)}.$$

$$tH = \lambda^2\mu^2 + \big((1-\chi_0)\lambda + t\big)\mu + \frac{\lambda}{2}.$$

From Heun $(\frac{3}{2}; \frac{3}{2})$ to doubly degenerate Painlevé III'

$$z^2\partial_z^2 + 2z\partial_z + \frac{1}{2}z + \frac{t}{2z}.$$

$$a(z) := \frac{\lambda z}{t(z-\lambda)}, \quad b(z) = -\frac{\lambda^2\mu}{t(z-\lambda)}.$$

$$tH = \lambda^2\mu^2 + \lambda\mu + \frac{\lambda}{2} + \frac{t}{2\lambda}$$

From Heun ($\underline{1}$;3) to Painlevé IV

$$z\partial_z^2 + \left(1 - \kappa_0 - tz - \frac{z^2}{2}\right)\partial_z + \frac{\theta_\infty}{2} z.$$

$$a(z) := \frac{2z}{(z-\lambda)}, \quad b(z) = -\frac{2\lambda\mu}{(z-\lambda)}.$$

$$H = 2\lambda\mu^2 - (\lambda^2 + 2t\lambda + 2\kappa_0)\mu + \theta_\infty \lambda.$$

From Heun ($\underline{1}$;$\frac{5}{2}$) to Painlevé 34

$$z\partial_z^2 + (1-\kappa_0)\partial_z - \frac{1}{2} z^2 - \frac{tz}{2}.$$

$$a(z) = \frac{z}{z-\lambda}, \quad b(z) = -\frac{\lambda\mu}{z-\lambda}.$$

$$H = \lambda\mu^2 - \kappa_0\mu - \frac{\lambda^2}{2} - \frac{t\lambda}{2}.$$

From Heun (;4) to Painlevé II

$$\partial_z^2 - (2z^2 + t)\partial_z - (2\alpha + 1)z.$$

$$a(z) := \frac{1}{2(z-\lambda)}, \quad b(z) = -\frac{\mu}{2(z-\lambda)}.$$

$$H = \frac{1}{2}\mu^2 - \left(\lambda^2 + \frac{t}{2}\right)\mu - \left(\alpha + \frac{1}{2}\right)\lambda.$$

From Heun (;$\frac{7}{2}$) to Painlevé I

$$\partial_z^2 - 4z^3 - 2tz.$$

$$a(z) := \frac{1}{2(z-\lambda)}, \quad b(z) = -\frac{\mu}{2(z-\lambda)}.$$

$$H = \frac{1}{2}\mu^2 - 2\lambda^3 - t\lambda.$$

From Heun class to Painlevé — an attempt of a unified treatment

(following D., Ishkhanyan, Latosiński).

Recall that the starting point of the derivation of Painlevé equations is the choice of a time-dependent family of Heun class equations:

$$\big(\sigma(z,t)\partial_z^2 + \tau(z,t)\partial_z + \eta(z,t)\big)u(z) = 0,$$

where where $\deg \sigma \leq 3$, $\deg \tau \leq 2$ and $\deg \sigma\eta \leq 4$. Then we need to find the compatibility functions a, b. From them we can compute the Painlevé Hamiltonian $H(t, \lambda, \mu)$.

We will try to describe this derivation in a way which is as unified as possible, using two similar ansatzes, called Case A and Case B. Together they cover all normal forms of Heun class and Painlevé.

In Case A among other conditions we need to assume that $\sigma(s) = 0$ for some $s = s(t)$, so that we can write $\sigma(z) = (z-s)\rho(z)$. Then for some $t \mapsto m(t)$

$$H = m(t)\Big((\lambda - s)\rho(\lambda)\mu^2 + \big(\tau(\lambda,t) - (\lambda - s)\rho'(\lambda)\big)\mu + \eta(\lambda,t)\Big).$$

In Case B among other things we suppose that $\deg\sigma \le 2$. Then for some $t \mapsto m(t)$

$$H = m(t)\Big(\sigma(\lambda)\mu^2 + \big(\tau(\lambda,t) - \sigma'(\lambda)\big)\mu + \eta(\lambda,t)\Big).$$

The resulting Painlevé Hamiltonians are proportional to appropriately interpreted symbols of the Heun class operator:

$$\begin{aligned} &(z-s)\partial_z\rho(z)\partial_z + \big(\tau(z,t) - (z-s)\rho'(z)\big)\partial_z + \eta(z,t), && \text{Case A;} \\ &\partial_z\sigma(z)\partial_z + \big(\tau(z,t) - \sigma'(z)\big)\partial_z + \eta(z,t), && \text{Case B.} \end{aligned}$$

(Replace z and ∂_z with λ, resp. μ).

Cases A and B need to be subdivided further into subcases.

Case A is subdivided into A1, Ap and Aq.

Case B is subdivided into Bp and Bq.

They differ by the choice of the time variable t.

- A1. The variable t is the position of one of Fuchsian singularities.
- Ap, Bp. The variable t is contained in $\tau(z)$.
- Aq, Bq. The variable t is contained in $\eta(z)$.

Note that Subcase A1 works generically. In particular it works for the standard Heun type $(\underline{1111})$, leading to Painlevé VI. However, it does not work for some other types. For instance, in most of the degenerate types one needs to use either Subcase Aq or Subcase Bq.

In the following list we informally describe when we can apply various subcases.

A1. $\sigma(t) = 0$, $\sigma'(t) \neq 0$, $\qquad \deg(z-t)\eta \leq 2$.

Ap. $\sigma(0) = \sigma'(0) = 0$, $\quad \tau(0) \neq 0$, $\quad \sigma\eta(0) = (\sigma\eta)'(0) = 0$.

Aq. $\sigma(0) = \sigma'(0) = 0$, $\quad \tau(0) = 0$, $\quad \sigma\eta(0) = 0$, $(\sigma\eta)'(0) \neq 0$.

Bp. $\deg\sigma \leq 2$, $\quad \deg\tau = 2$, $\quad \deg\sigma\eta \leq 2$.

Bq. $\deg\sigma \leq 2$, $\quad \deg\tau \leq 1$, $\quad \deg\sigma\eta = 3$.

One can derive Ap and Aq from Bp and Bq assuming $\sigma(0) = 0$ and making the change of variables $z \mapsto \frac{1}{z}$, $\lambda \to \frac{1}{\lambda}$, $\mu \to -\lambda^2\mu$.

Subcase A1.

$$\deg\rho \le 2, \quad \deg\phi, \deg\eta_0 \le 1, \quad \kappa, \alpha \in \mathbb{C};$$

$$\begin{aligned}
&(z-t)\rho(z)\partial_z^2 \\
&+ \big((1-\kappa)\rho(z) + \phi(z)(z-t)\big)\partial_z + \frac{\alpha\rho(t)}{z-t} + \eta_0(z),
\end{aligned}$$

$$\begin{aligned}
\rho(t)H :=& (\lambda - t)\rho(\lambda)\mu^2 \\
&+ \big((1-\kappa)\rho(\lambda) + (\phi(\lambda) - \rho'(\lambda))(\lambda - t)\big)\mu + \frac{\alpha\rho(t)}{\lambda - t} + \eta_0(\lambda).
\end{aligned}$$

Subcase Ap. $\deg \rho_1 \le 1, \quad \deg \tau_0, \deg \psi \le 2;$

$$z^2\rho_1(z)\partial_z^2 + \big(t\rho_1(z) + \tau_0(z)\big)\partial_z + \frac{\psi(z)}{\rho_1(z)};$$

$$\big(\tau_0(0) + t\rho_1(0)\big)H := \lambda^2\rho_1(\lambda)\mu^2$$
$$+ \big(t\rho_1(\lambda) + \tau_0(\lambda) - \rho_1'\lambda^2 - \rho_1(\lambda)\lambda\big)\mu + \frac{\psi(\lambda)}{\rho_1(\lambda)}.$$

Subcase Aq. $\deg \rho_1, \deg \phi \le 1, \quad \deg \psi_0 \le 2;$

$$z^2\rho_1(z)\partial_z^2 + z\phi(z)\partial_z + \frac{t}{z} + \frac{\psi_0(z)}{\rho_1(z)};$$

$$\big((\sigma\eta_0)'(0) + \rho_1(0)t\big)H := \lambda^2\rho_1(\lambda)\mu^2$$
$$+\big(\lambda(\phi(\lambda) - \rho_1(\lambda)) - \rho_1'\lambda^2\big)\mu + \frac{t}{\lambda} + \frac{\psi_0(\lambda)}{\rho_1(\lambda)}.$$

Subcase Bp.

$$\deg \sigma \le 2, \quad \deg \tau_0 \le 2, \quad \deg \psi \le 2;$$

$$\sigma(z)\partial_z^2 + \big(t\sigma(z) + \tau_0(z)\big)\partial_z + \frac{\psi(z)}{\sigma(z)};$$

$$\left(t\frac{\sigma''}{2} + \frac{\tau_0''}{2}\right)H := \sigma(\lambda)\mu^2 + \big(t\sigma(\lambda) + \tau_0(\lambda) - \sigma'(\lambda)\big)\mu + \frac{\psi(\lambda)}{\sigma(\lambda)}.$$

Subcase Bq.

$$\deg \sigma \le 2, \quad \deg \tau \le 1, \quad \deg \psi_0 \le 3;$$

$$\sigma(z)\partial_z^2 + \tau(z)\partial_z + tz + \frac{\psi_0(z)}{\sigma(z)};$$

$$\left(t\frac{\sigma''}{2} + \frac{(\sigma\eta_0)'''}{6}\right)H := \sigma(\lambda)\mu^2 + \big(\tau(\lambda) - \sigma'(\lambda)\big)\mu + t\lambda + \frac{\psi_0(\lambda)}{\sigma(\lambda)}.$$

THANK YOU FOR YOUR ATTENTION