HOMOGENEOUS SCHRÖDINGER OPERATORS ON HALFLINE

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Let U_τ be the group of dilations on $L^2[0,\infty[$, that is $(U_\tau f)(x) = \mathrm{e}^{\tau/2} f(e^\tau x)$. We say that A is homogeneous of degree ν if $U_\tau A U_\tau^{-1} = \mathrm{e}^{-\nu\tau} A$.

Consider the differential expression

$$-\partial_x^2 + cx^{-2} \tag{1}$$

on $C_{\mathrm{c}}^\infty]0,\infty[$. Clearly (1) is homogeneous of degree -2.

(1) is essentially self-adjoint on $C_{\mathrm{c}}^\infty]0,\infty[$ for $c \geq \frac{3}{4}$ and not essentially self-adjoint for $c < \frac{3}{4}$.

Theorem 1 *1. There is a unique holomorphic family $\{H_m\}_{\mathrm{Re}\, m>-1}$ such that H_m coincides with the the closure of $-\partial_x^2 + \left(m^2 - \frac{1}{4}\right) x^{-2}$ if $m \geq 1$.*

2. For each m with $\mathrm{Re}\, m > -1$, the operators H_m are homogeneous of degree -2 and satisfy $H_m^* = H_{\overline{m}}$, the spectrum of H_m is absolutely continuous and is equal to $[0, \infty[$.

3. H_m is self-adjoint if m is real.

4. If $0 \leq m < 1$, the operator H_m is the Friedrichs extension of $-\partial_x^2 + \left(m^2 - \frac{1}{4}\right) x^{-2}$.

5. If $-1 < m \leq 0$, the operator H_m is the Krein extension of $-\partial_x^2 + \left(m^2 - \frac{1}{4}\right) x^{-2}$.

The operators H_m are interesting for many reasons.

- They have several interesting physical applications, eg. they appear in the decomposition of the Aharonov-Bohm Hamiltonian.
- They have rather subtle and rich properties, illustrating various concepts of the operator theory in Hilbert spaces (eg. the Friedrichs and Krein extensions, holomorphic families of closed operators).
- Surprisingly rich is also the theory of the first order homogeneous operators $A_\alpha = \mathrm{i}\partial_x + \mathrm{i}\frac{\alpha}{x}$, which is closely related to the theory of H_m.
- Essentially all basic objects related to H_m, such as their resolvents, spectral projections, wave and scattering operators, can be explicitly computed.
- A number of nontrivial identities involving special functions, especially the Bessel functions, find an appealing operator-theoretical interpretation in terms of the operators H_m. Eg. the Barnes identity leads to the formula for wave operators.

For each complex number α let $\widetilde{A}_\alpha$ be the differential expression

$$\widetilde{A}_\alpha := -\mathrm{i}\partial_x + \mathrm{i}\frac{\alpha}{x}$$

acting on distributions on $\mathbb{R}_+$. Its restriction to $C_{\mathrm{c}}^\infty]0,\infty[$ is a closable operator in $L^2[0,\infty[$ whose closure will be denoted $A_\alpha^{\min}$. This is the minimal operator associated to $\widetilde{A}_\alpha$. The maximal operator $A_\alpha^{\max}$ associated to $\widetilde{A}_\alpha$ is defined as the restriction of $\widetilde{A}_\alpha$ to $\mathcal{D}(A_\alpha^{\max}) := \{f \in L^2[0,\infty[\mid \widetilde{A}_\alpha f \in L^2[0,\infty[\}$.

The following properties of the operators $A_\alpha^{\min}$ and $A_\alpha^{\max}$ are easy to check:

(i) $A_\alpha^{\min} \subset A_\alpha^{\max}$,

(ii) $(A_\alpha^{\min})^* = A_{-\overline{\alpha}}^{\max}$ and $(A_\alpha^{\max})^* = A_{-\overline{\alpha}}^{\min}$,

(iii) $A_\alpha^{\min}$ and $A_\alpha^{\max}$ are homogeneous of degree -1.

Proposition 2 *1. We have* $A_\alpha^{\min} = A_\alpha^{\max}$ *if and only if* $|\mathrm{Re}\,\alpha| \geq 1/2$*.*

2. Let $\mathrm{Re}\,\alpha > -1/2$*. Then*

(i) $\mathrm{rs}\,(A_\alpha^{\max}) = \mathbb{C}_-$*.*

(ii) The map $\alpha \mapsto A_\alpha^{\max}$ *is holomorphic in the region* $\mathrm{Re}\,\alpha > -1/2$*.*

(iii) If $\mathrm{Re}\,\alpha \geq 0$ *then* $\mathrm{i}A_\alpha^{\max}$ *is the generator of a* C^0*-semigroup of contractions*

3. Let $\mathrm{Re}\,\alpha < 1/2$*. Then*

(i) $\mathrm{rs}\,(A_\alpha^{\min}) = \mathbb{C}_+$*.*

(ii) The map $\alpha \mapsto A_\alpha^{\min}$ *is holomorphic in the region* $\mathrm{Re}\,\alpha < 1/2$*.*

(iii) if $\mathrm{Re}\,\alpha \leq 0$ *the operator* $-\mathrm{i}A_\alpha^{\min}$ *is the generator of a* C^0*-semigroup of contractions*

For an arbitrary complex number m we introduce the differential expression

$$\tilde{L}_m := -\partial_x^2 + \frac{m^2 - 1/4}{x^2}$$

acting on distributions on $\mathbb{R}_+$. Note that $\tilde{L}_m = \tilde{L}_{-m}$. We will however see that m, not m^2, is the natural parameter.

Let $L_m^{\min}$ and $L_m^{\max}$ be the minimal and maximal operators associated to it in $L^2(0, \infty)$. It is clear that they are homogeneous operators of degree -2 and

$$(L_m^{\min})^* = L_{\overline{m}}^{\max}.$$

If m is a complex number we set

$$\zeta_m(x) = x^{1/2+m}.$$

Note that $\zeta_{\pm m}$ are both square integrable at the origin if and only if $|\mathrm{Re}\, m| < 1$.

We also choose $\xi \in C^\infty(\mathbb{R}_+)$ such that $\xi = 1$ on $[0, 1]$ and 0 on $[2, \infty[$.

For $\mathrm{Re}\,(m) > -1$, we define H_m to be the operator $L_m^{\max}$ restricted to $\mathcal{D}(L_m^{\min}) + \mathbb{C}\xi\zeta_m$.

Theorem 3 *For any* $\mathrm{Re}\,(m) > -1$ *we have* $\mathrm{sp}\,(H_m) = [0, \infty[$*. If* $R_m(\lambda; x, y)$ *is the integral kernel of the operator* $(\lambda - H_m)^{-1}$*, then for* $\mathrm{Re}\, k > 0$ *we have*

$$R_m(-k^2; x, y) = \begin{cases} \sqrt{xy} I_m(kx) K_m(ky) & \text{if} \quad x < y, \\ \sqrt{xy} I_m(ky) K_m(kx) & \text{if} \quad x > y. \end{cases},$$

where I_m *is the modified Bessel function and* K_m *is the MacDonald function.*

Theorem 4 *Let $m > -1$. Then the operators H_m are positive, self-adjoint, homogeneous of degree 2 with $\operatorname{sp} H_m = [0, \infty[$. Besides we have the following table:*

$m \geq 1$:	$H_m = A^*_{1/2+m} A_{1/2+m} \ = \ A^*_{1/2-m} A_{1/2-m}$,	$H_0^1 = \mathcal{Q}(H_m)$, $H_m = L_m^{\min} = L_m^{\max}$;
$0 < m < 1$:	$H_m = A^*_{1/2+m} A_{1/2+m} \ = \ \left(A^{\min}_{1/2-m}\right)^* A^{\min}_{1/2-m}$	$H_0^1 = \mathcal{Q}(H_m)$, H_m *is the Friedrichs ext. of* $L_m^{\min}$;
$m = 0$:	$H_0 = A^*_{1/2} A_{1/2}$,	$H_0^1 + \mathrm{c}\xi\zeta_0$ *dense in* $\mathcal{Q}(H_0)$, H_0 *is the Friedrichs and Krein ext. of* $L_0^{\min}$;
$-1 < m < 0$:	$H_m = \left(A^{\max}_{1/2+m}\right)^* A^{\max}_{1/2+m}$,	$H_0^1 + \mathrm{c}\xi\zeta_m = \mathcal{Q}(H_m)$, H_m *is the Krein ext. of* $L_m^{\min}$.

In the region $-1 < m < 1$ (which is the most interesting one), it is quite remarkable that for strictly positive m one can factorize H_m in two different ways, whereas for $m \leq 0$ only one factorization appears.

As an example, let us consider the case of the Laplacian $-\partial_x^2$, i.e. $m^2 = 1/4$. The operators $H_{1/2}$ and $H_{-1/2}$ coincide with the Dirichlet and Neumann Laplacian respectively. One usually factorizes them as $H_{1/2} = P_{\min}^* P_{\min}$ and $H_{-1/2} = P_{\max}^* P_{\max}$, where $P_{\min}$ and $P_{\max}$ denote the usual momentum operator on the half-line with domain $\mathcal{H}_0^1[0, \infty[$ and $H^1[0, \infty[$ respectively. The above analysis says that, whereas for the Neumann Laplacian this is the only factorization of the form S^*S with S homogeneous, in the case of the Dirichlet Laplacian one can also factorize it in the rather unusual following way

$$H_{1/2} = \left(P_{\min} + \mathrm{i}x^{-1}\right)^* \left(P_{\min} + \mathrm{i}x^{-1}\right).$$

Proposition 5 *The family H_m has the following property:*

$$0 \le m \le m' \;\Rightarrow\; H_m \le H_{m'},$$
$$0 \le m < 1 \;\Rightarrow\; H_{-m} \le H_m.$$

Proposition 6 *Let $m \neq 0$.*

i) If $0 \le \arg m \le \pi/2$, then $\mathrm{Num}\,(H_m) = \{z \mid 0 \le \arg z \le 2\arg m\}$. Hence H_m is maximal sectorial and $\mathrm{i}H_m$ is dissipative.

ii) If $-\pi/2 \le \arg m \le 0$, then $\mathrm{Num}\,(H_m) = \{z \mid 2\arg m \le \arg z \le 0\}$. Hence H_m is maximal sectorial and $-\mathrm{i}H_m$ is dissipative.

iii) If $|\arg m| \le \pi/4$, then $-H_m$ is dissipative.

iv) If $\pi/2 < |\arg m| < \pi$, then $\mathrm{Num}\,(H_m) = \mathbb{C}$.

Proposition 7 *For $0 < a < b < \infty$, the integral kernel of $1\!\!1_{[a,b]}(H_m)$ is*

$$1\!\!1_{[a,b]}(H_m)(x,y) \;=\; \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_m(kx) J_m(ky) k \mathrm{d}k,$$

where J_m is the Bessel function.

Let $\mathcal{F}_m$ be the operator on $L^2(0,\infty)$ given by

$$\mathcal{F}_m : f(x) \mapsto \int_0^\infty J_m(kx)\sqrt{kx} f(x) \mathrm{d}x \tag{2}$$

Up to an inessential factor, $\mathcal{F}_m$ is the so-called Hankel transformation.

Theorem 8 *$\mathcal{F}_m$ is a unitary involution on $L^2(0,\infty)$ diagonalizing H_m, more precisely*

$$\mathcal{F}_m H_m \mathcal{F}_m^{-1} = x^2.$$

It satisfies $\mathcal{F}_m \mathrm{e}^{\mathrm{i}tD} = \mathrm{e}^{-\mathrm{i}tD} \mathcal{F}_m$ for all $t \in \mathbb{R}$.

Theorem 9 *If $m, k > -1$ are real then the wave operators associated to the pair H_m, H_k exist and*

$$\Omega^{\pm}_{m,k} := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_m}\mathrm{e}^{-\mathrm{i}tH_k} = \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\mathcal{F}_m\mathcal{F}_k = \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\frac{\Xi_k(D)}{\Xi_m(D)}. \quad (3)$$

where

$$\Xi_m(t) = \mathrm{e}^{\mathrm{i}\ln(2)t}\frac{\Gamma(\frac{m+1+\mathrm{i}t}{2})}{\Gamma(\frac{m+1-\mathrm{i}t}{2})}.$$

The scattering operator $S_{m,k}$ for the pair (H_m, H_k) is a scalar operator $S_{m,k} = \mathrm{e}^{\mathrm{i}\pi(m-k)}\mathbb{1}$.

(3) has been obtained independently by Richard and Pankrashkin.

The definition (or actually a number of equivalent definitions) of a holomorphic family of bounded operators is quite obvious and does not need to be recalled. In the case of unbounded operators the situation is more subtle.

Suppose that Θ is an open subset of $\mathbb{C}$, $\mathcal{H}$ is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on $\mathcal{H}$. We say that this is a holomorphic family of closed operators if for each $z_0 \in \Theta$ there exists a neighborhood Θ_0 of z_0, a Banach space $\mathcal{K}$ and a holomorphic family of injective bounded operators $\Theta_0 \ni z \mapsto A(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\mathrm{Ran}\, A(z) = \mathcal{D}(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)A(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

We have the following practical criterion:

Theorem 10 *Suppose that* $\{H(z)\}_{z\in\Theta}$ *is a function whose values are closed operators on* $\mathcal{H}$*. Suppose in addition that for any* $z \in \Theta$ *the resolvent set of* $H(z)$ *is nonempty. Then* $z \mapsto H(z)$ *is a holomorphic family of closed operators if and only if for any* $z_0 \in \Theta$ *there exists* $\lambda \in \mathrm{c}$ *and a neighborhood* Θ_0 *of* z_0 *such that* $\lambda \in \mathrm{rs}\,(H(z))$ *for* $z \in \Theta_0$ *and* $z \mapsto (H(z) - \lambda)^{-1} \in B(\mathcal{H})$ *is holomorphic on* Θ_0*.*

The above theorem indicates that it is more difficult to study holomorphic families of closed operators that for some values of the complex parameter have an empty resolvent set.

It is interesting to note that $\Xi_m(D)$ is a unitary operator for all real values of m and

$$\Xi_m^{-1}(D)x^{-2}\Xi_m(D) \tag{4}$$

is a function with values in self-adjoint operators for all real m. $\Xi_m(D)$ is bounded and invertible also for all m such that $\mathrm{Re}\, m \neq -1, -2, \dots$. Therefore, the formula (4) defines an operator for all $\{m \mid \mathrm{Re}\, m \neq -1, -2, \dots\} \cup \mathbb{R}$. Clearly, for $\mathrm{Re}\, m > -1$, this operator function coincides with the operator H_m studied in this paper. Its spectrum is always equal to $[0, \infty[$ and it is analytic in the interior of its domain.

One can then pose the following question: does this operator function extend to a holomorphic function of closed operators on the whole complex plane?