# Van Hove Hamiltonians—exactly solvable models of the infrared and ultraviolet problem

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**Abstract.** Quadratic bosonic Hamiltonians with a linear perturbation are studied. Depending on the infrared and ultraviolet behavior of the perturbation, their properties are described from the point of view of spectral and scattering theory.

## 1 Introduction

Some of the simplest exactly solvable models in classical and quantum physics are quadratic Hamiltonians with a linear perturbation. Following [Sch] we will call them van Hove Hamiltonians. They arise in classical and quantum field theory in many contexts. For example, the Hamiltonian of electrodynamics with prescribed external charges is a van Hove Hamiltonian.

In the case of finite degrees of freedom, an appropriate translation in phase space transforms a van Hove Hamiltonian into a purely quadratic Hamiltonian. In the quantum case this translation can be implemented by a unitary Bogolubov transformation.

If the phase space is infinite dimensional, the situation becomes more complicated, because the above mentioned Bogolubov transformation can become "improper" (not implementable by a unitary operator). Still, as it is well known, one can fully analyze the properties of van Hove Hamiltonians even in the case of the infinite dimensional

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phase space. These properties are quite interesting, both physically and mathematically. In fact, some types of van Hove Hamiltonians can be viewed as exactly solvable toy models of renormalization, both in the infrared and ultraviolet regime. Depending on the assumptions on the perturbation, one can distinguish several types of these Hamiltonians with distinct properties.

Let us describe some of the results of our paper in a somewhat informal language. We will restrict ourselves to the quantum case, since the classical case is very similar. (In the main part of the paper a different, more compact notation is used; moreover, both the classical and quantum case is treated).

Let $(K, \mathrm{d}k)$ be a space with a measure. (We use the notation $\mathrm{d}k$ to denote an arbitrary measure, not necessarily the Lebesgue measure). Let $K \ni k \mapsto a^*(k)$ and $K \ni k \mapsto a(k)$ be the corresponding bosonic creation and annihilation operators. Let $K \ni k \mapsto h(k)$ be an almost everywhere positive measurable function, describing the dispersion relation. Finally, suppose that $K \ni k \mapsto z(k)$ is a complex function describing the interaction. By a van Hove Hamiltonian we will mean a self-adjoint operator given by the formal expression

$$H := \int \big(h(k)a^*(k)a(k)\mathrm{d}k + z(k)a^*(k)\mathrm{d}k + \overline{z}(k)a(k)\big)\mathrm{d}k + c, \tag{1.1}$$

where $c$ is an arbitrary (maybe infinite) constant. It turns out that if

$$\int_{h(k)<1} |z(k)|^2\mathrm{d}k + \int_{h(k)\geq 1} \tfrac{|z(k)|^2}{h(k)^2}\mathrm{d}k < \infty. \tag{1.2}$$

then properly interpreted (1.1) defines a family of self-adjoint operators.

There are two natural choices of $c$ in (1.1). The first choice is $c = 0$. It is possible if we assume that

$$\int_{h(k)<1} |z(k)|^2\mathrm{d}k + \int_{h(k)\geq 1} \tfrac{|z(k)|^2}{h(k)}\mathrm{d}k < \infty, \tag{1.3}$$

Under this condition we can define the van Hove Hamiltonian of the first kind:

$$H_{\mathrm{I}} := \int \big(h(k)a^*(k)a(k)\mathrm{d}k + z(k)a^*(k)\mathrm{d}k + \overline{z}(k)a(k)\big)\mathrm{d}k. \tag{1.4}$$

The second choice is $c = \int \frac{|z(k)|^2}{h(k)}\mathrm{d}k$. It is possible if

$$\int_{h(k)<1} \tfrac{|z(k)|^2}{h(k)}\mathrm{d}k + \int_{h(k)\geq 1} \tfrac{|z(k)|^2}{h(k)^2}\mathrm{d}k < \infty. \tag{1.5}$$

Under this condition we can define the van Hove Hamiltonians of the second kind:

$$H_{\mathrm{II}} := \int h(k)\Big(a^*(k) + \frac{z(k)}{h(k)}\Big)\Big(a(k) + \frac{\overline{z}(k)}{h(k)}\Big)\mathrm{d}k. \tag{1.6}$$

The conditions (1.3) and (1.5) are true at the same time iff

$$\int \frac{|z(k)|^2}{h(k)}\mathrm{d}k < \infty, \tag{1.7}$$

and then the two types of van Hove Hamiltonians are well defined and differ by a constant:

$$H_{\mathrm{II}} = H_{\mathrm{I}} + \int \frac{|z(k)|^2}{h(k)} \mathrm{d}k. \tag{1.8}$$

Next we would like to describe various types of behavior of van Hove Hamiltonians. We will consider separately the case when the dispersion relation is separated away from zero and the case of a bounded dispersion relation. In the former case the infrared problem is absent and we can study the ultraviolet problem in its pure form. In the latter case the ultraviolet problem is absent and all the difficulties are due to the infrared problem. We will see that one can distinguish $3 \times 3 = 9$ distinct classes of van Hove Hamiltonians.

## 1.1 Ultraviolet problem

Assume that for some $0 < c_0$ we have $c_0 \leq h(k)$. We allow $h$ to be unbounded. (In quantum field theory this behavior is typical for massive particles). Under this assumption, we can always define $H_{\mathrm{II}}$. We can distinguish three cases:

(1) $\int |z(k)|^2 \mathrm{d}k < \infty$.
This is the most regular case. The perturbation is operator bounded. $H_{\mathrm{I}}$ and $H_{\mathrm{II}}$ are well defined by the Kato-Rellich theorem [Kato, RS2]. One can define a "dressing operator"

$$U := \exp\left( -\int \frac{z(k)}{h(k)} a^*(k) \mathrm{d}k + \int \frac{\overline{z}(k)}{h(k)} a(k) \mathrm{d}k \right) \tag{1.9}$$

which intertwines $H_{\mathrm{II}}$ with the quadratic Hamiltonian:

$$H_{\mathrm{II}} = U \int h(k) a^*(k) a(k) \mathrm{d}k U^*. \tag{1.10}$$

Because of (1.10), the Hamiltonians have a ground state.

(2) $\int \frac{|z(k)|^2}{h(k)} \mathrm{d}k < \infty, \qquad \int |z(k)|^2 \mathrm{d}k = \infty.$
This is also quite a regular case. The only difference with (1) is the fact that the perturbation is only form bounded and one needs to use the KLMN theorem to define $H_{\mathrm{I}}$ or $H_{\mathrm{II}}$ [Kato, RS2].

(3) $\int \frac{|z(k)|^2}{h(k)^2} \mathrm{d}k < \infty, \qquad \int \frac{|z(k)|^2}{h(k)} \mathrm{d}k = \infty.$
The operator $H_{\mathrm{I}}$ is not defined. This follows from the fact that the "counterterm" in (1.8) is infinite. But the dressing operator (1.9) is well defined, which can be used to define $H_{\mathrm{II}}$.

Note that all the difficulty stems from the ultraviolet behavior of the interaction. Thus this case provides an example of "the ultraviolet renormalization": one has to subtract an infinite counterterm from $H_{\mathrm{I}}$ to define a Hamiltonian.

## 1.2 Infrared problem

Let us assume that the function $k \mapsto h(k)$ is bounded, but we allow $h$ to have arbitrarily small positive values. (This is typical for zero-mass particles with an ultraviolet cut-off). Then the operator $H_{\mathrm{I}}$ is always well defined. We can distinguish the following three cases:

(1) $\int \frac{|z(k)|}{h(k)^2} \mathrm{d}k < \infty$.
Again, this is the most regular case. The perturbation is operator bounded, both $H_{\mathrm{I}}$ and $H_{\mathrm{II}}$ are well defined by the Kato-Rellich theorem [Kato, RS2]. One can define the dressing operator (1.9) and the Hamiltonians have a ground state.

(2) $\int \frac{|z(k)|}{h(k)} \mathrm{d}k < \infty, \qquad \int \frac{|z(k)|^2}{h(k)^2} \mathrm{d}k = \infty$.
In this case, the perturbation is operator bounded, but the dressing operator is not defined. $H_{\mathrm{I}}$, and consequently also $H_{\mathrm{II}}$, have no ground states, but are bounded from below. This is the case where the infrared problem manifests itself. In quantum field theory this case is typical for massless particles (e.g. photons) interacting by a linear term with an external classical source of non-zero charge, as noted by Kibble [Ki]. One can say that the vacuum "escapes from the Hilbert space", "soft photons are present in every state in the Hilbert space". The dressing operator (at least in the usual sense) is no longer well defined.

(3) $\int |z(k)|^2 \mathrm{d}k < \infty, \qquad \int \frac{|z(k)|^2}{h(k)} \mathrm{d}k = \infty$.
This is the case where the infrared problem is the most severe. In order to define $H_{\mathrm{I}}$ one cannot use the Kato-Rellich nor the KLMN theorem. Instead one needs Nelson's commutator theorem [RS2]. $H_{\mathrm{I}}$ is unbounded from below and the dressing operator is not well defined. The operator $H_{\mathrm{II}}$ is not defined at all.

## 1.3 Scattering theory

Van Hove Hamiltonians are also interesting from the point of view of scattering theory. It turns out that the usual formalism of scattering theory, used in the context of Schrödinger operators and described for instance in [RS3], does not apply in the case of van Hove Hamiltonians. Instead, one needs to use some other versions to scattering theory. We have in fact a choice of at leat two approaches.

One of them is based on replacing the usual definition of wave operators with a definition that uses the so-called Abelian limit [Ya]. In order to define unitary wave operators one has to perform the so-called "renormalization of the wave function". This approach breaks down if we have the infrared problem (unless one is willing to divide by zero).

The second approach to scattering for van Hove Hamiltonians is based on the notion of asymptotic fields. It goes back to the so-called LSZ formalism. In this approach there is no need for renormalization and the infrared problem is manifested by the non-Fock property of asymptotic fields. An essentially the same formalism works in a much more complicated context, e.g. [HK, DG1, DG2].

Both approaches lead essentially to the same wave operator equal to the dressing operator. Unfortunately, the scattering operator turns out to be equal to one—so physically, scattering theory of van Hove Hamiltonians turns out to be trivial.

### 1.4 Remarks about the literature

Many of the ideas of this paper are contained in the literature in one form or another.

The analysis of the ultraviolet problem can be found eg. in the book of Berezin [Be], section III.7.4 and [Sch], following earlier papers [vH, EP, To, GS].

The understanding of the infrared problem can be traced back to the paper by Bloch and Nordsieck [BN], and was then analyzed in a series of papers of Kibble [Ki]. The infrared problem is related to the so-called coherent state representations, which were discussed already in [Frie]. The infrared problem was also studied for some models similar to the van Hove Hamiltonian in [Mi, Fr] and recently in [AHH], Lemma 4.3.

In recent literature one can find analysis of operators similar to but more complicated than van Hove Hamiltonians—under the names of the spin-boson, the Pauli-Fierz or the non-relativistic QED Hamiltonian, see eg. [DG1, BFS1, DJ]. In fact, one can view van Hove Hamiltonians as a special, exactly solvable subclass of Pauli-Fierz Hamiltonians. We believe that it is useful to study van Hove Hamiltonians to gain intuition about properties of Pauli-Fierz Hamiltonians.

In mathematical literature, scattering theory based on asymptotic fields applied to models similar to the van Hove model goes back to [KM1, KM2, HK], and was recently devoloped in [DG1, DG2]. In particular, in [KM2], Theorem 3.1, scattering for the van Hove Hamiltonian (without the infrared problem) is studied.

We could not find a complete treatment of van Hove Hamiltonians in the literature. We think that a careful analysis of van Hove Hamiltonian helps to understand some of the concepts of quantum field theory. It is also an instructive exercise in the theory of unbounded operators.

## 2 Notation

### 2.1 Differentiation in a Hilbert space

Let $\mathcal{W}$ be a Hilbert space with the scalar product $(\cdot|\cdot)$.

Let $\mathrm{Dom}\, G$ be a subset of $\mathcal{W}$ and $G : \mathrm{Dom}\, G \to \mathbb{R}$. Let $w_0 \in \mathrm{Dom}\, G$ and $w \in \mathcal{W}$. We will say that the derivative of $G$ at $w_0$ in the direction of $w$ exists iff there exists $\epsilon > 0$ such that $\{w_0 + tw \ : \ |t| < \epsilon\} \subset \mathrm{Dom}\, G$ and there exists

$$\frac{\mathrm{d}}{\mathrm{d}t} G(w_0 + tw)\Big|_{t=0} =: w\nabla G(w_0).$$

We will say that $G$ is differentiable at $w_0$ iff

$$\mathcal{D} := \{w \in \mathcal{W} \ : \ w\nabla G(w_0) \ \text{ exists } \}$$

is a dense linear subspace of $\mathcal{W}$ and $\mathcal{D} \ni w \mapsto w\nabla G(w_0)$ is a bounded linear functional. If this is the case, then the gradient of $G$ at $w_0$ is denoted by $\nabla G(w_0) \in \mathcal{W}$ and defined by

$$\mathrm{Re}(w|\nabla G(w_0)) = w\nabla G(w_0), \quad w \in \mathcal{W}.$$

We will say that a map $\alpha : \mathcal{W} \to \mathcal{W}$ preserves the scalar product iff

$$(\alpha(w_1) - \alpha(w_2)|\alpha(w_3) - \alpha(w_4)) = (w_1 - w_2|w_3 - w_4), \quad w_1, \dots, w_4 \in \mathcal{W}.$$

## 2.2 Unbounded perators

Let $h$ be a self-adjoint operator on $\mathcal{W}$ with $h \geq 0$. $\mathrm{sp}h$ will denote the spectrum of $h$. Let $\mathrm{Dom}\, h$ denote its domain and $(\mathrm{Dom}\, h)^*$ the space of bounded antilinear functionals on $\mathrm{Dom}\, h$. If $w_1 \in \mathrm{Dom}\, h$ and $w_2 \in (\mathrm{Dom}\, h)^*$, then we will write $(w_1|w_2)$ for the action of $w_2$ on $w_1$.

Clearly, $\mathcal{W}$ can be embedded in a natural way in $(\mathrm{Dom}\, h)^*$. The operators $h$ and $h^{1/2}$ extend to bounded maps from $\mathcal{W}$ to $(\mathrm{Dom}\, h)^*$. We denote by $h\mathcal{W}$ and $h^{1/2}\mathcal{W}$ respectively the images of these extensions.

Clearly, the algebraic sums $\mathcal{W} + h^{1/2}\mathcal{W}$, $h^{1/2}\mathcal{W} + h\mathcal{W}$ and $\mathcal{W} + h\mathcal{W}$ are well defined subspaces of $(\mathrm{Dom}\, h)^*$. (Note, in particular, that $\mathcal{W} + h\mathcal{W}$ coincides with $(\mathrm{Dom}\, h)^*$).

## 2.3 Fock spaces

[RS2, BR] If $\mathcal{X}$ is a vector space, then $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$ will denote the algebraic symmetric Fock space over $\mathcal{X}$, that means the space of finite linear combinations of symmetric tensor products of elements of $\mathcal{X}$. $\Omega$ will denote the vacuum.

If $\mathcal{W}$ is a Hilbert space, then $\Gamma_{\mathrm{s}}(\mathcal{W})$ will denote the (complete) Fock space, that is the completion of $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{W})$.

If $u$ is a contraction on $\mathcal{W}$, then $\Gamma(u)$ denotes the contraction on $\Gamma_{\mathrm{s}}(\mathcal{W})$ that on the $n$ particle sector equals $u^{\otimes n}$.

If $h$ is a self-adjoint operator, then $\mathrm{d}\Gamma(h)$ denotes the self-adjoint operator that on the $n$ particle sector equals $h \otimes 1^{\otimes(n-1)} + \cdots + 1^{\otimes(n-1)} \otimes h$.

Let $b$ be a sesquilinear form on $\mathcal{W}$ with the domain $\widetilde{\mathcal{W}}$. Abusing the notation, we will use the symbol $\mathrm{d}\Gamma(b)$ to denote the sequilinear form on $\Gamma_{\mathrm{s}}(\mathcal{W})$ with the domain $\overset{\circ}{\Gamma}_{\mathrm{s}}(\widetilde{\mathcal{W}})$ that on the $n$ particle sector equals $b \otimes 1^{\otimes(n-1)} + \cdots + 1^{\otimes(n-1)} \otimes b$.

## 2.4 Creation and annihilation operators

The notion of creation and annihillation operators is standard in the context of Fock spaces and can be found eg. in [BR, RS2]. Nevertheless, we will need slight generalizations of these concepts.

Let $w$ be an antilinear form on $\mathcal{W}$ with the domain $\mathrm{Dom} w = \widetilde{\mathcal{W}} \subset \mathcal{W}$. We define the annihilation operator $\overline{w}(a)$ as an operator with the domain $\mathring{\Gamma}_{\mathrm{s}}(\widetilde{\mathcal{W}})$ satisfying

$$\overline{w}(a)\, z^{\otimes n} := \sqrt{n}(w|z) z^{\otimes(n-1)}, \quad z \in \widetilde{\mathcal{W}}.$$

(Note that vectors of the form $z^{\otimes n}$ span $\mathring{\Gamma}_{\mathrm{s}}(\widetilde{\mathcal{W}})$).

The operator $\overline{w}(a)$ is closable iff $w \in \mathcal{W}$. If this is the case, we denote its closure by the same symbol. Its adjoint is called the creation operator and denoted

$$w(a^*) := \overline{w}(a)^*.$$

If $f \in \mathring{\Gamma}_{\mathrm{s}}(\mathcal{Z})$, then $f(a^*)$ and $\overline{f}(a)$ have the obvious meaning as polynomials in $a^*$ and $a$. For instance $z^{\otimes n}(a^*) = (z(a^*))^n$ and $\overline{z}^{\otimes n}(a) = (\overline{z}(a))^n$.

We also introduce the field operator

$$\phi(w) := \frac{1}{\sqrt{2}}(w(a^*) + \overline{w}(a)), \tag{2.11}$$

which is self-adjoint on $\mathrm{Dom}\, \overline{w}(a)$. We also introduce the Weyl operators

$$W(w) := \mathrm{e}^{\mathrm{i}\phi(w)}. \tag{2.12}$$

If $w \notin \mathcal{W}$, then the annihilation operator $\overline{w}(a)$ is not closable. In this case the creation operator $w(a^*)$ is not even densely defined and is not of much use. On the other hand, we can define the "creation form", denoted $w(a^{*\mathrm{f}})$:

$$\mathring{\Gamma}_{\mathrm{s}}(\widetilde{\mathcal{W}}) \times \Gamma_{\mathrm{s}}(\mathcal{W}) \ni (\Phi, \Psi) \mapsto (\Phi|w(a^{*\mathrm{f}})\Psi) := (\overline{w}(a)\Phi|\Psi).$$

Note that $w(a^{*\mathrm{f}})$ is a different object from $w(a^*)$. In what follows, however, we will sometimes abuse the notation and we will write $w(a^*)$ instead of $w(a^{*\mathrm{f}})$.

# 3 Van Hove Hamiltonians

In Subsections 3.1 and 3.2 we consider the classical dynamics $\alpha$ generated by a Hamiltonian $G$ equal to a quadratic polynomial. In the remaining subsections we discuss their quantum counterparts—the dynamics $\beta$ and its Hamiltonian $H$.

## 3.1 Classical dynamics

Suppose that $h$ is a positive operator on $\mathcal{W}$. We will assume that $\mathrm{Ker} h = \{0\}$. Let

$$z \in \mathcal{W} + h\mathcal{W} \tag{3.13}$$

For $w \in \mathcal{W}$, $t \in \mathbb{R}$, we define

$$\alpha^t(w) := \mathrm{e}^{\mathrm{i}th} w + (\mathrm{e}^{\mathrm{i}th} - 1)h^{-1}z.$$

It is easy to see that $\mathbb{R} \ni t \mapsto \alpha^t$ is a 1-parameter group of transformations preserving the scalar product.

Let us note the following property of the dynamics $\alpha$. Let $p_1, p_2$ be two complementary orthogonal projections commuting with $h$. For $i = 1, 2$ let $\mathcal{W}_i := \mathrm{Ran} p_i$. For $w \in \mathcal{W}$, set $w_i := p_i w$. Set $h_i := p_i h$, treated as a self-adjoint operator on $\mathcal{W}_i$. Let $\alpha_i$ be the dynamics on $\mathcal{W}_i$ defined by $h_i$, $z_i$. Then the dynamics $\alpha$ splits:

$$\alpha^t(w_1, w_2) = (\alpha_1^t(w_1), \alpha_2^t(w_2)).$$

In particular, we can take

$$p_1 := 1_{[0,1]}(h), \quad p_2 := 1_{]1,\infty[}(h). \tag{3.14}$$

Then $h_1$ is bounded and $h_2$ is bounded from below by a positive constant. In the case of $h_1$, the ultraviolet problem is absent, but the infrared problem can show up. In the case of $h_2$ we have the opposite situation: the infrared problem is absent, but we can face the ultraviolet problem.

## 3.2 Classical Hamiltonian

The dynamics $\alpha$ preserves the symplectic form $\mathrm{Im}(\cdot|\cdot)$. Therefore, we should expect that $\alpha$ possesses a Hamiltonian $G$, that is a function satisfying

$$\mathrm{Im}(w_1|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha^t(w)) = \mathrm{Re}(w_1|\nabla G(\alpha^t(w))), \quad w_1 \in \mathcal{W}, \tag{3.15}$$

or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha^t(w) = \mathrm{i}\nabla G(\alpha^t(w)). \tag{3.16}$$

In this subsection we will construct such Hamiltonians. (Note, however, that in general these Hamiltonians will not be everywhere defined).

First we define Hamiltonians of $\alpha$ in two special cases.

If $z \in \mathcal{W} + h^{1/2}\mathcal{W}$, then we define $\mathrm{Dom}\, G_{\mathrm{I}} := \mathrm{Dom}\, h^{1/2}$ and for $w \in \mathrm{Dom}\, G_{\mathrm{I}}$ we set

$$G_{\mathrm{I}}(w) := \frac{1}{2}\left((w|hw) + (z|w) + (w|z)\right).$$

We will say that $G_{\rm I}$ is the Hamiltonian of $\alpha$ of the first kind.

If $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$, then we define $\mathrm{Dom}\, G_{\rm II} := \{w \in \mathcal{W} \ : \ h^{1/2}w + h^{-1/2}z \in \mathcal{W}\}$, and for $w \in \mathrm{Dom}\, G_{\rm II}$ we set

$$G_{\rm II}(w) := \frac{1}{2}\|h^{1/2}w + h^{-1/2}z\|^2.$$

We will say that $G_{\rm II}$ is the Hamiltonian of $\alpha$ of the second kind.

Clearly, both $G_{\rm I}$ and $G_{\rm II}$ are well defined iff $z \in h^{1/2}\mathcal{W}$, and then

$$G_{\rm II} = G_{\rm I} + \frac{1}{2}(z|h^{-1}z).$$

If $z$ is any functional satisfying (3.13) we split the dynamics $\alpha$ into two parts $\alpha_1$ and $\alpha_2$, as explained at the end of Subsection 3.1. We can then define the Hamiltonian $G_{1,\rm I}$ for the dynamics $\alpha_1$ and the Hamiltonian $G_{2,\rm II}$ for the dynamics $\alpha_2$. We will say that a function $G$ is a Hamiltonian of $\alpha$ iff its domain equals $\mathrm{Dom}\, G_{1,\rm I} \oplus \mathrm{Dom}\, G_{2,\rm II}$ and it is equal to

$$G(w_1, w_2) := G_{1,\rm I}(w_1) + G_{2,\rm II}(w_2) + c,$$

where $c \in \mathbb{R}$ is arbitrary.

Note that, in general, $(\mathcal{W} + h^{1/2}\mathcal{W}) \cup (h^{1/2}\mathcal{W} + h\mathcal{W})$ is strictly smaller than $\mathcal{W} + h\mathcal{W}$. Therefore, for some $z$, the dynamics $\alpha$ is well defined but neither the Hamiltonian of the first kind $G_{\rm I}$ nor the Hamiltonian of the second kind $G_{\rm II}$ are well defined.

**Theorem 3.1** *Let $z \in \mathcal{W} + h\mathcal{W}$ and let $G$ be a Hamiltonian for $\alpha^t$. Then the following statements are true:*

(1) *The function $G$ is differentiable at $w \in \mathcal{W}$ iff $hw + z$ belongs to $\mathcal{W}$. We then have*

$$\nabla G(w) = hw + z.$$

*The dynamics $t \mapsto \alpha^t(w)$ is differentiable wrt $t$ iff $h\alpha^t(w) + z \in \mathcal{W}$. We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha^t(w) = \mathrm{i}(h\alpha^t(w) + z),$$

*which can be written in the form (3.16). Note also that $\alpha^t$ leaves $\mathrm{Dom}\, G$ invariant and $G$ is constant along the trajectories.*

(2) *$0$ belongs to $\mathrm{Dom}\, G$ iff $z \in \mathcal{W} + h^{1/2}\mathcal{W}$. We have $G = G_{\rm I}$ iff $G_{\rm I}(0) = 0$.*

(3) *$G$ is bounded from below iff $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$. We have $G = G_{\rm II}$ iff $\inf G = 0$.*

(4) *$G$ has a minimum iff $z \in h\mathcal{W}$. This minimum is at $-h^{-1}z$, and then*

$$G_{\rm II}(w) = \frac{1}{2}\left(w + h^{-1}z|h(w + h^{-1}z)\right).$$

## 3.3 Quantum dynamics

Let $h$, $z$ be as above. Assume (3.13), that is $z \in \mathcal{W} + h\mathcal{W}$. Define the family of unitary operators on $\Gamma_{\mathrm{s}}(\mathcal{W})$

$$V(t) := \Gamma(\mathrm{e}^{\mathrm{i}th}) \exp\Big((1-\mathrm{e}^{-\mathrm{i}th})h^{-1}z(a^*) - (1-\mathrm{e}^{\mathrm{i}t\overline{h}})\overline{h}^{-1}\overline{z}(a)\Big).$$

Define for $B \in \Gamma_{\mathrm{s}}(\mathcal{W})$

$$\beta^t(B) := V(t)BV(t)^*. \tag{3.17}$$

It is easy to check that $\beta$ is a 1-parameter group of $*$-automorphisms of $B(\Gamma_{\mathrm{s}}(\mathcal{W}))$ continuous in the strong operator topology.

In order to make the relationship with the classical dynamics more clear, one can note that

$$\beta^t(w(a^*)) = ((\mathrm{e}^{\mathrm{i}th}-1)h^{-1}z|w) + (\mathrm{e}^{\mathrm{i}th}w)(a^*).$$

Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$, as in Subsection 3.1. Then we have the identification $\Gamma_{\mathrm{s}}(\mathcal{W}) = \Gamma_{\mathrm{s}}(\mathcal{W}_1) \otimes \Gamma_{\mathrm{s}}(\mathcal{W}_2)$. The dynamics $\beta$ factorizes

$$\beta^t(B_1 \otimes B_2) = \beta_1^t(B_1) \otimes \beta_2^t(B_2), \quad B_1 \in B(\Gamma_{\mathrm{s}}(\mathcal{W}_1)),\ B_2 \in B(\Gamma_{\mathrm{s}}(\mathcal{W}_2)). \tag{3.18}$$

## 3.4 Quantum Hamiltonian

We say that a self-adjoint operator $H$ is a Hamiltonian of the dynamics $\beta$ iff

$$\beta^t(B) = \mathrm{e}^{\mathrm{i}tH}B\mathrm{e}^{-\mathrm{i}tH}. \tag{3.19}$$

Clearly, (3.19) defines $H$ only up to an additive constant.

For $z \in \mathcal{W} + h^{1/2}\mathcal{W}$, define

$$U_{\mathrm{I}}(t) := \mathrm{e}^{\mathrm{i}\mathrm{Im}(h^{-1}z|\mathrm{e}^{\mathrm{i}th}h^{-1}z) - \mathrm{i}t(z|h^{-1}z)}V(t).$$

For $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$, define

$$U_{\mathrm{II}}(t) := \mathrm{e}^{\mathrm{i}\mathrm{Im}(h^{-1}z|\mathrm{e}^{\mathrm{i}th}h^{-1}z)}V(t).$$

We easily check that both $U_{\mathrm{I}}(t)$ and $U_{\mathrm{II}}(t)$ are 1-parameter strongly continuous unitary groups. Therefore, there exist self-adjoint operators $H_{\mathrm{I}}$ and $H_{\mathrm{II}}$ such that

$$U_{\mathrm{I}}(t) = \mathrm{e}^{\mathrm{i}tH_{\mathrm{I}}}, \qquad U_{\mathrm{II}}(t) = \mathrm{e}^{\mathrm{i}tH_{\mathrm{II}}}.$$

Clearly, both $H_{\mathrm{I}}$ and $H_{\mathrm{II}}$ are Hamiltonians of $\beta$. They are both well defined iff $z \in h^{1/2}\mathcal{W}$, and then

$$H_{\mathrm{II}} = H_{\mathrm{I}} + (z|h^{-1}z).$$

A Hamiltonian of $\beta$ will be called a van Hove Hamiltonian. The operator $H_{\mathrm{I}}$ will be called the van Hove Hamiltonian of the first kind and $H_{\mathrm{II}}$ will be called the van Hove Hamiltonian of the second kind.

**Theorem 3.2** *Let $z \in \mathcal{W} + h\mathcal{W}$. Then the following statements are true:*

(1) *There exist Hamiltonians of $\beta$.*

(2) *Let $H$ be a Hamiltonian of $\beta$. $\Omega$ belongs to $\mathrm{Dom}|H|^{1/2}$ (the form domain of $H$) iff $z \in \mathcal{W} + h^{1/2}\mathcal{W}$. Under this condition $H = H_{\mathrm{I}}$ iff $(\Omega|H\Omega) = 0$.*

(3) *The operator $H$ is bounded from below iff $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$. Under this condition $H = H_{\mathrm{II}}$ iff $\inf H = 0$.*

(4) *The operator $H$ has a ground state ($\inf H$ is an eigenvalue of $H$, where $\inf H$ denotes the infimum of the spectrum of $H$) iff $z \in h\mathcal{W}$. Then we can define the "dressing operator"*

$$U := \exp\left(-h^{-1}z(a^*) + \overline{h}^{-1}\overline{z}(a)\right),$$

*and*

$$H_{\mathrm{II}} = U\mathrm{d}\Gamma(h)U^*. \tag{3.20}$$

**Proof of (1).** We split the dynamics $\beta$ as in (3.18) with the projections $p_1$, $p_2$ given by (3.14). Then we can define the Hamiltonian $H_{1,\mathrm{I}}$ for $\beta_1$ and the Hamiltonian $H_{2,\mathrm{II}}$ for $\beta_2$. We set

$$H := H_{1,\mathrm{I}} \otimes 1 + 1 \otimes H_{2,\mathrm{II}}.$$

□

Easy calculations show that, at least formally,

$$H_{\mathrm{I}} = \mathrm{d}\Gamma(h) + z(a^*) + \overline{z}(a), \tag{3.21}$$

$$H_{\mathrm{II}} = \mathrm{d}\Gamma(h) + z(a^*) + \overline{z}(a) + (z|h^{-1}z). \tag{3.22}$$

Below we will make these formulas precise.

**Remark 3.3** *By the spectral theorem, one can find a measure space $(K, \mathrm{d}k)$ such that $\mathcal{W}$ is isomorphic to $L^2(K, \mathrm{d}k)$ and $h$ is a multiplication operator by a measurable function $K \ni k \mapsto h(k)$. Then we can introduce $K \ni k \mapsto a^*(k), a(k)$ and write $\int z(k)a^*(k)\mathrm{d}k$, $\int \overline{z}(k)a(k)$, $\int h(k)a^*(k)a(k)\mathrm{d}k$ instead of $z(a^*)$, $\overline{z}(a)$ and $\mathrm{d}\Gamma(h)$. We used this notation in the introduction. For example, Condition (1.2) of the introduction,*

$$\int_{h(k)<1} |z(k)|^2\mathrm{d}k + \int_{h(k)\geq 1} \tfrac{|z(k)|^2}{h(k)^2}\mathrm{d}k < \infty,$$

*corresponds to the condition $z \in \mathcal{W} + h\mathcal{W}$.*

*The notation involving the operator valued measures $a^*(k)$ and $a(k)$ is very common and often convenient, but it depends on a non-canonical choice of the measure space $K$, and therefore we do not use it.*

## 3.5 3 types of the ultraviolet problem

In this subsection we describe van Hove Hamiltonians without the infrared problem.

**Theorem 3.4** *Assume that $h$ is a self-adjoint (possibly unbounded) operator bounded from below by a positive constant. Then the dressing operator $U$ and the van Hove Hamiltonian of the second kind $H_{\mathrm{II}}$ are well defined. $H_{\mathrm{II}}$ possesses a unique ground state at* 0. *Moreover, we can distinguish 3 cases:*

(1) *Let $z \in \mathcal{W}$.*
*Then $z(a^*) + \overline{z}(a)$ is a $\mathrm{d}\Gamma(h)$-bounded operator with the infinitesimal bound. $H_{\mathrm{I}}$ and $H_{\mathrm{II}}$ are self-adjoint on $\mathrm{Dom}\,\mathrm{d}\Gamma(h)$ and can be defined by the formulas (3.21), (3.22) and by the Kato-Rellich theorem.*

(2) *Let $z \in h^{1/2}\mathcal{W}\backslash\mathcal{W}$.*
*Then $z(a^*) + \overline{z}(a)$ is not an operator. Instead of $z(a^*) + \overline{z}(a)$ we should consider the form with the domain $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathrm{Dom}\, z)$ equal to $z(a^{*\mathrm{f}}) + \overline{z}(a)$. This form is $\mathrm{d}\Gamma(h)$-form bounded with the infinitesimal bound. The operators $H_{\mathrm{I}}$ and $H_{\mathrm{II}}$ are bounded from below and their form domains equal $\mathrm{Dom}\,\mathrm{d}\Gamma(h)^{1/2}$. They can be defined by the formulas (3.21), (3.22) and by the KLMN theorem.*

(3) *Let $z \in h\mathcal{W}\backslash h^{1/2}\mathcal{W}$.*
*Then the form $z(a^{*\mathrm{f}}) + \overline{z}(a)$ is not $\mathrm{d}\Gamma(h)$-form bounded. $H_{\mathrm{I}}$ is not defined.*

## 3.6 3 types of the infrared problem

In this subsection we describe van Hove Hamiltonians without the ultraviolet problem.

**Theorem 3.5** *Assume that $h$ is a bounded positive operator. Then the formula (3.21) defines the operator $H_{\mathrm{I}}$ as an essentially self-adjoint operator on $\mathrm{Dom}\,\mathrm{d}\Gamma(h{+}1)$ by Nelson's commutator theorem. Moreover, we can distinguish the following three cases:*

(1) *Let $z \in h\mathcal{W}$*
*Then $z(a^*) + \overline{z}(a)$ is $\mathrm{d}\Gamma(h)$-bounded with the infinitesimal bound. The operators $H_{\mathrm{I}}$ and $H_{\mathrm{II}}$ are self-adjoint on $\mathrm{Dom}\,\mathrm{d}\Gamma(h)$ and they can be defined by the formulas (3.21), (3.22) and the Kato-Rellich theorem. They have ground states and the dressing operator $U$ is well defined.*

(2) *Let $z \in h^{1/2}\mathcal{W}\backslash h\mathcal{W}$*
*Then $z(a^*){+}\overline{z}(a)$ is $\mathrm{d}\Gamma(h)$-bounded with the infinitesimal bound. Again, the operators $H_{\mathrm{I}}$ and $H_{\mathrm{II}}$ are self-adjoint on $\mathrm{Dom}\,\mathrm{d}\Gamma(h)$ and they can be defined by the formulas (3.21), (3.22) and the Kato-Rellich theorem. They are bounded from below but have no ground state and the dressing operator $U$ is not defined.*

(3) *Let* $z \in \mathcal{W} \backslash h^{1/2}\mathcal{W}$.
*Then* $z(a^*) + \overline{z}(a)$ *is not* $\mathrm{d}\Gamma(h)$*-form bounded.* $H_{\mathrm{I}}$ *is not bounded from below and the operator* $H_{\mathrm{II}}$ *is not defined at all.*

In the following subsections we will show various elements of the above theorems.

## 3.7 Relative form boundedness of field operators

**Proposition 3.6** *Let* $h$ *be a positive operator on* $\mathcal{W}$ *and* $z \in h^{1/2}\mathcal{W}$. *Then*

(1)
$$\|\overline{z}(a)\Psi\|^2 \leq (z|h^{-1}z)(\Psi|\mathrm{d}\Gamma(h)\Psi).$$

(2) $z(a^{*\mathrm{f}}) + \overline{z}(a)$ *is form bounded wrt* $\mathrm{d}\Gamma(h)$ *with the infinitesimal bound. More precisely, for any* $\epsilon > 0$, *we have*
$$|(\Psi|(z(a^{*\mathrm{f}}) + \overline{z}(a))\Psi)| \leq \epsilon(z|h^{-1}z)(\Psi|\mathrm{d}\Gamma(h)\Psi) + \epsilon^{-1}\|\Psi\|^2.$$

**Proof.** (1) If $z$ is an antilinear functional on $\mathcal{W}$ with the domain $\mathrm{Dom}z$, then
$$\mathrm{Dom}\, z \times \mathrm{Dom}\, z \ni (w_1, w_2) \mapsto (w_1|z)(z|w_2)$$
defines a sesquilinear form, that we will denote by $|z)(z|$. Note that the following inequality is true:
$$|z)(z| \leq (z|h^{-1}z)h, \tag{3.23}$$

Clearly,
$$\|\overline{z}(a)\Psi\|^2 = (\Psi|\mathrm{d}\Gamma\left(|z)(z|\right)\Psi).$$
(3.23) implies
$$\mathrm{d}\Gamma\left(|z)(z|\right) \leq (z|h^{-1}z)\mathrm{d}\Gamma(h).$$

(2)
$$\begin{aligned} \pm(\Psi|(z(a^{*\mathrm{f}}) + \overline{z}(a))\Psi) &= \pm 2\mathrm{Re}\Big(\Psi|\overline{z}(a)\Psi) \\ &\leq 2\|\Psi\|\|\overline{z}(a)\Psi\| \leq \epsilon^{-1}\|\Psi\|^2 + \epsilon\|\overline{z}(a)\Psi\|^2. \end{aligned}$$

□

**Corollary 3.7** *If* $z \in h^{1/2}\mathcal{W}$, *then both (3.21) and (3.22) (with* $z(a^*)$ *replaced by* $z(a^{*\mathrm{f}})$*) define by the KLMN theorem the self-adjoint operators* $H_{\mathrm{I}}$ *and* $H_{\mathrm{II}}$ *with the form domains* $\mathrm{Dom}\,\mathrm{d}\Gamma(h)^{1/2}$.

## 3.8 Relative boundedness of field operators

**Proposition 3.8** *Let $h$ be a positive operator on $\mathcal{W}$ and $z \in h^{1/2}\mathcal{W} \cap \mathcal{W}$. Then*

(1)
$$\|(z(a^*) + \overline{z}(a))\Psi\|^2 \leq 4(z|h^{-1}z)(\Psi|\mathrm{d}\Gamma(h)\Psi) + 2\|z\|^2\|\Psi\|^2.$$

(2) *$z(a^*) + \overline{z}(a)$ is bounded wrt $\mathrm{d}\Gamma(h)$ with the infinitesimal bound. More precisely, for any $\epsilon > 0$, we have*

$$\|(z(a^*) + \overline{z}(a))\Psi\|^2 \leq 2\epsilon(z|h^{-1}z)\|\mathrm{d}\Gamma(h)\Psi\|^2 + \left(2\epsilon^{-1}(z|h^{-1}z) + 2\|z\|^2\right)\|\Psi\|^2.$$

**Proof.** (2) follows immediately from (1). To see (1) we compute using Proposition 3.6 (1):

$$\begin{aligned} \|(z(a^*) + \overline{z}(a))\Psi\|^2 &\leq 2\|z(a^*)\Psi\|^2 + \|\overline{z}(a)\Psi\|^2 \\ &= 4\|z(a^*)\Psi\|^2 + 2\|z\|^2\|\Psi\|^2 \\ &\leq 4(z|h^{-1}z)(\Psi|\mathrm{d}\Gamma(h)\Psi) + 2\|z\|^2\|\Psi\|^2. \end{aligned}$$

□

**Corollary 3.9** *If $z \in \mathcal{W} \cap h^{1/2}\mathcal{W}$, then both (3.21) and (3.22) define by the Kato-Rellich theorem the self-adjoint operators $H_{\mathrm{I}}$ and $H_{\mathrm{II}}$ with the domains* $\mathrm{Dom}\,\mathrm{d}\Gamma(h)$.

## 3.9 The infimum of van Hove Hamiltonians

**Proposition 3.10** *Assume $z \in \mathcal{W} + h^{1/2}\mathcal{W}$. Then the operator $H_{\mathrm{I}}$ satisfies*

$$\begin{aligned} &\inf H_{\mathrm{I}} = -(z|h^{-1}z), \\ &\mathrm{sp} H_{\mathrm{I}} = -(z|h^{-1}z) + \mathrm{sp}\,\mathrm{d}\Gamma(h). \end{aligned}$$

**Proof.** We drop I from $H_{\mathrm{I}}$. It is sufficient to assume that

$$0 \text{ is not an isolated point in the spectrum of } h. \tag{3.24}$$

This in particular implies that $\mathrm{sp}\,\mathrm{d}\Gamma(h) = [0, \infty[$. Thus we need to show that

$$\mathrm{sp} H = [-(z|h^{-1}z), \infty[.$$

**Step 1)** Setting $\epsilon = \pm(z|h^{-1}z)^{-1}$ in Proposition 3.6 (1) with we get

$$\mathrm{d}\Gamma(h) + z(a^*) + \overline{z}(a) \geq -(z|h^{-1}z). \tag{3.25}$$

**Step 2)** For any $n$ define $\mathcal{W}^n := 1_{[\frac{1}{n},\infty[}(h)\mathcal{W}$, $h^n := 1_{[\frac{1}{n},\infty[}(h)h$, $z^n := 1_{[\frac{1}{n},\infty[}(h)z$, and the operators on $\Gamma_{\mathrm{s}}(\mathcal{W}^n)$, $H^n := \mathrm{d}\Gamma(h^n) + z^n(a^*) + \overline{z^n}(a)$ and $U^n := \exp(-h^{-1}z^n(a^*) + \overline{h}^{-1}\overline{z}^n(a))$. Then $U^{n*}H^nU^n = \mathrm{d}\Gamma(h^n) - (z^n|h^{-1}z^n)$. Clearly, $\inf \mathrm{d}\Gamma(h^n) = 0$, hence

$$\begin{aligned} \inf H^n &= -(z^n|h^{-1}z^n), \\ \mathrm{sp}H^n &= -(z^n|h^{-1}z^n) + \mathrm{sp}\mathrm{d}\Gamma(h^n). \end{aligned} \tag{3.26}$$

**Step 3)** Likewise, define $\mathcal{W}_n := 1_{[0,\frac{1}{n}[}(h)\mathcal{W}$, $h_n := 1_{[0,\frac{1}{n}[}(h)h$, $z_n := 1_{[0,\frac{1}{n}[}(h)z$, and the operator on $\Gamma_{\mathrm{s}}(\mathcal{W}_n)$, $H_n := \mathrm{d}\Gamma(h_n) + z_n(a^*) + \overline{z}_n(a)$. We have $\Omega \in \mathrm{Dom}H_n$ and $(\Omega|H_n\Omega) = 0$. Hence,

$$\inf H_n \leq 0. \tag{3.27}$$

Arguing as in Step 1) we get

$$H_n \geq -(z_n|h^{-1}z_n). \tag{3.28}$$

**Step 4)** $\Gamma_{\mathrm{s}}(\mathcal{W})$ can be identified with $\Gamma_{\mathrm{s}}(\mathcal{W}_n) \otimes \Gamma_{\mathrm{s}}(\mathcal{W}^n)$ and

$$H = H_n \otimes 1 + 1 \otimes H^n. \tag{3.29}$$

Therefore,

$$\begin{aligned} \inf H &= \inf H_n + \inf H^n, \\ \mathrm{sp}H &= \mathrm{sp}H_n + \mathrm{sp}H^n. \end{aligned} \tag{3.30}$$

**Step 5)** If $t > 0$, then by (3.24), we will find a sequence $t_n \in \mathrm{sp}\,\mathrm{d}\Gamma(h_n)$ such that $t_n \to t$. By Step 4), $\inf H_n + t_n \in \mathrm{sp}\,H$. But by Step 3), $\inf H_n \to 0$. By the closedness of $\mathrm{sp}H$, $t \in \mathrm{sp}\,H$. □

## 3.10 Essential self-adjointness of van Hove Hamiltonians

**Proposition 3.11** *Suppose that $z \in \mathcal{W}$. Then $H_{\mathrm{I}}$ is essentially self-adjoint on* $\mathrm{Dom}\,\mathrm{d}\Gamma(1+h)$.

**Proof.** First assume that $h$ is bounded. We apply Nelson's commutator theorem with the comparison operator $B := 1 + \mathrm{d}\Gamma(1 + h)$ [RS2]. In fact,

$$\|(z(a^*) + \overline{z}(a))\Psi\| \leq c\|N\Psi\| \leq c\|B\Psi)\|,$$

$$\|\mathrm{d}\Gamma(h)\Psi\| \leq \|B\Psi\|.$$

Moreover,

$$[B, (z(a^*) + \overline{z}(a))] = (1 + h)z(a^*) + (1 + \overline{h})\overline{z}(a).$$

Hence

$$|(\Psi|[B, (z(a^*) + \overline{z}(a))]\Psi)| \leq (\Psi|B\Psi).$$

Hence $H$ is essentially self-adjoint on $\mathrm{Dom}B$.

Next consider an arbitrary $h$. As described in Subsection 3.3, we can split $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$, where $\mathcal{W}_1 = \mathrm{Ran}1_{[0,1]}(h)$. We can define the operator $H_{\mathrm{I}}$ on $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{W}) = \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{W}_1) \otimes \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{W}_2)$ and it splits as

$$H_{\mathrm{I}} = H_{\mathrm{I},1} \otimes 1 + 1 \otimes H_{\mathrm{I},2},$$

with

$$H_{\mathrm{I},i} = \mathrm{d}\Gamma(h_i) + z_i(a^*) + \overline{z}_i(a).$$

We proved above that $H_{\mathrm{I},1}$ is essentially self-adjoint on $\mathrm{Dom}\,\mathrm{d}\Gamma(h_1 + 1)$. By corollary 3.9, $H_{\mathrm{I},2}$ is self-adjoint on $\mathrm{Dom}\,\mathrm{d}\Gamma(h_2)$. This implies that $H_{\mathrm{I}}$ is essentially self-adjoint on $\mathrm{Dom}\,\mathrm{d}\Gamma(h_1+1)\overset{\circ}{\otimes}\mathrm{Dom}\,\mathrm{d}\Gamma(h_1)$, which is dense in $\mathrm{Dom}\,\mathrm{d}\Gamma(h+1)$. ($\overset{\circ}{\otimes}$ denotes the algebraic tensor product). □

## 3.11 Absence of a ground state

Let us recall the following well known result about coherent states:

**Theorem 3.12** *Suppose that $\widetilde{\mathcal{W}}$ is a dense subspace of $\mathcal{W}$ and $f$ is an antilinear functional on $\widetilde{\mathcal{W}}$. Let $\Psi \in \Gamma_{\mathrm{s}}(\mathcal{W})$, for any $w \in \widetilde{\mathcal{W}}$, $\Psi \in \mathrm{Dom}\overline{h}(a)$ and*

$$\overline{h}(a)\Psi = (h|f)\Psi.$$

*Then the following is true:*

(1) *If $f \in \mathcal{W}$, then $\Psi$ is proportional to $\exp(f(a^*) - \overline{f}(a))\Omega$.*

(2) *If $f \notin \mathcal{W}$, then $\Psi = 0$.*

**Proof.** By induction we show that for $w_1, \dots, w_n \in \widetilde{\mathcal{W}}$, $\overline{w}_{n-1}(a) \cdots \overline{w}_1(a)\Psi \in \mathrm{Dom}\overline{w}_n(a)$ and

$$\overline{w}_n(a) \cdots \overline{w}_1(a)\Psi = (w_1|f) \cdots (w_n|f)\Psi.$$

This implies

$$(w_1(a^*) \cdots w_n(a^*)\Omega|\Psi) = (w_1|f) \cdots (w_n|f)(\Omega|\Psi). \tag{3.31}$$

In particular,

$$(w|\Psi) = (w(a^*)\Omega|\Psi) = (w|f)(\Omega|\Psi), \quad w \in \widetilde{\mathcal{W}}.$$

Using the fact that $\widetilde{\mathcal{W}}$ is dense in $\mathcal{W}$ we see that $(\Omega|\Psi)f$ is a bounded functional on $\mathcal{W}$, hence it belongs to $\mathcal{W}$. Thus either $f \in \mathcal{W}$ or $(\Omega|\Psi) = 0$. In the latter case, (3.31) implies that $\Psi = 0$. □

**Proposition 3.13** *$H_{\mathrm{I}}$ has a ground state iff $z \in h\mathcal{W}$.*

**Proof.** If $z \in h\mathcal{W}$, then we can define the dressing transformation and $U\Omega$ is a ground state of $H_{\mathrm{I}}$.

Suppose that $z \notin h\mathcal{W}$. We use the notation of the proof of Proposition 3.10. Let $\Psi$ be a ground state of $H$. Then it is also a ground state of $H_n \otimes 1$ and of $1 \otimes H^n$. Being a ground state of $1 \otimes H^n$, it must be equal to $\Psi_n \otimes U^n\Omega$. Therefore, for $w \in \widetilde{\mathcal{W}} := \bigcup\limits_{n=1}^{\infty} \mathcal{W}^n$

$$\overline{w}(a)\Psi = (w|h^{-1}z)\Psi.$$

But $\widetilde{\mathcal{W}}$ is dense in $\mathcal{W}$. By Theorem 3.12, this means that either $h^{-1}z \in \mathcal{W}$ or $\Psi = 0$. □

# 4 Scattering theory

## 4.1 The usual formalism

The most common setup for scattering theory starts with a pair of self-adjoint operators $H_0$ and $H$. The wave operators $\Omega^{\pm}$ are defined by the formulas

$$\Omega^{\pm} := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}. \tag{4.32}$$

Note that $\Omega^{\pm}$ are automatically isometric and

$$\Omega^{\pm}H_0 = H\Omega^{\pm}. \tag{4.33}$$

The scattering operator is defined as

$$S := \Omega^{+*}\Omega^{-}. \tag{4.34}$$

It satisfies

$$S = \mathrm{w-}\lim_{t_+,t_-\to\infty} \mathrm{e}^{\mathrm{i}t_+H_0}\mathrm{e}^{-\mathrm{i}(t_++t_-)H}\mathrm{e}^{\mathrm{i}t_-H_0} \tag{4.35}$$

and commutes with $H_0$. If we have

$$\mathrm{Ran}\Omega^{+} = \mathrm{Ran}\Omega^{-},$$

then $S$ is unitary.

## 4.2 Wave operators defined by the Abelian limit—general formalism

The limits in (4.32) often do not exists. This happens, for instance, if $H_0$ has an eigenvector, which is not an eigenvector of $H$. This is the case of many models of quantum field theory, whose free vacuum (the unique eigenvector of $H_0$) is often different from the interacting vacuum (the unique eigenvector of $H$).

Nevertheless, sometimes even in this situation some kind of a scattering theory can be developed. In this subsection we will describe one of possible approaches to scattering theory, which, as we will see, works in the case of van Hove Hamiltonians. One can argue that this approach, or some its variation, is implicit in most textbook presentations of QFT.

Again, we start with a pair of self-adjoint operators $H_0$ and $H$. We suppose that there exists the Abelian limit

$$\Omega_{\rm ur}^{\pm} := \mathrm{s}-\lim_{\epsilon\downarrow 0} 2\epsilon \int_0^\infty \mathrm{e}^{\pm \mathrm{i}tH}\mathrm{e}^{\mp \mathrm{i}tH_0}\mathrm{e}^{-2\epsilon t}\mathrm{d}t. \tag{4.36}$$

Note that there is no guarantee that $\Omega_{\rm ur}^{\pm}$ are isometric. One knows only that $\Omega_{\rm ur}^{\pm}$ are contractions. One can easily see that $\Omega_{\rm ur}^{\pm}$ have the intertwining property:

$$\Omega_{\rm ur}^{\pm}H_0 = H\Omega_{\rm ur}^{\pm}. \tag{4.37}$$

We will call $\Omega_{\rm ur}^{\pm}$ the "unrenormalized wave operators". Of course, it may happen that $\Omega_{\rm ur}^{\pm} = 0$.

Define the "renormalization of wave function operator"

$$Z^{\pm} := \Omega_{\rm ur}^{\pm *}\Omega_{\rm ur}^{\pm}.$$

It is easy to see hat $Z^{\pm}$ commutes with $H_0$. Assume that $\mathrm{Ker}Z^{\pm} = \{0\}$. Then we define the "renormalized wave operators"

$$\Omega_{\rm rn}^{\pm} := \Omega_{\rm ur}^{\pm}(Z^{\pm})^{-1/2}.$$

Note that $\Omega_{\rm rn}^{\pm}$ are isometric and have the intertwining property:

$$\Omega_{\rm rn}^{\pm}H_0 = H\Omega_{\rm rn}^{\pm}. \tag{4.38}$$

The unrenormalized scattering operator is defined as

$$S_{\rm ur} := \Omega_{\rm ur}^{+*}\Omega_{\rm ur}^{-}. \tag{4.39}$$

Note that it can be also obtained as the following weak limit:

$$S_{\rm ur} = \mathrm{w}-\lim_{\epsilon_-,\epsilon_+\downarrow 0} 4\epsilon_-\epsilon_+ \int_0^\infty \mathrm{d}t_- \int_0^\infty \mathrm{d}t_+ \mathrm{e}^{\mathrm{i}t_+H_0}\mathrm{e}^{-\mathrm{i}(t_-+t_+)H}\mathrm{e}^{\mathrm{i}t_-H_0}\mathrm{e}^{-2(\epsilon_-+2\epsilon_+)t}.$$

$S_{\rm ur}$ is a contraction that commutes with $H_0$.

We can define the renormalized scattering operator as

$$S_{\rm rn} := (Z^+)^{-1/2}S_{\rm ur}(Z^-)^{-1/2} = \Omega_{\rm rn}^{+*}\Omega_{\rm rn}^{-}.$$

$S_{\rm rn}$ also commutes with $H_0$ and if

$$\mathrm{Ran}\Omega_{\rm rn}^{+} = \mathrm{Ran}\Omega_{\rm rn}^{-},$$

then it is unitary.

We will see in the following subsection that van Hove Hamiltonians provide an example where the formalism based on the Abelian limit is applicable.

## 4.3 Wave operators for the van Hove Hamiltonian

Let $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$. Let $H = H_{\rm II}$ be the van Hove Hamiltonian of the second kind and

$$H_0 := \mathrm{d}\Gamma(h).$$

It is easy to see that in the case of the van Hove Hamiltonian, the limit in (4.32) does not exist unless $H = H_0$. Hence the construction of wave operators cannot be based on the approach described in Subsection 4.1.

We will show, however, that the formalism of Subsection 4.2 works for van Hove Hamiltonians if $h$ has an absolutely continuous spectrum. It will turn out that the two renormalized wave operator coincide with one another and are equal to the dressing operator $U$. The operators $Z_\pm =: Z$ coincide and are just constants. The renormalized scattering operator equals identity.

If $z \notin h\mathcal{W}$, then $Z = 0$. This is one of the manifestations of the infrared problem.

All these statements are described in the following theorem.

**Theorem 4.1** *Suppose that $h$ has absolutely continuous spectrum and $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$. Then $\Omega_{\rm ur}^{\pm}$ exists and*

$$\Omega_{\rm ur}^{\pm} = Z^{1/2}U, \quad S_{\rm ur} = Z,$$

*where*

$$Z = \mathrm{e}^{-\|h^{-1}z\|^2}.$$

*$Z \neq 0$ iff $z \in h\mathcal{W}$, and then*

$$Z = (\Omega|U\Omega)^2.$$

*We can then define the renormalized operators, which are equal*

$$\Omega_{\rm rn}^{\pm} = U, \quad S_{\rm rn} = 1.$$

**Proof.** Let us assume that $z \in h\mathcal{W}$. (The general case can be obtained by the limiting argument). Set $g := h^{-1}z$.

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0} &= U\mathrm{e}^{\mathrm{i}tH_0}U^*\mathrm{e}^{-\mathrm{i}tH_0} \\
&= U\exp(-\mathrm{e}^{\mathrm{i}th}g(a^*) + \mathrm{e}^{-\mathrm{i}t\overline{h}}\overline{g}(a)) \\
&= U\mathrm{e}^{-\frac{1}{2}\|g\|^2}\exp(-\mathrm{e}^{\mathrm{i}th}g(a^*))\exp(\mathrm{e}^{-\mathrm{i}t\overline{h}}g(a)).
\end{aligned} \tag{4.40}$$

Let $P_m$ be the projection onto the states with $\leq m$ particles. Suppose that $P_m\Psi = \Psi$. Now

$$\begin{aligned} 2\epsilon \textstyle\int_0^\infty \mathrm{e}^{-2\epsilon t} \exp(-\mathrm{e}^{\mathrm{i}th} g(a^*)) \exp(\mathrm{e}^{-\mathrm{i}t\overline{h}} \overline{g}(a)) \Psi \mathrm{d}t = \Psi \\ +2\epsilon \textstyle\int_0^\infty \mathrm{e}^{-2\epsilon t} \big(\exp(-\mathrm{e}^{\mathrm{i}th} g(a^*)) - 1\big) \Psi \mathrm{d}t \\ +2\epsilon \textstyle\int_0^\infty \mathrm{e}^{-2\epsilon t} \exp(-\mathrm{e}^{\mathrm{i}th} g(a^*)) \big(\exp(\mathrm{e}^{-\mathrm{i}t\overline{h}} \overline{g}(a)) - 1\big) \Psi \mathrm{d}t. \end{aligned} \tag{4.41}$$

The norm of the third term can be estimated from above by

$$2\epsilon \int_0^\infty \mathrm{e}^{-2\epsilon t} \| \exp(-\mathrm{e}^{\mathrm{i}th} g(a^*)) P_m \| \| \big(\exp(\mathrm{e}^{-\mathrm{i}t\overline{h}} \overline{g}(a)) - 1\big) \Psi \| \mathrm{d}t.$$

Clearly, $\| \exp(-\mathrm{e}^{\mathrm{i}th} g(a^*)) P_m \|$ is bounded uniformly in time. Besides, $\| \big(\exp(\mathrm{e}^{-\mathrm{i}t\overline{h}} \overline{g}(a)) - 1\big) \Psi \| \to 0$, by the absolute continuity of $h$ and the Riemann-Lebesgue lemma. Therefore, the third term of (4.41) goes to zero.

The second term equals

$$\begin{aligned} \sum_{n=1}^{\infty} 2\epsilon \int_0^\infty \mathrm{e}^{-2\epsilon t} \tfrac{(-1)^n}{n!} (\mathrm{e}^{\mathrm{i}th} g)^{\otimes n} (a^*) \Psi \mathrm{d}t \\ = \sum_{n=1}^{\infty} \tfrac{(-1)^n}{n!} 2\epsilon \big( (2\epsilon - \mathrm{i}\mathrm{d}\Gamma(h))^{-1} g^{\otimes n} \big) (a^*) \Psi. \end{aligned} \tag{4.42}$$

Note that the $n$th term on the right goes to zero as $\epsilon \searrow 0$ and can be estimated by

$$\frac{\sqrt{(m+1)\cdots(m+n)}}{n!} \|g\|^n \|\Psi\|. \tag{4.43}$$

The series with elements (4.43) is convergent. Hence by the dominated convergence theorem, (4.42) goes to zero as $\epsilon \searrow 0$.

This shows that, for a finite particle $\Psi$, the left hand side of (4.41) goes to $\Psi$. By density, we can extend this to all $\Psi \in \Gamma_{\mathrm{s}}(\mathcal{W})$. $\square$

## 4.4 Asymptotic fields—general formalism

There exists an alternative approach to scattering in quantum field theory. Instead of starting from wave operators, one looks at the limits of certain observables in the interaction picture. There are various forms of this approach, some of them go under the name of the LSZ formalism, see e.g. [Schwa].

Let us present the abstract framework of one of the versions of this approach developed in [HK] and used eg. in [DG2].

Suppose that $H$ is a self-adjoint operator on the Fock space $\Gamma_{\mathrm{s}}(\mathcal{W})$ and $h$ is a self-adjoint operator on $\mathcal{W}$. Assume that for some subspace $\mathcal{W}_1 \subset \mathcal{W}$ there exists

$$\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} W(\mathrm{e}^{-\mathrm{i}th} w)\mathrm{e}^{-\mathrm{i}tH} =: W^{\pm}(w), \quad w \in \mathcal{W}_1.$$

Then

$$\mathcal{W}_1 \ni w \mapsto W^{\pm}(w) \in U(\Gamma_{\mathrm{s}}(\mathcal{W})) \tag{4.44}$$

are two representations of Canonical Commutation Relations (CCR), that means

$$W^{\pm}(w_1)W^{\pm}(w_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(w_1|w_2)} W^{\pm}(w_1 + w_2).$$

Moreover, they satisfy

$$\mathrm{e}^{\mathrm{i}tH} W^{\pm}(w)\mathrm{e}^{-\mathrm{i}tH} = W^{\pm}(\mathrm{e}^{\mathrm{i}th} w).$$

Suppose that the representations (4.44) are unitarily equivalent to the Fock representation, which means that there exist unitary operators $\Omega^{\pm}$ such that

$$W^{\pm}(w) = \Omega^{\pm} W(w) \Omega^{\pm *}. \tag{4.45}$$

Then the operators $\Omega^{\pm}$ are defined up to a phase factor. They are called wave operators.

The scattering operator is defined as $S := \Omega^{+*}\Omega^{-}$. Again, the scattering operator is defined up to a phase factor.

Suppose that both the formalism of Subsection 4.2 and of Subsection 4.4 apply. One can ask whether the renormalized wave operators $\Omega^{\pm}_{\mathrm{rn}}$, defined as in Subsection 4.2, and the wave operator $\Omega^{\pm}$ defined in this section coincide up to a phase factor. In general, there seems to be no guarantee for this to hold. Nevertheless we will see that this is true in the case of van Hove Hamiltonians.

## 4.5 Asymptotic fields for van Hove Hamiltonians

The formalism of asymptotic fields works very well in the case of van Hove Hamiltonians.

**Theorem 4.2** *Let $h$ have an absolutely continuous spectrum, $0 \le \beta \le 1$ and $z \in h^{1-\beta}\mathcal{W} + h\mathcal{W}$. Let $w \in \mathrm{Dom}\, h^{-\beta}$.*

(1) *There exists the norm limit*

$$\lim_{|t|\to\infty} \mathrm{e}^{\mathrm{i}tH} W(\mathrm{e}^{-\mathrm{i}th} w)\mathrm{e}^{\mathrm{i}tH} = W(w)\mathrm{e}^{\mathrm{i}2\mathrm{Re}(w|h^{-1}z)} =: W^{\mathrm{as}}(w).$$

(2) $\mathrm{Dom}\, h^{-\beta} \ni w \mapsto W^{\mathrm{as}}(w)$ *is a regular representation of CCR.*

(3) *This representation is unitarily equivalent to the Fock representation iff $z \in h\mathcal{W}$, and then*

$$W^{\mathrm{as}}(w) = UW(w)U^*,$$

*where $U$ is the dressing operator.*

**Proof.**

$$\mathrm{e}^{\mathrm{i}tH}W(\mathrm{e}^{-\mathrm{i}th}w)\mathrm{e}^{\mathrm{i}tH} = W(w)\exp\left(\mathrm{i}2\mathrm{Re}(w|(1-\mathrm{e}^{-\mathrm{i}th})h^{-1}z)\right).$$

Now $h^{-1}z \in h^{-\beta}\mathcal{W} + \mathcal{W}$ and $w \in \mathrm{Dom}\, h^{-\beta}$. Hence $\lim_{|t|\to\infty}(w|\mathrm{e}^{-\mathrm{i}th}h^{-1}z) = 0$ by the Riemann-Lebesgue lemma. This proves (1). □

# 5 Examples

## 5.1 Harmonic oscillators

In this section we describe van Hove Hamiltonians in a somewhat more concrete setting, typical for physical applications. We will restrict ourselves to the classical case, since it is parallel to the quantum case. We will describe a system of harmonic oscillators with a linear perturbing potential.

Up to now, we assumed that our system is described by phase space $\mathcal{W}$. There was no need to introduce the configuration space. For a system of harmonic oscillators it is however natural to start from a configuration space, which will be described by a real Hilbert space $\mathcal{X}$ with the scalar product denoted by the dot. The preliminary phase space is $\mathcal{X} \oplus \mathcal{X}$. It has the structure of a symplectic space with the symplectic form

$$(x_1, \xi_1)\omega(x_2, \xi_2) = x_1 \cdot \xi_2 - x_2 \cdot \xi_1. \tag{5.46}$$

Note, however, that we will have to take a slightly different phase space.

Let $r$ denote a positive operator on $\mathcal{X}$ and $q$ is a linear functional on $\mathcal{X}$ (possibly unbounded and not everywhere defined). A system of harmonic oscillators with a linear perturbing potential is described by the (classical) Hamiltonian

$$G(x, \xi) = \frac{1}{2}|rx|^2 + \frac{1}{2}|\xi|^2 + q \cdot x,$$

defined for $x \ni \mathrm{Dom}\, r \cap \mathrm{Dom}\, q$, $\xi \in \mathcal{X}$. It is easy to see that $\mathcal{X} \oplus \mathcal{X}$ is not an appropriate space for the Hamiltonian $G$. It is natural to replace it by the space $\mathcal{W} := r^{-1/2}\mathcal{X} \oplus r^{1/2}\mathcal{X}$. We keep the symplectic form (5.46)

We equip $\mathcal{W}$ with the complex structure

$$\mathrm{i}(x, \xi) := (-r^{-1}\xi, rx).$$

We can view $\mathcal{W}$ as a complex Hilbert space:

$$(w_1|w_2) = r^{1/2}x_1 \cdot r^{1/2}x_2 + r^{-1/2}\xi_1 \cdot r^{-1/2}\xi_2 + \mathrm{i}x_1 \cdot \xi_2 - \mathrm{i}x_2 \cdot \xi_1,$$
$$w_1 = (x_1, \xi_1), \quad w_2 = (x_2, \xi_2).$$

Note that the symplectic form (5.46) is the imaginary part of the scalar product.

Introduce $z := (r^{-1/2}q, 0)$ and a positive self-adjoint operator $h$ on $\mathcal{W}$ defined by $h := r \oplus r$. Then we can rewrite $G$ as

$$G(w) = \frac{1}{2}\big((w|hw) + (z|w) + (w|z)\big).$$

Note that the infrared and ultraviolet conditions expressed in terms of $q$ instead of $z$ have their power shifted by $1/2$. More precisely, $z \in h^\alpha \mathcal{W}$ iff $q \in r^{1/2+\alpha}\mathcal{X}$.

## 5.2 Scalar massless field theory

Suppose that $\mathcal{X} = L^2(\mathbb{R}^d)$, $r = |\mathrm{i}\nabla|$. Then the Hamiltonian $\frac{1}{2}|rx|^2 + \frac{1}{2}|\xi|^2$ describes the so-called scalar massless field theory. After taking the Fourier transformation, the operator $r$ becomes the multiplication by $|\xi|$, where $\xi$ is the momentum variable.

Suppose that we add a linear perturbation given by $q \in \mathcal{S}(\mathbb{R}^d)$. After taking the Fourier transformation we get $\hat{q} \in \mathcal{S}(\mathbb{R}^d)$ and we see that the ultraviolet problem is absent. The infrared problem will depend on whether $\hat{q}(0)$ equals zero or not. $\hat{q}(0)$ equals the integral of $q$ over the whole configuration space. Since in some physical examples $q$ can be interpreted as the density of the charge, we will call $\hat{q}(0)$ the total charge. Note that if $\hat{q}(0) = 0$, then $|\hat{q}(\xi)| = O(|\xi|)$ around zero.

Concerning the type of the infrared behavior, we easily get the following table (the number in the round brackets corresponds to the part of Theorem 3.5):

| Dimension of configuration space | Nonzero total charge | Zero total charge |
|---|---|---|
| $d = 1$ | Hamiltonian undefined | (2) |
| $d = 2$ | (3) | (1) |
| $d = 3$ | (2) | (1) |
| $d \geq 4$ | (1) | (1) |

**Remark 5.1** *As we see from the table, in dimension 3, in the nonzero charge case we get the infrared behavior of type (2). Thus the Hamiltonian is bounded from below, but the ground state is absent. This is the type of the infrared problem widely discussed in the literature [Ki].*

*Some authors say, however, that the type (2) behavior is an artifact of the model and disappears if one takes a more physical Hamiltonian. In fact, in [BFS2] it is proven that the (ultraviolet cut-off) Hamiltonian of QED with some types of a potential (e.g. the Coulomb potential) possesses a ground state. This is related to the fact that in above considerations we considered a scalar field, whereas photons in QED have spin one and are coupled to the charge by the minimal coupling prescription.*

# 6 Time dependent van Hove Hamiltonians

In this section we describe a certain class of strongly continuous dynamics on the Fock space. One can say that these are the dynamics generated by a time dependent family of van Hove Hamiltonians.

Let $\mathbb{R} \ni t \mapsto g^t \in \mathcal{W}$ be a continuous vector valued function and $\mathbb{R} \ni t \mapsto u^t \in U(\mathcal{W})$ be a strongly continuous function with values in unitary operators. We assume that $g^0 = 0$ and $u^0 = 1$. Set

$$V(t) := \Gamma(u^t) \exp\bigl(\mathrm{i}g^t(a^*) + \mathrm{i}\overline{g}^t(a)\bigr).$$

For $A \in B(\Gamma_{\mathrm{s}}(\mathcal{W}))$ we set

$$\beta^t(A) := V(t)AV(t)^*.$$

Note that

$$\beta^t(w(a^*)) = u^t w(a^*) + \mathrm{i}(g^t|w).$$

**Theorem 6.1** (1) *$V(t)$ is a strongly continuous family of unitary operators on $\Gamma_{\mathrm{s}}(\mathcal{W})$ such that $V(0) = 1$.*

(2) *$\beta^t$ is a pointwise strongly continuous family of $*$-automorphisms of $B(\Gamma_{\mathrm{s}}(\mathcal{W}))$ such that $\beta^0$ is the identity.*

(3) *$V(t)$ is the distinguished implementation of $\beta^t$ in the following sense: if $\tilde{V}(t)$ is a family of unitary operators such that $\beta^t(A) = \tilde{V}(t)A\tilde{V}(t)^*$ and $(\Omega|\tilde{V}(t)\Omega) > 0$, then $\tilde{V}(t) = V(t)$.*

One can ask what is the time-dependent generator of $V(t)$. To answer this question we proceed formally, without worrying about the exact meaning of various objects involved in our formulas.

Suppose that the dot denotes the temporal derivative. It is easy to check the following identities:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}g^t(a^*)+\mathrm{i}\overline{g}^t(a)} &= \bigl(\tfrac{\mathrm{i}}{2}\mathrm{Im}(\dot{g}^t|g^t) + \mathrm{i}\dot{g}^t(a^*) + \mathrm{i}\dot{\overline{g}}^t(a)\bigr)\mathrm{e}^{\mathrm{i}g^t(a^*)+\mathrm{i}\overline{g}^t(a)}, \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\Gamma(u^t) &= \mathrm{d}\Gamma(\dot{u}^t u^{t*})\Gamma(u^t). \end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = \mathrm{i}\Bigl(\frac{1}{2}\mathrm{Im}(\dot{g}^t|g^t) + u^t\dot{g}^t(a^*) + \overline{u}^t\dot{\overline{g}}^t(a) - \mathrm{i}\mathrm{d}\Gamma(\dot{u}^t u^{t*})\Bigr)V(t)$$

Now suppose that $t \mapsto z^t$ is a family of vectors and $t \mapsto h^t$ is a family of self-adjoint operators. Suppose that $u^t$ is the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}u^t = \mathrm{i}h^t u^t, \quad u^0 = 1;$$

and

$$
\begin{aligned}
g^t &:= \int_0^t u^{s*} z^s \mathrm{d}s, \\
\sigma^t &:= \tfrac{1}{2}\mathrm{Im}\int_0^t (z^t | u^t u^{s*} z^s)\mathrm{d}s.
\end{aligned}
$$

Then

$$
\frac{\mathrm{d}}{\mathrm{d}t}V(t) = \mathrm{i}H(t)V(t),
$$

where

$$
H(t) := \mathrm{d}\Gamma(h^t) + z^t(a^*) + \overline{z}^t(a) + \sigma^t.
$$

Thus, at least on a formal level, $V(t)$ is generated by van Hove Hamiltonians.

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