SYMMETRIES OF HYPERGEOMETRIC TYPE EQUATIONS

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J.Dereziński, P.Majewski: "From the conformal group to symmetries of hypergeometric type equations",
J.Dereziński: "Hypergeometric type functions and their symmetries", Annales Henri Poincare 15 (2014) 1569-1653,

Inspired by

W. Miller: "Lie theory and special functions",
W. Miller: "Symmetry and separation of variables",
Nikiforov-Uvarov: "Special functions of mathematical physics"

$$\left(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta\right) F(z) = 0$$

is called a hypergeometric type equation if $\eta \in \mathbb{C}$ and $\sigma(z),\ \tau(z)$ are polynomials with

$$\mathrm{deg}\sigma \leq 2,\ \mathrm{deg}\tau \leq 1.$$

It is given by the hypergeometric type operator

$$\mathcal{C}(z, \partial_z) := \sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta$$

Below we list normal forms of all nontrivial classes of hypergeometric type operators.

The hypergeometric operator $z(1-z)\partial_z^2+\big(c-(a+b+1)z\big)\partial_z-ab$.

The Gegenbauer operator $(1-z^2)\partial_z^2-(a+b+1)z\partial_z-ab$.

The confluent operator $z\partial_z^2+(c-z)\partial_z-a$.

The Hermite operator $\partial_z^2-2z\partial_z-2a$.

The ${}_0F_1$ operator (related to the Bessel operator) $z\partial_z^2+c\partial_z-1$.

We will see that all hypergeometric type equations can be obtained by separating the variables of a certain 2nd order PDE's with constant coefficients. Every such a PDE has a Lie algebra and a Lie group of generalized symmetries.

1. The orthogonal Lie algebra $\mathrm{so}(\mathbb{C}^{n+2})$ describes generalized symmetries of the Laplace equation $\Delta_{\mathbb{C}^n} f = 0$.
2. The Schrödinger Lie algebra $\mathrm{sch}(\mathbb{C}^{n-2}) = \mathbb{C}^2 \otimes \mathbb{C}^{n-2} \rtimes \big(\mathrm{sl}(\mathbb{C}^2) \oplus \mathrm{so}(\mathbb{C}^{n-2})\big)$ generalized symmetries of the heat equation $\big(\Delta_{\mathbb{C}^{n-2}} + \partial_t\big) f = 0$.
3. The affine orthogonal Lie algebra $\mathbb{C}^{n-1} \rtimes \mathrm{so}(\mathbb{C}^{n-1})$ describes symmetries of the Helmholtz equation $\big(\Delta_{\mathbb{C}^{n-1}} - 1\big) f = 0$.

All the equations from the above list can be derived by an appropriate reduction from the Laplace equation in $n+2$ dimensions

$$\Delta_{\mathbb{C}^{n+2}} K = 0.$$

On the $(n+2)$-dimensional level the corresponding symmetries are very straightforward. In particular, all the corresponding symmetries sit in the conformal symmetries of $\mathbb{C}^n$:

$$\begin{aligned} \mathrm{sch}(\mathbb{C}^{n-2}) \; &\subset \; \mathrm{so}(\mathbb{C}^{n+2}), \\ \mathbb{C}^{n-1} \rtimes \mathrm{so}(\mathbb{C}^{n-1}) \; &\subset \; \mathrm{so}(\mathbb{C}^{n+1}) \subset \mathrm{so}(\mathbb{C}^{n+2}) \end{aligned}$$

In the Lie algebra of generalized symmetries we fix a certain maximal commutative algebra, which we will call the “Cartan algebra”. The eigenvalues of the “Cartan algebra” correspond to parameters of the equation.

Operators whose adjoint action is diagonal in the “Cartan algebra” will be called “root operators”. Root operators correspond to commutation relations of the equation.

In the Lie group of generelized symmetries we will distinguish a discrete subgroup, which leaves invariant the “Cartan algebra”. It will be called the group of “Weyl symmetries”. Weyl symmetries correspond to discrete symmetries of the equation.

PDE	Lie algebra	discrete symmetries	equation	number of parameters	number of basic com. rel
$\Delta_{\mathbb{C}^4}$	$\mathrm{so}(\mathbb{C}^6)$	cube	hypergeometric	3	$12;$
$\Delta_{\mathbb{C}^3}$	$\mathrm{so}(\mathbb{C}^5)$	square	Gegenbauer	2	$8;$
$\Delta_{\mathbb{C}^2} + \partial_t$	$\mathrm{sch}(\mathbb{C}^2)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	confluent	2	$6;$
$\Delta_{\mathbb{C}} + \partial_t$	$\mathrm{sch}(\mathbb{C}^1)$	$\mathbb{Z}_4$	Hermite	1	$4;$
$\Delta_{\mathbb{C}^2} - 1$	$\mathbb{C}^2 \rtimes \mathrm{so}(\mathbb{C}^2)$	$\mathbb{Z}_2$	${}_0F_1$	1	$2.$

Thus to derive the hypergeometric and confluent equation together with all its symmetries one should start with

$$\Delta_{\mathbb{C}^6} K = 0.$$

To derive the Gegenbauer, Hermite and ${}_0F_1$ equation together with all its symmetries it is enough to start with

$$\Delta_{\mathbb{C}^5} K = 0.$$

It is easy to reduce the Laplace equation from 6 to 5 dimensions. Thus the Laplace equation in 6 dimensions is the mother of all hypergeometric type equations.

Every hypergeometric type operator can be written as

$$\mathcal{C}(z, \partial_z) = \rho^{-1}(z)\partial_z \rho(z)\partial_z + \eta,$$

which defines a function $\rho(z)$ called the weight. The operator

$$\mathcal{C}^{\mathrm{bal}}(z, \partial_z) := \rho(z)^{\frac{1}{2}}\mathcal{C}(z, \partial_z)\rho(z)^{-\frac{1}{2}},$$

will be called the balanced form of $\mathcal{C}$.

On the following slides we list all normal forms of hypergeometric type equations both in the standard form and in the balanced form. We also present two classes of identities whose group-theoretical derivation we would like to present: commutation relations and discrete symmetries.

We will always use special parameters. The special parameters and the balanced form will help us to visualize the symmetries.

The (standard) hypergeometric operator

$$\begin{aligned}\mathcal{F}_{\alpha,\beta,\mu}(w,\partial_w) &= w(1-w)\partial_w^2 + \big((1+\alpha)(1-w)-(1+\beta)w\big)\partial_w \\ &\quad +\frac{1}{4}\mu^2 - \frac{1}{4}(\alpha+\beta+1)^2.\end{aligned}$$

The balanced hypergeometric operator

$$\begin{aligned}\mathcal{F}^{\mathrm{bal}}_{\alpha,\beta,\mu}(w,\partial_w) &:= w^{\frac{\alpha}{2}}(1-w)^{\frac{\beta}{2}}\mathcal{F}_{\alpha,\beta,\mu}(w,\partial_w)w^{-\frac{\alpha}{2}}(1-w)^{-\frac{\beta}{2}} \\ &= \partial_w w(1-w)\partial_w - \frac{\alpha^2}{4w} - \frac{\beta^2}{4(1-w)} + \frac{\mu^2-1}{4}.\end{aligned}$$

Discrete symmetries, known as Kummer's table:

$\mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta,\,\mu}(w,\,\partial_w)$ does not change if we flip the signs of α, β, μ.
Besides, the following operators coincide with $\mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta,\,\mu}(w,\,\partial_w)$:

$$
\begin{aligned}
w &= z \;: && \mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta,\,\mu}(z,\,\partial_z),\\
w &= 1 - z \;: && \mathcal{F}^{\mathrm{bal}}_{\beta,\,\alpha,\,\mu}(z,\,\partial_z),\\
w &= \frac{1}{z} \;: && z^{\frac{1}{2}}\,(-z)\,\mathcal{F}^{\mathrm{bal}}_{\mu,\,\beta,\,\alpha}(z,\,\partial_z)\,z^{-\frac{1}{2}},\\
w &= 1 - \frac{1}{z} \;: && z^{\frac{1}{2}}\,(-z)\,\mathcal{F}^{\mathrm{bal}}_{\mu,\,\alpha,\,\beta}(z,\,\partial_z)\,z^{-\frac{1}{2}},\\
w &= \frac{1}{1-z} \;: && (1-z)^{\frac{1}{2}}\,(z-1)\,\mathcal{F}^{\mathrm{bal}}_{\beta,\,\mu,\,\alpha}(z,\,\partial_z)\,(1-z)^{-\frac{1}{2}},\\
w &= \frac{z}{z-1} \;: && (1-z)^{\frac{1}{2}}\,(z-1)\,\mathcal{F}^{\mathrm{bal}}_{\alpha,\,\mu,\,\beta}(z,\,\partial_z)\,(1-z)^{-\frac{1}{2}}.
\end{aligned}
$$

Commutation relations.

$$\sqrt{w(1-w)}\left(\partial_w - \frac{\alpha}{2w} + \frac{\beta}{2(1-w)}\right)\mathcal{F}^{\mathrm{bal}}_{\alpha,\beta,\mu}(w,\,\partial_w)$$

$$= \mathcal{F}^{\mathrm{bal}}_{\alpha+1,\beta+1,\mu}(w,\,\partial_w)\sqrt{w(1-w)}\left(\partial_w - \frac{\alpha}{2w} + \frac{\beta}{2(1-w)}\right),$$

$$\sqrt{w(1-w)}\left(\partial_w + \frac{\alpha}{2w} - \frac{\beta}{2(1-w)}\right)\mathcal{F}^{\mathrm{bal}}_{\alpha,\beta,\mu}(w,\,\partial_w)$$

$$= \mathcal{F}^{\mathrm{bal}}_{\alpha-1,\beta-1,\mu}(w,\,\partial_w)\sqrt{w(1-w)}\left(\partial_w + \frac{\alpha}{2w} - \frac{\beta}{2(1-w)}\right),$$

$$\sqrt{w(1-w)}\left(\partial_w - \frac{\alpha}{2w} - \frac{\beta}{2(1-w)}\right)\mathcal{F}^{\mathrm{bal}}_{\alpha,\beta,\mu}(w,\,\partial_w)$$

$$= \mathcal{F}^{\mathrm{bal}}_{\alpha+1,\beta-1,\mu}(w,\,\partial_w)\sqrt{w(1-w)}\left(\partial_w - \frac{\alpha}{2w} - \frac{\beta}{2(1-w)}\right),$$

$$\sqrt{w(1-w)}\left(\partial_w + \frac{\alpha}{2w} + \frac{\beta}{2(1-w)}\right)\mathcal{F}^{\mathrm{bal}}_{\alpha,\beta,\mu}(w,\,\partial_w)$$

$$= \mathcal{F}^{\mathrm{bal}}_{\alpha-1,\beta+1,\mu}(w,\,\partial_w)\sqrt{w(1-w)}\left(\partial_w + \frac{\alpha}{2w} + \frac{\beta}{2(1-w)}\right),$$

$$\sqrt{w}\left(2(1-w)\,\partial_w-\frac{\alpha}{w}-\mu-3\right)\,\mathcal{F}^{\text{bal}}_{\alpha,\,\beta,\,\mu}(w,\,\partial_w)$$
$$=\mathcal{F}^{\text{bal}}_{\alpha+1,\,\beta,\,\mu+1}(w,\,\partial_w)\,\sqrt{w}\left(2(1-w)\,\partial_w-\frac{\alpha}{w}-\mu-1\right),$$
$$\sqrt{w}\left(2(1-w)\,\partial_w+\frac{\alpha}{w}+\mu-3\right)\,\mathcal{F}^{\text{bal}}_{\alpha,\,\beta,\,\mu}(w,\,\partial_w)$$
$$=\mathcal{F}^{\text{bal}}_{\alpha-1,\,\beta,\,\mu-1}(w,\,\partial_w)\,\sqrt{w}\left(2(1-w)\,\partial_w+\frac{\alpha}{w}+\mu-1\right),$$
$$\sqrt{w}\left(2(1-w)\,\partial_w-\frac{\alpha}{w}+\mu-3\right)\,\mathcal{F}^{\text{bal}}_{\alpha,\,\beta,\,\mu}(w,\,\partial_w)$$
$$=\mathcal{F}^{\text{bal}}_{\alpha+1,\,\beta,\,\mu-1}(w,\,\partial_w)\,\sqrt{w}\left(2(1-w)\,\partial_w-\frac{\alpha}{w}+\mu-1\right),$$
$$\sqrt{w}\left(2(1-w)\,\partial_w+\frac{\alpha}{w}-\mu-3\right)\,\mathcal{F}^{\text{bal}}_{\alpha,\,\beta,\,\mu}(w,\,\partial_w)$$
$$=\mathcal{F}^{\text{bal}}_{\alpha-1,\,\beta,\,\mu+1}(w,\,\partial_w)\,\sqrt{w}\left(2(1-w)\,\partial_w+\frac{\alpha}{w}-\mu-1\right),$$

$$
\begin{aligned}
&\sqrt{1-w}\left(-2w\,\partial_w - \frac{\beta}{1-w} - \mu - 3\right)\,\mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta,\,\mu}(w,\,\partial_w)\\
&\qquad = \mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta+1,\,\mu+1}(w,\,\partial_w)\,\sqrt{1-w}\left(-2w\,\partial_w - \frac{\beta}{1-w} - \mu - 1\right),\\
&\sqrt{1-w}\left(-2w\,\partial_w + \frac{\beta}{1-w} + \mu - 3\right)\,\mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta,\,\mu}(w,\,\partial_w)\\
&\qquad = \mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta-1,\,\mu-1}(w,\,\partial_w)\,\sqrt{1-w}\left(-2w\,\partial_w + \frac{\beta}{1-w} + \mu - 1\right),\\
&\sqrt{1-w}\left(-2w\,\partial_w - \frac{\beta}{1-w} + \mu - 3\right)\,\mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta,\,\mu}(w,\,\partial_w)\\
&\qquad = \mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta+1,\,\mu-1}(w,\,\partial_w)\,\sqrt{1-w}\left(-2w\,\partial_w - \frac{\beta}{1-w} + \mu - 1\right),\\
&\sqrt{1-w}\left(-2w\,\partial_w + \frac{\beta}{1-w} - \mu - 3\right)\,\mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta,\,\mu}(w,\,\partial_w)\\
&\qquad = \mathcal{F}^{\mathrm{bal}}_{\alpha,\,\beta-1,\,\mu+1}(w,\,\partial_w)\,\sqrt{1-w}\left(-2w\,\partial_w + \frac{\beta}{1-w} - \mu - 1\right).
\end{aligned}
$$

The (standard) Gegenbauer operator

$$\mathcal{S}_{\alpha,\lambda}(w,\partial_w) \;:=\; (1-w^2)\partial_w^2 - 2(1+\alpha)w\partial_w + \lambda^2 - \Big(\alpha+\frac{1}{2}\Big)^2.$$

The balanced Gegenbauer operator

$$\begin{aligned}\mathcal{S}^{\mathrm{bal}}_{\alpha,\lambda}(w,\partial_w) \;&:=\; (w^2-1)^{\frac{\alpha}{2}}\mathcal{S}_{\alpha,\lambda}(w,\partial_w)(w^2-1)^{-\frac{\alpha}{2}}\\ &=\; \partial_w(1-w^2)\partial_w - \frac{\alpha^2}{1-w^2} + \lambda^2 - \frac{1}{4}.\end{aligned}$$

Discrete symmetries. $\mathcal{S}^{\mathrm{bal}}_{\alpha,\,\lambda}(w,\,\partial_w)$ does not change if we flip the signs of α, λ. Besides, the following operators coincide with $\mathcal{S}^{\mathrm{bal}}_{\alpha,\,\lambda}(w,\,\partial_w)$:

$$
\begin{aligned}
w &= z \ : && \mathcal{S}^{\mathrm{bal}}_{\alpha,\,\lambda}(z,\,\partial_z),\\
w &= \frac{z}{\sqrt{z^2-1}} \ : && (z^2-1)^{\frac14}\,(z^2-1)\,\mathcal{S}^{\mathrm{bal}}_{\lambda,\,\alpha}(z,\,\partial_z)\,(z^2-1)^{-\frac14}.
\end{aligned}
$$

Commutation relations.

$$\sqrt{1-w^2}\left(-\frac{5}{2}-w\,\partial_w-\frac{\alpha}{1-w^2}-\lambda\right)\mathcal{S}^{\text{bal}}_{\alpha,\,\lambda}(w,\,\partial_w)$$
$$=\mathcal{S}^{\text{bal}}_{\alpha+1,\,\lambda+1}(w,\,\partial_w)\,\sqrt{1-w^2}\left(-\frac{1}{2}-w\,\partial_w-\frac{\alpha}{1-w^2}-\lambda\right),$$

$$\sqrt{1-w^2}\left(-\frac{5}{2}-w\,\partial_w+\frac{\alpha}{1-w^2}+\lambda\right)\mathcal{S}^{\text{bal}}_{\alpha,\,\lambda}(w,\,\partial_w)$$
$$=\mathcal{S}^{\text{bal}}_{\alpha-1,\,\lambda-1}(w,\,\partial_w)\,\sqrt{1-w^2}\left(-\frac{1}{2}-w\,\partial_w+\frac{\alpha}{1-w^2}+\lambda\right),$$

$$\sqrt{1-w^2}\left(-\frac{5}{2}-w\,\partial_w-\frac{\alpha}{1-w^2}+\lambda\right)\mathcal{S}^{\text{bal}}_{\alpha,\,\lambda}(w,\,\partial_w)$$
$$=\mathcal{S}^{\text{bal}}_{\alpha+1,\,\lambda-1}(w,\,\partial_w)\,\sqrt{1-w^2}\left(-\frac{1}{2}-w\,\partial_w-\frac{\alpha}{1-w^2}+\lambda\right),$$

$$\sqrt{1-w^2}\left(-\frac{5}{2}-w\,\partial_w+\frac{\alpha}{1-w^2}-\lambda\right)\mathcal{S}^{\text{bal}}_{\alpha,\,\lambda}(w,\,\partial_w)$$
$$=\mathcal{S}^{\text{bal}}_{\alpha-1,\,\lambda+1}(w,\,\partial_w)\,\sqrt{1-w^2}\left(-\frac{1}{2}-w\,\partial_w+\frac{\alpha}{1-w^2}-\lambda\right),$$

$$
\begin{aligned}
w\left(-\frac{5}{2}+\frac{1-w^2}{w}\,\partial_w-\lambda\right)\mathcal{S}^{\mathrm{bal}}_{\alpha,\,\lambda}(w,\,\partial_w)&\\
&=\mathcal{S}^{\mathrm{bal}}_{\alpha,\,\lambda+1}(w,\,\partial_w)\,w\left(-\frac{1}{2}+\frac{1-w^2}{w}\,\partial_w-\lambda\right),\\
w\left(-\frac{5}{2}+\frac{1-w^2}{w}\,\partial_w+\lambda\right)\mathcal{S}^{\mathrm{bal}}_{\alpha,\,\lambda}(w,\,\partial_w)&\\
&=\mathcal{S}^{\mathrm{bal}}_{\alpha,\,\lambda-1}(w,\,\partial_w)\,w\left(-\frac{1}{2}+\frac{1-w^2}{w}\,\partial_w+\lambda\right),
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{1-w^2}\left(\partial_w+\frac{w}{1-w^2}\,\alpha\right)\mathcal{S}^{\mathrm{bal}}_{\alpha,\,\lambda}(w,\,\partial_w)&\\
&=\mathcal{S}^{\mathrm{bal}}_{\alpha+1,\,\lambda}(w,\,\partial_w)\,\sqrt{1-w^2}\left(\partial_w+\frac{w}{1-w^2}\,\alpha\right).\\
\sqrt{1-w^2}\left(\partial_w-\frac{w}{1-w^2}\,\alpha\right)\mathcal{S}^{\mathrm{bal}}_{\alpha,\,\lambda}(w,\,\partial_w)&\\
&=\mathcal{S}^{\mathrm{bal}}_{\alpha+1,\,\lambda}(w,\,\partial_w)\,\sqrt{1-w^2}\left(\partial_w-\frac{w}{1-w^2}\,\alpha\right).
\end{aligned}
$$

The (standard) confluent operator

$$\mathcal{F}_{\theta,\alpha}(w,\partial_w) \;=\; w\partial_w^2 + (1+\alpha-w)\partial_w - \frac{1}{2}(1+\theta+\alpha).$$

The balanced confluent operator

$$\begin{aligned}\mathcal{F}^{\mathrm{bal}}_{\theta,\alpha}(w,\partial_w) \;&=\; w^{\frac{\alpha}{2}}\mathrm{e}^{-\frac{w}{2}}\mathcal{F}_{\theta,\alpha}(w,\partial_w)w^{-\frac{\alpha}{2}}\mathrm{e}^{\frac{w}{2}}\\ &=\; \partial_w w\partial_w - \frac{w}{4} - \frac{\theta}{2} - \frac{\alpha^2}{4w}.\end{aligned}$$

Discrete symmetries. $\mathcal{F}^{\mathrm{bal}}_{\theta,\alpha}(w,\, \partial_w)$ does not change if we flip the sign of α. Besides, the following operators coincide with $\mathcal{F}^{\mathrm{bal}}_{\theta,\alpha}(w,\, \partial_w)$:

$$
\begin{aligned}
w = \quad z \ &: \ \mathcal{F}^{\mathrm{bal}}_{\theta,\alpha}(z, \partial_z), \\
w = -z \ &: \ \mathcal{F}^{\mathrm{bal}}_{-\theta,\alpha}(z, \partial_z).
\end{aligned}
$$

Commutation relations.

$$\frac{1}{\sqrt{w}}\left(w\,\partial_w+\frac{\alpha}{2}+\frac{w}{2}\right)\mathcal{F}^{\mathrm{bal}}_{\theta,\,\alpha}(w,\,\partial_w)$$

$$=\mathcal{F}^{\mathrm{bal}}_{\theta+1,\,\alpha-1}(w,\,\partial_w)\,\frac{1}{\sqrt{w}}\left(w\,\partial_w+\frac{\alpha}{2}+\frac{w}{2}\right),$$

$$\frac{1}{\sqrt{w}}\left(w\,\partial_w-\frac{\alpha}{2}+\frac{w}{2}\right)\mathcal{F}^{\mathrm{bal}}_{\theta,\,\alpha}(w,\,\partial_w)$$

$$=\mathcal{F}^{\mathrm{bal}}_{\theta+1,\,\alpha+1}(w,\,\partial_w)\,\frac{1}{\sqrt{w}}\left(w\,\partial_w-\frac{\alpha}{2}+\frac{w}{2}\right),$$

$$\frac{1}{\sqrt{w}}\left(w\,\partial_w+\frac{\alpha}{2}-\frac{w}{2}\right)\mathcal{F}^{\mathrm{bal}}_{\theta,\,\alpha}(w,\,\partial_w)$$

$$=\mathcal{F}^{\mathrm{bal}}_{\theta-1,\,\alpha-1}(w,\,\partial_w)\,\frac{1}{\sqrt{w}}\left(w\,\partial_w+\frac{\alpha}{2}-\frac{w}{2}\right),$$

$$\frac{1}{\sqrt{w}}\left(w\,\partial_w-\frac{\alpha}{2}-\frac{w}{2}\right)\mathcal{F}^{\mathrm{bal}}_{\theta,\,\alpha}(w,\,\partial_w)$$

$$=\mathcal{F}^{\mathrm{bal}}_{\theta-1,\,\alpha+1}(w,\,\partial_w)\,\frac{1}{\sqrt{w}}\left(w\,\partial_w-\frac{\alpha}{2}-\frac{w}{2}\right),$$

$$\left(-w\,\partial_w - \frac{\theta}{2} - \frac{w}{2} - \frac{3}{2}\right)\mathcal{F}^{\mathrm{bal}}_{\theta,\,\alpha}(w,\ \partial_w)$$
$$= \mathcal{F}^{\mathrm{bal}}_{\theta+2,\,\alpha}(w,\ \partial_w)\ \left(-w\,\partial_w - \frac{\theta}{2} - \frac{w}{2} - \frac{1}{2}\right),$$
$$\left(w\,\partial_w - \frac{\theta}{2} - \frac{w}{2} + \frac{3}{2}\right)\mathcal{F}^{\mathrm{bal}}_{\theta,\,\alpha}(w,\ \partial_w)$$
$$= \mathcal{F}^{\mathrm{bal}}_{\theta-2,\,\alpha}(w,\ \partial_w)\ \left(w\,\partial_w - \frac{\theta}{2} - \frac{w}{2} + \frac{1}{2}\right).$$

The (standard) Hermite operator

$$\mathcal{S}_\lambda(w, \partial_w) \;=\; \partial_w^2 - 2w\partial_w - 2\lambda - 1.$$

The balanced Hermite operator

$$\begin{aligned}\mathcal{S}_\lambda^{\mathrm{bal}}(w, \partial_w) \;&:=\; \mathrm{e}^{-\frac{w^2}{2}}\mathcal{S}_\lambda(w, \partial_w)\mathrm{e}^{\frac{w^2}{2}} \\ &=\; \partial_w^2 - w^2 - 2\lambda.\end{aligned}$$

Discrete symmetries. The following operators coincide with $\mathcal{S}_\lambda^{\mathrm{bal}}(w,\ \partial_w)$:

$$\begin{aligned}
w = \quad z &: \ \mathcal{S}_\lambda^{\mathrm{bal}}(z, \partial_z), \\
w = \quad \mathrm{i}z &: \ -\mathcal{S}_{-\lambda}^{\mathrm{bal}}(z, \partial_z), \\
w = -z &: \ \mathcal{S}_\lambda^{\mathrm{bal}}(z, \partial_z), \\
w = -\mathrm{i}z &: \ -\mathcal{S}_{-\lambda}^{\mathrm{bal}}(z, \partial_z).
\end{aligned}$$

Commutation relations.

$$\big(\,\partial_w + w\big)\mathcal{S}_\lambda^{\rm bal}(w,\,\partial_w)$$
$$= \mathcal{S}_{\lambda+1}^{\rm bal}(w,\,\partial_w)\ \big(\,\partial_w + w\big)\,,$$
$$\big(\,\partial_w - w\big)\mathcal{S}_\lambda^{\rm bal}(w,\,\partial_w)$$
$$= \mathcal{S}_{\lambda-1}^{\rm bal}(w,\,\partial_w)\ \big(\,\partial_w - w\big)\,,$$

$$\big(-w\,\partial_w - \lambda - w^2 - \tfrac{5}{2}\big)\mathcal{S}_\lambda^{\rm bal}(w,\,\partial_w)$$
$$= \mathcal{S}_{\lambda+2}^{\rm bal}(w,\,\partial_w)\ \big(-w\,\partial_w - \lambda - w^2 - \tfrac{1}{2}\big)\,,$$
$$\big(w\,\partial_w - \lambda - w^2 + \tfrac{5}{2}\big)\mathcal{S}_\lambda^{\rm bal}(w,\,\partial_w)$$
$$= \mathcal{S}_{\lambda-2}^{\rm bal}(w,\,\partial_w)\ \big(w\,\partial_w - \lambda - w^2 + \tfrac{1}{2}\big)\,.$$

The (standard) ${}_0F_1$ operator

$$\mathcal{F}_\alpha(w, \partial_w) \ := \ w\partial_w^2 + (\alpha+1)\partial_w - 1.$$

The balanced ${}_0F_1$ operator

$$\begin{aligned}\mathcal{F}_\alpha^{\mathrm{bal}}(w, \partial_w) \ &:= \ w^{\frac{\alpha}{2}}\mathcal{F}_\alpha(w, \partial_w)w^{-\frac{\alpha}{2}} \\ &= \ \partial_w w \partial_w - 1 - \frac{\alpha^2}{4w}.\end{aligned}$$

Discrete symmetries. $\mathcal{F}_\alpha(w,\partial_w)$ does not change if we flip the sign of α.

Commutation relations.

$$\frac{1}{\sqrt{w}}\left(w\ \partial_w - \frac{\alpha}{2}\right)\mathcal{F}^{\rm bal}_\alpha(w,\,\partial_w) = \mathcal{F}^{\rm bal}_{\alpha+1}(w,\,\partial_w)\,\frac{1}{\sqrt{w}}\left(w\ \partial_w - \frac{\alpha}{2}\right),$$
$$\frac{1}{\sqrt{w}}\left(w\ \partial_w + \frac{\alpha}{2}\right)\mathcal{F}^{\rm bal}_\alpha(w,\,\partial_w) = \mathcal{F}^{\rm bal}_{\alpha-1}(w,\,\partial_w)\,\frac{1}{\sqrt{w}}\left(w\ \partial_w + \frac{\alpha}{2}\right).$$

Symmetries of PDE's

Let $\mathcal{C}$ be a linear differential operator on a complex manifold U.

Let α be a biholomorphic transformation on U. Then α acts on analytic functions on U by

$$\alpha f(u) := f(\alpha^{-1}(u)).$$

We say that it is a symmetry of $\mathcal{C}$ iff

$$\alpha\mathcal{C} = \mathcal{C}\alpha.$$

One can also consider a pair of actions of α that involve multipliers

$$\alpha^\sharp f(u) \;:=\; m^\sharp(u) f(\alpha^{-1}(u)),$$
$$\alpha^\flat f(u) \;:=\; m^\flat(u) f(\alpha^{-1}(u)).$$

We say that a pair $(\alpha^\sharp, \alpha^\flat)$ is a generalized symmetry of $\mathcal{C}$ if

$$\alpha^\flat \mathcal{C} = \mathcal{C} \alpha^\sharp.$$

Clearly, $\mathcal{C}F = 0$ implies $\mathcal{C}\alpha^\sharp F = 0$. Generalized symmetries form a group.

Let A be a holomorphic vector field on U. A acts on analytic functions on U:

$$Af(u) := A^i(u)\partial_i f(u).$$

We say that it is an infinitesimal symmetry of $\mathcal{C}$ iff

$$A\mathcal{C} = \mathcal{C}A.$$

One can also consider a pair of actions of A that involve 0th order terms:

$$
\begin{aligned}
A^\sharp f(u) &:= A^i(u)\partial_i f(u) + M^\sharp(u) f(u), \\
A^\flat f(u) &:= A^i(u)\partial_i f(u) + M^\flat(u) f(u).
\end{aligned}
$$

We say that a pair $(A^\sharp, A^\flat)$ is a generalized infinitesimal symmetry of $\mathcal{C}$ if

$$
A^\flat \mathcal{C} = \mathcal{C} A^\sharp.
$$

Clearly, $\mathcal{C} F = 0$ implies $\mathcal{C} A^\sharp F = 0$. Infinitesimal generalized symmetries form a Lie algebra.

Generalized symmetries of 2nd order PDE's

PDE	infinitesimal symmetries
Laplace $\Delta_{\mathbb{C}^n}$	$\mathrm{so}(\mathbb{C}^{n+2})$ conformal transformations $(n > 2)$
Heat $\Delta_{\mathbb{C}^n} + \partial_t$	$\mathrm{sch}(n) = \mathbb{C}^2 \otimes \mathbb{C}^n \rtimes \big(\mathrm{sl}(\mathbb{C}^2) \oplus \mathrm{so}(n)\big)$ found by Schrödinger
Helmholz $\Delta_{\mathbb{C}^n} - 1$	$\mathrm{aso}(n) = \mathbb{C}^n \rtimes \mathrm{so}(n)$ affine orthogonal transformations

Conformal invariance

Consider $\mathbb{C}^{n+2}$ equipped with a bilinear nondegenerate form $\langle\cdot|\cdot\rangle$. Choose coordinates on $\mathbb{C}^{n+2}$ such that the scalar product has a "split" form:

$$
\begin{aligned}
z &= (z_{-1}, z_1, \ldots, z_{-m-1}, z_{m+1}),\\
\langle z|z\rangle &= 2z_{-1}z_1 + \cdots + z_{-m-1}z_{m+1}, \quad n = 2m,\\
z &= (z_0, z_{-1}, z_1, \ldots, z_{-m-1}, z_{m+1}),\\
\langle z|z\rangle &= z_0^2 + 2z_{-1}z_1 + \cdots + z_{-m-1}z_{m+1}, \quad n = 2m+1.
\end{aligned}
$$

$\mathrm{O}(\mathbb{C}^{n+2})$ and $\mathrm{so}(\mathbb{C}^{n+2})$ act on functions on $\mathbb{C}^{n+2}$ in the obvious way.

To a function $f(\dots, y_{-m}, y_m)$ on $\mathbb{C}^n$ we associate the function on $\mathbb{C}^{n+2}$

$$\Phi^\eta f := z_{m+1}^\eta f(\dots, \tfrac{z_{-m}}{z_{m+1}}, \tfrac{z_m}{z_{m+1}}).$$

To a function $K(\dots, z_{-m}, z_m, z_{-m-1}, z_{m+1})$ on $\mathbb{C}^{n+2}$ we associate the function on $\mathbb{C}^n$

$$\Psi K := f\Big(\dots, y_{-m}, y_m, -(\dots + y_{-m}y_m), 1\Big).$$

We have

$$\mathrm{id} = \Psi \circ \Phi^\eta.$$

For any η we have representations on functions on $\mathbb{C}^n$:

$$A^{\mathrm{fl},\eta} = \Psi A \Phi^\eta, \quad A \in \mathrm{so}(\mathbb{C}^{n+2}),$$

$$\alpha^{\mathrm{fl},\eta} = \Psi \alpha \Phi^\eta, \quad \alpha \in \mathrm{O}(\mathbb{C}^{n+2}).$$

We have the identity

$$\Delta_{\mathbb{C}^n} = \Psi \Delta_{\mathbb{C}^{n+2}} \Phi^{\frac{2-n}{2}}.$$

Therefore, for this special value of η we have a generalized symmetry of the Laplace equation:

$$\Delta_{\mathbb{C}^n} A^{\mathrm{fl},-\frac{n-2}{2}} = A^{\mathrm{fl},-\frac{n+2}{2}} \Delta_{\mathbb{C}^n}, \quad A \in \mathrm{so}(\mathbb{C}^{n+2}),$$

$$\Delta_{\mathbb{C}^n} \alpha^{\mathrm{fl},-\frac{n-2}{2}} = \alpha^{\mathrm{fl},-\frac{n+2}{2}} \Delta_{\mathbb{C}^n}, \quad \alpha \in \mathrm{O}(\mathbb{C}^{n+2}).$$

Note that it is important to consider the Laplacian $\Delta_{\mathbb{C}^{n+2}}$ on functions homogeneous of degree $-\frac{n-2}{2}$.

Geometric description: Introduce the null quadric in $\mathbb{C}^{n+2}$

$$\mathcal{V} := \{z \in \mathbb{C}^{n+2} \ : \ \langle z|z\rangle = 0, \quad z \neq 0\}$$

and the corresponding projective quadric

$$\mathcal{Y} := \mathcal{V}/\mathbb{C}^{\times}.$$

$\mathrm{O}(\mathbb{C}^{n+2})$ acts naturally on $\mathbb{C}^{n+2}$, hence also on $\mathcal{Y}$.

We have a map of an open dense subset of $\mathcal{Y}$:

$$\mathbb{C}^n \ni y \mapsto \big(y, -\frac{1}{2}\langle y|y\rangle, 1\big) \times \mathbb{C}^{\times} \in \mathcal{Y},$$

Such maps endow $\mathcal{Y}$ with a conformal structure, which is preserved by $\mathrm{O}(\mathbb{C}^{n+2})$.

On $\mathbb{C}^n$ (which can be viewed as an open dense subset of $\mathcal{Y}$) we have the Laplacian $\Delta_{\mathbb{C}^n}$ and a bilinear metric g. The action of $\alpha \in \mathrm{O}(\mathbb{C}^{n+2})$, resp. $A \in \mathrm{so}(\mathbb{C}^{n+2})$ on $\mathbb{C}^n$ will be denoted α^{fl}, resp. A^{fl}. This action is conformal, in other words, for some function ϕ,

$$\alpha^{\mathrm{fl}} g = \mathrm{e}^{2\phi} g, \quad A^{\mathrm{fl}} g = 2\phi g.$$

Identifying $\mathcal{Y}$ with $\mathbb{C}^n$, we obtain

$$\alpha^{\mathrm{fl},\eta} \;=\; \mathrm{e}^{\eta\phi}\alpha^{\mathrm{fl}}, \quad A^{\mathrm{fl},\eta} = A^{\mathrm{fl}} + \eta\phi$$

Set $t = y_{-m}$, so that $\Delta_{\mathbb{C}^n} = \Delta_{\mathbb{C}^{n-2}} + 2\partial_t \partial_{y_m}$. On functions of the form

$$f(\ldots, y_{m-1}, t, r) = \mathrm{e}^{y_m} h(\ldots, y_{m-1}, t). \tag{5}$$

the Laplace equation becomes the heat equation:

$$(2\partial_t + \Delta_{\mathbb{C}^{n-2}}) h(\ldots, y_{m-1}, t) = 0.$$

The Schrödinger group $\mathrm{Sch}(\mathbb{C}^{n-2})$ consists of transformations $\alpha \in \mathrm{O}(\mathbb{C}^{n+2})$ such that α^{fl} preserves the ansatz (5).

The Schrödinger Lie algebra $\mathrm{sch}(\mathbb{C}^{n-2})$ consists of transformations $A \in \mathrm{so}(\mathbb{C}^{n+2})$ such that A^{fl} preserves the ansatz (5).

For a function $h(\ldots, y_{m-1}, t)$ on $\mathbb{C}^{n-2} \times \mathbb{C}$ we define the function on $\mathbb{C}^{n-2} \times \mathbb{C}^2$

$$\theta h(\ldots, y_{m-1}, y_{-m}, y_m) := h(\ldots, y_{m-1}, y_{-m})\mathrm{e}^{y_m}.$$

For a function $f(\ldots, y_{m-1}, y_{-m}, y_m)$ on $\mathbb{C}^{n-2} \times \mathbb{C}^2$ we define the function on $\mathbb{C}^{n-2} \times \mathbb{C}$

$$\zeta f(\ldots, y_{m-1}, t) := f(\ldots, y_{m-1}, t, 0).$$

Clearly,

$$\zeta \circ \theta = \mathrm{id},$$
$$\zeta \Delta_{\mathbb{C}^n} \theta = 2\partial_t + \Delta_{\mathbb{C}^{n-2}}.$$

For any η we obtain representations

$$A^{\mathrm{sch},\eta} = \zeta A^{\mathrm{fl},\eta} \theta, \quad A \in \mathrm{sch}(\mathbb{C}^{n-2}),$$

$$\alpha^{\mathrm{sch},\eta} = \zeta \alpha^{\mathrm{fl},\eta} \theta, \quad \alpha \in \mathrm{Sch}(\mathbb{C}^{n-2}).$$

For a special η we obtain a generalized symmetry

$$(2\partial_t + \Delta_{\mathbb{C}^{n-2}}) A^{\mathrm{sch},-\frac{n-2}{2}} = A^{\mathrm{sch},-\frac{n+2}{2}} (2\partial_t + \Delta_{\mathbb{C}^{n-2}}), \quad A \in \mathrm{sch}(\mathbb{C}^{n-2}),$$

$$(2\partial_t + \Delta_{\mathbb{C}^{n-2}}) \alpha^{\mathrm{sch},-\frac{n-2}{2}} = \alpha^{\mathrm{sch},-\frac{n+2}{2}} (2\partial_t + \Delta_{\mathbb{C}^{n-2}}), \quad \alpha \in \mathrm{Sch}(\mathbb{C}^{n-2}).$$

Laplace equation in $\mathbf{C}^4$ – the hypergeometric equation

Extended space $\mathbb{C}^6$

Consider first the "extended space" $\mathbb{C}^6$ with scalar product defined by

$$\langle z|z\rangle = 2z_{-1}z_1 + 2z_{-2}z_2 + 2z_{-3}z_3,$$

where $z = (z_{-1}, z_1, z_{-2}, z_2, z_{-3}, z_3)$

The Lie algebra $\mathrm{so}(\mathbb{C}^6)$ is spanned by

$$N_i = z_{-i}\partial_{z_{-i}} - z_i\partial_{z_i}, \quad i = 1, 2, 3;$$

$$B_{i,j} := z_{-i}\partial_{z_j} - z_{-j}\partial_{z_i}, \quad 1 \le |i| < |j| \le 3.$$

Its Weyl group is generated by permutations $\sigma \in S_3$ and flips τ_i, $i = 1, 2, 3$. For instance,

$$\tau_1 K = K(z_1, z_{-1}, z_{-2}, z_2, z_{-3}, z_3),$$

$$\sigma_{(12)} K = K(z_{-2}, z_2, z_{-1}, z_1, z_{-3}, z_3).$$

We have relations

$$[N_k, B_{\pm k,j}] = \pm B_{\pm k,j},$$

$$\sigma N_j \sigma^{-1} = N_{\sigma_j},$$

$$\tau_i N_i \tau_i^{-1} = (-1)^{\delta_{ij}} N_i,$$

Choose coordinates

$$
\begin{aligned}
r &= \sqrt{2\left(z_{-1}z_1 + z_{-2}z_2\right)}\ , \\
p &= \sqrt{2z_3z_{-3},}\ \ , \\
w &= \frac{z_{-1}z_1}{z_{-1}z_1 + z_{-2}z_2}\ , \\
u_1 &= \sqrt{\frac{z_{-1}}{z_1}}\ , \\
u_2 &= \sqrt{\frac{z_{-2}}{z_2}}\ , \\
u_3 &= \sqrt{\frac{z_{-3}}{z_3}}\ .
\end{aligned}
$$

The Cartan algebra is especially simple in these coordinates: we use operators fromn the Cartan algebra

$$N_1^{\mathrm{sph}} = u_1\, \partial_{u_1}\,,$$
$$N_2^{\mathrm{sph}} = u_2\, \partial_{u_2}\,,$$
$$N_3^{\mathrm{sph}} = u_3\, \partial_{u_3}\,.$$

The null quadric in these coordinates is given by $r^2 + p^2 = 0$. The generator of dilations is $r\,\partial_r + p\,\partial_p$.

The Laplacian is

$$\Delta_{\mathbb{C}^6} = \frac{4}{r^2}\Bigg(\frac{1}{4}\Bigg((r\,\partial_r)^2 + 2(r\,\partial_r) + \frac{r^2}{p^2}(p\,\partial_p)^2\Bigg) + \partial_w\, w(1-w)\,\partial_w - \frac{(u_1\,\partial_{u_1})^2}{4w} - \frac{(u_1\,\partial_{u_2})^2}{4(1-w)} - \frac{r^2}{p^2}\frac{(u_3\,\partial_{u_3})^2}{4}\Bigg).$$

If we restrict to the null quadric given by $\frac{r^2}{p^2} = -1$ and to functions homogeneous of degree $-\frac{n-2}{2} = -1$, by setting $r\,\partial_r + p\,\partial_p = -1$, and finally fix the gauge (section of the null quadric) $r^2 = 4$, we obtain

$$\partial_w\, w(1-w)\,\partial_w - \frac{(N_1^{\mathrm{sph}})^2}{4w} - \frac{(N_2^{\mathrm{sph}})^2}{4(1-w)} + \frac{(N_3^{\mathrm{sph}})^2}{4} - \frac{1}{4}.$$

On functions of the form

$$f(w, u_1, u_2, u_3) = u_1^\alpha u_2^\beta u_3^\mu F(w)$$

we obtain the balanced hypergeometric operator

$$\partial_w w(1-w)\partial_w - \frac{\alpha^2}{4w} - \frac{\beta^2}{4(1-w)} + \frac{\mu^2-1}{4}.$$

Laplace equation in $\mathbf{C}^3$ – Gegenbauer equation

The "extended space" $\mathbb{C}^5$ has the scalar product given by

$$\langle z|z\rangle = z_0^2 + 2z_{-2}z_2 + 2z_{-3}z_3,$$

where $z = (z_0, z_{-2}, z_2, z_{-3}, z_3)$.

The Lie algebra $\mathrm{so}(\mathbb{C}^5)$ is spanned by

$$N_i = z_{-i}\partial_{z_{-i}} - z_i\partial_{z_i}, \quad i = 2, 3;$$

$$B_{i,j} := z_{-i}\partial_{z_j} - z_{-j}\partial_{z_i}, |i| < |j|, |i|, |j| \in \{0, 2, 3\}.$$

Here are examples of Weyl symmetries

$$\tau_0 K = K(-z_0, z_2, z_{-2}, z_{-3}, z_3),$$

$$\tau_1 K = K(z_0, z_2, z_{-2}, z_{-3}, z_3),$$

$$\sigma_{(12)} K = K(z_0, z_{-3}, z_3, z_{-2}, z_2).$$

We choose coordinates

$$r = \sqrt{z_0^2 + 2z_{-2}z_2}\ ,$$

$$p = \sqrt{2z_3z_{-3}}\ ,$$

$$w = \sqrt{\frac{z_0^2}{2z_{-2}z_2 + z_0^2}}\ ,$$

$$u_2 = \sqrt{\frac{z_{-2}}{z_2}}\ ,$$

$$u_3 = \sqrt{\frac{z_{-3}}{z_3}}\ .$$

Similarly as previously the null quadric in these coordinates is given by $r^2 + p^2 = 0$ and the generator of dilations is $r\,\partial_r + p\,\partial_p$.

Cartan operators

$$N_2^{\mathrm{sph}} = u_2\,\partial_{u_2}\,, \tag{6}$$

$$N_3^{\mathrm{sph}} = u_3\,\partial_{u_3}\,. \tag{7}$$

The Laplacian in these coordinates is

$$\Delta_{\mathbb{C}^5} \;=\; \frac{1}{r^2}\Big((r\,\partial_r)^2 + (r\,\partial_r) + \frac{r^2}{p^2}(p\,\partial_p)^2 \\ + \partial_w\,(1-w^2)\,\partial_w - \frac{(u_2\,\partial_{u_1})^2}{1-w^2} - \frac{r^2}{p^2}(u_3\,\partial_{u_3})^2\Big).$$

If we restrict to the null quadric by setting $\frac{r^2}{p^2} = -1$, to functions homogeneous of degree $-\frac{n-2}{2} = -\frac{1}{2}$ by setting $r\,\partial_r + p\,\partial_p = -\frac{1}{2}$, and finally fix the gauge $r^2 = 1$, we obtain

$$\partial_w\,(1-w^2)\,\partial_w - \frac{(N_2^{\mathrm{sph}})^2}{1-w^2} + (N_3^{\mathrm{sph}})^2 - \frac{1}{4}.$$

On functions of the form

$$f(w, u_2, u_3) = u_2^\alpha u_3^\lambda F(w)$$

we obtain the balanced Gegenbauer operator

$$\partial_w (1 - w^2) \partial_w - \frac{\alpha^2}{1 - w^2} + \lambda^2 - \frac{1}{4}$$

The heat equation in $\mathbf{C}^2 \oplus \mathbf{C}$ – the confluent equation

Extended space $\mathbb{C}^6$

We start from the same extended space $\mathbb{C}^6$, as in the case of the hypergeometric equation. $\mathrm{sch}(2)$ can be defined as the commutant of $B_{-3,2}$ inside $\mathrm{so}(6)$.

Here is its "Cartan algebra"

$$\begin{aligned} N_{23} &= z_{-2}\partial_{z_{-2}} - z_2\partial_{z_2} + z_{-3}\partial_{z_{-3}} - z_3\partial_{z_3} \\ N_1 &= z_{-1}\partial_{z_{-1}} - z_1\partial_{z_1}, \\ B_{-3,2} &= z_3\partial_{z_2} - z_{-2}\partial_{z_{-3}}. \end{aligned}$$

Here are the root operators of $\mathrm{sch}(2)$:

$$
\begin{aligned}
B_{2,-1} &= z_{-2}\partial_{z_{-1}} - z_1\partial_{z_2}, \\
B_{2,1} &= z_{-2}\partial_{z_1} - z_1\partial_{z_{-2}}, \\
B_{-3,-1} &= z_3\partial_{z_{-1}} - z_1\partial_{z_{-3}}, \\
B_{-3,1} &= z_3\partial_{z_1} - z_{-1}\partial_{z_{-3}}, \\
B_{-3,-2} &= z_3\partial_{z_{-2}} - z_2\partial_{z_{-3}}, \\
B_{3,2} &= z_{-3}\partial_{z_2} - z_{-2}\partial_{z_3}.
\end{aligned}
$$

The "Weyl group" of $\mathrm{Sch}(2)$ is generated by

$$\kappa K = K(z_{-1}, z_1, -z_3, -z_{-3}, z_2, z_{-2}),$$
$$\tau_1 K = K(z_1, z_{-1}, z_{-2}, z_2, z_{-3}, z_3).$$

We have relations

$$\begin{aligned}
[N_{23}, B_{2,\pm1}] &= B_{2,\pm1}, \\
[N_{23}, B_{-3,\pm1}] &= -B_{-3,\pm1}, \\
[N_{23}, B_{\pm(3,2)}] &= \pm 2 B_{\pm(3,2)}, \\
[N_1, B_{j,\pm1}] &= \pm B_{j,\pm1}, \\
\kappa N_{23} \kappa^{-1} &= -N_{23}, \\
\tau_1 N_1 \tau_1^{-1} &= -N_1.
\end{aligned}$$

Auxiliary space $\mathbb{C}^4$

$$
\begin{aligned}
N_{23}^{\mathrm{fl},\eta} &= y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + y_{-2}\partial_{y_{-2}} - \eta,\\
N_{1}^{\mathrm{fl},\eta} &= y_{-1}\partial_{y_{-1}} - y_1\partial_{y_1},\\
B_{2,-1}^{\mathrm{fl},\eta} &= y_{-2}\partial_{y_{-1}} - y_1\partial_{y_2},\\
B_{2,1}^{\mathrm{fl},\eta} &= y_{-2}\partial_{y_1} - y_1\partial_{y_{-2}},\\
B_{-3,-1}^{\mathrm{fl},\eta} &= \partial_{y_{-1}}, \quad B_{-3,1}^{\mathrm{fl},\eta} = \partial_{y_1}, \quad B_{-3,-2}^{\mathrm{fl},\eta} = \partial_{y_{-2}},\\
B_{3,2}^{\mathrm{fl},\eta} &= -y_{-1}y_1\partial_{y_2} + y_{-2}(y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + (y_{-2})\partial_{y_{-2}} - \eta);
\end{aligned}
$$

$$
\begin{aligned}
\tau_1^{\mathrm{fl},\eta} g &= g(y_1, y_{-1}, y_{-2}, y_2),\\
\kappa^{\mathrm{fl},\eta} g &= y_2^{\eta} g(\tfrac{y_1}{y_2}, \tfrac{2y_{-1}y_1+2y_{-2}y_2}{2y_2}, -\tfrac{1}{y_2}).
\end{aligned}
$$

Heat equation on $\mathbb{C}^2 \oplus \mathbb{C}$

A vector in $\mathbb{C}^2 \oplus \mathbb{C}$ will be denoted by (y_{-1}, y_1, t). The heat operator in $\mathbb{C}^2 \oplus \mathbb{C}$ is defined as

$$\mathcal{L}_{\mathbb{C}^2} := 2\partial_{y_{-1}}\partial_{y_1} + 2\partial_t.$$

$$\mathcal{L}_{\mathbb{C}^2} B^{\mathrm{sch},-1} = B^{\mathrm{sch},-3}\mathcal{L}_{\mathbb{C}^2}, \quad B \in \mathrm{sch}(\mathbb{C}^2),$$

$$\mathcal{L}_{\mathbb{C}^2} \alpha^{\mathrm{sch},-1} = \alpha^{\mathrm{sch},-3}\mathcal{L}_{\mathbb{C}^2}, \quad \alpha \in \mathrm{Sch}(\mathbb{C}^2).$$

$$
\begin{aligned}
N_{23}^{\mathrm{sch},\eta} &= y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + 2t\partial_t - \eta,\\
N_{1}^{\mathrm{sch},\eta} &= y_{-1}\partial_{y_{-1}} - y_1\partial_{y_1}\\
B_{2,-1}^{\mathrm{sch},\eta} &= t\partial_{y_{-1}} - y_1,\\
B_{2,1}^{\mathrm{sch},\eta} &= t\partial_{y_1} - y_{-1},\\
B_{-3,-1}^{\mathrm{sch},\eta} &= \partial_{y_{-1}},\\
B_{-3,1}^{\mathrm{sch},\eta} &= \partial_{y_1},\\
B_{-3,-2}^{\mathrm{sch},\eta} &= \partial_t,\\
B_{3,2}^{\mathrm{sch},\eta} &= t(y_{-1}\partial_{y_{-1}} + y_1\partial_{y_1} + t\partial_t - \eta) - y_{-1}y_1.
\end{aligned}
$$

$$
\begin{aligned}
\tau_1^{\mathrm{sch},\eta} h &= h(y_1, y_{-1}, t),\\
\kappa^{\mathrm{sch},\eta} h &= t^\eta \exp(\tfrac{y_{-1}y_1}{t}) h(\tfrac{y_{-1}}{t}, \tfrac{y_1}{t}, -\tfrac{1}{t}).
\end{aligned}
$$

We introduce the coordinates

$$w = \frac{y_{-1}y_1}{t}\,,$$
$$u = \sqrt{\frac{y_{-1}}{y_1}}\,,$$
$$s = \sqrt{t}\,.$$

We also sandwich all the operators

$$\hat{B} := \mathrm{e}^{-\frac{w}{2}} B \mathrm{e}^{\frac{w}{2}}.$$

The Cartan algebra becomes

$$
\begin{aligned}
\hat{N}_{23}^{\mathrm{sch},\eta} &= s\,\partial_s - \eta, \\
\hat{N}_{1}^{\mathrm{sch}} &= u\,\partial_u, \\
\hat{B}_{-3\,2}^{\mathrm{sch}} &= 1,
\end{aligned}
$$

Here is the heat opeator after these transformations:

$$
\hat{\mathcal{L}}_{\mathbb{C}^2} = \mathrm{e}^{-\frac{w}{2}}\,\mathcal{L}_{\mathbb{C}^2}\,\mathrm{e}^{\frac{w}{2}} = \frac{2}{s^2}\left(\partial_w\, w\,\partial_w - \frac{w}{4} - \frac{\left(\hat{N}_1^{\mathrm{sch}}\right)^2}{4\,w} + \frac{1}{2}\hat{N}_{23}^{\mathrm{sch},-1}\right)
$$

We make an ansatz

$$h(w,u,s) = u^{\alpha} s^{-\theta-1} F(w).$$

Then $\frac{2}{s^2}\hat{\mathcal{L}}_{\mathbb{C}^2}$ becomes the balanced confluent operator

$$\partial_w w \partial_w - \frac{w}{4} - \frac{\theta}{2} - \frac{\alpha^2}{4w}.$$

The heat equation in $\mathbf{C} \oplus \mathbf{C}$ – the Hermite equation

Extended space $\mathbb{C}^5$

We start from the same extended space $\mathbb{C}^5$ as in the case of the Gegenbauer equation. $\mathrm{sch}(\mathbb{C}^1)$ has a Cartan algebra

$$\begin{aligned} N &= z_{-2}\partial_{z_{-2}} - z_2\partial_{z_2} + z_{-3}\partial_{z_{-3}} - z_3\partial_{z_3} \\ B_{-3,2} &= z_3\partial_{z_2} - z_{-2}\partial_{z_{-3}}. \end{aligned}$$

and root operators

$$
\begin{aligned}
B_{2,0} &= z_{-2}\partial_{z_0} - z_0\partial_{z_2}, \\
B_{-3,0} &= z_3\partial_{z_0} - z_0\partial_{z_{-3}}, \\
B_{-3,-2} &= z_3\partial_{z_{-2}} - z_2\partial_{z_{-3}}, \\
B_{3,2} &= z_{-3}\partial_{z_2} - z_2\partial_{z_{-3}}.
\end{aligned}
$$

We consider also $\kappa \in \mathrm{Sch}(\mathbb{C}^1)$: given by

$$\kappa K = K(z_0, -z_2, -z_{-2}, z_1, z_{-1}).$$

It generates a group isomorphic to $\mathbb{Z}_4$.

We have relations

$$\begin{aligned} [N, B_{1,0}] &= B_{1,0}, \\ [N, B_{-2,0}] &= -B_{-2,0}, \\ [N, B_{\pm(2,1)}] &= \pm 2B_{\pm(2,1)}, \\ \\ \kappa N \kappa^{-1} &= -N. \end{aligned}$$

Auxiliary space $\mathbb{C}^3$

On $\mathbb{C}^3$ the Lie algebra $\mathrm{sch}(\mathbb{C}^1)$ acts as follows:

$$
\begin{aligned}
N_{23}^{\mathrm{fl},\eta} &= y_0\partial_{y_0} + 2y_{-2}\partial_{y_{-2}} - \eta, \\
B_{-3\,2}^{\mathrm{fl}} &= \partial_{y_2}, \\
B_{2,0}^{\mathrm{fl},\eta} &= y_{-2}\partial_{y_0} - y_0\partial_{y_2}, \\
B_{-2,0}^{\mathrm{fl},\eta} &= \partial_{y_0}, \\
B_{-2,-1}^{\mathrm{fl},\eta} &= \partial_{y_{-2}}, \\
B_{2,1}^{\mathrm{fl},\eta} &= y_{-2}y_0\partial_{y_0} + y_{-2}^2\partial_{y_{-2}} - \tfrac{1}{2}y_0^2\partial_{y_2} - \eta y_2, \\
\kappa^{\mathrm{fl},\eta} g &= y_2^\eta g\left(\tfrac{y_0}{y_2}, \tfrac{y_0^2+2y_{-2}y_2}{2y_2}, -\tfrac{1}{y_2}\right), \\
(\kappa^{\mathrm{fl},\eta})^2 g &= (\mathrm{i})^\eta g(-y_0, y_{-2}, y_2).
\end{aligned}
$$

Heat equation on $\mathbb{C} \oplus \mathbb{C}$

The heat operator in $\mathbb{C} \oplus \mathbb{C}$ is defined as

$$\mathcal{L}_{\mathbb{C}} := \partial_y^2 + 2\partial_t.$$

$$\mathcal{L}_{\mathbb{C}} A^{\mathrm{sch},-\frac{1}{2}} = A^{\mathrm{sch},-\frac{5}{2}} \mathcal{L}_{\mathbb{C}}, \quad A \in \mathrm{sch}(\mathbb{C}),$$

$$\mathcal{L}_{\mathbb{C}} \alpha^{\mathrm{sch},-\frac{1}{2}} = \alpha^{\mathrm{sch},-\frac{5}{2}} \mathcal{L}_{\mathbb{C}}, \quad \alpha \in \mathrm{Sch}(\mathbb{C}).$$

We have

$$
\begin{aligned}
&N^{\mathrm{sch},\eta} = y\partial_y + 2t\partial_t - \eta,\\
&B_{1,0}^{\mathrm{sch},\eta} = t\partial_y - y,\\
&B_{-2,0}^{\mathrm{sch},\eta} = \partial_y,\\
&B_{-2,-1}^{\mathrm{sch},\eta} = \partial_t,\\
&B_{2,1}^{\mathrm{sch},\eta} = t(y\partial_y + t\partial_t - \eta) - y^2,\\
&\kappa^{\mathrm{sch},\eta} h(y,t) = t^\eta \exp(\tfrac{y^2}{2t}) h(\tfrac{y}{t}, -\tfrac{1}{t}),\\
&(\kappa^{\mathrm{sch},\eta})^2 h(y,t) = (-\mathrm{i})^\eta h(-y,t).
\end{aligned}
$$

Space $\mathbb{C} \oplus \mathbb{C}$ in special coordinates

$$w = \frac{y}{\sqrt{2\,t}}\,,$$

$$s = \sqrt{t}\,.$$

We sandwich all operators with a Gaussian weight:

$$\hat{C} := \mathrm{e}^{-\frac{w^2}{2}} C \mathrm{e}^{\frac{w^2}{2}}.$$

Cartan operators.

$$\hat{N}_{23}^{\mathrm{sch},\eta} = s\,\partial_s - \eta,$$
$$\hat{B}_{-3\,2} = 1.$$

$$\hat{\mathcal{L}}_{\mathbb{C}} = \frac{1}{2s^2}\left(\partial_w^2 - w^2 - 2\hat{N}^{\mathrm{sch},-\frac{1}{2}}\right)$$

We make an ansatz

$$h(w,s) = s^{\lambda-\frac{1}{2}}F(w).$$

$2s^2\mathcal{L}_{\mathbb{C}}$ coincides on such functions with the balanced Hermite operator

$$\partial_w^2 - w^2 - 2\lambda.$$