

HOMOGENEOUS SCHRÖDINGER OPERATORS ON HALFLINE

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HOMOGENEOUS SCHRÖDINGER OPERATORS

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Consider a formal differential expression

$$L_\alpha = -\partial_x^2 + \left(-\frac{1}{4} + \alpha\right)\frac{1}{x^2}.$$

We would like to interpret it as a well-defined (unbounded) operator. To do this we need to specify its domain.

We will obtain operators with surprisingly rich mathematical phenomenology, which should be close to physicists' hearts: the “running coupling constant” flows under the action of the “renormalization group”, there are two “phase transitions”, attractive and repulsive fixed points, limit cycles, breakdown of conformal symmetry, etc.

I will discuss both the self-adjoint and non-self-adjoint cases. The latter have quite curious properties and I am looking for their physical applications.

Let U_τ be the **group of dilations** on $L^2[0, \infty[$, that is

$$(U_\tau f)(x) = e^{\tau/2} f(e^\tau x).$$

We say that B is **homogeneous of degree ν** if

$$U_\tau B U_\tau^{-1} = e^{\nu\tau} B.$$

Clearly, L_α is **homogeneous of degree -2** .

Here are two natural questions:

1. If $\alpha \in \mathbb{R}$, how to interpret L_α as a **self-adjoint operator** on $L^2[0, \infty[$? When is it homogeneous of degree -2 ?
2. If $\alpha \in \mathbb{C}$, how to interpret L_α as a **closed operator** on $L^2[0, \infty[$? When is it homogeneous of degree -2 ?

L_α , and closely related operators H_m , which we introduce shortly, are interesting for many reasons.

- They appear as the **radial** part of the Laplacian in all dimensions, in the decomposition of the **Aharonov-Bohm Hamiltonian**, in the membranes with **conical singularities**, in the theory of **many body systems with contact interactions** and in the **Efimov effect**.
- They have rather subtle and rich properties illustrating various concepts of the operator theory in Hilbert spaces (eg. the **Friedrichs and Krein extensions**, **holomorphic families of closed operators**).

- Essentially all basic objects related to H_m , such as their **resolvents, spectral projections, wave and scattering operators**, can be explicitly computed.
- A number of nontrivial identities involving special functions, especially from the **Bessel family**, find an appealing operator-theoretical interpretation in terms of the operators H_m . Eg. the **Barnes identity** leads to the formula for wave operators.

Two naive interpretations of L_α :

1. The **minimal** operator L_α^{\min} : We start from L_α on $C_c^\infty[0, \infty[$, and then we take its closure.

2. The **maximal** operator L_α^{\max} : We consider the domain consisting of all

$$f \in L^2[0, \infty[\text{ such that } L_\alpha f \in L^2[0, \infty[.$$

Clearly, $\text{Dom}(L_{\min}) \subset \text{Dom}(L_{\max})$ and

$$L_{\max} \Big|_{\text{Dom}(L_{\min})} = L_{\min}.$$

In other words $L_{\min} \subset L_{\max}$.

We will see that it is often natural to write $\alpha = m^2$

Theorem 1 .

1. For $1 \leq \operatorname{Re} m$, $L_{m^2}^{\min} = L_{m^2}^{\max}$.
2. For $-1 < \operatorname{Re} m < 1$, $L_{m^2}^{\min} \subsetneq L_{m^2}^{\max}$, and the codimension of their domains is 2.
3. $(L_{\alpha}^{\min})^* = L_{\bar{\alpha}}^{\max}$. Hence, for $\alpha \in \mathbb{R}$, L_{α}^{\min} is **Hermitian**.
4. L_{α}^{\min} and L_{α}^{\max} are homogeneous of degree -2 .

Notice that

$$Lx^{\frac{1}{2}\pm m} = 0.$$

Let ξ be a compactly supported cutoff equal 1 around 0.

Let $-1 < \operatorname{Re} m$. Note that $x^{\frac{1}{2}+m}\xi$ belongs to $\operatorname{Dom}L_{m^2}^{\max}$.

This suggests to define the operator H_m to be the restriction of $L_{m^2}^{\max}$ to

$$\operatorname{Dom}L_{m^2}^{\min} + \mathbb{C}x^{\frac{1}{2}+m}\xi.$$

Theorem 2 .

- 1. For $1 \leq \operatorname{Re} m$, $L_{m^2}^{\min} = H_m = L_{m^2}^{\max}$.*
- 2. For $-1 < \operatorname{Re} m < 1$, $L_{m^2}^{\min} \subsetneq H_m \subsetneq L_{m^2}^{\max}$ and the codimension of the domains is 1.*
- 3. $H_m^* = H_{\bar{m}}$. Hence, for $m \in]-1, \infty[$, H_m is self-adjoint.*
- 4. H_m is homogeneous of degree -2 .*
- 5. $\operatorname{sp} H_m = [0, \infty[$.*
- 6. $\{\operatorname{Re} m > -1\} \ni m \mapsto H_m$ is a holomorphic family of closed operators.*

Theorem 3 .

1. For $\alpha \geq 1$, $L_{\alpha}^{\min} = H_{\sqrt{\alpha}}$ is *essentially self-adjoint* on $C_c^{\infty}[0, \infty[$.
2. For $\alpha < 1$, L_{α}^{\min} is not essentially self-adjoint on $C_c^{\infty}[0, \infty[$.
3. For $0 \leq \alpha < 1$, the operator $H_{\sqrt{\alpha}}$ is the *Friedrichs extension* and $H_{-\sqrt{\alpha}}$ is the *Krein extension* of L_{α}^{\min} .
4. $H_{\frac{1}{2}}$ is the *Dirichlet Laplacian* and $H_{-\frac{1}{2}}$ is the *Neumann Laplacian* on halfline.
5. For $\alpha < 0$, L_{α}^{\min} has no homogeneous selfadjoint extensions.

It is easy to see that

$$\begin{aligned} & x^{-\frac{1}{2}} \left(-\partial_x^2 + \left(-\frac{1}{4} + \alpha \right) \frac{1}{x^2} \pm 1 \right) x^{\frac{1}{2}} \\ &= -\partial_x^2 - \frac{1}{x} \partial_x + \left(-\frac{1}{4} + \alpha \right) \frac{1}{x^2} \pm 1, \end{aligned}$$

which is the **(modified) Bessel operator**.

Therefore, it is not surprising that various objects related to H_m can be computed with help of functions from the Bessel family.

Theorem 4. *If $R_m(\lambda; x, y)$ is the integral kernel of the operator $(\lambda - H_m)^{-1}$, then for $\operatorname{Re} k > 0$ we have*

$$R_m(-k^2; x, y) = \begin{cases} \sqrt{xy} I_m(kx) K_m(ky) & \text{if } x < y, \\ \sqrt{xy} I_m(ky) K_m(kx) & \text{if } x > y, \end{cases}$$

where I_m is the **modified Bessel function** and K_m is the **MacDonald function**.

Proposition 5 . For $0 < a < b < \infty$, the integral kernel of $\mathbb{1}_{[a,b]}(H_m)$ is

$$\mathbb{1}_{[a,b]}(H_m)(x, y) = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_m(kx) J_m(ky) k dk,$$

where J_m is the **Bessel function**.

Let \mathcal{F}_m be the operator on $L^2(0, \infty)$ given by

$$\mathcal{F}_m : f(x) \mapsto \int_0^\infty J_m(kx) \sqrt{kx} f(x) dx$$

\mathcal{F}_m is the so-called **Hankel transformation**. Define also the operator $Xf(x) := xf(x)$.

Theorem 6. \mathcal{F}_m is a bounded invertible **involution** on $L^2[0, \infty[$ **diagonalizing** H_m and anticommuting with the self-adjoint **generator of dilations** $A = \frac{1}{2i}(x\partial_x + \partial_x x)$:

$$\mathcal{F}^2 = \mathbb{1},$$

$$\mathcal{F}_m H_m \mathcal{F}_m^{-1} = X^2,$$

$$\mathcal{F}_m A = -A \mathcal{F}_m.$$

Theorem 7 Set

$$\mathcal{I}f(x) = x^{-1}f(x^{-1}), \quad \Xi_m(t) = e^{i \ln(2)t} \frac{\Gamma(\frac{m+1+it}{2})}{\Gamma(\frac{m+1-it}{2})}.$$

Then

$$\mathcal{F}_m = \Xi_m(A)\mathcal{I}.$$

Therefore, we have the identity

$$H_m := \Xi_m^{-1}(A)X^{-2}\Xi_m(A)$$

(Result obtained independently by Bruneau, Georgescu, D, and by Richard and Pankrashkin).

Theorem 8 . The *wave operators* associated to the pair H_m, H_k exist and

$$\begin{aligned}\Omega_{m,k}^{\pm} &:= \lim_{t \rightarrow \pm\infty} e^{itH_m} e^{-itH_k} \\ &= e^{\pm i(m-k)\pi/2} \mathcal{F}_m \mathcal{F}_k \\ &= e^{\pm i(m-k)\pi/2} \frac{\Xi_k(A)}{\Xi_m(A)}.\end{aligned}$$

The formula

$$H_m := \Xi_m^{-1}(A)X^{-2}\Xi_m(A) \quad (1)$$

valid for $\operatorname{Re} m > -1$ can be used as an alternative definition of the family H_m also beyond this domain. It defines a family of closed operators for the parameter m that belongs to

$$\{m \mid \operatorname{Re} m \neq -1, -2, \dots\} \cup \mathbb{R}. \quad (2)$$

Its spectrum is always equal to $[0, \infty[$ and it is analytic in the interior of (2).

In fact, $\Xi_m(A)$ is a unitary operator for all real values of m . Therefore, for $m \in \mathbb{R}$, (1) is well-defined and self-adjoint.

$\Xi_m(A)$ is bounded and invertible also for all m such that $\operatorname{Re} m \neq -1, -2, \dots$. Therefore, the formula (1) defines an operator for all such m .

One can then pose various questions:

- What happens with this operator along the lines $\operatorname{Re} m = -1, -2, \dots$?
- What is the meaning of the operator to the left of $\operatorname{Re} = -1$? (It is not a differential operator!)

The definition (or actually a number of equivalent definitions) of a **holomorphic family of bounded operators** is quite obvious and does not need to be recalled. In the case of unbounded operators the situation is more subtle.

Suppose that Θ is an open subset of \mathbb{C} , \mathcal{H} is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on \mathcal{H} . We say that this is a **holomorphic family of closed operators** if for each $z_0 \in \Theta$ there exists a neighborhood Θ_0 of z_0 , a Banach space \mathcal{K} and a holomorphic family of injective bounded operators $\Theta_0 \ni z \mapsto B(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\text{Ran } B(z) = \mathcal{D}(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)B(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

We have the following practical criterion:

Theorem 9 . *Suppose that $\{H(z)\}_{z \in \Theta}$ is a function whose values are closed operators on \mathcal{H} . Suppose in addition that for any $z \in \Theta$ the resolvent set of $H(z)$ is nonempty. Then $z \mapsto H(z)$ is a **holomorphic family of closed operators** if and only if for any $z_0 \in \Theta$ there exists $\lambda \in \mathbb{C}$ and a neighborhood Θ_0 of z_0 such that λ belongs to the resolvent set of $H(z)$ for $z \in \Theta_0$ and $z \mapsto (H(z) - \lambda)^{-1} \in B(\mathcal{H})$ is holomorphic on Θ_0 .*

The above theorem indicates that it is more difficult to study holomorphic families of closed operators that for some values of the complex parameter have an **empty resolvent set**.

Conjecture 10 . *It is impossible to extend*

$$\{\operatorname{Re} m > -1\} \ni m \mapsto H_m$$

to a holomorphic family of closed operators on a larger connected open subset of \mathbb{C} .

ALMOST HOMOGENEOUS SCHRÖDINGER OPERATORS (in collaboration with SERGE RICHARD)

For any $\kappa \in \mathbb{C} \cup \{\infty\}$ let $H_{m,\kappa}$ be the restriction of $L_{m^2}^{\max}$ to the domain

$$\text{Dom}(H_{m,\kappa}) = \left\{ f \in \text{Dom}(L_{m^2}^{\max}) \mid \text{for some } c \in \mathbb{C}, \right. \\ \left. f(x) - c(x^{1/2-m} + \kappa x^{1/2+m}) \in \text{Dom}(L_{m^2}^{\min}) \right. \\ \left. \text{around } 0 \right\}, \quad \kappa \neq \infty;$$

$$\text{Dom}(H_{m,\infty}) = \left\{ f \in \text{Dom}(L_{m^2}^{\max}) \mid \text{for some } c \in \mathbb{C}, \right. \\ \left. f(x) - cx^{1/2+m} \in \text{Dom}(L_{m^2}^{\min}) \text{ around } 0 \right\}.$$

For $\nu \in \mathbb{C} \cup \{\infty\}$, let H_0^ν be the restriction of L_0^{\max} to

$$\text{Dom}(H_0^\nu) = \left\{ f \in \text{Dom}(L_0^{\max}) \mid \text{for some } c \in \mathbb{C}, \right. \\ \left. f(x) - c(x^{1/2} \ln x + \nu x^{1/2}) \in \text{Dom}(L_0^{\min}) \right. \\ \left. \text{around } 0 \right\}, \quad \nu \neq \infty;$$

$$\text{Dom}(H_0^\infty) = \left\{ f \in \text{Dom}(L_0^{\max}) \mid \text{for some } c \in \mathbb{C}, \right. \\ \left. f(x) - cx^{1/2} \in \text{Dom}(L_0^{\min}) \text{ around } 0 \right\}.$$

Proposition 11 .

1. For any $|\operatorname{Re}(m)| < 1$, $\kappa \in \mathbb{C} \cup \{\infty\}$

$$H_{m,\kappa} = H_{-m,\kappa^{-1}}.$$

2. $H_{0,\kappa}$ does not depend on κ , and these operators coincide with H_0^∞ .

Proposition 12. *For any m with $|\operatorname{Re}(m)| < 1$ and any $\kappa, \nu \in \mathbb{C} \cup \{\infty\}$, we have*

$$U_\tau H_{m,\kappa} U_{-\tau} = e^{-2\tau} H_{m, e^{-2\tau m \kappa}},$$

$$U_\tau H_0^\nu U_{-\tau} = e^{-2\tau} H_0^{\nu+\tau}.$$

In particular, only

$$H_{m,0} = H_{-m},$$

$$H_{m,\infty} = H_m,$$

$$H_0^\infty = H_0$$

are homogeneous.

Proposition 13 .

$$H_{m,\kappa}^* = H_{\overline{m},\overline{\kappa}} \quad \text{and} \quad H_0^{\nu*} = H_0^{\overline{\nu}}.$$

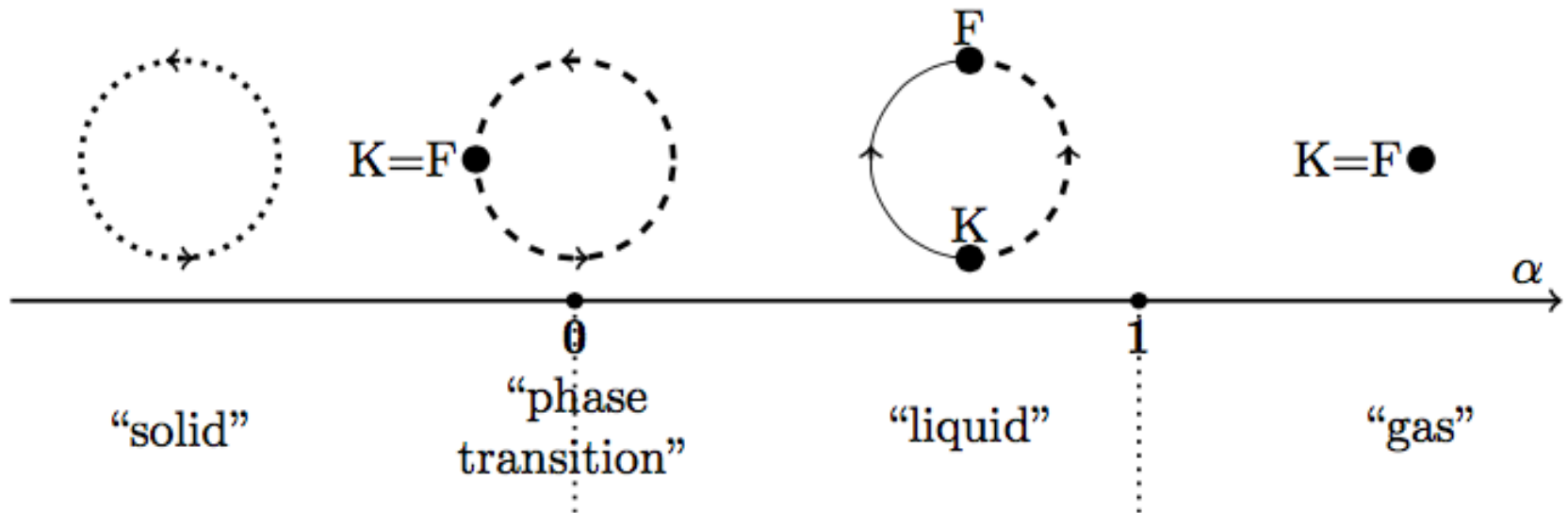
In particular,

- (i) $H_{m,\kappa}$ is self-adjoint for $m \in] - 1, 1[$ and $\kappa \in \mathbb{R} \cup \{\infty\}$,
and for $m \in i\mathbb{R}$ and $|\kappa| = 1$.*
- (ii) H_0^ν is self-adjoint for $\nu \in \mathbb{R} \cup \{\infty\}$.*

Self-adjoint extensions of the Hermitian operator

$$L_\alpha = -\partial_x^2 + \left(-\frac{1}{4} + \alpha\right)\frac{1}{x^2}.$$

K—Krein, F—Friedrichs, dashed line—single bound state, dotted line—infinite sequence of bound states.



The essential spectrum of $H_{m,\kappa}$ and H_0^ν is $[0, \infty[$.

Theorem 14 .

1. $z \in \mathbb{C} \setminus [0, \infty[$ belongs to the point spectrum of $H_{m,\kappa}$ iff

$$(-z)^{-m} = \kappa \frac{\Gamma(m)}{\Gamma(-m)}.$$

2. H_0^ν possesses an eigenvalue iff $-\pi < \operatorname{Im} 2\nu < \pi$, and then it is $z = -e^{-2\nu}$.

For a given m, κ all eigenvalues form a **geometric sequence** that lies on a **logarithmic spiral**, which should be viewed as a curve on the Riemann surface of the logarithm. Only its **“physical sheet”** gives rise to eigenvalues.

For m which are not purely imaginary, only a finite piece of the spiral is on the “physical sheet” and therefore the number of eigenvalues is finite.

If m is purely imaginary, this spiral degenerates to a half-line starting at the origin.

If m is real, the spiral degenerates to a circle. But then the operator has at most one eigenvalue.

Theorem 15 . Let $m = m_r + im_i \in \mathbb{C}^\times$ with $|m_r| < 1$.

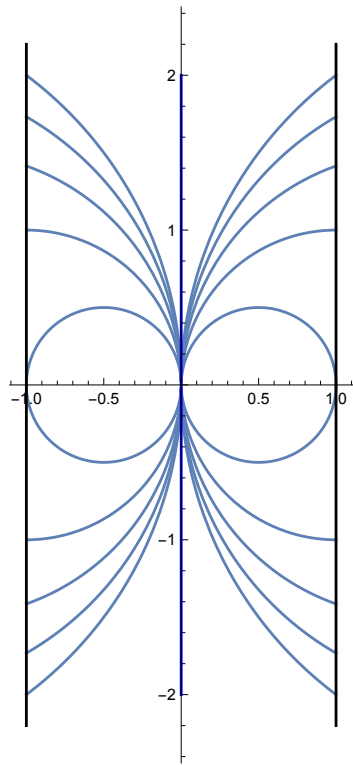
(i) Let $m_r = 0$.

(a) If $\frac{\ln \left| \kappa \frac{\Gamma(m)}{\Gamma(-m)} \right|}{m_i} \in] - \pi, \pi [$, then $\#\sigma_p(H_{m,\lambda}) = \infty$,

(a) if $\frac{\ln \left| \kappa \frac{\Gamma(m)}{\Gamma(-m)} \right|}{m_i} \notin] - \pi, \pi [$ then $\#\sigma_p(H_{m,\lambda}) = 0$.

(ii) If $m_r \neq 0$ and if $N \in \mathbb{N}$ satisfies $N < \frac{m_r^2 + m_i^2}{|m_r|} \leq N + 1$,
then

$$\#\sigma_p(H_{m,\lambda}) \in \{N, N + 1\}.$$



HOMOGENEOUS RANK ONE PERTURBATIONS

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+)$ and operator X

$$Xf(x) := xf(x).$$

Let $m \in \mathbb{C}$, $\lambda \in \mathbb{C} \cup \{\infty\}$. We consider a family of operators formally given by

$$H_{m,\lambda} := X + \lambda |x^{\frac{m}{2}}\rangle \langle x^{\frac{m}{2}}|.$$

Note that X is homogeneous of degree 1.

$|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ is homogeneous of degree $1 + m$. However strictly speaking, it is not a well defined operator, because $x^{\frac{m}{2}}$ is never square integrable.

If $-1 < \operatorname{Re} m < 0$, the perturbation $|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ is **form bounded** relatively to X and then $H_{m,\lambda}$ can be defined.

The perturbation is formally rank one. Therefore,

$$\begin{aligned}
 (z - H_{m,\lambda})^{-1} &= (z - X)^{-1} \\
 &+ \sum_{n=0}^{\infty} (z - X)^{-1} |x^{\frac{m}{2}}\rangle (-\lambda)^{n+1} \langle x^{\frac{m}{2}}| (z - X)^{-1} |x^{\frac{m}{2}}\rangle^n \langle x^{\frac{m}{2}}| (z - X)^{-1} \\
 &= (z - X)^{-1} \\
 &+ \left(\lambda^{-1} - \langle x^{\frac{m}{2}}| (z - X)^{-1} |x^{\frac{m}{2}}\rangle \right)^{-1} (z - X)^{-1} |x^{\frac{m}{2}}\rangle \langle x^{\frac{m}{2}}| (z - X)^{-1}.
 \end{aligned}$$

By straightforward complex analysis methods we obtain

$$\begin{aligned} & \langle x^{\frac{m}{2}} | (z - X)^{-1} | x^{\frac{m}{2}} \rangle \\ &= \int_0^\infty x^m (z - x)^{-1} dx = (-z)^m \frac{\pi}{\sin \pi m}. \end{aligned}$$

Therefore, the resolvent of $H_{m,\lambda}$ can be given in a closed form:

$$\begin{aligned} & (z - H_{m,\lambda})^{-1} = (z - X)^{-1} \\ & + \left(\lambda^{-1} - (-z)^m \frac{\pi}{\sin \pi m} \right)^{-1} (z - X)^{-1} | x^{\frac{m}{2}} \rangle \langle x^{\frac{m}{2}} | (z - X)^{-1}. \end{aligned}$$

The above formula defines a resolvent of a closed operator for all $-1 < \operatorname{Re} m < 1$ and $\lambda \in \mathbb{C} \cup \{\infty\}$. This allows us to define a **holomorphic family of closed operators** $H_{m,\lambda}$.

Note that $H_{m,0} = X$.

$m = 0$ is special: $H_{0,\lambda} = X$ for all λ .

We introduce H_0^ρ for any $\rho \in \mathbb{C} \cup \{\infty\}$ by

$$(z - H_0^\rho)^{-1} = (z - X)^{-1} - (\rho + \ln(-z))^{-1} (z - X)^{-1} |x^0\rangle \langle x^0| (z - X)^{-1}.$$

In particular, $H_0^\infty = X$.

The group of dilations (“the renormalization group”) acts on our operators in a simple way:

$$U_\tau H_{m,\lambda} U_\tau^{-1} = e^\tau H_{m,e^{\tau m} \lambda},$$
$$U_\tau H_0^\rho U_\tau^{-1} = e^\tau H_0^{\rho+\tau}.$$

Define the unitary operator

$$(If)(x) := x^{-\frac{1}{4}} f(2\sqrt{x}).$$

Its inverse is

$$(I^{-1}f)(x) := \left(\frac{y}{2}\right)^{\frac{1}{2}} f\left(\frac{y^2}{4}\right).$$

Note that

$$I^{-1}XI = \frac{X^2}{4},$$
$$I^{-1}AI = \frac{A}{2}.$$

We change slightly notation: the almost homogeneous Schrödinger operators H_m , $H_{m,\kappa}$ and H_0^ν will be denoted \tilde{H}_m , $\tilde{H}_{m,\kappa}$ and \tilde{H}_0^ν

Recall that we introduced the Hankel transformation \mathcal{F}_m , which is a bounded invertible involution satisfying

$$\begin{aligned}\mathcal{F}_m \tilde{H}_m \mathcal{F}_m^{-1} &= X^2, \\ \mathcal{F}_m A \mathcal{F}_m^{-1} &= -A.\end{aligned}$$

Theorem 16 .

1.

$$\mathcal{F}_m^{-1} I^{-1} H_{m,\lambda} I \mathcal{F}_m = \frac{1}{4} \tilde{H}_{m,\kappa},$$

where

$$\lambda \frac{\pi}{\sin(\pi m)} = \kappa \frac{\Gamma(m)}{\Gamma(-m)},$$

2.

$$\mathcal{F}_m^{-1} I^{-1} H_0^\rho I \mathcal{F}_m = \frac{1}{4} \tilde{H}_0^\nu,$$

where $\rho = -2\nu$.