QUADRATIC BOSONIC HAMILONIANS AND THEIR RENORMALIZATION

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INTRODUCTION

Bogoliubov Hamiltonians formally given by

$$
\begin{aligned}
H &= \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi \\
&\quad + \frac{1}{2} \int g(\xi, \xi') a_\xi^* a_{\xi'}^* \mathrm{d}\xi + \frac{1}{2} \int \overline{g}(\xi, \xi') a_\xi a_{\xi'} \mathrm{d}\xi + c.
\end{aligned}
$$

Van Hove Hamiltonians formally given by

$$
H = \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi + \int \overline{z}(\xi) a_\xi \mathrm{d}\xi + \int z(\xi) a_\xi^* \mathrm{d}\xi + c.
$$

a_ξ^* / a_ξ **are bosonic creation/annihilation operators,** $h(\xi)$ **is positive,** c **is a constant, which can be infinite.**

When does the above expression define a self-adjoint operator on a bosonic Fock space?

PLAN OF THE LECTURE

(1) **Formalism of second quantization.**
(2) **Van Hove Hamiltonians J.D.**
(3) **Scattering theory of Van Hove Hamiltonians J.D.**
(4) **Bogoliubov Hamiltonians L.Bruneau, J.D.**

FORMALISM OF SECOND QUANTIZATION

1-particle Hilbert space: $\mathcal{Z}$.

Bosonic Fock space: $\Gamma_{\mathrm{s}}(\mathcal{Z}) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathcal{Z}$.

Vacuum vector: $\Omega = 1 \in \otimes_{\mathrm{s}}^{0} \mathcal{Z} = \mathbb{C}$.

If $\Psi \in \otimes_{\mathrm{s}}^{n} \mathcal{Z}$, $\Phi \in \otimes_{\mathrm{s}}^{m} \mathcal{Z}$, **then we define the symmetric tensor product**

$$\Psi \otimes_{\mathrm{s}} \Phi := \Theta_{\mathrm{s}} \, \Psi \otimes \Phi \in \otimes_{\mathrm{s}}^{n+m} \mathcal{Z},$$

where Θ_{s} **is the symmetrizing operator.**

Creation and annihilation operators

For $z \in \mathcal{Z}$ we define the creation operator

$$a^*(z)\Psi := \sqrt{n+1} z \otimes_{\mathrm{s}} \Psi, \quad \Psi \in \otimes_{\mathrm{s}}^n \mathcal{Z},$$

and the annihilation operator $a(z) := (a^*(z))^*$.

Traditional notation**: identify $\mathcal{Z}$ with $L^2(\Xi)$ for some measure space $(\Xi, \mathrm{d}\xi)$. If z equals a function $\Xi \ni \xi \mapsto z(\xi)$, then:**

$$a^*(z) = \int z(\xi) a_\xi^* \mathrm{d}\xi, \quad a(z) = \int \overline{z}(\xi) a_\xi \mathrm{d}\xi.$$

2-particle creation and annihilation operators

For $g \in \otimes_{\mathrm{s}}^{2}\mathcal{Z}$ we define the 2-particle creation operator

$$a^*(g)\Psi := \sqrt{(n+2)(n+1)}\, g \otimes_{\mathrm{s}} \Psi, \quad \Psi \in \otimes_{\mathrm{s}}^{n}\mathcal{Z},$$

and the annihilation operator $a(g) = a^*(g)^*$.
Traditional notation**: if g equals a function $g(\xi, \xi')$:**

$$a^*(g) = \int g(\xi,\xi') a_{\xi}^* a_{\xi'}^* \mathrm{d}\xi \mathrm{d}\xi', \quad a(g) = \int \overline{g}(\xi,\xi') a_{\xi} a_{\xi'} \mathrm{d}\xi \mathrm{d}\xi'.$$

Second quantization

For an operator q **on** $\mathcal{Z}$ **we define the operator** $\Gamma(q)$ **on** $\Gamma_{\rm s}(\mathcal{Z})$ **by**

$$\Gamma(q)\Big|_{\otimes_{\rm s}^n\mathcal{Z}} = q\otimes\cdots\otimes q.$$

For an operator h **on** $\mathcal{Z}$ **we define the operator** ${\rm d}\Gamma(h)$ **on** $\Gamma_{\rm s}(\mathcal{Z})$ **by**

$${\rm d}\Gamma(h)\Big|_{\otimes_{\rm s}^n\mathcal{Z}} = h\otimes 1^{(n-1)\otimes}+\cdots 1^{(n-1)\otimes}\otimes h.$$

Traditional notation: If h **is the multiplication operator by** $h(\xi)$**, then**

$${\rm d}\Gamma(h) = \int h(\xi)a_\xi^* a_\xi {\rm d}\xi.$$

Note the identity $\Gamma({\rm e}^{{\rm i}th}) = {\rm e}^{{\rm i}t{\rm d}\Gamma(h)}$**.**

VAN HOVE HAMILTONIANS

J.D.

We assume that $h(\xi) \geq 0$.

$$\begin{aligned} H &:= \mathrm{d}\Gamma(h) + a^*(z) + a(z) + c \\ &= \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi + \int \overline{z}(\xi) a_\xi \mathrm{d}\xi + \int z(\xi) a_\xi^* \mathrm{d}\xi + c. \end{aligned}$$

Note that c can be infinite.
When does the above expression define a self-adjoint operator?

Projective 1-parameter unitary group

When

$$\int_{h<1} |z(\xi)|^2 \mathrm{d}\xi + \int_{h\geq 1} \frac{|z(\xi)|^2}{h(\xi)^2} \mathrm{d}\xi < \infty,$$

we can define a family of unitary operators

$$V(t) := \Gamma(\mathrm{e}^{\mathrm{i}th}) \exp\left(a^*((1-\mathrm{e}^{-\mathrm{i}th})h^{-1}z) - \mathrm{hc} \right).$$

One can check that

$$V(t_1)V(t_2) = c(t_1, t_2)V(t_1 + t_2).$$

Rigorous definition of a van Hove operator

For an operator $B \in B(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ define

$$\beta_t(B) := V(t)BV(t)^*.$$

Then β is a 1-parameter group of $*$-automorphisms of the algebra of bounded operators on the Fock space. continuous in the strong operator topology.
By a general theorem, there exists a self-adjoint operator H such that

$$\beta_t(B) = \mathrm{e}^{\mathrm{i}tH} B \, \mathrm{e}^{-\mathrm{i}tH}.$$

H is definded uniquely up to an additive constant. We call it a van Hove Hamiltonian. Formally it is given by the expression from one of previous slides.

Type I van Hove Hamiltonians

Theorem. **Let**

$$\int_{h(\xi)<1} |z(\xi)|^2 \mathrm{d}\xi + \int_{h(\xi)\geq 1} \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi < \infty.$$

Then

$$U_{\mathrm{I}}(t) := \exp\left(\mathrm{i} \int |z(\xi)|^2 \frac{\sin th(\xi) - th(\xi)}{h^2(\xi)} \mathrm{d}\xi \right) V(t)$$

is a strongly continuous unitary group.

We define the type I van Hove operator by

$$U_{\mathrm{I}}(t) = \mathrm{e}^{\mathrm{i}tH_{\mathrm{I}}}.$$

Formally,

$$H_{\mathrm{I}} = \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi + \int \overline{z}(\xi) a_\xi \mathrm{d}\xi + \int z(\xi) a_\xi^* \mathrm{d}\xi.$$

Properties of type I van Hove Hamiltonians

It satisfies $\Omega \in \mathrm{Dom} H_{\mathrm{I}},\ (\Omega|H_{\mathrm{I}}\Omega) = 0$**,**

$$\inf \mathrm{sp} H_{\mathrm{I}} = -\int \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi,$$

(which can be $-\infty$**).**
Perturbation is an operator iff $\int |z(\xi)|^2 \mathrm{d}\xi < \infty$**, otherwise it is a form.**

Type II van Hove Hamiltonians

Theorem. **Let**

$$\int_{h(\xi)<1} \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi + \int_{h(\xi)\geq 1} \frac{|z(\xi)|^2}{h^2(\xi)} \mathrm{d}\xi < \infty.$$

Then

$$U_{\mathrm{II}}(t) := \exp\left(\mathrm{i} \int |z(\xi)|^2 \frac{\sin t h(\xi)}{h^2(\xi)} \mathrm{d}\xi\right) V(t)$$

is a strongly continuous unitary group.

We define the type II van Hove operator by

$$U_{\mathrm{II}}(t) = \mathrm{e}^{\mathrm{i}tH_{\mathrm{II}}}.$$

Formally,

$$H_{\mathrm{II}} = \int h(\xi) \left(a_\xi^* + \frac{\overline{z}(\xi)}{h(\xi)}\right) \left(a_\xi + \frac{z(\xi)}{h(\xi)}\right) \mathrm{d}\xi.$$

Properties of type II van Hove Hamiltonians

It satisfies $\inf \mathrm{sp} H_{\mathrm{II}} = 0$**.**
The dressing operator

$$U := \exp\left(-a^*(\frac{z}{h}) + a(\frac{z}{h})\right)$$

is well defined iff

$$\int \frac{|z(\xi)|^2}{h^2(\xi)} \mathrm{d}\xi < \infty.$$

It intertwines H_{II} **and the free van Hove Hamiltonian:**

$$H_{\mathrm{II}} = U \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi \; U^*.$$

Hence, in this case H_{II} **has a ground state. Otherwise** H_{II} **has no ground state.**

Both H_{I} and H_{II} are well defined iff

$$\int \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi < \infty,$$

and then

$$H_{\mathrm{II}} = H_{\mathrm{I}} + \int \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi < \infty.$$

If

$$\int_{h(\xi)<1} \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi = \int_{h(\xi)\geq 1} \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi = \infty,$$

then neither H_{I} nor H_{II} is well defined.

	$\int_{h>1} \lvert z\rvert^2 < \infty$	$\int_{h>1} \lvert z\rvert^2 = \infty$ $\int_{h>1} \frac{\lvert z\rvert^2}{h} < \infty$	$\int_{h>1} \frac{\lvert z\rvert^2}{h} = \infty$ $\int_{h>1} \frac{\lvert z\rvert^2}{h^2} < \infty$	
$\int_{h<1} \frac{\lvert z\rvert^2}{h^2} < \infty$				**H_{II} defined gr. st. exists**
$\int_{h<1} \frac{\lvert z\rvert^2}{h^2} = \infty$ $\int_{h<1} \frac{\lvert z\rvert^2}{h} < \infty$				**H_{II} defined no gr. st.**
$\int_{h<1} \frac{\lvert z\rvert^2}{h} = \infty$ $\int_{h<1} \lvert z\rvert^2 < \infty$				**unbounded from below**
	H_{I} defined pert. is an operator	**H_{I} defined pert. is not an operator**	**infinite renormalization**	

Massless scalar QFT with a linear perturbation
Infrared problem in various dimensions.

$$H = \frac{1}{2}\int :\Big(\pi(x)^2 + (\nabla_x \phi(x))^2\Big): \mathrm{d}x + \int q(x)\phi(x)\mathrm{d}x.$$

"Total charge": $\int q(x)\mathrm{d}x$.

Dimension of configuration space	Nonzero total charge	Zero total charge
$d = 1$	**Hamiltonian undefined**	**(2)**
$d = 2$	**(3)**	**(1)**
$d = 3$	**(2)**	**(1)**
$d \geq 4$	**(1)**	**(1)**

(1), (2), (3) denote the three kinds of the infrared condition.

Massive scalar QFT with a point-like linear perturbation Ultraviolet problem in various dimensions.

$$H = \frac{1}{2}\int :\Big(\pi(x)^2 + (\nabla_x \phi(x))^2 + m^2\phi(x)^2\Big):\mathrm{d}x + \phi(0).$$

Dimension of configuration space	
$d = 1$	**(2)**
$d = 2$	**(3)**
$d \geq 3$	**Hamiltonian undefined**

(1), (2), (3) denote the three kinds of the ultraviolet condition.

SCATTERING THEORY OF VAN HOVE HAMILTONIANS

General scattering theory – the standard approach

We are given two self-adjoint operators: H_0 and H.
The wave operators: $\Omega^\pm := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_0}$.
They satisfy $\Omega^\pm H_0 = H\Omega^\pm$ **and are isometric.**
If $\mathrm{Ran}\Omega^+ = \mathrm{Ran}\Omega^-$, **then the scattering operator**

$$S = \Omega^{+*}\Omega^-$$

is unitary and $H_0 S = S H_0$.

General scattering theory – the Abelian approach

Define the Abelian unrenormalized wave operators:

$$\Omega_{\rm ur}^{\pm} := {\rm s-}\lim_{\epsilon\searrow 0} 2\epsilon\int_0^\infty {\rm e}^{-2\epsilon t}\,{\rm e}^{\pm {\rm i}tH}\,{\rm e}^{\mp{\rm i}tH_0}\,{\rm d}t.$$

They satisfy $\Omega_{\rm ur}^{\pm}H_0 = H\Omega_{\rm ur}^{\pm}$ but do not have to be isometric

Let $Z^{\pm} := \Omega_{\rm ur}^{\pm *}\Omega_{\rm ur}^{\pm}$ have a zero kernel. Then we can define the renormalized wave operators

$$\Omega_{\rm rn}^{\pm} := \Omega_{\rm ur}^{\pm}(Z^{\pm})^{-1/2}.$$

They also satisfy $\Omega_{\rm rn}^{\pm}H_0 = H\Omega_{\rm rn}^{\pm}$ and are isometric.

If ${\rm Ran}\Omega_{\rm rn}^{+} = {\rm Ran}\Omega_{\rm rn}^{-}$, then the renormalized scattering operator

$$S_{\rm rn} = \Omega_{\rm rn}^{+*}\Omega_{\rm rn}^{-}$$

is unitary and $H_0 S_{\rm rn} = S_{\rm rn}H_0$.

Scattering theory for van Hove Hamiltonians

Let

$$H_0 = \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi,$$

$$H = \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi + \int \overline{z}(\xi) a_\xi \mathrm{d}\xi + \int z(\xi) a_\xi^* \mathrm{d}\xi + \int \frac{|\boldsymbol{z}(\xi)|^2}{h(\xi)} \mathrm{d}\xi.$$

Suppose that h **has an absolutely continuous spectrum and the assumption for the existence of** H_{II} **is satisfied. Let** U **be the dressing operator and**

$$Z = \exp \int \frac{|\boldsymbol{z}(\xi)|^2}{h(\xi)} \mathrm{d}\xi.$$

Then the Abelian wave operators exist, but after renormalization the scattering operator is trivial:

$$\Omega_{\mathrm{ur}}^{\pm} = Z^{1/2} U, \quad \Omega_{\mathrm{rn}}^{\pm} = U, \qquad S_{\mathrm{rn}} = 1.$$

BOGOLIUBOV HAMILTONIANS

We assume that $h(\xi) > 0$. We want to interpret the following formal expression

$$\begin{aligned} H = & \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi \\ & + \frac{1}{2} \int g(\xi,\xi') a_\xi^* a_{\xi'}^* \mathrm{d}\xi + \frac{1}{2} \int \overline{g}(\xi,\xi') a_\xi a_{\xi'} \mathrm{d}\xi + c. \end{aligned}$$

as a self-adjoint operator.

Classical phase space of a bosonic system

$\overline{\mathcal{Z}}$ **denotes the space complex conjugate to** $\mathcal{Z}$**.**
The real vector space

$$\mathcal{Y} := \{(z, \overline{z}) \ : \ z \in \mathcal{Z}\} \subset \mathcal{Z} \oplus \overline{\mathcal{Z}}.$$

equipped with a natural symplectic form

$$(z_1, \overline{z}_1)\omega(z_2, \overline{z}_2) := \mathrm{Im}(z_1|z_2).$$

has the meaning of the dual of the classical phase space of the quantum system described by the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$**.**

Canonical commutation relations

For $y = (z, \overline{z}) \in \mathcal{Y}$ **we define the corresponding** **Weyl operator**

$$W(y) := \mathrm{e}^{\mathrm{i}a^*(z)+\mathrm{i}a(z)}.$$

Note that $W(y_1)W(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\omega y_2}W(y_1+y_2)$.

A map r **on** $\mathcal{Y}$ **is called symplectic if**

$$(ry_1)\omega(ry_2) = y_1\omega y_2.$$

For such r**,**

$$W(ry_1)W(ry_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\omega y_2}W(r(y_1+y_2)).$$

Matrix represention of symplectic maps

Every linear map r **on** $\mathcal{Y}$ **can be uniquely extended to a complex linear map on** $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ **and written as**

$$r = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}.$$

r **is symplectic iff**

$$\begin{aligned} p^* p - \overline{q}^* \overline{q} = 1, \quad & -\overline{p}^* \overline{q} + q^* p = 0, \\ p p^* - q q^* = 1, \quad & \overline{q} p^* - \overline{p} q^* = 0. \end{aligned}$$

We have the decomposition

$$r = \begin{bmatrix} 1 & 0 \\ d^* & 1 \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & \overline{p}^{*-1} \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix},$$

with symmetric operators $d := q\overline{p}^{-1}$**,** $c := p^{-1}q$**.**

1-parameter symplectic groups

If h is a self-adjoint operator on $\mathcal{Z}$

$$h = h^*$$

and g is a bounded symmetric operator from $\overline{\mathcal{Z}}$ to $\mathcal{Z}$

$$\overline{g} = g^*,$$

then

$$r(t) = \exp \mathrm{i}t \begin{bmatrix} h & g \\ \overline{g} & \overline{h} \end{bmatrix}$$

is a 1-parameter symplectic group.
Clearly, in finite dimension every symplectic group is of this form.

Classical quadratic Hamiltonians

Consider a classical quadratic Hamiltonian

$$H(\overline{z}, z) = \int h(\xi)\overline{z}_\xi z_\xi \mathrm{d}\xi \\ +\frac{1}{2}\int g(\xi,\xi')\overline{z}_\xi\overline{z}_\xi \mathrm{d}\xi\mathrm{d}\xi' + \frac{1}{2}\int \overline{g}(\xi,\xi') z_\xi z_{\xi'}\mathrm{d}\xi\mathrm{d}\xi'.$$

It is a function on the classical phase space

$$\overline{\mathcal{Y}} := \{(\overline{z}, z) \ : \ z \in \mathcal{Z}\} \subset \overline{\mathcal{Z}} \oplus \mathcal{Z}.$$

Bogoliubov Hamiltonians for a finite number of degrees of freedom

Suppose that $\dim \mathcal{Z} < \infty$**. The Weyl quantization of** $H(\overline{z}, z)$ **equals**

$$H = \frac{1}{2}\int h(\xi)a_\xi^* a_\xi \mathrm{d}\xi + \frac{1}{2}\int h(\xi)a_\xi a_\xi^* \mathrm{d}\xi \\ +\frac{1}{2}\int g(\xi,\xi')a_\xi^* a_{\xi'}^* \mathrm{d}\xi\mathrm{d}\xi' + \frac{1}{2}\int \overline{g}(\xi,\xi')a_\xi a_{\xi'} \mathrm{d}\xi\mathrm{d}\xi'$$

and corresponds to the choice

$$c = \frac{1}{2}\int h(\xi,\xi)\mathrm{d}\xi = \frac{1}{2}\mathrm{Tr}h,$$

H **is essentially self-adjoint on finite particle vectors. We have**

$$\mathrm{e}^{\mathrm{i}tH} = \det(p(t))^{-\frac{1}{2}}\, \mathrm{e}^{-\frac{1}{2}a^*(d(t))}\, \Gamma(p(t)^{*-1})\, \mathrm{e}^{\frac{1}{2}a(c(t))}\,.$$

Metaplectic group

Operators of the form

$$\det p^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}a^*(d)}\,\Gamma(p^{*-1})\,\mathrm{e}^{\frac{1}{2}a(c)}$$

are closed wrt the multiplication and consitute the metaplectic group $Mp(\mathcal{Y})$**.**

They are well defined also if $\dim\mathcal{Z}=\infty$ **provided that** $p-1$ **is trace class, or equivalently,** $r-1$ **is trace class. Operators of this form are also closed wrt multiplication. Thus, as noticed by Lundberg, the metaplectic group can be defined also in the case of an infinite number of degrees of freedom.**

Bogoliubov $*$-automorphisms

Shale Theorem. **Let r be symplectic. There exists a unitary U, which we call a Bogoliubov implementer, such that**

$$UW(y)U^* = W(ry), \quad y \in \mathcal{Y},$$

iff $\mathrm{Tr} q^* q < \infty$.

The map $B(\Gamma_{\mathrm{s}}(\mathcal{Z})) \ni A \mapsto UAU^*$, where U is a Bogoliubov implementer, will be called a Bogoliubov automorphism. For a given r, a Bogoliubov implementer is determined up to a phase. There exists a distinguished choice, denoted U_{nat}, satisfying $(\Omega | U_{\mathrm{nat}} \Omega) > 0$, given by

$$U_{\mathrm{nat}} := |\det pp^*|^{-\frac{1}{4}} \mathrm{e}^{-\frac{1}{2} a^*(d)} \Gamma(p^{*-1}) \mathrm{e}^{\frac{1}{2} a(c)} \,.$$

1-parameter groups of Bogoliubov $*$-automorphisms

We say that a strongly continuous 1-parameter group of symplectic transformations

$$t \mapsto r(t) = \begin{bmatrix} p(t) & q(t) \\ \overline{q}(t) & \overline{p}(t) \end{bmatrix}$$

is implementable iff there exists a strongly continuous 1-parameter unitary group $t \mapsto U(t)$ such that

$$U(t)W(y)U^*(t) = W(r(t)y), \quad y \in \mathcal{Y}.$$

If $U(t)$ is a 1-parameter unitary group satisfying the above condition, then $H := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U(t)\Big|_{t=0}$ will be called a Bogoliubov Hamiltonian.

Theorem. $t \mapsto r(t)$ **is implementable iff** $\mathrm{Tr} q^*(t)q(t) < \infty$ **and** $\lim_{t\to 0} \mathrm{Tr} q^*(t)q(t) = 0$**.**

Type I Bogoliubov Hamiltonians

Let $t \mapsto r(t)$ be implementable. We say that it is of type I iff

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t)\Big|_{t=0} = \mathrm{i}h,$$

$p(t)\,\mathrm{e}^{-\mathrm{i}th} - 1$ **is trace class and**
$\|p(t)\,\mathrm{e}^{-\mathrm{i}th} - 1\|_1 \to 0$**.**

Theorem. **In the type I case**

$$U_{\mathrm{I}}(t) := \det(p(t)\,\mathrm{e}^{-\mathrm{i}th})^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}a^*(d(t))}\,\Gamma(p(t)^{*-1})\,\mathrm{e}^{\frac{1}{2}a(c(t))}$$

is a 1-parameter group.

A type I Bogoliubov Hamiltonian is defined as

$$H_{\mathrm{I}} := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U_{\mathrm{I}}(t)\Big|_{t=0}.$$

Type II Bogoliubov Hamiltonians

Let $t \mapsto r(t)$ be implementable. We say that it is of type II iff the implementing 1-parameter group has a generator, which is bounded from below. In this case we define the type II Hamiltonian to be

$$H_{\mathrm{II}} := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U_{\mathrm{II}}(t)\Big|_{t=0}.$$

such that $\inf \mathrm{sp} H_{\mathrm{II}} = 0$ and $U_{\mathrm{II}}(t)$ implements $r(t)$.

Type I and II Bogoliubov Hamiltonians in a finite number of degrees of freedom

Let $\mathcal{Z}$ be finite dimensional.
Then $r(t)$ is always type I and

$$H_{\mathrm{I}} = \mathrm{d}\Gamma(h) + \frac{1}{2}a^*(g) + \frac{1}{2}a(g).$$

$r(t)$ is type II iff its classical Hamiltonian is positive definite

$$\overline{z}hz + \frac{1}{2}\overline{z}g\overline{z} + \frac{1}{2}z\overline{g}z \geq 0,$$

and then

$$H_{\mathrm{II}} = H_{\mathrm{I}} - \frac{1}{4}\mathrm{Tr}\left[\begin{pmatrix} \overline{h}^2 - \overline{g}g & \overline{h}\overline{g} - \overline{g}h \\ hg - g\overline{h} & h^2 - g\overline{g} \end{pmatrix}^{1/2} - \begin{pmatrix} \overline{h} & 0 \\ 0 & h \end{pmatrix}\right].$$

Essential self-adjointness of type I Bogoliubov Hamiltonians

Theorem. **Let g be Hilbert-Schmidt. Then**

$$H_{\rm I} = {\rm d}\Gamma(h) + \frac{1}{2}a^*(g) + \frac{1}{2}a(g)$$

is essentially self-adjoint on the algebraic Fock space over ${\rm Dom}(h)$ and ${\rm e}^{{\rm i}tH_{\rm I}}$ implements $r(t)$.

Bogoliubov Hamiltonians defined by the relative boundedness technique

Theorem. **Let h be positive,**

$$\|h^{-1/2}\otimes h^{-1/2}\ g\|_{\Gamma_{\mathrm{s}}^2(\mathcal{Z})} < 1,$$
$$\|h^{-1/2}g\|_{B(\overline{\mathcal{Z}},\mathcal{Z})} \leq \infty.$$

Then $\frac{1}{2}a^*(g)+\frac{1}{2}a(g)$ **is relatively** $\mathrm{d}\Gamma(h)$**-bounded with the bound less than** 1**. Therefore, in this case both the type I and type II Bogoliubov Hamiltonians are well defined.**

Case of commuting h and g

Suppose that $g\overline{h} = hg$**. Without loss of generality we can assume that they are diagonal in a common orthonormal basis** $e_1, e_2, \ldots$**:**

$$he_n = h_n e_n,\ \ h_n \in \mathbb{R}; \quad g\overline{e}_n = g_n e_n, \quad g_n \in \mathbb{C}.$$

Theorem.

(1) $r(t)$ **is well defined iff for some** b**,** $a < 1$**,** $|g_n| \le ah_n + b$**.**

(2) $r(t)$ **is implementable iff** $\sum_n \frac{|g_n|^2}{1+h_n^2} < \infty.$

(3) $r(t)$ **is type I iff** $\sum_n \frac{|g_n|^2}{1+h_n} < \infty.$

(4) $r(t)$ **is type II iff** $\sum_n \frac{|g_n|^2}{h_n+h_n^2} < \infty.$

Conclusion

Infrared problem. **There exist implementable 1-parameter symplectic groups, which are not type II, even though their classical Hamiltonian is positive definite. Thus there exist Bogoliubov Hamiltonians unbounded from below with a positive classical symbol.**

Ultraviolet problem. **There exist implementable 1-parameter symplectic groups, which are not type I. This means that in order to express them in terms of creation and annihilation operators one needs to add an infinite constant – perform an appropriate renormalization.**

Open question. **Give sufficient and necessary conditions for the symplectic group $r(t)$ to be of type II.**