EXTENDED WEAK COUPLING LIMIT

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Based on joint work with Wojciech De Roeck

2. Various levels of description used in physics

- **More exact fundamental description;**
- **More approximate effective description.**

One of the aims of theoretical and mathematical physics is to justify effective models as limiting cases of more fundamental theories.

3. Small quantum system weakly interacting with a large reservoir.

We are interested in a class of dynamics generated by a Hamiltonian (self-adjoint operator) H_λ of the form

Hamiltonian of the small system

+ Hamiltonian of the large reservoir

+ λ times interaction.

There are a number of varieties of such Hamiltonians used in quantum physics and they go under various names. We use the name

Pauli-Fierz Hamiltonians.

4. Reduced weak coupling limit
(Pauli, van Hove,...,Davies)

- **Reduce the dynamics to the small system.**
- **Consider weak coupling $\lambda \to 0$.**
- **Rescale time as $\frac{t}{\lambda^2}$.**
- **Subtract the dynamics of the small system.**

In the limit one obtains a dynamics given by a completely positive Markov semigroup. It is an irreversible non-Hamiltonian dynamics.

5. Extended weak coupling limit
(Accardi-Frigerio-Lu, D.-De-Roeck)

Known also as stochastic limit.

- **Consider weak coupling $\lambda \to 0$.**
- **Rescale time as $\frac{t}{\lambda^2}$.**
- **Rescale the reservoir energy by the factor of λ^2 around the Bohr frequencies.**
- **Subtract the dynamics of the small system.**

In the limit one obtains a (reversible) quantum Langevin dynamics, which gives a dilation of the completely positive semigroup obtained in the reduced weak coupling limit.

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7. DILATIONS OF CONTRACTIVE SEMIGROUPS

8. Dilations of contractive semigroups

Let $\mathcal{K}$ be a Hilbert space and $\mathrm{e}^{-\mathrm{i}t\Upsilon}$ a contractive semigroup. This implies that $\mathrm{i}\Upsilon$ is dissipative:

$$-\mathrm{i}\Upsilon + \mathrm{i}\Upsilon^* \leq 0.$$

Let $\mathcal{Z}$ be a Hilbert space containing $\mathcal{K}$, $I_{\mathcal{K}}$ the embedding of $\mathcal{K}$ in $\mathcal{Z}$ and $\mathrm{e}^{-\mathrm{i}tZ}$ a unitary group on $\mathcal{Z}$. We say that $(\mathcal{Z}, I_{\mathcal{K}}, \mathrm{e}^{-\mathrm{i}tZ})$ is a dilation of $\mathrm{e}^{-\mathrm{i}t\Upsilon}$ iff

$$I_{\mathcal{K}}^* \mathrm{e}^{-\mathrm{i}tZ} I_{\mathcal{K}} = \mathrm{e}^{-\mathrm{i}t\Upsilon}, \quad t \geq 0.$$

This clearly implies

$$I_{\mathcal{K}}^* \mathrm{e}^{-\mathrm{i}tZ} I_{\mathcal{K}} = \mathrm{e}^{-\mathrm{i}t\Upsilon^*}, \quad t \leq 0.$$

We say that the dilation is minimal if $\{\mathrm{e}^{-\mathrm{i}tZ}\mathcal{K} \ : \ t \in \mathbb{R}\}$ is total in $\mathcal{Z}$.

9. Standard construction of a dilation I

We define the vector space $\tilde{\mathcal{F}}$ of functions f from $\mathbb{R}$ to $\mathcal{K}$, such that

$$\{s \in \mathbb{R} | f(s) \neq 0\} \text{ is a finite set.}$$

We equip $\tilde{\mathcal{F}}$ with a bilinear form

$$(f|f') := \sum_{t \geq s} (f(s)|\, \mathrm{e}^{-\mathrm{i}\Upsilon|t-s|} f'(t))_{\mathcal{K}} + \sum_{t<s} (f(s)|\, \mathrm{e}^{\mathrm{i}\Upsilon^*|t-s|} f'(t))_{\mathcal{K}}$$

One checks that the form $(\cdot|\cdot)$ is positive definite.
Let $\mathcal{N}$ denote the subspace of f, for which $(f|f) = 0$.
Let $\mathcal{F}$ denote the completion of the pre-Hilbert space $\tilde{\mathcal{F}}/\mathcal{N}$.

10. Standard construction of a dilation II

For $u \in \mathcal{K}$ define $\tilde{P}u(s) := \delta_{s,0}u$, where $\delta_{s,0}$ is Kronecker's delta. Then $Pu := [Pu] \in \mathcal{F}$ defines an isometric embedding of $P : \mathcal{K} \to \mathcal{F}$.
Define now

$$\tilde{W}_t f(s) = f(s-t).$$

$\tilde{W}_t$ is a one-parameter group on $\tilde{\mathcal{F}}$ that preserves the form $(\cdot|\cdot)$. Therefore, it defines a one-parameter unitary group W_t on $\mathcal{F}$. W_t dilates the semigroup $\mathrm{e}^{-\mathrm{i}t\Upsilon}$:

$$PW_tP = \mathrm{e}^{-\mathrm{i}t\Upsilon}.$$

In fact, it is a minimal dilation of $\mathrm{e}^{-\mathrm{i}t\Upsilon}$.

11. Construction of a dilation

Let $\mathfrak{h}$ be an auxiliary space and $\nu : \mathcal{K} \to \mathfrak{h}$ satisfy

$$\frac{1}{2\mathrm{i}}(\Upsilon - \Upsilon^*) = -\pi\nu^*\nu.$$

Let $(1|$ be a linear functional with domain $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$:

$$(1|f = \int f(x)\mathrm{d}x.$$

Let Z_{R} be the operator of multiplication on $L^2(\mathbb{R})$ by the variable x. Define $\mathcal{Z} := \mathcal{K} \ \oplus \ \mathfrak{h} \otimes L^2(\mathbb{R})$. Introduce the singular Friedrichs operator given by the following formal expression:

$$Z := \begin{bmatrix} \frac{1}{2}(\Upsilon + \Upsilon^*) & (2\pi)^{-\frac{1}{2}}\nu^* \otimes (1| \\ (2\pi)^{-\frac{1}{2}}\nu \otimes |1) & Z_{\mathrm{R}} \end{bmatrix}$$

Then $(\mathcal{Z}, I_{\mathcal{K}}, \mathrm{e}^{-\mathrm{i}tZ})$ is a dilation of $\mathrm{e}^{-\mathrm{i}t\Upsilon}$.

12. Construction of a dilation – the unitary group

$$U_t = I_{\mathrm{R}}^* \, \mathrm{e}^{-\mathrm{i}tZ_{\mathrm{R}}} \, I_{\mathrm{R}} \;+\; I_{\mathcal{K}}^* \, \mathrm{e}^{-\mathrm{i}t\Upsilon} \, I_{\mathcal{K}}$$
$$-\mathrm{i}(2\pi)^{-\frac{1}{2}} I_{\mathcal{K}}^* \int_0^t \mathrm{d}u \; \mathrm{e}^{-\mathrm{i}(t-u)\Upsilon} \, \nu^* \otimes (1| \, \mathrm{e}^{-\mathrm{i}uZ_{\mathrm{R}}} \, I_{\mathrm{R}}$$
$$-(2\pi)^{-\frac{1}{2}} \mathrm{i} I_{\mathrm{R}}^* \int_0^t \mathrm{d}u \; \mathrm{e}^{-\mathrm{i}(t-u)Z_{\mathrm{R}}} \, \nu \otimes |1) \, \mathrm{e}^{-\mathrm{i}u\Upsilon} \, I_{\mathcal{K}}$$
$$-(2\pi)^{-1} I_{\mathrm{R}}^* \int\limits_{0\le u_1, u_2, \, u_1+u_2 \le t} \mathrm{d}u_1 \mathrm{d}u_2 \; \mathrm{e}^{-\mathrm{i}u_2 Z_{\mathrm{R}}} \, \nu \otimes |1) \, \mathrm{e}^{-\mathrm{i}(t-u_2-u_1)\Upsilon} \, \nu^* \otimes (1| \, \mathrm{e}^{-\mathrm{i}u_1 Z_{\mathrm{R}}} \, I_{\mathrm{R}}.$$

We check that U_t is a strongly continuous unitary group. Therefore, we can define Z as its unitary generator: $U_t = \mathrm{e}^{-\mathrm{i}tZ}$. (Here I_{R} is the embedding of $\mathfrak{h}\otimes L^2(\mathbb{R})$ in $\mathcal{Z}$). .

13. Construction of a dilation – resolvent of the generator

For $z \in \mathbb{C}_+$**, we define**

$$\begin{aligned} R(z) &:= I_{\mathrm{R}}^*(z - Z_{\mathrm{R}})^{-1} I_{\mathrm{R}} + I_{\mathcal{K}}^*(z - \Upsilon)^{-1} I_{\mathcal{K}} \\ &\quad +(2\pi)^{-\frac{1}{2}} I_{\mathcal{K}}^*(z - \Upsilon)^{-1} \nu^* \otimes (1|(z - Z_{\mathrm{R}})^{-1} I_{\mathrm{R}} \\ &\quad +(2\pi)^{-\frac{1}{2}} I_{\mathrm{R}}^*(z - Z_{\mathrm{R}})^{-1} \nu \otimes |\nu)(z - \Upsilon)^{-1} I_{\mathcal{K}} \\ &\quad +(2\pi)^{-1} I_{\mathrm{R}}^*(z - Z_{\mathrm{R}})^{-1} \nu \otimes |1)(z - \Upsilon)^{-1} \nu^* \otimes (\nu|(z - Z_{\mathrm{R}})^{-1} I_{\mathrm{R}}; \\ R(\overline{z}) &:= R(z)^*. \end{aligned}$$

We can check that $R(z_1) - R(z_2) = (z_2 - z_1) R(z_1) R(z_2)$**,** $\mathrm{Ker} R(z) = \{0\}$**. Therefore, we can define** Z **as the self-adjoint operator** Z **satisfying** $R(z) = (z - Z)^{-1}$**.**

14. Construction of a dilation – removing a cutoff

Z **is the norm resolvent limit for** $r \to \infty$ **of the following regularized operators:**

$$Z_r := \begin{bmatrix} \frac{1}{2}(\Upsilon + \Upsilon^*) & (2\pi)^{-\frac{1}{2}}\nu^* \otimes (1|1_{[-r,r]}(Z_{\mathrm{R}}) \\ (2\pi)^{-\frac{1}{2}}\nu \otimes 1_{[-r,r]}(Z_{\mathrm{R}})|1) & 1_{[-r,r]}(Z_{\mathrm{R}})Z_{\mathrm{R}} \end{bmatrix}$$

(Note that it is important to remove the cut-off in a symmetric way).

15. False quadratic form of the generator of dilations

On $\mathcal{D} := \mathcal{K} \oplus \mathfrak{h} \otimes (L^2(\mathbb{R}) \cap L^1(\mathbb{R}))$ we can define the (non-self-adjoint) quadratic form

$$Z^+ := \begin{bmatrix} \Upsilon & (2\pi)^{-\frac{1}{2}}\nu^* \otimes (1| \\ (2\pi)^{-\frac{1}{2}}\nu \otimes |1) & Z_{\mathrm{R}} \end{bmatrix}$$

One can say that it is a "false form" of Z. In fact, for $\psi, \psi' \in \mathcal{D}$, the function $\mathbb{R} \ni t \mapsto (\psi| \, \mathrm{e}^{-\mathrm{i}tZ} \, \psi')$ is differentiable away from $t = 0$, its derivative $t \mapsto \frac{\mathrm{d}}{\mathrm{d}t}(\psi| \, \mathrm{e}^{-\mathrm{i}tZ} \, \psi')$ is continuous away from 0 and at $t = 0$ it has the right limit equal to

$$-\mathrm{i}(\psi|Z^+\psi') = \lim_{t \downarrow 0} t^{-1} \left(\psi|(\mathrm{e}^{-\mathrm{i}tZ} - 1)\psi' \right).$$

16. Scaling invariance

For $\lambda \in \mathbb{R}$, introduce the following unitary operator on $\mathcal{Z}$

$$j_\lambda u = u, \quad u \in \mathcal{K}; \qquad j_\lambda g(y) := \lambda^{-1} g(\lambda^{-2} y), \quad g \in \mathcal{Z}_{\mathrm{R}}.$$

Note that

$$j_\lambda^* Z_{\mathrm{R}} j_\lambda = \lambda^2 Z_{\mathrm{R}}, \qquad j_\lambda^* |1) = \lambda |1).$$

Therefore, the operator Z is invariant with respect to the following scaling:

$$Z = \lambda^{-2} j_\lambda^* \begin{bmatrix} \lambda^2 \frac{1}{2}(\Upsilon + \Upsilon^*) & \lambda (2\pi)^{-\frac{1}{2}} \nu^* \otimes (1| \\ \lambda (2\pi)^{-\frac{1}{2}} \nu \otimes |1) & Z_{\mathrm{R}} \end{bmatrix} j_\lambda.$$

17. WEAK COUPLING LIMIT FOR FRIEDRICHS OPERATORS

18. Friedrichs operators

Let $\mathcal{H} := \mathcal{K} \oplus \mathcal{H}_{\mathrm{R}}$ be a Hilbert space, where $\mathcal{K}$ is finite dimensional. Let $I_{\mathcal{K}}$ be the embedding of $\mathcal{K}$ in $\mathcal{H}$. Let K be a self-adjoint operator on $\mathcal{K}$ and H_{R} be a self-adjoint operator on $\mathcal{H}_{\mathrm{R}}$. Let $V : \mathcal{K} \to \mathcal{H}_{\mathrm{R}}$. Define the Friedrichs Hamiltonian

$$H_\lambda := \begin{bmatrix} K & \lambda V^* \\ \lambda V & H_{\mathrm{R}} \end{bmatrix}.$$

19. Reduced weak coupling limit for Friedrichs operators

Assume that $\int \|V^* \mathrm{e}^{-\mathrm{i}tH_\mathrm{R}} V\|\mathrm{d}t < \infty$. **Define the Level Shift Operator**

$$\Upsilon := \sum_k \int_0^\infty 1_k(K) V^* \mathrm{e}^{-\mathrm{i}t(H_\mathrm{R}-k)} V 1_k(K) \mathrm{d}t.$$

Note that $\Upsilon K = K\Upsilon$.

Theorem.

$$\lim_{\lambda\to 0} \mathrm{e}^{\mathrm{i}tK/\lambda^2} I_{\mathcal{K}}^* \mathrm{e}^{-\mathrm{i}tH_\lambda/\lambda^2} I_{\mathcal{K}} = \mathrm{e}^{-\mathrm{i}t\Upsilon}.$$

20. Continuity of spectrum

Assumption. **We suppose that for any $k \in \mathrm{sp} K$ there exists an open $I_k \subset \mathbb{R}$ such that $k \in I_k$,**

$$\mathrm{Ran} 1_{I_k}(H_{\mathrm{R}}) \simeq \mathfrak{h}_k \otimes L^2(I_k, \mathrm{d}x),$$

$1_{I_k}(H_{\mathrm{R}}) H_{\mathrm{R}}$ **is the multiplication operator by the variable $x \in I_k$ and**

$$1_{I_k}(H_{\mathrm{R}}) V \simeq \int_{I_k}^{\oplus} v(x) \mathrm{d}x.$$

We assume that I_k are disjoint for distinct k and $x \mapsto v(x) \in B(\mathcal{K}, \mathfrak{h}_k)$ is continuous at k.

21. Asymptotic space

Define $\mathfrak{h} := \bigoplus\limits_{k} \mathfrak{h}_k$, $\mathcal{Z} := \mathcal{K} \oplus \mathfrak{h} \otimes L^2(\mathbb{R})$. **Let** $\nu : \mathcal{K} \to \mathfrak{h}$ **be defined as**

$$\nu := (2\pi)^{\frac{1}{2}} \bigoplus_{k} v(k) 1_k(K).$$

Note that it satisfies

$$\nu^* \nu = \frac{1}{\mathrm{i}}(\Upsilon - \Upsilon^*).$$

As before, we set Z_{R} **to be the multiplication by** x **on** $L^2(\mathbb{R})$ **and**

$$Z := \begin{bmatrix} \frac{1}{2}(\Upsilon + \Upsilon^*) & (2\pi)^{-\frac{1}{2}} \nu^* \otimes (1| \\ (2\pi)^{-\frac{1}{2}} \nu \otimes |1) & Z_{\mathrm{R}} \end{bmatrix},$$

so that $(\mathcal{Z}, I_{\mathcal{K}}, \mathrm{e}^{-\mathrm{i}tZ})$ **is a dilation of** $\mathrm{e}^{-\mathrm{i}t\Upsilon}$**.**

22. Scaling

For $\lambda > 0$, we define the family of partial isometries $J_{\lambda,k} : \mathfrak{h}_k \otimes L^2(\mathbb{R}) \to \mathfrak{h}_k \otimes L^2(I_k)$:

$$(J_{\lambda,k} g_k)(y) = \begin{cases} \frac{1}{\lambda} g_k\left(\frac{y-k}{\lambda^2}\right), & \textbf{if } y \in I_k; \\ 0, & \textbf{if } y \in \mathbb{R} \backslash I_k. \end{cases}$$

We set $J_\lambda : \mathcal{Z} \to \mathcal{H}$, defined for $g = (g_k) \in \mathcal{Z}_{\mathrm{R}}$ by

$$J_\lambda g := \sum_k J_{\lambda,k} g_k,$$

and on $\mathcal{K}$ equal to the identity. Note that J_λ are partial isometries and

$$\mathrm{s-}\lim_{\lambda \searrow 0} J_\lambda^* J_\lambda = 1.$$

23. Extended weak coupling limit for Friedrichs operators

On $\mathcal{Z} = \mathcal{K} \oplus \bigoplus_k \mathfrak{h}_k \otimes L^2(\mathbb{R})$**. we define the renormalizing Hamiltonian** $Z_{\mathrm{ren}} := K \oplus \bigoplus_k k$.

Theorem.

$$\mathrm{s}^* - \lim_{\lambda \searrow 0} \mathrm{e}^{\mathrm{i}\lambda^{-2} t Z_{\mathrm{ren}}} J_\lambda^* \, \mathrm{e}^{-\mathrm{i}\lambda^{-2} t H_\lambda} J_\lambda = \mathrm{e}^{-\mathrm{i}tZ} .$$

Here we used the strong* limit: $s^* - \lim_{\lambda \searrow 0} A_\lambda = A$ **means that for any vector** ψ

$$\lim_{\lambda \searrow 0} A_\lambda \psi = A\psi,$$

$$\lim_{\lambda \searrow 0} A_\lambda^* \psi = A^* \psi.$$

24. COMPLETELY POSITIVE MAPS

25. Positive maps

Let $\mathcal{K}_1, \mathcal{K}_2$ be Hilbert spaces. We say that a map

$$\Lambda : B(\mathcal{K}_1) \to B(\mathcal{K}_2)$$

is positive iff $A \geq 0$ implies $\Lambda(A) \geq 0$.
We say that Λ is Markov iff $\Lambda(1) = 1$.

26. n-positive maps

Let $\mathcal{K}_1, \mathcal{K}_2$ be Hilbert spaces. We say that a map Λ is n-positive iff

$$\Lambda \otimes \mathrm{id} : B(\mathcal{K}_1 \otimes \mathbb{C}^n) \to B(\mathcal{K}_2 \otimes \mathbb{C}^n)$$

is positive. We say that it is completely positive, or c.p. for short iff it is n-positive for any n.

There are many positive but not completely positive maps. For instance, the transposition is positive but not 2-positive.

27. The Stinespring dilation of a c.p. map

Theorem.

1. **Let $\mathfrak{h}$ be a Hilbert space and $\nu \in B(\mathcal{K}_1, \mathcal{K}_2 \otimes \mathfrak{h})$. Then**

$$\Lambda(A) := \nu^* \ A \otimes 1 \ \nu \qquad\qquad (*)$$

is c.p.

2. **Conversely, if Λ is c.p., then there exist a Hilbert space $\mathfrak{h}$ and $\nu \in B(\mathcal{K}_1, \mathcal{K}_2 \otimes \mathfrak{h})$ such that (*) is true and $B(\mathcal{K}_2) \otimes 1 \ \nu \ \mathcal{K}_1$ is dense in $\mathcal{K}_2 \otimes \mathfrak{h}$.**

3. **If $\mathfrak{h}'$ and ν' also satisfy the above properties, then there exists a $U \in U(\mathfrak{h}, \mathfrak{h}')$ such that $\nu' = 1_{\mathcal{K}_2} \otimes U \ \nu$.**

28. Construction of the Stinespring dilation I

We equip the algebraic tensor product $B(\mathcal{K}_1)\otimes\mathcal{K}_2$ with the scalar product:

$$\tilde{v} = \sum_i X_i \otimes v_i, \qquad \tilde{w} = \sum_i Y_i \otimes w_i,$$

$$(\tilde{v}|\tilde{w}) \; = \; \sum_{i,j} (v_i|\Lambda(X_i^* Y_j) w_j).$$

By the complete positivity, it is positive.

29. Construction of the Stinespring dilation II

Define

$$\pi_0(A)\tilde{v} := \sum_i AX_i \otimes v_i.$$

We check that

$$\begin{aligned} (\pi_0(A)\tilde{v}|\pi_0(A)\tilde{v}) &\leq \|A\|^2(\tilde{v}|\tilde{v}), \\ \pi_0(AB) &= \pi_0(A)\pi_0(B), \\ \pi_0(A^*) &= \pi_0(A)^*. \end{aligned}$$

30. Construction of the Stinespring dilation III

Let $\mathcal{N}$ be the set of $\tilde{v}$ with $(\tilde{v}|\tilde{v}) = 0$. Then the completion of $\mathcal{H} := B(\mathcal{K}_1) \otimes \mathcal{K}_2/\mathcal{N}$ is a Hilbert space. There exists a nondegenerate $*$-representation π of $B(\mathcal{K}_1)$ in $\mathcal{H}$ such that

$$\pi(A)(\tilde{v} + \mathcal{N}) = \pi_0(A)\tilde{v}.$$

For every such a representation we can identify $\mathcal{H}$ with $\mathcal{K}_1 \otimes \mathfrak{h}$ for some Hilbert space $\mathfrak{h}$ and $\pi(A) = A \otimes 1$.
We set

$$\nu v := 1 \otimes v + \mathcal{N}.$$

We check that

$$\Lambda(A) = \nu^* A \otimes 1 \ \nu.$$

31. Uniqueness of the Stinespring dilation

If $\mathfrak{h}'$, ν' is another pair. We check that

$$\left\| \sum_i X_i \otimes 1_{\mathfrak{h}} \; \nu \; v_i \right\| = \left\| \sum_i X_i \otimes 1_{\mathfrak{h}'} \; \nu' \; v_i \right\| .$$

Therefore, there exists a unitary $U_0 : \mathcal{K}_2 \otimes \mathfrak{h} \to \mathcal{K}_2 \otimes \mathfrak{h}'$ such that $U_0 \nu = \nu'$ and $U_0 \; A \otimes 1_{\mathfrak{h}} = A \otimes 1_{\mathfrak{h}'} \, U_0$. Therefore, there exists a unitary $U : \mathfrak{h} \to \mathfrak{h}'$ such that $U_0 = 1 \otimes U$.

32. Kadison-Schwarz inequality

Theorem. **If Λ is c.p. and $\Lambda(1)$ is invertible, then**

$$\Lambda(A)^*\Lambda(1)^{-1}\Lambda(A) \leq \Lambda(A^*A).$$

Proof.

$$\begin{aligned}\Lambda(A)^*\Lambda(1)^{-1}\Lambda(A) &= \nu^* A^* \otimes 1\nu(\nu^*\nu)^{-1}\nu^* A\otimes 1\nu \\ &\leq \nu^* \; A^* A\otimes 1 \; \nu.\end{aligned}$$

33. COMPLETELY POSITIVE SEMIGROUPS

34. C.p. semigroups

Let $\mathcal{K}$ be a finite dimensional Hilbert space. We will consider a c.p. semigroup on $B(\mathcal{K})$. We will always assume the semigroup to be continuous, so that it can be written as e^{tM} for a bounded operator M on $B(\mathcal{K})$ It is called Markov if it preserves the identity.

If M_1, M_2 are generators of (Markov) c.p. semigroups and $c_1, c_2 \geq 0$, then $c_1M_1 + c_2M_2$ is a generator of a (Markov) c.p. semigroup. This follows by the Trotter formula.

35. Examples of semigroups

Example 1. **Let $\Upsilon = \Theta + \mathrm{i}\Delta$ be an operator on $\mathcal{K}$. Then**

$$M(A) := \mathrm{i}\Upsilon A - \mathrm{i}A\Upsilon^* = \mathrm{i}[\Theta, A] - [\Delta, A]_+$$

is a generator of a c.p. semigroup and

$$\mathrm{e}^{tM}(A) = \mathrm{e}^{\mathrm{i}t\Upsilon} A \,\mathrm{e}^{-\mathrm{i}t\Upsilon^*}\,.$$

Example 2. **Let Λ be a c.p. map on $\mathcal{K}$. Then it is the generator of a c.p. semigroup and**

$$\mathrm{e}^{t\Lambda}(A) = \sum_{j=0}^{\infty} \frac{t^j}{j!}\Lambda^j(A).$$

36. Generators of c.p. semigroups I

Theorem. **Let** e^{tM} **be a c.p. semigroup on a finite dimensional space** $\mathcal{K}$**. Then there exists self-adjoint operators** Θ, Δ **on** $\mathcal{K}$**, an auxiliary Hilbert space** $\mathfrak{h}$ **and an operator** $\nu \in B(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$ **such that** M **can be written in the so-called Lindblad form**

$$M(S) = \mathrm{i}[\Theta, A] - [\Delta, A]_+ + \nu^* \; A\otimes 1 \; \nu, \qquad\qquad A \in B(\mathcal{K}).$$

We can choose Θ **and** ν **so that**

$$\mathrm{Tr}\Theta = 0, \quad \mathrm{Tr}\nu = 0.$$

e^{tM} **is Markov iff** $2\Delta = \nu^*\nu$**.**

37. Generators of c.p. semigroups II

Remark. **If we identify** $\mathfrak{h} = \mathbb{C}^n$**, then we can write**

$$\nu^* A \otimes 1 \nu = \sum_{j=1}^{n} \nu_j^* A \nu_j.$$

Then $\mathrm{Tr}\nu = 0$ **means** $\mathrm{Tr}\nu_j = 0$**,** $j = 1, \ldots, n$**.**

38. Construction of the Lindblad form I

The unitary group on $\mathcal{K}$, denoted $U(\mathcal{K})$, is compact. Therefore, there exists the Haar measure on $U(\mathcal{K})$, which we denote $\mathrm{d}U$. Note that

$$\int UXU^*\mathrm{d}U = \mathrm{Tr}X.$$

Define

$$\mathrm{i}\Theta - \Delta_0 := \int M(U^*)U\mathrm{d}U,$$

where Θ and Δ_0 are self-adjoint.

Lemma. $\int M(XU^*)U\mathrm{d}U = (\mathrm{i}\Theta - \Delta_0)X.$

Proof. **First check this identity for unitary X, which follows by the invariance of the measure. But every operator is a linear combination of unitaries.**

39. Construction of the Lindblad form II

Differentiating the inequality

$$\mathrm{e}^{tM}(X)^* \,\mathrm{e}^{tM}(1)^{-1}\,\mathrm{e}^{tM}(X) \leq \mathrm{e}^{tM}(X^*X)$$

we obtain

$$M(X^*X) + X^*M(1)X - M(X^*)X - X^*M(X) \geq 0.$$

Replacing X **with** UXU**, where** U **is unitary, we obtain**

$$M(X^*X)+X^*U^*M(1)UX-M(X^*U^*)UX-X^*U^*M(UX) \geq 0.$$

Integrating over $U(\mathcal{K})$ **we obtain**

$$M(X^*X)+X^*X\mathrm{Tr}M(1)-(\mathrm{i}\Theta-\Delta_0)X^*X-X^*X(-\mathrm{i}\Theta-\Delta_0)^* \geq 0.$$

40. Construction of the Lindblad form III

Define

$$\Delta_1 := \Delta_0 + \frac{1}{2}\mathrm{Tr} M(1),$$
$$\Lambda(A) := M(A) - (\mathrm{i}\Theta - \Delta_1)A - A(-\mathrm{i}\Theta - \Delta_1)$$

Arguing as above we see that Λ is completely positive. Hence it can be written as

$$\Lambda(A) = \nu_1^* \; A \otimes 1 \; \nu_1.$$

41. The Hamiltonian part of the Lindblad form

The operator Θ has trace zero, because

$$\begin{aligned}
\mathrm{i}\mathrm{Tr}\Theta + \mathrm{Tr}\Delta_0 &= \int U_1 M(U^*) U U_1^* \mathrm{d}U \mathrm{d}U_1 \\
&= \int U_2 U M(U^*) U_2^* \mathrm{d}U \mathrm{d}U_2 \\
&= -\mathrm{i}\mathrm{Tr}\Theta + \mathrm{Tr}\Delta_0.
\end{aligned}$$

We will say that the generator of a c.p. semigroup is purely dissipative **if $\Theta = 0$.**

42. Non-uniqueness of the Lindblad form

Let w **be an arbitrary vector in** $\mathfrak{h}$ **and**

$$\begin{aligned} \Delta &:= \Delta_1 + \nu^* 1 \otimes |w) + \frac{1}{2}(w|w), \\ \nu &:= \nu_1 + 1 \otimes |w). \end{aligned}$$

Then the same generator of a c.p. semigroup can be written in two Lindblad forms:

$$\begin{aligned} &(\mathrm{i}\Theta - \Delta_1)A + A(-\mathrm{i}\Theta - \Delta_1) + \nu_1^* A \nu_1 \\ = &\ (\mathrm{i}\Theta - \Delta)A + A(-\mathrm{i}\Theta - \Delta) + \nu^* A \nu. \end{aligned}$$

In particular, choosing $w := -\mathrm{Tr}\nu_1$**, we can make sure that** $\mathrm{Tr}\nu = 0$**.**

43. Scalar product given by a density matrix

Let ρ be a nondegenerate density matrix. On $B(\mathcal{K})$ we introduce the scalar product

$$(A|B)_\rho := \mathrm{Tr}\rho^{1/2}A^*\rho^{1/2}B$$

If M is a map on $B(\mathcal{K})$, then $M^{*\rho}$ will denote the adjoint for this scalar product. Clearly,

$$M^{*\rho}(A) = \rho^{-1/2}M^*(\rho^{1/2}A\rho^{1/2})\rho^{-1/2}.$$

44. Detailed Balance Condition I

Let M be a generator of a c.p. semigroup. Recall that it can be uniquely reresented as

$$M = \mathrm{i}[\Theta, \cdot] + M_{\mathrm{d}},$$

where M_{d} is its purely dissipative part and $\mathrm{i}[\Theta, \cdot]$ its Hamiltonian part. We say that M satisfies the Detailed Balance Condition for ρ iff M_{d} is self-adjoint and $\mathrm{i}[\Theta, \cdot]$ is anti-self-adjoint for $(\cdot|\cdot)_\rho$.

Proposition. If M, the generator of a Markov c.p. semigroup, satisfies the Detailed Balance Condition for ρ, then

$$[\Theta, \rho] = 0, \qquad M_{\mathrm{d}}(\rho) = 0.$$

45. Detailed Balance Condition II

Theorem. **Suppose that δ is a positive operator and ϵ is an antiunitary operator on a Hilbert space $\mathfrak{h}$ such that $\epsilon^2 = 1, \quad \epsilon\delta\epsilon = \delta^{-1/2}$. Let $\nu \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$. Assume that**

$$\rho^{-1/2} \otimes 1 \ \nu \rho^{1/2} = 1 \otimes \delta \ \nu,$$

$$(\phi \otimes w | \nu\psi) = (\nu\phi | \psi \otimes \delta\epsilon w), \quad \phi, \psi \in \mathcal{K}, \quad w \in \mathfrak{h}.$$

Then $M(A) := -\frac{1}{2}[\nu^*\nu, A]_+ + \nu^* A \otimes 1\nu$
is a purely dissipative generator of a c.p. Markov semigroup satisfying the Detailed Balance Condition for ρ and $\nu^*\nu\rho^{1/2} = \rho^{1/2}\nu^*\nu$.

46. PAULI FIERZ OPERATORS

47. Bosonic Fock spaces. Creation/annihilation operators

1-particle Hilbert space: $\mathcal{H}_{\mathrm{R}}$**.**

Fock space: $\Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}}) := \bigoplus\limits_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n}\mathcal{H}_{\mathrm{R}}$**.**

Vacuum vector: $\Omega = 1 \in \otimes_{\mathrm{s}}^{0}\mathcal{H}_{\mathrm{R}} = \mathbb{C}$**.**

If $z \in \mathcal{H}_{\mathrm{R}}$**, then**

$$a(z)\Psi := \sqrt{n}(z|\otimes 1^{(n-1)\otimes}\Psi \in \otimes_{\mathrm{s}}^{n-1}\mathcal{H}_{\mathrm{R}}, \quad \Psi \in \otimes_{\mathrm{s}}^{n}\mathcal{H}_{\mathrm{R}}$$

is the annihilation operator of z **and** $a^*(z) := a(z)^*$ **the corresponding creation operator. They are closable operators on** $\Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$**.**

48. Second quantization

For an operator q **on** $\mathcal{H}_{\mathrm{R}}$ **we define the operator** $\Gamma(q)$ **on** $\Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$ **by**

$$\Gamma(q)\Big|_{\otimes_{\mathrm{s}}^n \mathcal{H}_{\mathrm{R}}} = q \otimes \cdots \otimes q.$$

For an operator h **on** $\mathcal{H}_{\mathrm{R}}$ **we define the operator** $\mathrm{d}\Gamma(h)$ **on** $\Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$ **by**

$$\mathrm{d}\Gamma(h)\Big|_{\otimes_{\mathrm{s}}^n \mathcal{H}_{\mathrm{R}}} = h \otimes 1^{(n-1)\otimes} + \cdots 1^{(n-1)\otimes} \otimes h.$$

Note the identity $\Gamma(\mathrm{e}^{\mathrm{i}th}) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}$**.**

49 Pauli-Fierz operators

Let $\mathcal{K}$, $\mathcal{Z}_{\mathrm{R}}$ be Hilbert spaces. Consider a Hilbert space $\mathcal{H} := \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$, where $\mathcal{H}_{\mathrm{R}}$ is the 1-particle space of the reservoir and $\Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$ is the corresponding bosonic Fock space. The composite system is described by the self-adjoint operator

$$\begin{aligned} H_\lambda = \ & K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(H_{\mathrm{R}}) \\ & + \lambda(a^*(V) + a(V)) \end{aligned}$$

Here K describes the Hamiltonian of the small system, $\mathrm{d}\Gamma(H_{\mathrm{R}})$ describes the dynamics of the reservoir expressed by the second quantization of H_{R}, and $a^*(V)/a(V)$ are the creation/annihilation operators of an operator $V \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{H}_{\mathrm{R}})$.

49. Creation/annihilation operators in coupled spaces

If $\mathcal{K}$ **is a Hilbert space and** $V \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{H}_{\mathrm{R}})$**, then for** $\Psi \in \mathcal{K} \otimes \otimes_{\mathrm{s}}^{n} \mathcal{H}_{\mathrm{R}}$ **we set**

$$a(V)\Psi := \sqrt{n} V^* \otimes 1^{(n-1)\otimes} \Psi \in \mathcal{K} \otimes \otimes_{\mathrm{s}}^{n-1} \mathcal{H}_{\mathrm{R}}.$$

$a(V)$ **is called the annihilation operator of** V **and** $a^*(V) := a(V)^*$ **the corresponding creation operator. They are closable operators on** $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$**.**

51. Alternative notation

Identify $\mathcal{H}_{\mathrm{R}}$ with $L^2(\Xi, \mathrm{d}\xi)$, for some measure space $(\Xi, \mathrm{d}\xi)$, so that one can introduce a_ξ^*/a_ξ – the usual creation/annihilation operators. Let h be the multiplication operator by $x(\xi)$. Then V can be identified with a function $\Xi \ni \xi \mapsto v(\xi) \in B(\mathcal{K})$ and we have an alternative notation:

$$\begin{aligned}
\mathrm{d}\Gamma(H_{\mathrm{R}}) &= \int x(\xi) a_\xi^* a_\xi \mathrm{d}\xi, \\
a^*(V) &= \int v(\xi) a_\xi^* \mathrm{d}\xi, \\
a(V) &= \int v^*(k) a_\xi \mathrm{d}\xi, \\
H &= K + \int x(\xi) a_\xi^* a_\xi \mathrm{d}\xi + \lambda \int \left(v(\xi) a_\xi^* + v^*(\xi) a_\xi \right) \mathrm{d}\xi.
\end{aligned}$$

52. LANGEVIN DYNAMICS OF MARKOV SEMIGROUPS

53. C.p. Markov semigroups

Let $\mathcal{K}$ be a finite dimensional Hilbert space. Suppose that we are given M, the generator of a c.p. Markov semigroup on $B(\mathcal{K})$. Recall that there exists an operator Υ, an auxiliary Hilbert space $\mathfrak{h}$ and an operator ν from $\mathcal{K}$ to $\mathcal{K} \otimes \mathfrak{h}$ such that

$$-\mathrm{i}\Upsilon + \mathrm{i}\Upsilon^* = -\nu^*\nu$$

and M can be written in the Lindblad form

$$M(A) = -\mathrm{i}(\Upsilon A - A\Upsilon^*) + \nu^* \; A\otimes 1 \; \nu, \qquad A \in B(\mathcal{K}).$$

54. Quantum Langevin dynamics I

Let $(1|$ **denote the (unbounded) linear form on** $L^2(\mathbb{R})$**:**

$$(1|f := \int f(x)\mathrm{d}x.$$

$|1)$ **will denote the adjoint form. We define the 1-particle space** $\mathcal{Z}_{\mathrm{R}} := \mathfrak{h} \otimes L^2(\mathbb{R})$**. The full Hilbert space is** $\mathcal{Z} := \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{R}})$**.** Z_{R} **is the operator of multiplication by the variable** x **on** $L^2(\mathbb{R})$**.**

55. Quantum Langevin dynamics II

We choose a basis (b_j) **in** $\mathfrak{h}$ **and write**

$$\nu = \sum \nu_j \otimes |b_j).$$

Set

$$\begin{aligned} \nu_j^+ &= \nu_j, \\ \nu_j^- &= \nu_j^*. \end{aligned}$$

We will denote by $I_{\mathcal{K}}$ **the embedding of** $\mathcal{K} \simeq \mathcal{K} \otimes \Omega$ **in** $\mathcal{Z}$**.**

56. Quantum Langevin dynamics III

For $t \geq 0$ we define the quadratic form

$$
\begin{aligned}
U_t \;:=\; & \mathrm{e}^{-\mathrm{i}\mathrm{d}\Gamma(Z_\mathrm{R})} \sum_{n=0}^{\infty} \int\limits_{t\geq t_n\geq\cdots\geq t_1\geq 0} \mathrm{d}t_n \cdots \mathrm{d}t_1 \\
& \times (2\pi)^{-\frac{n}{2}} \sum_{j_1,\dots,j_n} \sum_{\epsilon_1,\dots,\epsilon_n\in\{+,-\}} \\
& \times (-\mathrm{i})^n \, \mathrm{e}^{-\mathrm{i}(t-t_n)\Upsilon} \, \nu_{j_n}^{\epsilon_n} \, \mathrm{e}^{-\mathrm{i}(t_n-t_{n-1})\Upsilon} \cdots \nu_{j_1}^{\epsilon_1} \, \mathrm{e}^{-\mathrm{i}(t_1-0)\Upsilon} \\
& \times \prod_{k=1,\dots,n:\ \ \epsilon_k=+} a^*(\mathrm{e}^{\mathrm{i}t_k Z_R}|1) \otimes b_{j_k}) \\
& \times \prod_{k'=1,\dots,n:\ \ \epsilon_{k'}=-} a(\mathrm{e}^{\mathrm{i}t_{k'} Z_R}|1) \otimes b_{j_{k'}}); \\
U_{-t} \;:=\; & U_t^*.
\end{aligned}
$$

57. Quantum Langevin dynamics IV

Theorem. U_t **is a strongly continuous unitary group on** $\mathcal{Z}$**, and hence can be written as** $U_t = \mathrm{e}^{-\mathrm{i}tZ}$ **for some self-adjoint operator** Z**. We have**

$$\begin{aligned} 1_{\mathcal{K}}^* \,\mathrm{e}^{-\mathrm{i}tZ}\, 1_{\mathcal{K}} &= \mathrm{e}^{-\mathrm{i}t\Upsilon}, \\ 1_{\mathcal{K}}^* \,\mathrm{e}^{\mathrm{i}tZ} \; A \otimes 1 \; \mathrm{e}^{-\mathrm{i}tZ}\, 1_{\mathcal{K}} &= \mathrm{e}^{tM}(A). \end{aligned}$$

Formally (and also rigorously with an appropriate regularization)

$$\begin{aligned} Z \;=\; & \frac{1}{2}(\Upsilon + \Upsilon^*) + \mathrm{d}\Gamma(Z_{\mathrm{R}}) \\ & +(2\pi)^{-\frac{1}{2}} a^* \left(\nu \otimes |1)\right) + (2\pi)^{-\frac{1}{2}} a \left(\nu \otimes |1)\right) \end{aligned}$$

58. Quantum Langevin equation I
(Hudson - Parthasaraty)

The cocycle $W_t := \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(Z_{\mathrm{R}})}\,\mathrm{e}^{-\mathrm{i}tZ}$ **solves**

$$\begin{aligned}
&\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}W_t\\
=\ &\Big(\frac{1}{2}(\Upsilon+\Upsilon^*)\\
&+(2\pi)^{-\frac{1}{2}}a^*\left(\nu\otimes|\,\mathrm{e}^{-\mathrm{i}tZ_{\mathrm{R}}}\,1)\right)+(2\pi)^{-\frac{1}{2}}a\left(\nu\otimes|\,\mathrm{e}^{-\mathrm{i}tZ_{\mathrm{R}}}\,1)\right)\Big)W_t,
\end{aligned}$$

59. Quantum Langevin equation II

Apply the Fourier transformation on $L^2(\mathbb{R})$, so that $(2\pi)^{-\frac{1}{2}}|1)$ will correspond to $|\delta_0)$. Writing $\hat{W}_t$ for W_t after this transformation, we obtain the quantum Langevin equation in a more familiar form:

$$\begin{aligned} & \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{W}_t \\ = \ & \left(\frac{1}{2}(\Upsilon + \Upsilon^*) + a^*\left(\nu \otimes |\delta_t)\right) + a\left(\nu \otimes |\delta_t)\right)\right)\hat{W}_t. \end{aligned}$$

60. Stochastic Schrödinger equation

Let $\mathcal{D}_0 := \mathfrak{h} \otimes (C(\mathbb{R}) \cap L^2(\mathbb{R}))$**. Let** $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D}_0)$**, denote the corresponding algebraic Fock space and** $\mathcal{D} := \mathcal{K} \otimes \overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D}_0)$**. For** $\psi, \psi' \in \mathcal{D}$**, and** $t > 0$**, the cocycle** $\hat{W}(t)$ **solves**

$$
\begin{aligned}
& \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(\psi|\hat{W}(t)\psi') \\
= & \left(\psi|(\Upsilon + a^*(\nu\otimes|\delta_t))\hat{W}_t\psi'\right) \\
& + \sum_i \left(\psi|\nu_i \hat{W}_t a(|b_i)\otimes|\delta_t))\psi'\right).
\end{aligned}
$$

61. The “age” of observables

For any Borel set $I \subset \mathbb{R}$, the space $L^2(I)$ can be treated as a subspace of $L^2(\mathbb{R})$. Therefore, we have the decomposition

$$\Gamma_{\mathrm{s}}(\mathfrak{h}\otimes L^2(I)) \otimes \Gamma_{\mathrm{s}}(\mathfrak{h}\otimes L^2(\mathbb{R}\setminus I)).$$

Therefore,

$$\begin{aligned} \mathfrak{M}_{\mathrm{R}}(I) &:= 1_{\mathcal{K}} \otimes B\left(\Gamma_{\mathrm{s}}(\mathfrak{h}\otimes L^2(I))\right), \\ \mathfrak{M}(I) &:= B\left(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathfrak{h}\otimes L^2(I))\right), \end{aligned}$$

are well defined as von Neumann subalgebras of $B(\mathcal{Z})$.

62. Quantum Langevin dynamics and the observables

A quantum Langevin dynamics makes the bosons “older”. At the time $t = 0$ they may become entangled with the small system.

Theorem. **If $t > 0$ and $I \subset \mathbb{R}\backslash]-t, 0[$, then**

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}tZ}\, \mathfrak{M}_{\mathrm{R}}(I)\, \mathrm{e}^{-\mathrm{i}tZ} &= \mathfrak{M}_{\mathrm{R}}(I+t), \\
\mathrm{e}^{\mathrm{i}tZ}\, \mathfrak{M}([-t,0])\, \mathrm{e}^{-\mathrm{i}tZ} &= \mathfrak{M}([0,t]).
\end{aligned}
$$

63. WEAK COUPLING LIMIT FOR PAULI-FIERZ OPERATORS

64. Reduced weak coupling limit (E.B.Davies)

We consider a Pauli-Fierz operator

$$H_\lambda = K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(H_{\mathrm{R}}) + \lambda(a^*(V) + a(V))$$

We assume that $\mathcal{K}$ is finite dimensional and for any $A \in B(\mathcal{K})$ we have $\int \|V^* A \otimes 1 \ \mathrm{e}^{-\mathrm{i}tH_0} V\| \mathrm{d}t < \infty$.

Theorem. **There exists a Markov semigroup e^{tM} such that**

$$\lim_{\lambda \searrow 0} \mathrm{e}^{-\mathrm{i}tK/\lambda^2} I_{\mathcal{K}}^* \mathrm{e}^{\mathrm{i}tH_\lambda/\lambda^2} \ A \otimes 1 \ \mathrm{e}^{-\mathrm{i}tH_\lambda/\lambda^2} I_{\mathcal{K}} \mathrm{e}^{\mathrm{i}tK/\lambda^2} = \mathrm{e}^{tM}(A),$$

and a contractive semigroup $\mathrm{e}^{-\mathrm{i}t\Upsilon}$ such that

$$\lim_{\lambda \searrow 0} \mathrm{e}^{\mathrm{i}tK/\lambda^2} I_{\mathcal{K}}^* \mathrm{e}^{-\mathrm{i}tH_\lambda/\lambda^2} I_{\mathcal{K}} = \mathrm{e}^{-\mathrm{i}t\Upsilon}.$$

65. Continuity of spectrum

Assumption. **Suppose that for any** $\omega \in \mathrm{sp}K - \mathrm{sp}K$ **there exists open** $I_\omega \subset \mathbb{R}$ **such that** $\omega \in I_\omega$ **and**

$$\mathrm{Ran} 1_{I_\omega}(H_{\mathrm{R}}) \simeq \mathfrak{h}_\omega \otimes L^2(I_\omega, \mathrm{d}x),$$

$1_{I_\omega}(H_{\mathrm{R}})H_{\mathrm{R}}$ **is the multiplication operator by the variable** $x \in I_\omega$ **and**

$$1_{I_\omega}(H_{\mathrm{R}})V \simeq \int_{I_\omega}^{\oplus} v(x)\mathrm{d}x.$$

We assume that I_ω **are disjoint for distinct** ω **and** $x \mapsto v(x) \in B(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}_\omega)$ **is continuous at** ω**.**

66. Formula for the Davies generator I

The operator $\Upsilon : \mathcal{K} \to \mathcal{K}$ arising in the weak coupling limit is

$$\Upsilon := -\mathrm{i} \sum_{\omega} \sum_{k-k'=\omega} \int_0^\infty 1_k(K) V^* 1_{k'}(K)\, \mathrm{e}^{-\mathrm{i}t(H_\mathrm{R}-\omega)}\, V\, 1_k(K) \mathrm{d}t.$$

Let $\mathfrak{h} := \bigoplus_{\omega} \mathfrak{h}_\omega$. We define $\nu_\omega : \mathcal{K} \to \mathcal{K} \otimes \mathfrak{h}_\omega$

$$\nu_\omega := (2\pi)^{\frac{1}{2}} \sum_{\omega=k-k'} 1_k(K) v(\omega) 1_{k'}(K),$$

$\nu : \mathcal{K} \to \mathcal{K} \otimes \mathfrak{h}$

$$\nu := \sum_{\omega} \nu_\omega.$$

67. Formula for the Davies generator II

Note that

$$
\begin{aligned}
\mathrm{i}\Upsilon - \mathrm{i}\Upsilon^* &= \sum_{\omega} \sum_{k-k'=\omega} \int_{-\infty}^{\infty} 1_k(K) V^* 1_{k'}(K)\, \mathrm{e}^{-\mathrm{i}t(H_{\mathrm{R}}-\omega)}\, V\, 1_k(K) \mathrm{d}t \\
&= \sum_{\omega} \sum_{k-k'=\omega} 1_k(K) v^*(\omega) 1_{k'}(K) v(\omega)\, 1_k(K) \\
&= \nu^* \nu.
\end{aligned}
$$

The generator of a c.p. Markov semigroup that arises in the reduced weak coupling limit, called sometimes the Davies generator is

$$
M(A) = -\mathrm{i}(\Upsilon A - A\Upsilon^*) + \nu^* A \otimes 1 \nu, \qquad A \in B(\mathcal{K}).
$$

68. Asymptotic space and dynamics

Recall that given $(\Upsilon, \nu, \mathfrak{h})$ **we can define the space** $\mathcal{Z}_{\mathrm{R}}$ **and the Langevin dynamics** $\mathrm{e}^{-\mathrm{i}tZ}$ **on the space** $\mathcal{Z} := \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{R}})$**. Recall that**

$$\mathcal{Z}_{\mathrm{R}} = \bigoplus_{\omega} \mathfrak{h}_{\omega} \otimes L^2(\mathbb{R}).$$

We will need the renormalizing Hamiltonian on $\mathcal{Z}$

$$Z_{\mathrm{ren}} := E + \mathrm{d}\Gamma(\bigoplus_{\omega} \omega).$$

69. Scaling

For $\lambda > 0$, we define the family of partial isometries $J_{\lambda,\omega} : \mathfrak{h}_\omega \otimes L^2(\mathbb{R}) \to \mathfrak{h}_\omega \otimes L^2(I_\omega)$:

$$(J_{\lambda,\omega} g_\omega)(y) = \begin{cases} \frac{1}{\lambda} g_\omega(\frac{y-\omega}{\lambda^2}), & \textbf{if } y \in I_\omega; \\ 0, & \textbf{if } y \in \mathbb{R} \backslash I_\omega. \end{cases}$$

We set $J_\lambda : \mathcal{Z}_{\mathrm{R}} \to \mathcal{H}_{\mathrm{R}}$, defined for $g = (g_\omega)$ by

$$J_\lambda g := \sum_\omega J_{\lambda,\omega} g_\omega.$$

Note that J_λ are partial isometries and

$$\mathrm{s}-\lim_{\lambda \searrow 0} J_\lambda^* J_\lambda = 1.$$

70. Extended weak coupling limit

(Inspired by Accardi-Frigerio-Lu).

Theorem.

$$\mathrm{s}^* - \lim_{\lambda \searrow 0} \mathrm{e}^{\mathrm{i}\lambda^{-2} t Z_{\mathrm{ren}}} \, \Gamma(J_\lambda^*) \, \mathrm{e}^{-\mathrm{i}\lambda^{-2} t H_\lambda} \, \Gamma(J_\lambda) = \mathrm{e}^{-\mathrm{i}tZ} \, .$$

Theorem. (Convergence of the interaction picture).

$$\mathrm{s}^* - \lim_{\lambda \searrow 0} \Gamma(J_\lambda^*) \, \mathrm{e}^{\mathrm{i}\lambda^{-2} t H_0} \, \mathrm{e}^{-\mathrm{i}\lambda^{-2}(t-t_0) H_\lambda} \, \mathrm{e}^{\mathrm{i}\lambda^{-2} t_0 H_0} \, \Gamma(J_\lambda)$$

$$= \mathrm{e}^{\mathrm{i}tZ_0} \, \mathrm{e}^{-\mathrm{i}(t-t_0)Z} \, \mathrm{e}^{-\mathrm{i}t_0 Z_0} \, .$$

71. Asymptotics of correlation functions

Corrolary **Let** $A_\ell, \dots, A_1 \in B(\mathcal{Z})$ **and** $t, t_\ell, \dots, t_1, t_0 \in \mathbb{R}$. **Then**

$$
\begin{aligned}
& \mathrm{s}^* - \lim_{\lambda \searrow 0} I_{\mathcal{K}}^* \, \mathrm{e}^{\mathrm{i}\lambda^{-2} t H_0} \, \mathrm{e}^{-\mathrm{i}\lambda^{-2}(t - t_\ell) H_\lambda} \, \mathrm{e}^{-\mathrm{i}\lambda^{-2} t_\ell H_0} \\
& \times \Gamma(J_\lambda) A_\ell \Gamma(J_\lambda^*) \cdots \Gamma(J_\lambda) A_1 \Gamma(J_\lambda^*) \\
& \mathrm{e}^{\mathrm{i}\lambda^{-2} t_1 H_0} \, \mathrm{e}^{-\mathrm{i}\lambda^{-2}(t_1 - t_0) H_\lambda} \, \mathrm{e}^{-\mathrm{i}\lambda^{-2} t_0 H_0} \, I_{\mathcal{K}} \\
= \; & I_{\mathcal{K}}^* \, \mathrm{e}^{\mathrm{i} t Z_0} \, \mathrm{e}^{-\mathrm{i}(t - t_\ell) Z} \, \mathrm{e}^{-\mathrm{i} t_\ell Z_0} \, A_\ell \\
& \cdots A_1 \, \mathrm{e}^{\mathrm{i} t_1 Z_0} \, \mathrm{e}^{-\mathrm{i}(t_1 - t_0) Z} \, \mathrm{e}^{-\mathrm{i} t_0 Z_0} \, I_{\mathcal{K}}.
\end{aligned}
$$

72. CANONICAL COMMUTATION RELATIONS

73. Representations of the CCR I

Let $\mathcal{Y}$ be a real vector space equipped with an antisymmetric form ω. (We call $\mathcal{Y}$ a symplectic space if ω is nondegenerate). Let $U(\mathcal{H})$ denote the set of unitary operators on a Hilbert space $\mathcal{H}$. We say that

$$\mathcal{Y} \ni y \mapsto W^\pi(y) \in U(\mathcal{H})$$

is a representation of the CCR over $\mathcal{Y}$ in $\mathcal{H}$ if

$$W^\pi(y_1)W^\pi(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\omega y_2}\, W^\pi(y_1+y_2), \qquad y_1, y_2 \in \mathcal{Y}.$$

This implies the canonical commutation relation in the Weyl form

$$W^\pi(y_1)W^\pi(y_2) \,=\, \mathrm{e}^{-\mathrm{i}y_1\omega y_2}\, W^\pi(y_2)W^\pi(y_1).$$

74. Representations of the CCR II

Let $\mathcal{Y} \ni y \mapsto W^\pi(y)$ be a representation of the CCR. We say that $\Psi_0 \in \mathcal{H}$ is cyclic if $\mathrm{Span}\{W^\pi(y)\Psi \ : \ y \in \mathcal{Y}\}$ is dense in $\mathcal{H}$.

Clearly,

$$\mathbb{R} \ni t \mapsto W^\pi(ty) \in U(\mathcal{H})$$

is a 1-parameter group. We say that a representation of the CCR) is regular if this group is strongly continuous for each $y \in \mathcal{Y}$.

75. Field operators

Assume that $y \mapsto W^\pi(y)$ be a regular representation of the CCR.

$$\phi^\pi(y) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}W^\pi(ty)\Big|_{t=0}.$$

$\phi^\pi(y)$ will be called the field operator corresponding to $y \in \mathcal{Y}$. We have Heisenberg canonical commutation relation

$$[\phi^\pi(y_1), \phi^\pi(y_2)] = \mathrm{i}y_1\omega y_2$$

We can extend the definition of field operators to the complexification $\mathbb{C}\mathcal{Y}$ of $\mathcal{Y}$:

$$\phi(y_\mathrm{R} + \mathrm{i}y_\mathrm{I}) = \phi(y_\mathrm{R}) + \mathrm{i}\phi(y_\mathrm{I}).$$

76. Quasi-free representations

Let $\mathcal{Y} \ni y \mapsto W^{\pi}(y)$ be a representation of the CCR. We say that $\Psi \in \mathcal{H}$ is a quasi-free vector iff there exists a quadratic form η such that

$$(\Psi|W(y)\Psi) = \exp(-\frac{1}{4}y\eta y).$$

Note that η is necessarily positive, that is $y\eta y \geq 0$ for $y \in \mathcal{Y}$.
A representation is called quasi-free if there exists a cyclic quasi-free vector in $\mathcal{H}$.

It is easy to see that quasi-free representations are regular. Therefore, in a quasi-free representation we can define the corresponding field operators, denoted $\phi(y)$.

77. Correlation functions

Theorem. Let $\Psi \in \mathcal{H}$**. Suppose we are given a regular representation of the CCR**

$$\mathcal{Y} \ni y \mapsto \mathrm{e}^{\mathrm{i}\phi(y)} \in U(\mathcal{H}).$$

Then the following statements are equivalent:

(1) **For any** $n = 1, 2, \dots$**,** $y_1, \dots y_n \in \mathcal{Y}$**,** $\Psi \in \mathrm{Dom}\,(\phi(y_1)\cdots\phi(y_n))$**, and**

$$(\Psi|\phi(y_1)\cdots\phi(y_{2m-1})\Psi) = 0;$$

$$(\Psi|\phi(y_1)\cdots\phi(y_{2m})\Psi)$$

$$= \sum_{\sigma\in\mathrm{Pairings}(2m)} \prod_{j=1}^{m} (\Psi|\phi(y_{\sigma(2j-1)})\phi(y_{\sigma(2j)})\Psi).$$

(2) Ψ **is a quasi-free vector.**

78. Conjugate Hilbert space

Let $\mathcal{Z}$ be a (complex) Hilbert space. The space conjugate to $\mathcal{Z}$ is a Hilbert space $\overline{\mathcal{Z}}$ equipped with an antilinear map

$$\mathcal{Z} \ni z \mapsto \overline{z} \in \overline{\mathcal{Z}}$$

such that $(\overline{z}_1|\overline{z}_2) = \overline{(z_1|z_2)}$. We will write $\overline{\overline{z}} = z$.

Natural model of a conjugate space: take $\overline{\mathcal{Z}} = \mathcal{Z}$ as a real vector space; $\overline{z} = z$; the new multiplication by the imaginary unit changes the sign:

$$\mathrm{i} \overline{\cdot} z := -\mathrm{i} \cdot z.$$

79. Symplectic space built on a complex Hilbert space

For a Hilbert space $\mathcal{Z}$ we define

$$\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) := \{(z, \overline{z}) \ : \ z \in \mathcal{Z}.\}.$$

$\mathcal{Y}$ is equipped with symplectic form

$$(z, \overline{z})\omega(w, \overline{w}) = 2\mathrm{Im}(z|w).$$

Note that $\mathbb{C}\mathcal{Y}$ can be identified with $\mathcal{Z} \oplus \overline{\mathcal{Z}}$.

80. Creation/annihilation operators

Suppose that

$$\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) \ni y \mapsto W(y) \in U(\mathcal{H}).$$

is a regular representation of the CCR.
For $z \in \mathcal{Z} \subset \mathbb{C}\mathcal{Y}$ we introduce creation/annihilation operators

$$a(z) := \phi(0, \overline{z}), \quad a^*(z) = \phi(z, 0).$$

They satisfy the usual relations

$$[a(z_1), a(z_2)] = 0, \quad [a(z_1), a(z_2)] = 0,$$
$$[a(z_1), a^*(z_2)] = (z_1|z_2).$$

81. Identifying a symplectic space with a Hilbert space

Often we identify $\mathcal{Y}$ with $\mathcal{Z}$ by

$$\mathcal{Z} \ni z \mapsto \frac{1}{\sqrt{2}}(z, \overline{z}) \ni \mathcal{Y}$$

so that $z\omega w = \mathrm{Im}(z|w)$**. Then**

$$\begin{aligned}
\phi(w) &= \frac{1}{\sqrt{2}}\left(a^*(w) + a(w)\right), \\
a^*(w) &= \frac{1}{\sqrt{2}}\left(\phi(w) - \mathrm{i}\phi(w)\right), \\
a(w) &= \frac{1}{\sqrt{2}}\left(\phi(w) + \mathrm{i}\phi(w)\right).
\end{aligned}$$

82. REPRESENTATIONS OF THE CCR IN FOCK SPACES

83. Fock representation of the CCR

Let $\mathcal{Z}$ be a Hilbert space and consider the creation/annihilation operators acting on the Fock space $\Gamma_{\rm s}(\mathcal{Z})$. Then

$$\phi(w) := \frac{1}{\sqrt{2}}(a^*(w) + a(w))$$

are self-adjoint and

$$\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) \simeq \mathcal{Z} \ni z \mapsto \exp \mathrm{i}\phi(w)$$

is a regular representation of the CCR called the Fock representation. We have

$$(\Omega| \, \mathrm{e}^{\mathrm{i}\phi(w)} \, \Omega) = \mathrm{e}^{-\frac{1}{4}(w|w)} \, .$$

$$a(z)\Omega = 0.$$

It is an example of a quasi-free representation.

84. Double Fock space

Let $\mathcal{Z}$ be a Hilbert space and consider the Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. We have creation/annihilation operators $a^*(z_1, z_2)$, $a(z_1, z_2)$, satisfying

$$\begin{aligned} [a^*(z_1, \overline{z}_2), a^*(w_1, \overline{w}_2)] &= [a(z_1, \overline{z}_2), a(w_1, \overline{w}_2)] = 0, \\ [a(z_1, \overline{z}_2), a^*(w_1, \overline{w}_2)] &= (z_1|w_1) + \overline{(z_2|w_2)}. \end{aligned}$$

The antiunitary involution

$$\mathcal{Z} \oplus \overline{\mathcal{Z}} \ni (z_1, \overline{z}_2) \mapsto \epsilon(z_1, \overline{z}_2) := (z_2, \overline{z}_1) \in \mathcal{Z} \oplus \overline{\mathcal{Z}},$$

will be useful. Note that

$$\begin{aligned} \Gamma(\epsilon)a(z_1, \overline{z}_2)\Gamma(\epsilon) &= a(z_2, \overline{z}_1), \\ \Gamma(\epsilon)a^*(z_1, \overline{z}_2)\Gamma(\epsilon) &= a^*(z_2, \overline{z}_1). \end{aligned}$$

85. Parametrization of Araki-Woods representation of the CCR

Fix a self-adjoint operator γ **on** $\mathcal{Z}$ **satisfying** $0 \leq \gamma \leq 1$**,** $\mathrm{Ker}(\gamma - 1) = \{0\}$**. We will also use a positive operator** ρ **on** $\mathcal{Z}$ **called the 1-particle density related to** γ **by**

$$\gamma := \rho(1+\rho)^{-1}, \quad \rho = \gamma(1-\gamma)^{-1}.$$

86. Left Araki-Woods representation of the CCR

$\mathcal{Z} \supset \mathrm{Dom}(\rho^{\frac12}) \ni z \mapsto W_{\gamma,\mathrm{l}}(z) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}))$ **is a regular representation of the CCR, called the left Araki-Woods representation, where**

$$a^*_{\gamma,\mathrm{l}}(z) := a^*\Big((\rho+1)^{\frac12} z, 0\Big) + a\Big(0, \overline{\rho}^{\frac12}\overline{z}\Big),$$

$$a_{\gamma,\mathrm{l}}(z) := a\Big((\rho+1)^{\frac12}\ z, 0\Big) + a^*\Big(0, \overline{\rho}^{\frac12}\overline{z}\Big),$$

$$\phi_{\gamma,\mathrm{l}}(z) := 2^{-\frac12}(a^*_{\gamma,\mathrm{l}}(z) + a_{\gamma,\mathrm{l}}(z)), \qquad W_{\gamma,\mathrm{l}}(z) := \mathrm{e}^{\mathrm{i}\phi_{\gamma,\mathrm{l}}(z)}.$$

In fact, $W_{\gamma,\mathrm{l}}(z_1)W_{\gamma,\mathrm{l}}(z_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(z_1|z_2)}\, W_{\gamma,\mathrm{l}}(z_1+z_2)$. **We will write** $\mathfrak{M}^{\mathrm{AW}}_{\gamma,\mathrm{l}}$ **for the von Neumann algebra generated by** $W_{\gamma,\mathrm{l}}(z)$**.**

87. Right Araki-Woods representation of the CCR

$\overline{\mathcal{Z}} \supset \mathrm{Dom}(\overline{\rho}^{\frac{1}{2}}) \ni z \mapsto W_{\gamma,\mathrm{r}}(z) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}))$ **is a regular representation of the CCR, called the right Araki-Woods representation, where**

$$a^*_{\gamma,\mathrm{r}}(\overline{z}) := a^*\Big(0, (\rho+1)^{\frac{1}{2}} z\Big) + a\Big(\overline{\rho}^{\frac{1}{2}}\overline{z}, 0\Big),$$

$$a_{\gamma,\mathrm{r}}(\overline{z}) := a\Big(0, (\rho+1)^{\frac{1}{2}}\ z\Big) + a^*\Big(\overline{\rho}^{\frac{1}{2}}\overline{z}, 0\Big),$$

$$\phi_{\gamma,\mathrm{r}}(\overline{z}) := 2^{-\frac{1}{2}}(a^*_{\gamma,\mathrm{r}}(\overline{z}) + a_{\gamma,\mathrm{r}}(\overline{z})), \qquad W_{\gamma,\mathrm{r}}(\overline{z}) := \mathrm{e}^{\mathrm{i}\phi_{\gamma,\mathrm{r}}(\overline{z})}.$$

In fact, $W_{\gamma,\mathrm{r}}(\overline{z}_1)W_{\gamma,\mathrm{r}}(\overline{z}_2) = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(z_1|z_2)}W_{\gamma,\mathrm{l}}(\overline{z}_1+\overline{z}_2)$. **We will write** $\mathfrak{M}^{\mathrm{AW}}_{\gamma,\mathrm{r}}$ **for the von Neumann generated by** $W_{\gamma,\mathrm{r}}(\overline{z})$**.**

88. Araki-Woods representation of the CCR as a quasi-free representation

The vacuum Ω is a bosonic quasi-free vector for $W_{\gamma,\mathrm{l}}$. Its expectation value for the Weyl operators is equal to

$$\Big(\Omega|W_{\gamma,\mathrm{l}}(z)\Omega\Big) = \exp\Big(-\frac{1}{4}(z|z) - \frac{1}{2}(z|\rho z)\Big)$$

and the "two-point functions" are equal to

$$\begin{aligned}
\Big(\Omega|a_{\gamma,\mathrm{l}}^*(z_1)a_{\gamma,\mathrm{l}}(z_2)\Omega\Big) &= (z_2|\rho z_1),\\
\Big(\Omega|a_{\gamma,\mathrm{l}}^*(z_1)a_{\gamma,\mathrm{l}}^*(z_2)\Omega\Big) &= 0,\\
\Big(\Omega|a_{\gamma,\mathrm{l}}(z_1)a_{\gamma,\mathrm{l}}(z_2)\Omega\Big) &= 0.
\end{aligned}$$

89. Araki-Woods representation of the CCR and von Neumann algebras

$\Gamma(\epsilon)a_{\gamma,\mathrm{l}}^{\mathrm{AW}}\Gamma(\epsilon) = a_{\gamma,\mathrm{r}}^{\mathrm{AW}}$, **etc. Hence,** $\Gamma(\epsilon)\mathfrak{M}_{\gamma,\mathrm{l}}^{\mathrm{AW}}\Gamma(\epsilon) = \mathfrak{M}_{\gamma,\mathrm{r}}^{\mathrm{AW}}$**.**
$\mathfrak{M}_{\gamma,\mathrm{l}}^{\mathrm{AW}}$ **is a factor.**
If $\gamma = 0$**, then it is of type I.**
If γ **has some continuous spectrum, it is of type III.**

Proposition. $\mathrm{Ker}\gamma = \{0\}$
iff Ω **is a cyclic vector for** $\mathfrak{M}_{\gamma,\mathrm{l}}^{\mathrm{AW}}$
iff Ω **is a separating vector for** $\mathfrak{M}_{\gamma,\mathrm{l}}^{\mathrm{AW}}$
iff $(\Omega|\cdot\Omega)$ **is a faithful state on** $\mathfrak{M}_{\gamma,\mathrm{l}}^{\mathrm{AW}}$**.**
In this case, the Tomita-Takesaki theory yields the modular conjugation $J = \Gamma(\epsilon)$ **and the modular operator** $\Delta = \Gamma(\gamma \oplus \overline{\gamma}^{-1})$**.**

90. Araki-Woods representation of the CCR and free dynamics

Let h **be a positive self-adjoint operator on** $\mathcal{Z}$ **commuting with** γ **and**

$$\tau^t(W_{\gamma,1}(z)) := W_{\gamma,1}(\mathrm{e}^{ith}\, z).$$

Then $t \mapsto \tau^t$ **extends to a** W^***-dynamics on** $\mathfrak{M}_{\gamma,1}^{\mathrm{AW}}$ **and**

$$L = \mathrm{d}\Gamma(h \oplus (-\overline{h}))$$

is its Liouvillean, that means

$$\begin{aligned} \tau^t(A) &= \mathrm{e}^{\mathrm{i}tL}\, A\, \mathrm{e}^{-\mathrm{i}tL}, \quad A \in \mathfrak{M}_{\gamma,1}^{\mathrm{AW}}, \\ L\Omega &= 0. \end{aligned}$$

$(\Omega|\cdot\Omega)$ **is a** (τ,β)**-KMS state iff** $\gamma = \mathrm{e}^{-\beta h}$**, or equivalently, the density satisfies the Planck law** $\rho = (\mathrm{e}^{\beta h} - 1)^{-1}$.

91. Confined Bose gas

Assume that γ (and equivalently ρ) is trace class. Then $\Gamma(\gamma)$ is trace class with

$$\mathrm{Tr}\Gamma(\gamma) = \det(1-\gamma)^{-1} = \det(1+\rho).$$

Define the state ω_γ on the W^*-algebra $B(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ given by the density matrix

$$\Gamma(\gamma)/\mathrm{Tr}\Gamma(\gamma).$$

Then

$$\omega_\gamma(W(z)) = \exp\Big(-\tfrac{1}{4}(z|z) - \tfrac{1}{2}(z|\rho z)\Big)$$

Thus we obtain the same expectation values as for the Araki-Woods representation.

92. Confined Bose gas in terms of a Araki-Woods representations

There exists a unitary operator

$$R_\gamma : \Gamma_{\mathrm{s}}(Z) \otimes \Gamma_{\mathrm{s}}(\overline{\mathcal{Z}}) \to \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$$

(a Bogoliubov transformation) such that

$$\begin{aligned} W_{\gamma,\mathrm{l}}(z) &= R_\gamma W(z) \otimes 1 R_\gamma^*, \\ \mathfrak{M}_{\gamma,\mathrm{l}}^{\mathrm{AW}} &= R_\gamma \ B(\Gamma_{\mathrm{s}}(\mathcal{Z})) \otimes 1 \ R_\gamma^*. \end{aligned}$$

93. SMALL SYSTEM IN CONTACT WITH BOSE GAS

94. Small quantum system in contact with Bose gas at zero density

Hilbert space of the small quantum system: $\mathcal{K} = \mathbb{C}^n$.
The Hamiltonian of the small system: K.
The free Pauli-Fierz Hamiltonian:

$$H_{\mathrm{fr}} := K \otimes 1 + 1 \otimes \int |\xi| a^*(\xi) a(\xi) \mathrm{d}\xi.$$

$$\mathbb{R}^d \ni \xi \mapsto v(\xi) \in B(\mathcal{K})$$

describes the interaction:

$$V := \int v(\xi) \otimes a^*(\xi) \mathrm{d}\xi + \mathrm{hc}$$

The full Pauli-Fierz Hamiltonian: $H := H_{\mathrm{fr}} + \lambda V$. The Pauli-Fierz system at zero density:

$$\left(B(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d)), \mathrm{e}^{\mathrm{i}tH} \cdot \mathrm{e}^{-\mathrm{i}tH} \right).$$

95. Small quantum system in contact with Bose gas at density ρ.

The algebra of observables of the composite system:

$$\mathfrak{M}_\gamma := B(\mathcal{K}) \otimes \mathfrak{M}^{\mathrm{AW}}_{\gamma,\mathrm{l}} \subset B\left(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))\right).$$

The free Pauli-Fierz semi-Liouvillean at density ρ:

$$L^{\mathrm{semi}}_{\mathrm{fr}} := K \otimes 1 + 1 \otimes \left(\int |\xi| a^*_{\mathrm{l}}(\xi) a_{\mathrm{l}}(\xi) \mathrm{d}\xi - \int |\xi| a^*_{\mathrm{r}}(\xi) a_{\mathrm{r}}(\xi) \mathrm{d}\xi\right).$$

The interaction: $V_\gamma := \int v(\xi) \otimes a^*_{\gamma,\mathrm{l}}(\xi)\mathrm{d}\xi + \mathrm{hc}.$

The full Pauli-Fierz semi-Liouvillean at density ρ:

$$L^{\mathrm{semi}}_\gamma := L^{\mathrm{semi}}_{\mathrm{fr}} + \lambda V_\gamma.$$

The Pauli-Fierz W^*-dynamical system at density ρ:

$$(\mathfrak{M}_\gamma, \tau_\gamma), \quad \textbf{where } \tau_{\gamma,t}(A) := \mathrm{e}^{\mathrm{i}tL^{\mathrm{semi}}_\gamma} A\, \mathrm{e}^{-\mathrm{i}tL^{\mathrm{semi}}_\gamma}.$$

96. Relationship between the dynamics at zero density and at density ρ.

Set $\gamma = 0$ (equivalently, $\rho = 0$).

$$\mathfrak{M}_0 \simeq B(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d)) \otimes 1.$$

$$L_0^{\mathrm{semi}} \simeq H \otimes 1 - 1 \otimes \int |\xi| a_{\mathrm{r}}^*(\xi) a_{\mathrm{r}}(\xi) \mathrm{d}\xi.$$

$$\tau_{0,t}(A \otimes 1) = \mathrm{e}^{\mathrm{i}tH} A \, \mathrm{e}^{-\mathrm{i}tH} \otimes 1.$$

If we formally replace $a_{\mathrm{l}}(\xi)$, $a_{\mathrm{r}}(\xi)$ with $a_{\gamma,\mathrm{l}}(\xi)$, $a_{\gamma,\mathrm{r}}(\xi)$ (the CCR do not change!) then $\mathfrak{M}_0$, L_0^{semi}, τ_0 transform into $\mathfrak{M}_\gamma$, L_γ^{semi}, τ_γ. In the case of a finite number of degrees of freedom this can be implemented by a unitary Bogoliubov transformation. $(\mathfrak{M}_\gamma, \tau_\gamma)$ can be viewed as a thermodynamical limit of $(\mathfrak{M}_0, \tau_0)$.

97. Thermal reservoirs

Theorem. If $\gamma = \mathrm{e}^{-\beta|\xi|}$, then

1. In the reduced weak coupling limit we obtain a c.p. Markov semigroup satisfying the Detailed Balance Condition wrt the state given by the density matrix $\mathrm{e}^{-\beta K}/\mathrm{Tr}\,\mathrm{e}^{-\beta K}$;
2. For any λ there exists a normal KMS state for the W^*-dynamical system τ_γ; Araki, D.-Jakšić-Pillet
3. Under some conditions on the interaction saying that it is sufficiently regular and effective, there exists $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$, this state is a unique normal stationary state (Jakšić-Pillet, D-Jakšić, Bach-Fröhlich-Sigal, Fröhlich-Merkli).

98. Standard representation of $\mathfrak{M}_\gamma$.

Often one uses the so-called standard representation:

$$\pi : \mathfrak{M}_\gamma \to B(\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)),$$

$$\pi(A \otimes B) = A \otimes 1 \otimes B,$$

$$J\Phi_1 \otimes \overline{\Phi}_2 \otimes \Psi = \Phi_2 \otimes \overline{\Phi}_1 \otimes \Gamma(\epsilon)\Psi.$$

The free Pauli-Fierz Liouvillean:

$$\begin{aligned} L_{\mathrm{fr}} &:= K \otimes 1 \otimes 1 - 1 \otimes \overline{K} \otimes 1 \\ &\quad +1 \otimes 1 \otimes \int \left(|\xi| \left(a_{\mathrm{l}}^*(\xi) a_{\mathrm{l}}(\xi) - a_{\mathrm{l}}^*(\xi) a_{\mathrm{l}}(\xi)\right)\right) \mathrm{d}\xi, \end{aligned}$$

$$\pi(V_\gamma) = \int v(\xi) \otimes 1 \otimes a_{\gamma,\mathrm{l}}^*(\xi) \mathrm{d}\xi + \mathrm{hc},$$

$$J\pi(V_\gamma)J = \int 1 \otimes \overline{v}(\xi) \otimes 1 \otimes a_{\gamma,\mathrm{r}}^*(\xi) \mathrm{d}\xi + \mathrm{hc}.$$

The full Pauli-Fierz Liouvillean at density ρ:

$$L_\gamma = L_{\mathrm{fr}} + \lambda\pi(V_\gamma) - \lambda J\pi(V_\gamma)J.$$