

Measure theory

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1 Measurability

1.1 Notation

2^X denotes the family of subsets of the set X . The symmetric difference is defined as

$$A\Delta B := (A \cup B) \setminus (A \cap B).$$

Let $A_1, A_2, \dots \in X$.

We write $A_n \nearrow A$, if $A_n \subset A_{n+1}$, $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n = A$.

We write $A_n \searrow A$, if $A_n \supset A_{n+1}$, $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} A_n = A$.

1.2 Rings and fields

Definition 1.1 $\mathcal{R} \subset 2^X$ is called a ring if

- (1) $A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}$;
- (2) $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$

Proposition 1.2 Let \mathcal{R} be a ring. Then $A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}$.

Proof. $A \cap B = A \setminus (A \setminus B)$. \square

If $(\mathcal{R}_i)_{i \in I}$ is a family of rings in X , then so is $\bigcap_{i \in I} \mathcal{R}_i$. Hence for any $\mathcal{T} \subset 2^X$ there exists the smallest ring containing \mathcal{T} . We denote it by $\text{Ring}(\mathcal{T})$.

Definition 1.3 $\mathcal{R} \subset 2^X$ is called a field if

- (1) $\emptyset \in \mathcal{R}$;
- (2) $A \in \mathcal{R} \Rightarrow X \setminus A \in \mathcal{R}$;
- (3) $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$.

Equivalently, a field is a ring containing X . (Field is a ring, because $A \setminus B = X \setminus ((X \setminus A) \cup B)$).

For $\mathcal{T} \subset 2^X$, $\text{Field}(\mathcal{T})$ denotes the smallest field of sets containing \mathcal{T} .

1.3 Ordered spaces

Suppose that (X, \leq) is an ordered set. Let U be a nonempty subset of X .

We say that u_0 is a largest minorant of U if

- (1) $u \in U$ implies $u_0 \leq u$
- (2) $u_1 \leq u$ for all $u \in U$ implies $u_1 \leq u_0$

If U possesses a largest minorant, then it is uniquely defined. The largest minorant of a set $\{x_1, x_2\}$ is often denoted $x_1 \wedge x_2$ and of a set U is denoted $\bigwedge_{x \in U} x$.

Analogously we define the smallest majorant of U . The smallest majorant of a set $\{x_1, x_2\}$ is often denoted $x_1 \vee x_2$ and of a set U is denoted $\bigvee_{x \in U} x$.

We say that (X, \leq) is a lattice if every two-element (hence every finite) set of elements of X possess the smallest majorant and the largest minorant. It is a countably complete lattice if every countable subset that has a majorant and a minorant has the smallest majorant and the largest minorant. It is a complete lattice if every countable subset that has a majorant and a minorant has the smallest majorant and the largest minorant.

Let \mathcal{X} be a vector space. (\mathcal{X}, \leq) is an ordered vector space iff

- (1) $x, y, z \in \mathcal{X}, x \leq y \Rightarrow x + z \leq y + z$;
- (2) $x \in \mathcal{X}, x \geq 0, \lambda \in \mathbb{R}, \lambda \geq 0 \Rightarrow \lambda x \geq 0$.

$\mathcal{X}_+ := \{x \in \mathcal{X} : x \geq 0\}$ is a cone called the positive cone.

We say that an ordered vector space (\mathcal{X}, \leq) is a Riesz space if it is a lattice. It is enough to check that it has \vee of two elements, since

$$x \wedge y := -(-x) \vee (-y).$$

1.4 Elementary functions

Definition 1.4 Let (X, \mathcal{R}) be a space with a ring. $u : X \rightarrow \mathbb{R}$ is called an elementary function if $u(X)$ is a finite set and $u^{-1}(\alpha) \in \mathcal{R}, \alpha \in \mathbb{R} \setminus \{0\}$. The set of elementary functions is denoted by $\mathcal{E}(X, \mathcal{R})$ or $\mathcal{E}(X)$. Positive elementary functions will be denoted $\mathcal{E}_+(X)$.

Lemma 1.5 (1) Let $u, v \in \mathcal{E}(X)$ and $\alpha \in \mathbb{R}$. Then

$$\alpha u, u + v, uv, \max(u, v), \min(u, v) \in \mathcal{E}(X).$$

In particular, $\mathcal{E}(X)$ is an algebra and a lattice.

- (2) $1 \in \mathcal{E}(X)$ iff \mathcal{R} is a field.

1.5 σ -rings and σ -fields

Definition 1.6 $\mathcal{F} \subset 2^X$ is called a σ -ring if

- (1) $A, B \in \mathcal{F} \Rightarrow A \setminus B \in \mathcal{F}$;
- (2) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Clearly, every σ -ring is a ring.

Proposition 1.7 Let \mathcal{F} be a σ -ring. Then

- (1) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcap_{j=1}^{\infty} A_j \in \mathcal{F}$,
- (2) $A_1, A_2, \dots \in \mathcal{F}, A_n \setminus A \Rightarrow A \in \mathcal{F}$,

(3) $A_1, A_2, \dots \in \mathcal{F}, A_n \nearrow A \Rightarrow A \in \mathcal{F}$,

Proof. Let us prove (1). Clearly, $A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$. Now by the de Morgan's law

$$\bigcap_{i=1}^{\infty} A_i = A \setminus \bigcup_{i=1}^{\infty} (A \setminus A_i) \in \mathcal{F}.$$

□

For $\mathcal{T} \subset 2^X$, $\sigma\text{-Ring}(\mathcal{T})$ denotes the smallest σ -ring of sets containing \mathcal{T} .

Theorem 1.8 *Let $\mathcal{T} \subset 2^X$ and $A \in \sigma\text{-Ring}(\mathcal{T})$. Then there exists a countable $\mathcal{T}_0 \subset \mathcal{T}$ such that $A \in \sigma\text{-Ring}(\mathcal{T}_0)$.*

Proof. Let \mathcal{F} be the family of $A \subset X$ such that there exists a countable $\mathcal{T}_0 \subset \mathcal{T}$ with $A \in \sigma\text{-Ring}(\mathcal{T}_0)$. Then $\mathcal{T} \subset \mathcal{F}$ and \mathcal{F} is a σ -ring. Hence $\sigma\text{-Ring}(\mathcal{T}) \subset \mathcal{F}$. □

Definition 1.9 $\mathcal{F} \subset 2^X$ is called a σ -field if

- (1) $\emptyset \in \mathcal{F}$;
- (2) $A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$;
- (3) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$.

Equivalently, a σ -field is a σ -ring containing X . Clearly, every σ -field is a field.
For $\mathcal{T} \subset 2^X$, $\sigma\text{-Field}(\mathcal{T})$ denotes the smallest σ -field of sets containing \mathcal{T} .

1.6 Transport of subsets

Let $F : X \rightarrow X'$ be a transformation. As usual, for $A \subset X$, $F(A)$ denotes the image of A , and for $A' \subset X'$, $F^{-1}(A')$ denotes the preimage of A' . Thus we have two maps

$$\begin{aligned} 2^X \ni A &\mapsto F(A) \in 2^{X'}, \\ 2^{X'} \ni A' &\mapsto F^{-1}(A') \in 2^X. \end{aligned} \tag{1.1}$$

Theorem 1.10 (1) *For $A \subset X$, $F^{-1}F(A) \supset A$ and we have the equality for all A iff F is injective.*
(2) *For $A' \in X'$, $FF^{-1}(A') \subset A'$ and we have the equality for all A' iff F is surjective.*
(3) $F^{-1}(\emptyset) = \emptyset$, $F^{-1}(X') = X$, $F^{-1}(A' \cup B') = F^{-1}(A') \cup F^{-1}(B')$, $F^{-1}(A' \setminus B') = F^{-1}(A') \setminus F^{-1}(B')$,

Let $F^* : 2^{X'} \rightarrow 2^X$ be the map given by (1.1). (We prefer not to denote it by F^{-1} to avoid ambiguous notation).

For $\mathcal{C}' \subset 2^{X'}$, we can write

$$F^*(\mathcal{C}') = \{F^{-1}(A') : A' \in \mathcal{C}'\}.$$

Let $\mathcal{C} \subset 2^X$. We will write

$$F_*(\mathcal{C}) := (F^*)^{-1}(\mathcal{C}) = \{A' \in 2^{X'} : F^{-1}(A') \in \mathcal{C}\}.$$

The following facts follow from Theorem 1.10 (1), (2) applied to F^* :

Theorem 1.11 (1) $F_*F^*(\mathcal{C}') \supset \mathcal{C}'$;
(2) $F^*F_*(\mathcal{C}) \subset \mathcal{C}$.

1.7 Transport of σ -rings

Theorem 1.12 (1) If \mathcal{F}' is a σ -ring over X' , then $F^*(\mathcal{F}')$ is a σ -ring over X .

(2) If \mathcal{F} is a σ -ring over X , then $F_*(\mathcal{F})$ is a σ -ring over X' .

(3) If $\mathcal{C}' \subset 2^{X'}$, then

$$F^*(\sigma\text{-Ring}(\mathcal{C}')) = \sigma\text{-Ring}(F^*(\mathcal{C}')).$$

Proof. To see (1) and (2), we use Theorem 1.10 (3), which says that F^* is a homomorphism for set-theoretical operations.

Let us prove (3). By (1), $F^*(\sigma\text{-Ring}(\mathcal{C}'))$ is a σ -ring. It contains $F^*(\mathcal{C}')$. Hence

$$F^*(\sigma\text{-Ring}(\mathcal{C}')) \supset \sigma\text{-Ring}(F^*(\mathcal{C}')).$$

By (2), $F_*(\sigma\text{-Ring}(F^*(\mathcal{C}')))$ is a σ -ring. Clearly

$$F_*(\sigma\text{-Ring}(F^*(\mathcal{C}'))) \supset F_*(F^*(\mathcal{C}')) \supset \mathcal{C}'.$$

Hence $F_*(\sigma\text{-Ring}(F^*(\mathcal{C}'))) \supset \sigma\text{-Ring}(\mathcal{C}')$. Hence,

$$\sigma\text{-Ring}(F^*(\mathcal{C}')) \supset F^*F_*(\sigma\text{-Ring}(F^*(\mathcal{C}'))) \supset F^*(\sigma\text{-Ring}(\mathcal{C}')).$$

□

For $A \in 2^X$ and $\mathcal{C} \subset 2^X$, we set

$$\mathcal{C} \Big|_A := \{A \cap C : C \in \mathcal{C}\}.$$

Theorem 1.13 If $\mathcal{T} \subset 2^X$ and $A \subset X$, then

$$\sigma\text{-Ring}(\mathcal{T}) \Big|_A = \sigma\text{-Ring}(\mathcal{T} \Big|_A).$$

Proof. Consider the inclusion map $J : A \rightarrow X$. If $C \in 2^X$, then $J^{-1}(C) = C \cap A$. Hence if $\mathcal{C} \subset 2^X$, then $J^*(\mathcal{C}) = \mathcal{C} \Big|_A$. Thus it is sufficient to apply Theorem 1.12 (3). □

1.8 Measurable transformations

Definition 1.14 Let (X, \mathcal{F}) , (X', \mathcal{F}') be spaces with σ -rings and $F : X \rightarrow X'$. Then F is called a $\mathcal{F} - \mathcal{F}'$ -measurable transformation if

$$F^*(\mathcal{F}') \subset \mathcal{F}.$$

Proposition 1.15 The composition of measurable transformations is measurable.

Theorem 1.16 Let $\mathcal{C}' \subset 2^{X'}$. If $\mathcal{F}' = \sigma\text{-Ring}(\mathcal{C}')$, then $F : X \rightarrow X'$ is $\mathcal{F} - \mathcal{F}'$ -measurable iff $F^*(\mathcal{C}') \subset \mathcal{F}$.

Proof.

$$F^*(\mathcal{F}') = F^*(\sigma\text{-Ring}(\mathcal{C}')) = \sigma\text{-Ring}(F^*(\mathcal{C}')) \subset \sigma\text{-Ring}(\mathcal{F}) = \mathcal{F},$$

where we used Theorem 1.12 (3) in the second equality. □

1.9 Measurable real functions

$\mathbb{R} \cup \{-\infty, \infty\} =: [-\infty, \infty]$ is a topological space in the obvious way. We can extend the addition to $[-\infty, \infty]$ except that $\infty - \infty$ is undefined. We extend the multiplication to $[-\infty, \infty]$, adopting the convention $0(\pm\infty) = 0$. Let $\text{Borel}([-\infty, \infty])$ denote the σ -field of Borel subsets of $[-\infty, \infty]$, that is the σ -field generated by open subsets of $[-\infty, \infty]$. If $Y \subset \mathbb{R}$, then $\text{Borel}(Y)$ will denote the σ -field in X generated by open subsets in X . Note in particular that $\text{Borel}([-\infty, 0] \cup [0, \infty])$ is generated by the sets $[-\infty, -\alpha[$ and $]\alpha, \infty[$ for $0 \leq \alpha$.

Let (X, \mathcal{F}) be a space with a σ -ring. We say that

$$f : X \rightarrow [-\infty, \infty]$$

is a \mathcal{F} -measurable function iff for any $A \in \text{Borel}([-\infty, 0] \cup [0, \infty])$, $f^{-1}(A) \in \mathcal{F}$. The set of such functions will be denoted $\mathcal{M}(X, \mathcal{F})$, or for shortness, $\mathcal{M}(X)$. The set of measurable functions with values in $[0, \infty]$ will be called $\mathcal{M}_+(X)$.

Let $A \subset X$. Its characteristic function is denoted by

$$1_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \in X \setminus A, \end{cases}$$

1_A is \mathcal{F} -measurable iff $A \in \mathcal{F}$.

If f, g are real functions on X , we will write

$$\{f \geq g\} := \{x \in X : f(x) \geq g(x)\}.$$

Similarly, we define $\{f > g\}$, etc.

Lemma 1.17 $f : X \rightarrow [-\infty, \infty]$ is \mathcal{F} -measurable if

$$\{\pm f > \pm\alpha\} \in \mathcal{F}, \quad 0 \leq \alpha < \infty.$$

Lemma 1.18 Let $f, g \in \mathcal{M}(X)$. Then

- (1) $\alpha f \in \mathcal{M}(X)$
- (2) $f + g \in \mathcal{M}(X)$ (if defined);
- (3) $fg \in \mathcal{M}(X)$
- (4) $1 \in \mathcal{M}(X)$ iff \mathcal{F} is a σ -field.

Proof. (2) For simplicity, we assume in addition that \mathcal{F} is a σ -field. Using the countability of \mathbb{Q} we see that

$$\{f + g > \alpha\} = \bigcup_{\beta \in \mathbb{Q}} \{f > \alpha + \beta\} \cap \{g > -\beta\}.$$

(3) First we show that $f \in \mathcal{M}(X)$ implies $f^2 \in \mathcal{M}(X)$.

By (1) and (2), $f - g$ is measurable.

Finally

$$fg = \frac{1}{4}(f + g)^2 - \frac{1}{4}(f - g)^2.$$

implies that fg is measurable. \square

Proposition 1.19 Let $f_1, f_2, \dots \in \mathcal{M}(X)$. Then

$$\sup_n f_n, \inf_n f_n, \limsup_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} f_n$$

are measurable. If there exists the pointwise limit of f_n , then also $\lim_{n \rightarrow \infty} f_n$ is measurable.

Proof. Let $f := \sup f_n$. Then

$$\{f \leq \alpha\} = \bigcap_{n=1}^{\infty} \{f_n \leq \alpha\} \in \mathcal{F}.$$

Hence f is measurable. $\inf f_n$ is treated similarly.

Then we use

$$\limsup_{n \rightarrow \infty} f_n = \inf_{n \in \mathbb{N}} \sup_{m \geq n} f_m, \quad \liminf_{n \rightarrow \infty} f_n = \sup_{n \in \mathbb{N}} \inf_{m \geq n} f_m$$

Finally,

$$\lim_{n \rightarrow \infty} f_n = \limsup_{n \rightarrow \infty} f_n = \liminf_{n \rightarrow \infty} f_n.$$

□

Theorem 1.20 Let $f : X \rightarrow \mathbb{R}$. Then $f \in \mathcal{M}_+(X)$ iff there exists an increasing sequence $u_n \in \mathcal{E}_+(X)$ such that

$$f = \sup_{n \in \mathbb{N}} u_n$$

Proof. \Leftarrow is obvious. Let us prove the converse statement.

Let $f \in \mathcal{M}_+(X)$. The sets

$$A_{in} := \begin{cases} \left\{ \frac{i}{2^n} \leq f < \frac{i+1}{2^n} \right\} & i = 0, 1, \dots, n2^n - 1, \\ \{n \leq f\} & i = n2^n \end{cases}$$

are disjoint and measurable. Hence

$$u_n := \sum_{j=0}^{n2^n} \frac{j}{2^n} 1_{A_{jn}} \in \mathcal{E}_+(X).$$

The sequence u_n is increasing and $\sup_{n \in \mathbb{N}} u_n = f$. □

1.10 Spaces L^∞

Assume that $\mathcal{I} \subset \mathcal{F} \subset 2^X$ are rings. We say that \mathcal{I} is an ideal in \mathcal{F} if

$$A \in \mathcal{I}, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{I}.$$

In what follows let $\mathcal{I} \subset \mathcal{F} \subset 2^X$ be σ -rings and \mathcal{I} be an ideal in \mathcal{F} . Then

Proposition 1.21 (1) $\mathcal{M}(X, \mathcal{I}) \subset \mathcal{M}(X, \mathcal{F})$

(2) $\mathcal{M}(X, \mathcal{I}) := \{f \in \mathcal{M}(X, \mathcal{F}) : \text{there exists } N \in \mathcal{I} \text{ such that } f = 0 \text{ on } X \setminus N\}$

(3) $f \in \mathcal{M}(X, \mathcal{F}), g \in \mathcal{M}(X, \mathcal{I})$ implies $fg \in \mathcal{M}(X, \mathcal{I})$.

For $f \in \mathcal{M}(X, \mathcal{F})$ we set

$$\|f\|_\infty := \inf \{ \sup \{|f(x)| : x \in X \setminus N\} : N \in \mathcal{I} \}.$$

Theorem 1.22 (1) Given $f \in \mathcal{M}(X, \mathcal{F})$, we can always find $N \in \mathcal{I}$ such that $\sup |f| \Big|_{X \setminus N} = \|f\|_\infty$;

$$(2) \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty;$$

$$(3) \|\alpha f\| = |\alpha| \|f\|_\infty.$$

Proof. (2) We can find $N, M \in \mathcal{I}$ such that

$$\sup |f|_{X \setminus N} = \|f\|_\infty,$$

$$\sup |g|_{X \setminus M} = \|g\|_\infty.$$

Then

$$\begin{aligned} \|f + g\|_\infty &\leq \sup |f + g| \Big|_{X \setminus (N \cup M)} \\ &\leq \sup (|f| + |g|) \Big|_{X \setminus (N \cup M)} \\ &\leq \sup |f| \Big|_{X \setminus (N \cup M)} + \sup |g| \Big|_{X \setminus (N \cup M)} \\ &\leq \sup |f| \Big|_{X \setminus N} + \sup |g| \Big|_{X \setminus M} = \|f\|_\infty + \|g\|_\infty. \end{aligned}$$

Let

$$\mathcal{L}^\infty(X, \mathcal{F}, \mathcal{I}) := \{f \in \mathcal{M}(X, \mathcal{F}) : \|f\|_\infty < \infty\},$$

Theorem 1.23 (Riesz-Fischer) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{L}^\infty(\mu)$ satisfying the Cauchy condition, that is for any $\epsilon > 0$ there exists N such that for $n, m \geq N$

$$\|f_n - f_m\|_\infty \leq \epsilon.$$

Then there exists $f \in \mathcal{L}^\infty(\mu)$ such that

$$\|f - f_n\|_\infty \rightarrow 0$$

We can also find a subsequence of $(f_n)_{n \in \mathbb{N}}$ pointwise convergent μ -a.e. to f .

Proof. There exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq 2^{-k}$, for any k . We set

$$g_k := f_{n_{k+1}} - f_{n_k}, \quad g := \sum_{k=1}^{\infty} |g_k|.$$

Then

$$\|g\|_\infty \leq \sum_{k=1}^{\infty} \|g_k\|_\infty \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Hence $g \in \mathcal{L}^\infty$ and therefore g is finite outside of a set N in \mathcal{I} . Hence the series $\sum_{k=1}^{\infty} g_k$ is convergent outside of N . This means that the sequence $(f_{n_k})_{k \in \mathbb{N}}$ is convergent to a function f outside of N . Inside N we set $f := 0$. We check that $f \in \mathcal{L}^\infty$ and $\|f - f_n\|_\infty \rightarrow 0$. \square

Theorem 1.24 (1) $\mathcal{M}(X, \mathcal{I}) = \{f \in \mathcal{L}^\infty(X, \mathcal{F}, \mathcal{I}) : \|f\|_\infty = 0\}$

(2) Therefore, $\|f\|_\infty$ defines a norm in $L^\infty(X, \mathcal{F}, \mathcal{I}) := \mathcal{L}^\infty(X, \mathcal{F}, \mathcal{I}) / \mathcal{M}(X, \mathcal{I})$.

(3) $L^\infty(X, \mathcal{F}, \mathcal{I})$ is a Banach space.

(4) Elementary functions are dense in $L^\infty(X, \mathcal{F}, \mathcal{I})$.

(5) $f, g \in L^\infty(X, \mathcal{F}, \mathcal{I}), 0 \leq f \leq g$ a.e. $\Rightarrow \|f\|_\infty \leq \|g\|_\infty$.

(6) $L^\infty(X, \mathcal{F}, \mathcal{I})$ is a countably complete lattice.

2 Measure and integral

2.1 Contents

Let \mathcal{R} be a ring. $\nu : \mathcal{R} \rightarrow [0, \infty]$ is a content if

- (1) $\nu(\emptyset) = 0$;
- (2) $A_1, A_2 \in \mathcal{R}, A_1 \cap A_2 = \emptyset \Rightarrow \nu(A_1 \cup A_2) = \nu(A_1) + \nu(A_2)$.

Theorem 2.1 *Let (X, \mathcal{R}, ν) be a content on a ring. Then if $A_1, A_2, \dots \in \mathcal{R}$ are disjoint and $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{R}$, then*

$$\nu(A) \geq \sum_{i=1}^{\infty} \nu(A_i).$$

Proof. For any n ,

$$\nu(A) \geq \nu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \nu(A_j).$$

Passing to the limit $n \rightarrow \infty$, we obtain the inequality. \square

2.2 Measures

Let (X, \mathcal{F}) be a space with a σ -ring. A function $\mu : \mathcal{F} \rightarrow [0, \infty]$ is called a measure if

- (1) $\mu(\emptyset) = 0$,
- (2) $A_1, A_2, \dots \in \mathcal{F}, A_i \cap A_j = \emptyset$ for $i \neq j \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

The triple (X, \mathcal{F}, μ) is called a space with a measure.

Proposition 2.2 (1) *If $A \subset B$, then $\mu(A) \leq \mu(B)$.*

(2) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

(3) $A_1, A_2, \dots \in \mathcal{F}, A_n \nearrow A \Rightarrow \mu(A_n) \nearrow \mu(A)$.

(4) *If $A_1, A_2, \dots \in \mathcal{F}, A_n \searrow A$ and for some n , $\mu(A_n) < \infty$, then $\mu(A_n) \searrow \mu(A)$.*

Definition 2.3 *Let (X, \mathcal{F}, μ) be a space with a measure. and $P(x)$ be a property defined on X . We say that $P(x)$ is true μ -almost everywhere (μ -a.e.) if*

$$\mu(\{x \in X : P(x) \text{ is not true}\}) = 0.$$

2.3 μ - σ -finite sets

Let (X, \mathcal{F}, μ) be a measure. A set $A \in \mathcal{F}$ is called μ -null if $\mu(A) = 0$. It is μ -finite if $\mu(A) < \infty$. It is called μ - σ -finite iff there exist a sequence of μ -finite sets $A_1, A_2, \dots \in \mathcal{F}$ such that $A_n \nearrow A$. Set

$$\mathcal{F}_\mu^0 := \{A \in \mathcal{F} : \mu(A) = 0\}.$$

$$\mathcal{F}_\mu^f := \{A \in \mathcal{F} : A \text{ is } \mu\text{-finite}\}.$$

$$\mathcal{F}_\mu^{\sigma f} := \{A \in \mathcal{F} : A \text{ is } \mu\text{-}\sigma\text{-finite}\}.$$

We say that μ is σ -finite iff $\mathcal{F} = \mathcal{F}_\mu^{\sigma f}$ and μ is finite iff $\mathcal{F} = \mathcal{F}_\mu^f$. We say that μ is probabilistic iff $\mu(X) = 1$.

Theorem 2.4 (1) \mathcal{F}_μ^0 is a σ -ring and an ideal in $\mathcal{F}_\mu^f, \mathcal{F}_\mu^{\sigma f}, \mathcal{F}$.

(2) $\mathcal{F}_\mu^{\sigma f}$ is a σ -ring and an ideal in \mathcal{F}

(3) \mathcal{F}_μ^f is a ring and an ideal in $\mathcal{F}_\mu^{\sigma f}, \mathcal{F}$. We have $\mathcal{F}_\mu^{\sigma f} = \sigma\text{-Ring}(\mathcal{F}_\mu^f)$.

(4) If \mathcal{F} is a σ -field, then μ is σ -finite iff X is μ - σ -finite; μ is finite if X is μ -finite.

Note that if (X, \mathcal{F}, μ) is any measure, then $X, \mathcal{F}^{\mu^f}, \mu|_{\mathcal{F}^{\mu^f}}$ is a σ -finite measure. We will call say that the latter measure has been obtained from the former by restricting to μ - σ -finite sets.

2.4 Integral on elementary functions I

Let (X, \mathcal{R}, μ) be a space with a ring and a content. For $f \in \mathcal{E}_+(X)$ we set

$$\int f d\mu := \sum_{t \in \mathbb{R}} t\mu(f^{-1}\{t\}).$$

Theorem 2.5 The function

$$\mathcal{E}_+(X) \ni u \mapsto \int u d\mu \in [0, \infty]$$

satisfies

(1) $\int 1_A d\mu = \mu(A)$

(2) $\alpha \geq 0$ implies $\int (\alpha u) d\mu = \alpha \int u d\mu$;

(3) $\int (u + v) d\mu = \int u d\mu + \int v d\mu$.

(4) $u \leq v$ implies $\int u d\mu \leq \int v d\mu$.

2.5 Integral on elementary functions II

Assume now that (X, \mathcal{F}, μ) is a set with a σ -ring and a measure. We define the integral on elementary functions as in the previous subsection.

Lemma 2.6 Let $(u_n)_{n \in \mathbb{N}}$ be an increasing sequence in $\mathcal{E}_+(X)$ and $v \in \mathcal{E}_+(X)$. Then

$$v \leq \sup_{n \in \mathbb{N}} u_n \Rightarrow \int v d\mu \leq \sup_{n \in \mathbb{N}} \int u_n d\mu.$$

Proof. It is sufficient to assume that

$$\{v \neq 0\} =: A \neq \emptyset.$$

Let $\alpha := \inf v(A), \beta := \sup v, 0 < \epsilon < \alpha$. Set

$$A_n := \{u_n \geq v - \epsilon\} \cap A.$$

Then $A_n \in \mathcal{F}$ and $A_n \nearrow A$. Hence $\mu(A_n) \nearrow \mu(A)$.

Consider two cases:

1) $\mu(A) = \infty$. Then

$$(\alpha - \epsilon)1_{A_n} \leq (v - \epsilon)1_{A_n} \leq u_n.$$

Hence

$$(\alpha - \epsilon)\mu(A_n) \leq \int u_n d\mu$$

But the lhs tends to $(\alpha - \epsilon)\infty = \infty$. Therefore,

$$\lim_{n \rightarrow \infty} \int u_n d\mu = \infty.$$

2) $\mu(A) < \infty$. Set $B_n := A \setminus A_n$. Then $B_n \in \mathcal{F}$, $\mu(B_n) < \infty$ and $B_n \searrow \emptyset$. Thus $\mu(B_n) \searrow 0$. Adding $v1_{A_n} \leq u_n + \epsilon 1_{A_n}$ and $1_{B_n}v \leq \beta 1_{B_n}$ we get

$$v \leq \epsilon 1_{A_n} + \beta 1_{B_n} + u_n$$

Hence

$$\int v d\mu \leq \epsilon \mu(A_n) + \beta \mu(B_n) + \int u_n d\mu.$$

After passing to the limit we get

$$\int v d\mu \leq \epsilon \mu(A) + \sup_{n \in \mathbb{N}} \int u_n d\mu.$$

ϵ can be taken arbitrarily close to zero, therefore,

$$\int v d\mu \leq \sup_{n \in \mathbb{N}} \int u_n d\mu.$$

□

Lemma 2.7 Let $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ be increasing sequences from $\mathcal{E}_+(X)$. Then

$$\sup_{n \in \mathbb{N}} u_n = \sup_{n \in \mathbb{N}} v_n \Rightarrow \sup_{n \in \mathbb{N}} \int u_n d\mu = \sup_{n \in \mathbb{N}} \int v_n d\mu.$$

Proof. For any $m = 1, 2, \dots$ we have $v_m \leq \sup u_n$. Therefore,

$$\int v_m d\mu \leq \sup_{n \in \mathbb{N}} \int u_n d\mu.$$

Thus

$$\sup_{m \in \mathbb{N}} \int v_m d\mu \leq \sup_{n \in \mathbb{N}} \int u_n d\mu.$$

□

2.6 Integral on positive measurable functions I

For $f \in \mathcal{M}_+(X)$ we define

$$\int f d\mu := \sup \left\{ \int u d\mu : u \in \mathcal{E}_+(X), u \leq f \right\}.$$

Theorem 2.8 The function

$$\mathcal{M}_+(X) \ni f \mapsto \int f d\mu \in [0, \infty]$$

satisfies

- (1) If $u_n \in \mathcal{E}_+(X)$ is an increasing sequence such that $f = \sup u_n$ (which always exists), then $\int u_n d\mu \rightarrow \int f d\mu$.
- (2) $\int 1_A d\mu = \mu(A)$;
- (3) $\int \lambda f d\mu = \lambda \int f d\mu$, $f \in \mathcal{M}_+(X)$, $\lambda \in [0, \infty[$;
- (4) $\int (f + g) d\mu = \int f d\mu + \int g d\mu$;
- (5) on $\mathcal{E}_+(X)$ it coincides with the previously defined integral.
- (6) if $f, g \in \mathcal{M}_+(X)$, $f \leq g$, then $\int f d\mu \leq \int g d\mu$.

Theorem 2.9 (Beppo Levi) Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence from $\mathcal{M}_+(X)$. Then $\sup_{n \in \mathbb{N}} f_n \in \mathcal{M}_+(X)$ and

$$\int \sup_{n \in \mathbb{N}} f_n d\mu = \sup_{n \in \mathbb{N}} \int f_n d\mu.$$

Proof. Set $f := \sup_{n \in \mathbb{N}} f_n$. Using $f_n \leq f$, we see that

$$\int f_n d\mu \leq \int f d\mu.$$

Hence

$$\sup \int f_n d\mu \leq \int f d\mu.$$

Let us prove the converse inequality.

We can find $u_{mn} \in \mathcal{E}_+(X)$ such that the sequences $(u_{mn})_{m \in \mathbb{N}}$ are increasing and $\sup_{m \in \mathbb{N}} u_{mn} = f_n$. Set

$$v_m := \sup\{u_{m1}, \dots, u_{mm}\} = \sup\{u_{ij} : i, j \leq m\}$$

Then $v_m \in \mathcal{E}_+(X)$, $(v_m)_{m \in \mathbb{N}}$ is increasing and $\sup v_m = f$. Hence

$$\int f d\mu = \sup \int v_n d\mu.$$

Using $v_n \leq f_n$, we obtain

$$\int v_n d\mu \leq \int f_n d\mu.$$

Hence

$$\int f d\mu \leq \sup_n \int f_n d\mu.$$

□

2.7 Integral on positive measurable functions II

Theorem 2.10 (The Fatou lemma) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}_+(X)$. Then

$$\int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

Proof. Set $f := \liminf_{n \rightarrow \infty} f_n$, $g_n := \inf_{m \geq n} f_m$. We have $f, g_n \in \mathcal{M}_+(X)$ and $g_n \nearrow f$. Hence

$$\int f d\mu = \sup_{n \in \mathbb{N}} \int g_n d\mu.$$

$g_n \leq f_m$ for $m \geq n$, therefore

$$\int g_n d\mu \leq \inf_{m \geq n} \int f_m d\mu.$$

□

Proposition 2.11 For any $f \in \mathcal{M}_+(X)$

$$\int f d\mu = 0 \Leftrightarrow f = 0 \text{ } \mu\text{-almost everywhere.}$$

Proof. Set

$$M := \{f \neq 0\}.$$

\Rightarrow Let

$$M_n := \{f \geq n^{-1}\}.$$

Then $M_n \in \mathcal{F}$ and

$$n^{-1} \mu(M_n) \leq \int f d\mu = 0.$$

Thus $\mu(M_n) = 0$. But $M_n \nearrow M$, so $\mu(M_n) \nearrow \mu(M)$. Hence $\mu(M) = 0$.

\Leftarrow Set

$$f_n := \inf\{f, n\}$$

Then $f_n \in \mathcal{M}_+(\mathcal{F})$ and $f_n \nearrow f$. So $\int f_n d\mu \nearrow \int f d\mu$. But

$$f_n \leq n 1_M,$$

therefore

$$\int f_n d\mu \leq n \mu(M) = 0.$$

Hence $\int f d\mu = 0$. □

Theorem 2.12 Let $f \in \mathcal{M}_+(X)$ and

$$\int f d\mu < \infty.$$

Then $f < \infty$ μ -a.e. and $\{f \neq 0\} \in \mathcal{F}_\mu^{\sigma f}$.

Proof. Let $A := \{f = \infty\}$. Then $0 \leq \infty 1_A \leq f$. Hence $\infty \mu(A) \leq \int f d\mu$. □

2.8 Integral of functions with a varying sign

For $f : X \rightarrow [-\infty, \infty]$ we set

$$f_+ := \sup(f, 0), \quad f_- := -\inf(f, 0).$$

Thus

$$f = f_+ - f_-, \quad |f| = f_+ + f_-.$$

Clearly, $f \in \mathcal{M}(X)$ iff $f_+, f_- \in \mathcal{M}_+(X)$.

Definition 2.13 *Let $f \in \mathcal{M}(X)$. Assume that one of the numbers $\int f_+ d\mu, \int f_- d\mu$ is finite. Then we say that the integral of f is well defined and*

$$\int f d\mu := \int f_+ d\mu - \int f_- d\mu.$$

Theorem 2.14 *Let $f, g \in \mathcal{M}_+(X)$ and $f = g$ μ -a.e. Then*

$$\int f d\mu = \int g d\mu.$$

2.9 Transport of a measure—the change of variables in an integral

If $(X, \mathcal{F}), (X', \mathcal{F}')$ are spaces with σ -rings, μ is a measure on (X, \mathcal{F}) and $T : X \rightarrow X'$ is a measurable transformation, then $T_*\mu$ is the measure on (X', \mathcal{F}') defined as

$$T_*\mu(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{F}'.$$

Clearly, we then have the formula for $f' \in \mathcal{M}_+(X')$:

$$\int f' dT_*\mu = \int f' \circ T d\mu.$$

If T is injective, and μ' is a measure on (X', \mathcal{F}') , then we define the measure $T^*\mu'$ on (X, \mathcal{F}) by

$$T^*\mu'(A) := \mu'(T(A)), \quad A \in \mathcal{F}.$$

and for $f \in \mathcal{M}_+(X)$:

$$\int f d\mu' = \int f \circ T dT^*\mu'.$$

2.10 Integrability

Definition 2.15 *Let $f \in \mathcal{M}(X)$. If*

$$\int f_+ d\mu < \infty, \quad \int f_- d\mu < \infty,$$

or equivalently, if $\int |f| d\mu < \infty$, then we say that $f \in \mathcal{L}^1$, or integrable (in the sense of \mathcal{L}^1) and we write

$$f \in \mathcal{L}^1(\mu), \quad \int f d\mu := \int f_+ d\mu - \int f_- d\mu.$$

Proposition 2.16 (1) *If $f \in \mathcal{L}^1(\mu), g \in \mathcal{M}(X)$ and $f = g$ μ -a.e., then $g \in \mathcal{L}^1(\mu)$.*

(2) *$f \in \mathcal{L}^1(\mu)$, then $|f| < \infty$ μ -a.e. and $\{f \neq 0\} \in \mathcal{F}_\mu^{\sigma f}$.*

(3) If $f \in \mathcal{M}(X)$, $g \in \mathcal{L}^1(\mu)$ and $|f| \leq g$ μ -a.e., then $f \in \mathcal{L}^1(\mu)$.

Lemma 2.17 If $u, v \in \mathcal{L}^1(\mu)$, $u, v \geq 0$ and

$$f = u - v,$$

then $f \in \mathcal{L}^1(\mu)$.

Proof. We have

$$f \leq u \leq u + v, \quad -f \leq v \leq u + v.$$

Therefore,

$$|f| \leq u + v \in \mathcal{L}^1(\mu).$$

□

Proposition 2.18 Let $f, g \in \mathcal{L}^1(\mu)$, $\alpha \in \mathbb{R}$. Then

- (1) $\alpha f \in \mathcal{L}^1(\mu)$;
- (2) $f + g \in \mathcal{L}^1(\mu)$;
- (3) $\sup(f, g), \inf(f, g) \in \mathcal{L}^1(\mu)$;
- (4) $f \leq g \Rightarrow \int f d\mu \leq \int g d\mu$
- (5) $|\int f d\mu| \leq \int |f| d\mu$.

Proof. (1) We have,

$$(\alpha f)_+ = \alpha f_+, \quad (\alpha f)_- = \alpha f_-, \quad \alpha > 0$$

$$(\alpha f)_+ = |\alpha| f_-, \quad (\alpha f)_- = |\alpha| f_+, \quad \alpha < 0.$$

Hence $\alpha f \in \mathcal{L}^1(\mu)$.

(2) Next we write

$$f + g = f_+ - f_- + g_+ - g_-, \tag{2.2}$$

put

$$u := f_+ + g_+ \in \mathcal{L}^1(\mu), \quad v := f_- + g_- \in \mathcal{L}^1(\mu),$$

and use Lemma 2.17, which shows that (2.2) belongs to $\mathcal{L}^1(\mu)$.

(3) The estimates

$$|\sup(f, g)| \leq |f| + |g|, \quad |\inf(f, g)| \leq |f| + |g|$$

and $|f| + |g| \in \mathcal{L}^1(\mu)$ show that $\sup(f, g), \inf(f, g) \in \mathcal{L}^1(\mu)$.

(4) $f \leq g$ implies that $f_+ \leq g_+$ and $f_- \geq g_-$. Hence

$$\int f d\mu \leq \int g d\mu. \tag{2.3}$$

(5) We have $f \leq |f|$ and $-f \leq |f|$. Therefore, if we put in (2.3) $g = |f|$ we get

$$|\int f d\mu| \leq \int |f| d\mu.$$

□

2.11 The Hölder and Minkowski inequalities

Let $1 \leq p \leq \infty$.

Definition 2.19 Let $f \in \mathcal{M}(X)$. Put

$$\|f\|_p := (\int |f|^p d\mu)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty := \inf\{\sup\{|f(x)| : x \in X \setminus N\} : N \in \mathcal{F}_\mu^0\}.$$

We define $\mathcal{L}^p(X, \mu)$ as the space of $f \in \mathcal{M}(X)$ such that $\|f\|_p < \infty$.

Theorem 2.20 (The Hölder inequality) Let $1 \leq p, q \leq \infty$,

$$\int |fg| d\mu \leq \|f\|_p \|g\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Assume first that $1 < p, q < \infty$. By the convexity of e^x ,

$$\frac{a}{p} + \frac{b}{q} \geq a^{\frac{1}{p}} b^{\frac{1}{q}}.$$

We substitute

$$a = \frac{|f|^p(x)}{\|f\|_p^p}, \quad b = \frac{|g|^q(x)}{\|f\|_q^q}.$$

We get

$$\frac{1}{p} \frac{|f|^p(x)}{\|f\|_p^p} + \frac{1}{q} \frac{|g|^q(x)}{\|f\|_q^q} \geq \frac{|f|(x)|g|(x)}{\|f\|_p \|g\|_q}.$$

We integrate

$$1 \geq \frac{\int |f||g| d\mu}{\|f\|_p \|g\|_q}.$$

The case $p = 1, q = \infty$ is straightforward. \square

Theorem 2.21 Let $1 \leq r \leq \infty, 0 \leq \alpha \leq 1, \frac{\alpha}{q} + \frac{1-\alpha}{r} = \frac{1}{p}$. Then

$$\|f\|_p \leq \|f\|_q^\alpha \|f\|_r^{1-\alpha} \leq \alpha \|f\|_q + (1-\alpha) \|f\|_r.$$

Proof.

$$\int |f|^p = \int |f|^{p\alpha} |f|^{p(1-\alpha)} \leq \| |f|^{p\alpha} \|_{\frac{q}{\alpha}} \| |f|^{p(1-\alpha)} \|_{\frac{r}{1-\alpha}}.$$

Theorem 2.22 (The generalized Minkowski inequality) Let X, Y be spaces with measures μ and $\nu, 1 \leq p < \infty$.

$$\left(\int d\nu(y) \left| \int f(x, y) d\mu(x) \right|^p \right)^{\frac{1}{p}} \leq \int d\mu(x) \left(\int |f|^p(x, y) d\nu(y) \right)^{\frac{1}{p}}$$

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$. It suffices to assume that $f \geq 0$. We will restrict ourselves to the case $p > 1$.

$$\begin{aligned}
& \int dy \left(\int f(x, y) dx \right)^p \\
&= \int dy \left(\int f(x_1, y) dx_1 \right)^{p-1} \left(\int f(x_2, y) dx_2 \right) \\
&= \int dx_2 \left(\int dy \left(\int f(x_1, y) dx_1 \right)^{p-1} f(x_2, y) \right) \\
&\leq \int dx_2 \left(\int dy_1 \left(\int f(x_1, y_1) dx_1 \right)^{q(p-1)} \right)^{\frac{1}{q}} \left(\int f^p(x_2, y_2) dy_2 \right)^{\frac{1}{p}} \quad (\text{the Hölder inequality}) \\
&= \left(\int dy_1 \left(\int f(x_1, y_1) dx_1 \right)^p \right)^{1-\frac{1}{p}} \left(\int dx_2 \left(\int f^p(x_2, y_2) dy_2 \right)^{\frac{1}{p}} \right).
\end{aligned}$$

Then we divide by the first factor on the left. \square

Corollary 2.23 *Setting $X = \{1, 2\}$ with the counting measure we get*

$$\|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p.$$

2.12 Dominated Convergence Theorem

Theorem 2.24 (Lebesgue) *Assume that $1 \leq p < \infty$, $g, f_n \in \mathcal{L}^p(\mu)$, f_n is μ -a.e. pointwise convergent and*

$$|f_n| \leq g.$$

Then there exists $f \in \mathcal{L}^p(\mu)$ such that $f_n \rightarrow f$ μ -a.e. and

$$\|f - f_n\|_p \rightarrow 0.$$

Proof. We define

$$f(x) := \begin{cases} \lim_{n \rightarrow \infty} f_n(x) & \text{if } \lim_{n \rightarrow \infty} f_n(x) \text{ exists} \\ 0 & \text{if } \lim_{n \rightarrow \infty} f_n(x) \text{ does not exist} \end{cases}$$

Then $f \in \mathcal{M}(X)$ and

$$|f| \leq g \quad \mu\text{-a.e.}$$

hence $f \in \mathcal{L}^p(\mu)$.

Set

$$h_n := |f - f_n|^p.$$

Then

$$0 \leq h_n \leq (|f_n| + |f|)^p \leq |2g|^p =: h.$$

Clearly, h and therefore also h_n are integrable. Besides, μ -a.e.

$$h = \lim_{n \rightarrow \infty} (h - h_n)$$

Therefore, by the Fatou Lemma applied to the sequence $h - h_n$ we get

$$\begin{aligned}
\int h d\mu &= \int \lim_{n \rightarrow \infty} (h - h_n) d\mu \\
&\leq \liminf_{n \rightarrow \infty} \int (h - h_n) d\mu \\
&= \int h d\mu - \limsup_{n \rightarrow \infty} \int h_n d\mu.
\end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} \int h_n d\mu \leq 0$$

Using $h_n \geq 0$ we get

$$\lim_{n \rightarrow \infty} \int h_n d\mu = 0.$$

□

Theorem 2.25 (Scheffe's lemma) *Let $f, f_1, f_2, \dots \in \mathcal{L}^1(\mu)$ and $f_n \rightarrow f$ a.e. Then*

$$\int |f_n - f| d\mu \rightarrow 0 \Leftrightarrow \int |f_n| d\mu \rightarrow \int |f| d\mu.$$

2.13 L^p spaces

Theorem 2.26 (Riesz-Fischer) *Let $1 \leq p \leq \infty$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{L}^p(\mu)$ satisfying the Cauchy condition, that is for any $\epsilon > 0$ there exists N such that for $n, m \geq N$*

$$\|f_n - f_m\|_p \leq \epsilon.$$

Then there exists $f \in \mathcal{L}^p(\mu)$ such that

$$\|f - f_n\|_p \rightarrow 0$$

We will also find a subsequence of $(f_n)_{n \in \mathbb{N}}$ pointwise convergent μ -a.e. to f .

Proof. There exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}$, for any k . We set

$$g_k := f_{n_{k+1}} - f_{n_k}, \quad g := \sum_{k=1}^{\infty} |g_k|.$$

Then

$$\|g\|_p \leq \sum_{k=1}^{\infty} \|g_k\|_p \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Hence $g \in \mathcal{L}^p(\mu)$ and therefore g is finite μ -a.e. Hence the series $\sum_{k=1}^{\infty} g_k$ is μ -a.e. convergent. This means that the sequence $(f_{n_k})_{k \in \mathbb{N}}$ is μ -a.e. convergent.

In the case $p = \infty$ it is sufficient to take the limit and to check that it is the limit in the \mathcal{L}_{∞} sense.

In the case $1 \leq p < \infty$, we need to apply the Lebesgue theorem. We first check that

$$|f_{n_k}| \leq |f_{n_1} + g_1 + \dots + g_{k-1}| \leq |f_{n_1}| + g$$

and $g + |f_{n_1}| \in \mathcal{L}^p(\mu)$. Therefore, we will find $f \in \mathcal{L}^p(\mu)$ such that

$$\lim_{k \rightarrow \infty} \|f_{n_k} - f\|_p = 0,$$

$$\lim_{k \rightarrow \infty} f_{n_k} = f \quad \mu\text{-a.e.}$$

Using the Cauchy condition we get

$$\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0.$$

□

Let $\mathcal{N}(\mu)$ be the space of functions in $\mathcal{M}(X)$ equal 0 μ -a.e. (That is, $\mathcal{N}(\mu) = \mathcal{M}(X, \mathcal{F}_{\mu}^0)$).

Theorem 2.27 Let $1 \leq p \leq \infty$

(1) $\mathcal{N}(\mu)$ is a vector subspace of $\mathcal{L}^p(\mu)$ such that $\|f\|_p = 0$, $f \in \mathcal{L}^p(\mu)$ iff $f \in \mathcal{N}(\mu)$.

(2) Therefore,

$$\|f + \mathcal{N}(\mu)\|_p := \|f\|_p$$

defines a norm in

$$L^p(\mu) := \mathcal{L}^p(\mu)/\mathcal{N}(\mu).$$

(3) $L^p(\mu)$ is a Banach space.

(4) Elementary functions are dense in $L^p(\mu)$.

(5) $f, g \in L^p(\mu)$, $0 \leq f \leq g$ a.e. $\Rightarrow \|f\|_p \leq \|g\|_p$.

(6) For $1 \leq p < \infty$, if we restrict the measure to $\mu - \sigma$ -finite sets, we obtain the same $L^p(\mu)$ space.

(7) For $1 \leq p < \infty$, $L^p(\mu)$ is a complete lattice. $L^\infty(\mu)$ is a countably complete lattice. (Later on, we will show that under some additional conditions it is also a complete lattice).

Proof. (4) Let $f \in \mathcal{L}^p(\mu)$. Then $f_+, f_- \in \mathcal{L}^p(\mu)$. We know that there exist sequences $u_{\pm, n} \in \mathcal{E}_+(\mu)$ with $u_{\pm, n} \nearrow f_{\pm}$. By the Lebesgue theorem, $\|f_{\pm} - u_{\pm, n}\|_p \rightarrow 0$. \square

2.14 Egorov theorem

Theorem 2.28 Let $f, f_1, f_2, \dots \in \mathcal{M}(X)$. Consider the following statements:

(1) $f_n(x) \rightarrow f(x)$ for a.a. $x \in X$.

(2) For all $\epsilon > 0$

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} |f_j - f| \geq \epsilon\right) = 0.$$

(3) For all $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \mu\left(\bigcup_{j=n}^{\infty} |f_j - f| \geq \epsilon\right) = 0.$$

(4) For every $\delta > 0$, there exists $A \in \mathcal{F}$ with $\mu(A) < \delta$ and

$$\lim_{n \rightarrow \infty} \sup_{x \in X \setminus A} |f - f_n| = 0.$$

Then (1) \Leftrightarrow (2) \Leftarrow (3) \Leftarrow (4). If $\mu(X) < \infty$, then (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4). (The implication (1) \Rightarrow (4) is called the **Egorov theorem**).

Proof. Only (3) \Rightarrow (4) is not immediate, and we are going to prove this implication. Let $\delta > 0$ and $k \in \mathbb{N}$. Then by (3) there exists n_k such that for

$$B_k := \bigcup_{j=n_k}^{\infty} \{|f - f_j| > 1/k\},$$

we have $\mu(B_k) < \delta 2^{-k}$. Set $A := \bigcup_{k=1}^{\infty} B_k$. We have $\mu(A) < \delta$ and on $X \setminus A$, $|f(x) - f_j(x)| \leq 1/k$ for $j \geq n_k$. Hence on $X \setminus A$, f_n converges uniformly to f . \square

3 Extension of a measure

3.1 Hereditary families

$\mathcal{T} \subset 2^X$. We say that \mathcal{T} is hereditary if $A \subset B \in \mathcal{T}$ implies $A \in \mathcal{T}$. For any \mathcal{T} we denote by $\text{Her}(\mathcal{T})$ the smallest hereditary family containing \mathcal{T} .

Theorem 3.1 (1) Let \mathcal{R} be a ring. Then $\text{Her}(\mathcal{R})$ is a ring.

(2) Let $\mathcal{T} \subset 2^X$. Then $Q \in \text{Her}(\sigma\text{-Ring}(\mathcal{T}))$ iff there exist $A_1, \dots, A_n \in \mathcal{T}$ such that $Q \subset A_1 \cup \dots \cup A_n$.

Theorem 3.2 (1) If \mathcal{I} is a σ -ring, then $\text{Her}(\mathcal{I})$ is a σ -ring.

(2) If $\mathcal{T} \subset 2^X$, then $Q \in \text{Her}(\mathcal{T})$ iff there exist $A_1, A_2, \dots \in \mathcal{T}$ such that $Q \subset \bigcup_{i=1}^{\infty} A_i$.

Theorem 3.3 If $\mathcal{I} \subset \mathcal{F}$ are σ -rings and \mathcal{I} is an ideal in \mathcal{F} , then

$$\begin{aligned} \sigma\text{-Ring}(\mathcal{F} \cup \text{Her}(\mathcal{I})) &= \{A \cup N : A \in \mathcal{F}_1, N \in \text{Her}(\mathcal{I})\} \\ &= \{A \setminus N : A \in \mathcal{F}, N \in \text{Her}(\mathcal{I})\}. \end{aligned}$$

3.2 Extension of a measure by null sets

Let X be a set and $(\mathcal{F}_2, \mu_2), (\mathcal{F}_1, \mu_1)$ be measures on X . We say that (\mathcal{F}_2, μ_2) extends (\mathcal{F}_1, μ_1) by null sets iff

(1) $\sigma\text{-Ring}(\mathcal{F}_1 \cup (\mathcal{F}_2)_{\mu_2}^0) = \mathcal{F}_2$;

(2) $\mu_2 \Big|_{\mathcal{F}_1} = \mu_1$.

Theorem 3.4 Suppose that (\mathcal{F}_2, μ_2) extends (\mathcal{F}_1, μ_1) by null sets. Then

(1) $\mathcal{F}_2 = \{A \cup N : A \in \mathcal{F}_1, N \in (\mathcal{F}_2)_{\mu_2}^0\}$.

(2) $\mathcal{N}(\mu_2) \cap \mathcal{M}(\mathcal{F}_1) = \mathcal{N}(\mu_2)$. Hence, we can identify $\mathcal{M}(X, \mathcal{F}_1)/\mathcal{N}(\mu_1)$ with $\mathcal{M}(X, \mathcal{F}_2)/\mathcal{N}(\mu_2)$ in the obvious way.

(3) We can identify $L^p(\mu_1)$ with $L^p(\mu_2)$.

3.3 Complete measures

Let (X, \mathcal{F}, μ) be a space with a measure. We say that the measure μ is complete if \mathcal{F}_{μ}^0 is hereditary.

Let μ be a not necessarily complete measure. Set

$$\mathcal{F}_{\mu}^{\text{cp}} := \{A \cup N : A \in \mathcal{F}, N \in \text{Her}(\mathcal{F}_{\mu}^0)\} = \sigma\text{-Ring}(\mathcal{F} \cup \text{Her}(\mathcal{F}_{\mu}^0)).$$

Define $\mu^{\text{cp}} : \mathcal{F}_{\mu}^{\text{cp}} \rightarrow [0, \infty]$,

$$\mu^{\text{cp}}(A \cup N) := \mu(A), \quad A \in \mathcal{F}, \quad N \in \text{Her}(\mathcal{F}_{\mu}^0).$$

Theorem 3.5 (1) $\mathcal{F}_{\mu}^{\text{cp}}$ is a σ -ring and μ^{cp} is a complete measure.

(2) μ^{cp} is an extension of μ by null sets.

(3) μ^{cp} is the unique extension of μ to a content on \mathcal{F}^{cp} .

(4) Every extension of (\mathcal{F}, μ) to a complete measure is an extension of $(\mathcal{F}^{\text{cp}}, \mu^{\text{cp}})$.

We will call $(X, \mathcal{F}^{\text{cp}}, \mu^{\text{cp}})$ the completion of μ .

3.4 External measures

Definition 3.6 A function $\mu^* : 2^X \rightarrow [0, \infty]$ is called an external measure if

- (1) $\mu^*(\emptyset) = 0$,
- (2) $Q_1, Q_2, \dots \in 2^X \Rightarrow \mu^*(\cup_{n=1}^{\infty} Q_n) \leq \sum_{n=1}^{\infty} \mu^*(Q_n)$.
- (3) $Q \subset P, Q, P \in 2^X, \Rightarrow \mu^*(Q) \leq \mu^*(P)$.

Clearly, every measure on $(X, 2^X)$ is an external measure.

For any set X the function that assigns 0 to \emptyset and 1 to a nonempty set is an external measure on X . It is not a measure if X contains more than one element.

Definition 3.7 Let μ^* be an external measure. We say that $A \in 2^X$ is measurable wrt μ^* , if one of the following two equivalent conditions holds

$$\mu^*(Q) \geq \mu^*(Q \cap A) + \mu^*(Q \setminus A), \quad Q \in 2^X; \quad (3.4)$$

$$\mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A), \quad Q \in 2^X. \quad (3.5)$$

(The equivalence of the conditions (3.5) follows from (2) of the definition of the external measure applied to $Q \cap A$ and $Q \setminus A$.)

The family of sets measurable wrt μ^* will be denoted \mathcal{F}^{ms} . μ^* restricted to \mathcal{F}^{ms} will be denoted μ^{ms} .

Theorem 3.8 Let μ^* be an external measure on X . Let \mathcal{F}^{ms} and μ^{ms} be defined as above. Then

- (1) \mathcal{F}^{ms} is a σ -field
- (2) $(X, \mathcal{F}^{\text{ms}}, \mu^{\text{ms}})$ is a complete measure;
- (3)

$$A \in 2^X, \quad \mu^*(A) = 0 \Leftrightarrow A \in \mathcal{F}^{\text{ms}}, \quad \mu^{\text{ms}}(A) = 0.$$

Proof. Step 0. Clearly, $\emptyset, X \in \mathcal{F}^{\text{ms}}$.

Step 1. $A, B \in \mathcal{F}^{\text{ms}} \Rightarrow A \setminus B \in \mathcal{F}^{\text{ms}}$.

Let $Q \in 2^X$.

Applying the measurability condition to $Q \cap A$ and B we get

$$\mu^*(Q \cap A) = \mu^*(Q \cap A \cap B) + \mu^*(Q \cap A \setminus B). \quad (3.6)$$

(Note that $(Q \cap B) \setminus A = Q \cap (B \setminus A)$). Then we apply it to $Q \setminus A$ and B to get

$$\mu^*(Q \setminus A) = \mu^*(Q \cap B \setminus A) + \mu^*(Q \setminus (A \cup B)). \quad (3.7)$$

Thus by (3.5)

$$\mu^*(Q) = \mu^*(Q \cap A \cap B) + \mu^*(Q \cap A \setminus B) + \mu^*(Q \cap B \setminus A) + \mu^*(Q \setminus (A \cup B)). \quad (3.8)$$

Applying the measurability condition to $Q \setminus (A \setminus B)$ and B gives

$$\mu^*(Q \setminus (A \setminus B)) = \mu^*(Q \setminus (A \cup B)) + \mu^*(Q \cap B). \quad (3.9)$$

Applying it to $Q \cap B$ and A we get

$$\mu^*(Q \cap B) = \mu^*(Q \cap B \setminus A) + \mu^*(Q \cap A \cap B). \quad (3.10)$$

Inserting (3.10) into (3.9) gives

$$\mu^*(Q \setminus (A \setminus B)) = \mu^*(Q \setminus (A \cup B)) + \mu^*(Q \cap B \setminus A) + \mu^*(Q \cap A \cap B). \quad (3.11)$$

Thus, by (3.8),

$$\mu^*(Q) = \mu^*(Q \setminus (A \setminus B)) + \mu^*(Q \cap (A \setminus B)). \quad (3.12)$$

Hence $A \setminus B \in \mathcal{F}^{\text{ms}}$.

Step 2. $A, B \in \mathcal{F}^{\text{ms}} \Rightarrow B \cup A \in \mathcal{F}^{\text{ms}}$.

We have, applying the measurability condition to $Q \cap (A \cup B)$ and A .

$$\mu^*(Q \cap (A \cup B)) = \mu^*(Q \cap A) + \mu^*(Q \cap B \setminus A). \quad (3.13)$$

Inserting (3.6) into (3.13) we get

$$\mu^*(Q \cap (A \cup B)) = \mu^*(Q \cap A \cap B) + \mu^*(Q \cap A \setminus B) + \mu^*(Q \cap B \setminus A). \quad (3.14)$$

Hence

$$\mu^*(Q) = \mu^*(Q \setminus (A \cup B)) + \mu^*(Q \cap (A \cup B)). \quad (3.15)$$

Therefore $A \cup B \in \mathcal{F}^{\text{ms}}$. Thus we proved that \mathcal{F}^{ms} is a field.

Step 3.

$$A_1, A_2, \dots \in \mathcal{F}^{\text{ms}} \Rightarrow \bigcup_{j=1}^{\infty} A_j =: A \in \mathcal{F}^{\text{ms}}. \quad (3.16)$$

It suffices to assume that A_j are disjoint. For any n we have

$$\begin{aligned} \mu^*(Q) &\geq \mu^*(\bigcup_{j=1}^n (Q \cap A_j) \cup Q \setminus A) \\ &= \sum_{j=1}^n \mu^*(Q \cap A_j) + \mu^*(Q \setminus A). \end{aligned} \quad (3.17)$$

Since n was arbitrary,

$$\begin{aligned} \mu^*(Q) &\geq \sum_{j=1}^{\infty} \mu^*(Q \cap A_j) + \mu^*(Q \setminus A). \\ &\geq \mu^*(Q \cap A) + \mu^*(Q \setminus A), \end{aligned} \quad (3.18)$$

Hence, by the equivalence of (3.4) and (3.5) we get

$$\mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A),$$

which shows $A \in \mathcal{F}^{\text{ms}}$.

Step 4. As a by-product we get

$$\mu^*(Q \cap A) = \sum_{j=1}^{\infty} \mu^*(Q \cap A_j).$$

Putting $Q = A$ we see that

$$\mu^*(A) = \sum_{j=1}^{\infty} \mu^*(A_j),$$

hence μ^* restricted to \mathcal{F}^{ms} is a measure.

Step 5. Let $A \in 2^X$ and $\mu^*(A) = 0$. Let $Q \in 2^X$. Then $Q \cap A \subset A$, hence $\mu^*(Q \cap A) = 0$. Moreover, $Q \setminus A \subset Q$, hence $\mu^*(Q \setminus A) \leq \mu^*(Q)$. Therefore,

$$\mu^*(Q) \geq \mu^*(Q \cap A) + \mu^*(Q \setminus A).$$

Hence $A \in \mathcal{F}^{\text{ms}}$ and $\mu^{\text{ms}}(A) = 0$. This proves (3), which implies the completeness of the measure μ^{ms} . \square

3.5 External measure generated by a measure

Theorem 3.9 Let (X, \mathcal{F}, μ) be a measure. For $Q \in 2^X$ define

$$\mu^*(Q) := \inf \{ \mu(A) : A \in \mathcal{F}, A \supset Q \}$$

Then

- (1) μ^* is an external measure;
- (2) $\mu^* = \mu$ on \mathcal{F} .
- (3) Let $\mathcal{F}_\mu^{\text{ms}}, \mu^{\text{ms}}$ be defined from μ^* as in the previous subsection. Then $\mathcal{F}_\mu^{\text{ms}}$ is a σ -field containing \mathcal{F} .
- (4) In the definition of μ^* we can replace \mathcal{F} with $\mathcal{F}_\mu^{\text{f}}, \mathcal{F}_\mu^{\text{sf}}, \mathcal{F}_\mu^{\text{cp}}, \mathcal{F}_\mu^{\text{ms}}$, etc., obtaining the same μ^* .

Proof. (1) The properties (1) and (3) of the definition of an external measure are obvious. Let us show the property (2).

Let $Q_1, Q_2, \dots \in 2^X$. For any $\epsilon > 0$ we will find a sequence $A_1, A_2, \dots \in \mathcal{F}$ such that

$$\begin{aligned} Q_n &\subset A_n, \\ \mu(A_n) &\leq \mu^*(Q_n) + 2^{-n}\epsilon. \end{aligned}$$

Then

$$\mu^*(\cup_{n=1}^{\infty} Q_n) \leq \mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n) \leq \sum_{n=1}^{\infty} \mu^*(Q_n) + \epsilon.$$

Hence

$$\mu^*(\cup_{n=1}^{\infty} Q_n) \leq \sum_{n=1}^{\infty} \mu^*(Q_n).$$

(2) is obvious.

Let us prove (3). Let $B \in \mathcal{F}, Q \in 2^X$. For any $\epsilon > 0$ and suitable $A \in \mathcal{F}$ such that $Q \subset A$ we have

$$\begin{aligned} \mu^*(Q) &\geq \mu(A) - \epsilon \\ &= \mu(A \cap B) + \mu(A \setminus B) - \epsilon \geq \mu^*(Q \cap B) + \mu^*(Q \setminus B) - \epsilon. \end{aligned}$$

Therefore,

$$\mu^*(Q) \geq \mu^*(Q \cap B) + \mu^*(Q \setminus B).$$

Thus, $B \in \mathcal{F}_\mu^{\text{ms}}$. \square

The measure $(X, \mathcal{F}^{\text{ms}}, \mu^{\text{ms}})$ is called the Caratheodory completion of the measure (X, \mathcal{F}, μ) .

A measure (X, \mathcal{F}, μ) is called Caratheodory complete if it coincides with its Caratheodory completion.

The Caratheodory completion of a measure is always Caratheodory complete.

Theorem 3.10 Suppose that (X, \mathcal{F}, μ) be a set with a σ -field and a finite measure. Let μ^* be the corresponding outer measure and let $S \subset X$ satisfy $\mu^*(S) = \mu(X)$. For $A \in \mathcal{F}|_S$ set

$$\mu_S(A) := \mu^*(A).$$

Then $(S, \mathcal{F}|_S, \mu_S)$ is a measure, which is isomorphic to μ modulo sets of measure zero,

3.6 Extension of a measure to localizable sets

Let $\mathcal{F} \subset 2^X$ be a σ -ring. Let

$$\mathcal{F}^{\text{loc}} := \{A \in 2^X : B \in \mathcal{F} \text{ implies } A \cap B \in \mathcal{F}\}.$$

We say that \mathcal{F}^{loc} is the family of sets localizable in \mathcal{F} . It is a σ -field and \mathcal{F} is its ideal.

Let (X, \mathcal{F}, μ) be a measure. We can extend canonically μ to \mathcal{F}^{loc} by setting

$$\mu^{\text{loc}}(A) := \begin{cases} \mu(A) & A \in \mathcal{F} \\ \infty & A \in \mathcal{F}^{\text{loc}} \setminus \mathcal{F}. \end{cases}$$

Note that $(\mathcal{F}_{\text{loc}})_{\mu_{\text{loc}}}^{\sigma\text{f}} = \mathcal{F}_{\mu}^{\sigma\text{f}}$, so the L^p spaces for both measures are the same.

Theorem 3.11 *Suppose (X, \mathcal{F}, μ) and $(X, \mathcal{F}_{\mu}^{\text{ms}}, \mu^{\text{ms}})$ are as in Theorem 3.9. Then $(\mathcal{F}_{\mu}^{\text{ms}}, \mu^{\text{ms}})$ can be obtained by applying consecutively to (\mathcal{F}, μ) the following constructions:*

1. restricting to σ -finite subsets,
2. completion
3. extending to localizable sets.

3.7 Sum-finite measures

Let (X, \mathcal{F}, μ) be a measure. Define the set of locally μ -measurable sets by

$$\begin{aligned} \mathcal{F}_{\mu}^{\text{loc}} &:= \{A \in 2^X : A \cap B \in \mathcal{F}, B \in \mathcal{F}_{\mu}^{\sigma\text{f}}\} \\ &= \{A \in 2^X : A \cap B \in \mathcal{F}, B \in \mathcal{F}_{\mu}^{\text{f}}\}. \end{aligned}$$

In other words, $\mathcal{F}_{\mu}^{\text{loc}}$ is the family of all sets localizable in $\mathcal{F}_{\mu}^{\sigma\text{f}}$ (or in $\mathcal{F}_{\mu}^{\text{f}}$). Clearly, $\mathcal{F}_{\mu}^{\text{loc}}$ is a σ -field containing \mathcal{F} as an ideal.

A family $\{X_i : i \in I\}$ of disjoint elements of $\mathcal{F}_{\mu}^{\text{f}}$ such that $\bigcup_{i \in I} X_i = X$ and

$$\mu(A) = \sum_{i \in I} \mu(A \cap X_i), \quad A \in \mathcal{F},$$

is called a localizing family for μ .

We say that the measure (X, \mathcal{F}, μ) is sum-finite if

- (1) $\mathcal{F} = \mathcal{F}_{\mu}^{\text{loc}}$;
- (2) There there exists a localizing family for μ .

Clearly, every σ -finite measure is sum-finite.

Theorem 3.12 *If $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ are localizing families for μ , then so is $\{X_i \cap Y_j : (i, j) \in I \times J\}$*

Theorem 3.13 *If (X, \mathcal{F}, μ) is a measure possessing a localizing family $\{X_i : i \in I\}$, then if for $A \in \mathcal{F}_{\mu}^{\text{loc}}$ we set*

$$\mu^{\text{loc}}(A) = \sum_{i \in I} \mu(A \cap X_i),$$

then $(X, \mathcal{F}_{\mu}^{\text{loc}}, \mu^{\text{loc}})$ is a sum-finite measure.

3.8 Boolean rings

We say that $(\mathcal{R}, \Delta, \emptyset, \cap)$ is a Boolean ring if it is an additive ring where all its elements are idempotent, that is $A \cap A = A$, $A \in \mathcal{R}$. We then set

$$A \cup B := (A \Delta B) \Delta (A \cap B), \quad A \setminus B := A \Delta (A \cap B),$$

$$A \subset B \Leftrightarrow B \supset A \Leftrightarrow A = A \cap B.$$

If there exists an identity element for \cap , called X , then $(\mathcal{R}, \Delta, \emptyset, \cap, X)$ is called a Boolean field.

Clearly, every ring/field in 2^X is a Boolean ring/field.

In the obvious way we introduce the notion of Boolean σ -rings, Boolean σ -fields, etc. In what follows we concentrate on σ -rings/fields.

If $\mathcal{I} \subset \mathcal{F}$ are σ -rings and \mathcal{I} is an ideal in \mathcal{F} , then \mathcal{F}/\mathcal{I} is a Boolean σ -ring.

Theorem 3.14 *Let $\mathcal{F} \subset \mathcal{F}_1$ be σ -rings. Let $\mathcal{I}_1 \subset \mathcal{F}_1$ be a σ -ring, which is an ideal in \mathcal{F}_1 . Let $\mathcal{I} := \mathcal{I}_1 \cap \mathcal{F}$, which is clearly a σ -ring and an ideal in \mathcal{F} . Then the σ -rings \mathcal{F}/\mathcal{I} and $\mathcal{F}_1/\mathcal{I}_1$ are in the obvious way isomorphic to one another.*

If \mathcal{F} is a Boolean σ -ring, we can define the space $\mathcal{M}(\mathcal{F})$ as the set of all function

$$[-\infty, 0[\cup]0, \infty] \ni \alpha \mapsto F(\alpha) \in \mathcal{F}$$

such that for $\alpha \neq \beta$, $F(\alpha) \cap F(\beta) = \emptyset$. If $\mathcal{F} \subset 2^X$, then we identify $f \in \mathcal{M}(X, \mathcal{F})$ with $F \in \mathcal{M}(\mathcal{F})$ where $F(\alpha) = f^{-1}(\{\alpha\})$.

If \mathcal{F} is a Boolean σ -ring and $F \in \mathcal{M}(\mathcal{F})$, we set $\|F\|_\infty := \sup\{|\alpha| : F(\alpha) \neq \emptyset\}$ and define the space $L^\infty(\mathcal{F})$. If $\mathcal{I} \subset \mathcal{F} \subset 2^X$ are σ -rings and \mathcal{I} is an ideal in \mathcal{F} , then we can identify $L^\infty(X, \mathcal{F}, \mathcal{I})$ with $L^\infty(\mathcal{F}/\mathcal{I})$.

3.9 Measures on Boolean rings

We can consider measures on Boolean σ -rings as well. Clearly, one can define L^p spaces for measures on Boolean rings.

If (\mathcal{F}, μ) is a measure on a Boolean σ -ring, we say that it is faithful if $A \in \mathcal{F}$, $\mu(A) = 0$ implies $A = \emptyset$.

If \mathcal{F}_μ^0 is the family of zero sets, then we can define $\tilde{\mathcal{F}} := \mathcal{F}/\mathcal{F}_\mu^0$ and $\tilde{\mu}(A \Delta N) := \mu(A)$ for $N \in \mathcal{F}_\mu^0$. Then $(\tilde{\mathcal{F}}, \tilde{\mu})$ is a faithful measure.

Theorem 3.15 *Let X be a set and $\mathcal{F} \subset \mathcal{F}_1$ are σ -rings over X . Let (\mathcal{F}, μ) and (\mathcal{F}_1, μ_1) be measures such that*

- (1) $\mu = \mu_1$ on \mathcal{F} ;
- (2) σ -Ring $(\mathcal{F} \cup (\mathcal{F}_1)_{\mu_1}^0) = \mathcal{F}_1$.

If $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}_1$ are faithful measures defined as above, then they are isomorphic.

4 Construction and uniqueness of a measure

4.1 Dynkin classes

We say that $\mathcal{T} \subset 2^X$ is \cap -stable if $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$.

We say that \mathcal{D} is a Dynkin class if

- (1) $A, B \in \mathcal{D}$, $A \subset B \Rightarrow B \setminus A \in \mathcal{D}$;

(2) $A_1, A_2 \in \mathcal{D}, A_1 \cap A_2 = \emptyset \Rightarrow A_1 \cup A_2 \in \mathcal{D}$

Theorem 4.1 *Let $\mathcal{R} \subset 2^X$ satisfy*

(1) $A, B \in \mathcal{R}, A \subset B \Rightarrow B \setminus A \in \mathcal{R};$

(2) $A_1, A_2 \in \mathcal{R}, A_1 \cap A_2 = \emptyset \Rightarrow A_1 \cup A_2 \in \mathcal{R}$

(3) $A, B \in \mathcal{R}, \Rightarrow B \cap A \in \mathcal{R}.$

Then \mathcal{R} is a ring. In other words, a \cap -stable Dynkin class is a ring.

For $\mathcal{A} \subset 2^X$ let $\text{Dyn}(\mathcal{A})$ denote the smallest Dynkin class containing \mathcal{A} .

Theorem 4.2 *Let \mathcal{C} be a \cap -stable family. Then $\text{Dyn}(\mathcal{C}) = \text{Ring}(\mathcal{C})$.*

Proof. By Theorem 4.1 every ring is a Dynkin class. Hence

$$\text{Dyn}(\mathcal{C}) \subset \text{Ring}(\mathcal{C}).$$

Let us prove the converse inclusion. For $B \in 2^X$. Set

$$\mathbf{K}(B) := \{A \in 2^X : A \cap B \in \text{Dyn}(\mathcal{C})\}.$$

Note that

$$A \in \mathbf{K}(B) \Leftrightarrow B \in \mathbf{K}(A). \quad (4.19)$$

Using the fact that $\text{Dyn}(\mathcal{C})$ is a Dynkin class we check that $\mathbf{K}(B)$ is a Dynkin class.

Using the fact that \mathcal{C} is \cap -stable we see that

$$B \in \mathcal{C} \Rightarrow \mathcal{C} \subset \mathbf{K}(B) \quad (4.20)$$

Hence,

$$B \in \mathcal{C} \Rightarrow \text{Dyn}(\mathcal{C}) \subset \mathbf{K}(B) \quad (4.21)$$

From (4.19) and (4.21) we get

$$A \in \text{Dyn}(\mathcal{C}) \Rightarrow \mathcal{C} \subset \mathbf{K}(A) \quad (4.22)$$

Hence

$$A \in \text{Dyn}(\mathcal{C}) \Rightarrow \text{Dyn}(\mathcal{C}) \subset \mathbf{K}(A) \quad (4.23)$$

Therefore, $\text{Dyn}(\mathcal{C})$ is \cap -stable. Hence, by Theorem 4.1 it is a ring. \square

4.2 Semirings

Definition 4.3 $\mathcal{T} \subset 2^X$ is called a semiring if

(1) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T};$

(2) $A, B \in \mathcal{T} \Rightarrow A \setminus B = \bigcup_{i=1}^n C_i$, where C_i are disjoint elements of \mathcal{T}

Theorem 4.4 *Let \mathcal{T} be a semiring. Then $A \in \text{Ring}(\mathcal{T})$ iff A is a disjoint union of elements of \mathcal{T} .*

Proof. Let \mathcal{R} be the family of finite unions of disjoint elements of \mathcal{T} . It is obvious that $\mathcal{R} \subset \text{Ring}(\mathcal{T})$.

Let us prove the converse inclusion. To this end it is enough to prove that \mathcal{R} is a ring.

Step 1. Let $A \in \mathcal{R}$, $B \in \mathcal{T}$. Then $A = \bigcup_{i=1}^n A_i$ with disjoint $A_i \in \mathcal{T}$. Now

$$A \setminus B = \bigcup_{i=1}^n (A_i \setminus B),$$

where $A_i \setminus B \in \mathcal{T}$ are disjoint and each $A_i \setminus B$ is a finite union of disjoint elements of \mathcal{T} . Hence $A \setminus B \in \mathcal{R}$.

Step 2. Let $A \in \mathcal{R}$, $B \in \mathcal{R}$. Then $B = \bigcup_{i=1}^n B_i$ with disjoint $B_i \in \mathcal{T}$. Now

$$A \setminus B = (\cdots (A \setminus B_1) \cdots \setminus B_n).$$

Hence, by Step 1, $A \setminus B \in \mathcal{R}$.

Step 3. Let $A \in \mathcal{R}$, $B \in \mathcal{R}$. Then $A = \bigcup_{i=1}^n A_i$ with disjoint $A_i \in \mathcal{T}$ and $B = \bigcup_{j=1}^m B_j$ with disjoint $B_j \in \mathcal{T}$.

Now

$$A \cap B = \bigcup_{i=1}^n \bigcup_{j=1}^m A_i \cap B_j,$$

where $A_i \cap B_j \in \mathcal{T}$ are disjoint Hence $A \cap B \in \mathcal{R}$.

Step 4. Let $A \in \mathcal{R}$, $B \in \mathcal{R}$. Then

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A).$$

thus by Steps 2 and 3, the left hand side is a union of three disjoint elements of \mathcal{R} . Therefore, it is a union of a finite family of elements of \mathcal{T} . Hence, $A \cup B \in \mathcal{R}$.

Thus we proved that \mathcal{R} is a ring. \square

4.3 σ -Dynkin classes

We say that \mathcal{D} is a σ -Dynkin class if

- (1) $A, B \in \mathcal{D}$, $A \subset B \Rightarrow B \setminus A \in \mathcal{D}$;
- (2) $A_1, A_2, \dots \in \mathcal{D}$, $A_i \cap A_j = \emptyset$, $i \neq j$, $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{D}$

Theorem 4.5 Let $\mathcal{R} \subset 2^X$ satisfy

- (1) $A, B \in \mathcal{R}$, $A \subset B \Rightarrow B \setminus A \in \mathcal{R}$;
- (2) $A_1, A_2, \dots \in \mathcal{R}$, $A_i \cap A_j = \emptyset$, $i \neq j \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{R}$
- (3) $A, B \in \mathcal{R}$, $\Rightarrow B \cap A \in \mathcal{R}$.

Then \mathcal{R} is a σ -ring. In other words, a \cap -stable σ -Dynkin class is a σ -ring.

For $\mathcal{A} \subset 2^X$ let $\sigma\text{-Dyn}(\mathcal{A})$ denote the smallest σ -Dynkin class containing \mathcal{A} .

Theorem 4.6 Let \mathcal{C} be a \cap -stable family. Then $\sigma\text{-Dyn}(\mathcal{C}) = \sigma\text{-Ring}(\mathcal{C})$.

4.4 Monotone classes

Let $\mathcal{M} \subset 2^X$, We say that \mathcal{M} is a monotone class if

- (1) $A_1, A_2, \dots \in \mathcal{M}, A_n \searrow A \Rightarrow A \in \mathcal{M}$;
- (2) $A_1, A_2, \dots \in \mathcal{M}, A_n \nearrow A \Rightarrow A \in \mathcal{M}$.

Proposition 4.7 (1) *A σ -ring is a monotone class;*
(2) *A monotone ring is a σ -ring*

For $\mathcal{T} \subset 2^X$, we denote by $\text{Mon}(\mathcal{T})$ the smallest monotone class containing \mathcal{T} .

Theorem 4.8 *Let \mathcal{R} be a ring. Then*

$$\text{Mon}(\mathcal{R}) = \sigma\text{-Ring}(\mathcal{R}).$$

Proof. Since a σ -ring is a monotone class and since $\mathcal{R} \subset \sigma\text{-Ring}(\mathcal{R})$, it follows that

$$\text{Mon}(\mathcal{R}) \subset \sigma\text{-Ring}(\mathcal{R}).$$

Let us prove the converse inclusion. Let $A \in 2^X$. Set

$$\text{K}(A) := \{B \in 2^X : A \setminus B, B \setminus A, A \cup B \in \text{Mon}(\mathcal{R})\}.$$

Note that

$$A \in \text{K}(B) \Leftrightarrow B \in \text{K}(A). \quad (4.24)$$

We easily check that for every $A \in 2^X$, $\text{K}(A)$ is a monotone class. Clearly,

$$A \in \mathcal{R} \Rightarrow \mathcal{R} \subset \text{K}(A). \quad (4.25)$$

Hence

$$A \in \mathcal{R} \Rightarrow \text{Mon}(\mathcal{R}) \subset \text{K}(A). \quad (4.26)$$

From (4.24) and (4.26), we get

$$A \in \text{Mon}(\mathcal{R}) \Rightarrow \mathcal{R} \subset \text{K}(A).$$

Hence,

$$A \in \text{Mon}(\mathcal{R}) \Rightarrow \text{Mon}(\mathcal{R}) \subset \text{K}(A).$$

Therefore, $\text{Mon}(\mathcal{R})$ is a ring. By Proposition 4.7 (2) it is a σ -ring. Hence,

$$\text{Mon}(\mathcal{R}) \supset \sigma\text{-Ring}(\mathcal{R}).$$

□

4.5 Extension and uniqueness of contents

We will need a generalization of the notion of a content to the case of $\mathcal{T} \subset 2^X$ with $\emptyset \in \mathcal{T}$. We say that $\nu : \mathcal{T} \rightarrow [0, \infty]$ is a content if

- (1) $\nu(\emptyset) = 0$;
- (2) $A_1, \dots, A_n \in \mathcal{T}, A_i \cap A_j = \emptyset, i \neq j, A_1 \cup \dots \cup A_n \in \mathcal{R} \Rightarrow \nu(A_1 \cup \dots \cup A_n) = \nu(A_1) + \dots + \nu(A_n)$.

Theorem 4.9 *Let \mathcal{T} be a \cap -stable family containing \emptyset and $\mathcal{R} = \text{Ring}(\mathcal{T})$. Let ν_1, ν_2 be finite contents on \mathcal{R} coinciding on \mathcal{T} . Then $\nu_1 = \nu_2$.*

Proof. Let $\mathcal{W} := \{A \in \mathcal{R} : \nu_1(A) = \nu_2(A)\}$. Then

$$A, B \in \mathcal{W}, A \subset B \Rightarrow B \setminus A \in \mathcal{W}$$

$$A_1, A_2 \in \mathcal{W}, A_1 \cap A_2 = \emptyset \Rightarrow A_1 \cup A_2 \in \mathcal{W}$$

Hence \mathcal{W} is a Dynkin system. Hence it contains $\text{Dyn}(\mathcal{T})$. But by Theorem 4.2, $\text{Dyn}(\mathcal{T}) = \mathcal{R}$. Hence $\mathcal{W} = \mathcal{R}$. \square

Theorem 4.10 *Suppose that \mathcal{T} is a semiring and ν is a content on \mathcal{T} . Then there exists a unique content on $\text{Ring}(\mathcal{T})$ extending ν .*

Proof. Every $A \in \text{Ring}(\mathcal{T})$ can be written as $A = \bigcup_{i=1}^n B_i$ for some disjoint $B_i \in \mathcal{T}$. Then we set

$$\nu(A) := \sum_{i=1}^n \nu(B_i).$$

Suppose now that $A = \bigcup_{i=1}^n B_i = \bigcup_{j=1}^m C_j$ are decompositions of the above type. Then

$$A = \bigcup_{i=1}^n \bigcup_{j=1}^m B_i \cap C_j$$

is also a decomposition into disjoint elements of the semiring, and

$$\nu(B_i) = \sum_{j=1}^m \nu(B_i \cap C_j), \quad \nu(C_j) = \sum_{i=1}^n \nu(B_i \cap C_j).$$

Therefore,

$$\sum_{i=1}^n \nu(B_i) = \sum_{i=1}^n \sum_{j=1}^m \nu(B_i \cap C_j) = \sum_{j=1}^m \nu(C_j).$$

Hence the definition is correct.

It is easy to check that the extended ν is a content. \square

4.6 Uniqueness of a measure

Theorem 4.11 *Let (X, \mathcal{F}) be a set with a σ -ring. Let μ_1 and μ_2 be two measures defined on (X, \mathcal{F}) . Suppose that $\mathcal{T} \subset \mathcal{F}$ is a \cap -stable family such that $\mu_1 = \mu_2$ on \mathcal{T} and is finite on \mathcal{T} . Then $\mu_1 = \mu_2$ on $\sigma\text{-Ring}(\mathcal{T})$.*

Proof. Let $\mathcal{W} := \{A \in \mathcal{F} : \mu_1(A) = \mu_2(A)\}$.

Step 0. By Theorem 4.9, $\mu_1 = \mu_2$ on $\text{Ring}(\mathcal{T})$. Note that $\sigma\text{-Ring}(\mathcal{T}) = \sigma\text{-Ring}(\text{Ring}(\mathcal{T}))$. Hence in what follows it suffices to assume that \mathcal{T} is a ring.

Step 1. Assume that μ_1 is finite. Clearly, \mathcal{W} is a σ -Dynkin class and $\mathcal{T} \subset \mathcal{W}$. Hence $\sigma\text{-Ring}(\mathcal{T}) \subset \mathcal{W}$ in this case.

Step 2. Assume that $A \in \mathcal{F}$. Then μ_1 restricted to $\sigma\text{-Ring}(\mathcal{T}|_A) = \sigma\text{-Ring}(\mathcal{T})|_A$ is finite and $\mu_1 = \mu_2$ on $\mathcal{T}|_A$. Hence, by Step 1, we have $\mu_1 = \mu_2$ on $\sigma\text{-Ring}(\mathcal{T})|_A$.

Step 3. Let $A \in \sigma\text{-Ring}(\mathcal{T})$. Then by Theorem 3.2, there exist $A_1, A_2, \dots \in \mathcal{T}$ such that $A_n \nearrow A$ and $\mu(A_n) < \infty$. Then

$$\mu_2(A) = \lim_{n \rightarrow \infty} \mu_2(A_n) = \lim_{n \rightarrow \infty} \mu_1(A_n) = \mu_1(A).$$

\square

4.7 Dense subsets in L^p spaces

Theorem 4.12 *Let $\mathcal{T} \subset \mathcal{F}$ be a semiring such that $\sigma\text{-Ring}(\mathcal{T}) = \mathcal{F}$ and μ is finite on \mathcal{T} . Assume that there exists a localizing family $\{X_i : i \in I\}$ contained in \mathcal{T} . If $1 \leq p < \infty$, then the span of characteristic functions of \mathcal{T} is dense in $L^p(\mu)$.*

Proof. Let \mathcal{W} be the family of sets whose characteristic functions can be approximated in $L^p(\mu)$ by linear combinations of characteristic functions of elements in \mathcal{T} .

Step 1. Assume first that μ is finite. Clearly, \mathcal{W} is then a σ -Dynkin class. Hence $\mathcal{F} \subset \mathcal{W}$.

Step 2. Let μ be arbitrary. Let $A \in \mathcal{F}_\mu^f$. Let $\{X_i : i \in I\}$ be a localizing family for μ contained in \mathcal{T} . Then there exists a sequence $i_1, i_2, \dots \in I$ such that $\mu(A) = \sum_{j=1}^{\infty} \mu(A \cap X_{i_j})$. We apply Step 1. to μ restricted to $\mathcal{T}|_{X_{i_j}}$. We conclude that $A \in \mathcal{W}$. Hence $\mathcal{F}_\mu^f \subset \mathcal{W}$. Consequently, linear combinations of characteristic functions of elements in \mathcal{T} are dense in $\mathcal{E} \cap \mathcal{L}^p(\mu)$.

Step 3. Let $f \in \mathcal{L}^p(\mu)$. There exist sequences $u_n^\pm \in \mathcal{E}_+$ such that $u_n^\pm \nearrow f_\pm$. Clearly, $u_n^\pm \in \mathcal{L}^p(\mu)$, $u_n^+ - u_n^-$ is dominated by $|f| \in \mathcal{L}^p(\mu)$, hence by the Lebesgue dominated convergence theorem $u_n^+ - u_n^- \rightarrow f$ in the $\mathcal{L}^p(\mu)$ sense. \square

4.8 Premeasures

Definition 4.13 *Let (X, \mathcal{R}) be a set with a ring. A function $\nu : \mathcal{R} \rightarrow [0, \infty]$ is called a premeasure if*

- (1) $\nu(\emptyset) = 0$,
- (2) $A_1, A_2, \dots \in \mathcal{R}$, $\cup_{n=1}^{\infty} A_n \in \mathcal{R}$, $A_i \cap A_j = \emptyset$ for $i \neq j \Rightarrow \nu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$.

Clearly, every premeasure is a content.

If (X, \mathcal{F}, μ) is a measure and $\mathcal{R} \subset \mathcal{F}$ is a ring, then $(X, \mathcal{R}, \mu|_{\mathcal{R}})$ is a premeasure.

Theorem 4.14 *Let (X, \mathcal{R}, ν) be a premeasure, $A_1, A_2, \dots \in \mathcal{R}$, $A \in \mathcal{R}$ and $A \subset \bigcup_{i=1}^{\infty} A_i$. Then*

$$\nu(A) \leq \sum_{i=1}^{\infty} \nu(A_i).$$

Proof. $B_n := (A_n \setminus \bigcup_{i=1}^{n-1} A_i) \cap A$ are disjoint elements of \mathcal{R} , $B_i \subset A_i$ and $A = \bigcup_{i=1}^{\infty} B_i$. Hence

$$\nu(A) = \sum_{i=1}^{\infty} \nu(B_i) \leq \sum_{i=1}^{\infty} \nu(A_i).$$

\square

4.9 Extending a premeasure to a measure

Theorem 4.15 *Let (X, \mathcal{R}) be a set with a ring. Let (X, \mathcal{R}, ν) be a premeasure. For any $Q \in 2^X$ define*

$$\mu^*(Q) := \inf \left\{ \sum_{i=1}^{\infty} \nu(A_i) : A_i \in \mathcal{R}, Q \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$

Then

- (1) μ^* is an external measure;
(2) $\nu = \mu^*$ on \mathcal{R} ;
(3) Let \mathcal{F}^{ms} be the σ -field of μ^* -measurable sets and μ^{ms} the corresponding measure. Then $(X, \mathcal{F}^{\text{ms}}, \mu^{\text{ms}})$ is a complete measure extending the premeasure (X, \mathcal{R}, ν) .
(4) Let $\mathcal{F} := \sigma\text{-Ring}(\mathcal{R})$. Let the restriction of μ^* to \mathcal{F} be denoted μ . Then for $Q \in 2^X$

$$\mu^*(Q) = \inf\{\mu(A) : Q \subset A, A \in \mathcal{F}\}.$$

Proof. (1) The properties (1) and (3) of the definition of an external measure are obvious. Let us show the property (2). Let $Q_1, Q_2, \dots \in 2^X$. For any $\epsilon > 0$ we will find a double sequence $(A_{nm})_{m \in \mathbb{N}}$ such that

$$Q_n \subset \bigcup_{m=1}^{\infty} A_{nm},$$

$$\sum_{m=1}^{\infty} \mu(A_{nm}) \leq \mu^*(Q_n) + 2^{-n}\epsilon.$$

Then

$$\mu^*(\bigcup_{n=1}^{\infty} Q_n) \leq \sum_{n,m=1}^{\infty} \mu(A_{nm}) \leq \sum_{n=1}^{\infty} \mu^*(Q_n) + \epsilon.$$

Hence

$$\mu^*(\bigcup_{n=1}^{\infty} Q_n) \leq \sum_{n=1}^{\infty} \mu^*(Q_n).$$

(2) It is obvious that $\mu^*(A) \leq \mu(A)$. The converse inequality follows by Theorem 4.14

(3) Let $A \in \mathcal{R}$, $Q \in 2^X$. For any $\epsilon > 0$ and suitable $A_1, A_2, \dots \in \mathcal{R}$ such that $Q \subset \bigcup_{i=1}^{\infty} A_i$, we have

$$\begin{aligned} \mu^*(Q) &\geq \sum_{j=1}^{\infty} \mu(A_j) - \epsilon \\ &= \sum_{j=1}^{\infty} \mu(A_j \cap A) + \sum_{j=1}^{\infty} \mu(A_j \setminus A) - \epsilon \\ &\geq \mu^*(Q \cap A) + \mu^*(Q \setminus A) - \epsilon. \end{aligned}$$

Therefore,

$$\mu^*(Q) \geq \mu^*(Q \cap A) + \mu^*(Q \setminus A).$$

Thus, $A \in \mathcal{F}^{\text{ms}}$. \square

5 Tensor product of measures

5.1 Tensor product of σ -rings

Theorem 5.1 Let \mathcal{T}_i be semirings over X_i , $i = 1, 2$. Set

$$\mathcal{T}_1 * \mathcal{T}_2 := \{A_1 \times A_2 : A_i \in \mathcal{T}_i, i = 1, 2\}.$$

Then $\mathcal{T}_1 * \mathcal{T}_2$ is a semiring.

Now assume that \mathcal{F}_i are σ -rings. Set $\mathcal{F}_1 \otimes \mathcal{F}_2 := \sigma\text{-Ring}(\mathcal{F}_1 * \mathcal{F}_2)$.

Definition 5.2 Let $B \subset X_1 \times X_2$, $x_i \in X_i$.

$$\pi_2^{x_1}(B) = \{x_2 \in X_2 : (x_1, x_2) \in B\},$$

$$\pi_1^{x_2}(B) = \{x_1 \in X_1 : (x_1, x_2) \in B\}.$$

Proposition 5.3 Let $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$, $x_i \in X_i$. Then $\pi_1^{x_2}(B) \in \mathcal{F}_1$ and $\pi_2^{x_1}(B) \in \mathcal{F}_2$.

Proof. Note that

$$\pi_1^{x_2}(\emptyset) = \emptyset,$$

$$\pi_1^{x_2}(A \setminus B) = \pi_1^{x_2}(A) \setminus \pi_1^{x_2}(B),$$

$$\pi_1^{x_2}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} \pi_1^{x_2}(A_i).$$

Hence

$$\mathcal{W} := \{B \subset X_1 \times X_2 : \pi_1^{x_2}(B) \in \mathcal{F}_1\}$$

is a σ -ring. Clearly \mathcal{W} contains $\mathcal{F}_1 * \mathcal{F}_2$. Hence $\mathcal{F}_1 \otimes \mathcal{F}_2 \subset \mathcal{W}$. \square

5.2 Tensor product of measures

Let $(X_i, \mathcal{F}_i, \mu_i)$ be set with σ -rings and measures. Let $\mathcal{F}_{\mu_i}^{\sigma f}$ be the σ -ring of μ_i - σ -finite sets.

Proposition 5.4 Let $B \in \mathcal{F}_1 \otimes \mathcal{F}_{\mu_2}^{\sigma f}$. Then the map

$$X_1 \ni x_1 \mapsto \mu_2(\pi_2^{x_1}(B)) \text{ is } \mathcal{F}_1\text{-measurable.}$$

Proof. Set

$$s_B(x_1) := \mu_2(\pi_2^{x_1}(B)).$$

Set

$$\mathcal{W} := \{B \subset X_1 \times X_2 : s_B \text{ is measurable}\}.$$

Step 1. Assume that $\mu(X_2) < \infty$. Clearly, $\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma f} \subset \mathcal{W}$. If $A, B \in \mathcal{W}$ with $A \subset B$, then $s_{B \setminus A} = s_B - s_A$, Hence $B \setminus A \in \mathcal{W}$. Let $B_1, B_2, \dots \in \mathcal{W}$ be disjoint and $B = \bigcup_{j=1}^{\infty} B_j$. Then $s_B = \sum_{j=1}^{\infty} s_{B_j}$. Hence $\bigcup_{j=1}^{\infty} B_j \in \mathcal{W}$. Therefore, \mathcal{W} is a σ -Dynkin class. Hence it contains $\sigma\text{-Ring}(\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma f})$.

Alternative version of Step 1. Assume that $\mu(X_2) < \infty$. If disjoint $B_1, B_2, \dots \subset \mathcal{W}$ and $B = \bigcup_{j=1}^{\infty} B_j$, then $s_{\bigcup_{j=1}^{\infty} B_j} = \sum_{j=1}^{\infty} s_{B_j}$. We know that $\text{Ring}(\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma f})$ are disjoint unions of elements in

$\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma f} \subset \mathcal{W}$. Hence $\text{Ring}(\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma f}) \subset \mathcal{W}$.

Clearly, if $A_1, A_2, \dots \in \mathcal{W}$ and $A_n \nearrow A$, then $s_{A_n} \nearrow s_A$. Hence, $A \in \mathcal{W}$.

Likewise, if $A_1, A_2, \dots \in \mathcal{W}$ and $A_n \searrow A$, then $s_{A_n} \searrow s_A$. Hence, using the finiteness of X , $A \in \mathcal{W}$.

Therefore, \mathcal{W} is a monotone class. Hence it contains $\sigma\text{-Ring}(\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma f}) = \mathcal{F}_1 \otimes \mathcal{F}_{\mu_2}^{\sigma f}$.

Step 2. Now drop the assumption $\mu_2(X_2) < \infty$. Let $B \in \mathcal{F}_1 \otimes \mathcal{F}_{\mu_2}^{\sigma f}$. Then there exists a disjoint family $A_1, A_2, \dots \in \mathcal{F}_2$ such that $\mu_2(A_i) < \infty$ and $B \subset X_1 \times \bigcup_{j=1}^{\infty} A_j$. Set $B_j := B \cap X_1 \times A_j$. Then

$$s_B = \sum_{j=1}^{\infty} s_{B_j},$$

and each s_{B_j} is measurable. Hence s_B is measurable. \square

Now we assume that both measures are σ -finite. If $A \in \mathcal{F}_{\mu_1} \otimes \mathcal{F}_{\mu_2}$, we define

$$\mu_1 \otimes \mu_2(A) := \int \mu_2(\pi_2^{x_1}(A)) d\mu_1(x_1). \quad (5.27)$$

Theorem 5.5 (1) $\mu_1 \otimes \mu_2$ is a measure.

(2) If $X_1 \times X_2 \ni (x_1, x_2) \mapsto \tau(x_1, x_2) := (x_2, x_1) \in X_2 \times X_1$ is the flip, then for $A \in \mathcal{F}_1 \otimes \mathcal{F}_2$,

$$\mu_1 \otimes \mu_2(A) = \mu_2 \otimes \mu_1(\tau A). \quad (5.28)$$

In particular, for $A \in \mathcal{F}_{\mu_1} \otimes \mathcal{F}_{\mu_2}$ (5.28) equals

$$\int \mu_2(\pi_2^{x_1}(A)) d\mu_1(x_1) = \int \mu_1(\pi_1^{x_2}(A)) d\mu_2(x_2)$$

(3) If ν is a measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$ satisfying

$$\nu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2), \quad A_1 \in \mathcal{F}_{\mu_1}, A_2 \in \mathcal{F}_{\mu_2},$$

then it coincides with $\mu_1 \otimes \mu_2$.

(4) $(\mu_1 \otimes \mu_2)^{\text{cp}} = (\mu_1^{\text{cp}} \otimes \mu_2^{\text{cp}})^{\text{cp}}$.

Proof. The formula (5.27) is well defined by Proposition 5.4. Then we check that it is a measure.

The uniqueness follows by Theorem 4.11, because both measures coincide on the semiring $\mathcal{F}_{\mu_1} * \mathcal{F}_{\mu_2}$.

\square

5.3 Multiple integrals

For any $x_2 \in X_2$, the function

$$X_1 \ni x_1 \mapsto (x_1, x_2) \in X_1 \times X_2$$

is measurable. Hence if $f \in \mathcal{M}(X_1 \times X_2)$, then

$$X_1 \ni x_1 \mapsto f(x_1, x_2) \in [-\infty, \infty]$$

belongs to $\mathcal{M}(X_1)$.

Theorem 5.6 Let $(X_i, \mathcal{F}_i, \mu_i)$ be spaces with measures. Let $f \in \mathcal{M}_+(X_1 \times X_2)$.

(1)

$$x_1 \mapsto \int f(x_1, x_2) d\mu_2(x_2)$$

belongs to $\mathcal{M}_+(X_1)$,

(2)

$$\int f d(\mu_1 \otimes \mu_2) = \int (\int f(x_1, x_2) d\mu_2(x_2)) d\mu_1(x_1) = \int (\int f(x_1, x_2) d\mu_1(x_1)) d\mu_2(x_2). \quad (5.29)$$

Proof. For elementary functions the theorem is obvious. For an arbitrary function from $\mathcal{M}_+(X_1 \times X_2)$ we use the monotone convergence. \square

Theorem 5.7 (Fubini) Let $(X_i, \mathcal{F}_i, \mu_i)$ be spaces with measures and

$$f \in \mathcal{L}^1(\mu_1 \otimes \mu_2).$$

The map

$$x_2 \mapsto f(x_1, x_2)$$

for μ_1 -almost all x_1 belongs to $\mathcal{L}^1(\mu_2)$. Let N_1 be the set of x_1 for which this is not true. Define

$$f_1(x_1) := \begin{cases} \int f(x_1, x_2) d\mu_2(x_2) & x_1 \in X_1 \setminus N_1 \\ 0 & x_1 \in N_1. \end{cases}$$

Then f_1 belongs to $\mathcal{L}^1(\mu_1)$ and

$$\int f d(\mu_1 \otimes \mu_2) = \int f_1 d\mu_1.$$

Proof. We have $f_{\pm} \in \mathcal{L}^1(\mu_1 \otimes \mu_2)$. Hence

$$\infty > \int f_+ d\mu_1 \otimes \mu_2 = \int \left(\int f_+(x_1, x_2) d\mu_2(x_2) \right) d\mu_1(x_1). \quad (5.30)$$

Thus $x_1 \mapsto \int f_+(x_1, x_2) d\mu_2(x_2)$ belongs to $L^1(\mu_1)$. Hence, by Theorem 2.12, for μ_1 -a.a. x_1 ,

$$\int f_+(x_1, x_2) d\mu_2(x_2) < \infty.$$

In other words, for μ_1 -a.a. x_1 $f_+(x_1, \cdot) \in \mathcal{L}^1(\mu_2)$.

Of course, the same is true for f_- . \square

Loosely speaking, the above theorem says that for $f \in \mathcal{L}^1(\mu_1 \otimes \mu_2)$

$$\int f d(\mu_1 \otimes \mu_2) = \int \left(\int f(x_1, x_2) d\mu_2(x_2) \right) d\mu_1(x_1) = \int \left(\int f(x_1, x_2) d\mu_1(x_1) \right) d\mu_2(x_2).$$

Theorem 5.8 If \otimes denotes the tensor product in the sense of Hilbert spaces, then $L^2(\mu_1) \otimes L^2(\mu_2) = L^2(\mu_1 \otimes \mu_2)$.

5.4 Layer-cake representation

Let (X, \mathcal{F}, μ) be a measure. Let ν be a Borel measure on $[0, \infty[$ and $f \in \mathcal{M}_+(X)$. For $a, t \geq 0$, set

$$\phi(t) := \int_0^t d\nu(s), \quad u(a) := \mu\{f > a\}.$$

Proposition 5.9

$$\int \phi(f(x)) d\mu(x) = \int_0^\infty u(a) d\nu(a).$$

Proof.

$$\begin{aligned} \int_0^\infty u(a) d\nu(a) &= \int \left(\int 1_{\{f > a\}}(x) d\mu(x) \right) d\nu(a) \\ &= \int \left(\int 1_{\{f > a\}}(x) d\nu(a) \right) d\mu(x) = \int \left(\int_0^{f(x)} d\nu(a) \right) d\mu(x) = \int \phi(f(x)) d\mu(x). \end{aligned}$$

\square

Corollary 5.10 (1) $\int |f(x)|^p d\mu(x) = p \int_0^\infty u(a) a^{p-1} da.$

(2) $f(x) = \int 1_{\{f>a\}}(x) da.$

Proof. For (1) we set $\phi(t) = t^p$, $d\nu(t) = pt^{p-1} dt.$

For (2) we put $\phi(t) := t$, $d\nu(t) = dt$, and μ is the Dirac delta at x . \square

6 Measures in \mathbb{R}^n

6.1 Regular contents

Suppose that X is a topological space. Let \mathcal{R} be a ring over X and ν a content on \mathcal{R} . We say that ν is regular iff for $F \in \mathcal{R}$ the following two conditions hold:

$$\begin{aligned} \nu(F) &= \sup\{\nu(G) : G^{\text{cl}} \subset F, G^{\text{cl}} \in \text{Compact}(X)\} \\ &= \inf\{\nu(H) : F \subset H^{\circ}\}. \end{aligned}$$

Theorem 6.1 *Every regular content is a premeasure.*

Let $F_1, F_2, \dots \in \mathcal{R}$ be disjoint and $F := \bigcup_{j=1}^{\infty} F_j \in \mathcal{R}$. We know by Theorem 2.1 that

$$\nu(F) \geq \sum_{j=1}^{\infty} \nu(F_j).$$

Let us show the converse inequality. Let $\epsilon > 0$. For any $j = 1, \dots, n$ we can find $H_j \in \mathcal{R}$ such that $F_j \subset H_j^{\circ}$ and $\nu(H_j \setminus F_j) < \epsilon 2^{-j-1}$. Likewise, we can find $G \in \mathcal{R}$ such that $G^{\text{cl}} \subset F$, G^{cl} is compact and $\nu(F \setminus G) < \epsilon/2$. Thus $\{H_j^{\circ} : j = 1, 2, \dots\}$ is an open cover of the compact set G^{cl} . We can choose a finite subcover $\{H_{j_k}^{\circ} : k = 1, \dots, m\}$, so that

$$G^{\text{cl}} \subset \bigcup_{k=1}^m H_{j_k}^{\circ}.$$

Consequently,

$$G \subset \bigcup_{k=1}^m H_{j_k}.$$

Thus

$$\nu(F) \leq \nu(G) + \epsilon/2 \leq \sum_{k=1}^m \nu(H_{j_k}) + \epsilon/2 \leq \sum_{j=1}^{\infty} \nu(F_j) + \epsilon.$$

\square

6.2 Borel sets in \mathbb{R}

Let $\mathcal{T} := \{]a, b[: a, b \in \mathbb{R}, a \leq b\}$.

Theorem 6.2 \mathcal{T} is a semiring. Moreover, let $A_1, \dots, A_n \in \mathcal{T}$ be disjoint, $A \in \mathcal{T}$ and $\bigcup_{i=1}^n A_i = A$. Then, after a possible renumbering of A_i , $A_i =]a_{i-1}, a_i[$, where $a_0 \leq a_1 \leq \dots \leq a_n$. $\sigma\text{-Field}(\mathcal{T})$ equals the σ -field of Borel subsets of \mathbb{R} and will be denoted by $\text{Borel}(\mathbb{R})$.

6.3 Borel premeasures on \mathbb{R}

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function. Define $\nu : \mathcal{T} \rightarrow [0, \infty[$ as

$$\nu(]a, b]) := f(b) - f(a).$$

Theorem 6.3 ν is a content on \mathcal{T} . Hence it extends uniquely to a content on $\text{Ring}(\mathcal{T})$.

Assume now in addition that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous from the right, that means

$$\lim_{t \downarrow t_0} f(t) = f(t_0), \quad t_0 \in \mathbb{R}.$$

Lemma 6.4 Let $F \in \text{Ring}(\mathcal{T})$ and $\epsilon > 0$. Then there exist $H, G \in \text{Ring}(\mathcal{T})$ such that

$$G^{\text{cl}} \subset F \subset H^{\circ}, \quad \nu(F \setminus G) < \epsilon, \quad \nu(H \setminus F) < \epsilon.$$

In other words, ν is a regular content.

Proof. We can assume that $F = \bigcup_{i=1}^n]a_{2i-1}, a_{2i}]$ and $0 < \delta < \min\{a_{j+1} - a_j : j = 0, \dots, 2n-1\}$. By decreasing δ we can demand in addition that $f(a_i + \delta) - f(a_i) < \epsilon/n$. Then we can set

$$G := \bigcup_{j=1}^n]a_{2j-1} + \delta, a_{2j}], \quad H := \bigcup_{j=1}^n]a_{2j-1}, a_{2j} + \delta].$$

□

By Theorem 6.1, we get:

Theorem 6.5 ν is a premeasure on $\text{Ring}(\mathcal{T})$.

6.4 Borel measures on \mathbb{R}

Theorem 6.6 (1) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function continuous from the right. Then there exists a unique measure μ_f on $(\mathbb{R}, \text{Borel}(\mathbb{R}))$ such that

$$\mu_f(]a, b]) = f(b) - f(a).$$

This measure is σ -finite.

(2) Let $(\mathbb{R}, \text{Borel}(\mathbb{R}), \mu)$ be a measure such that $\mu(A) < \infty$ for compact $A \subset \mathbb{R}$. Set

$$f(x) := \begin{cases} -\mu(]x, 0]), & x < 0 \\ \mu(]0, x]), & x \geq 0. \end{cases}$$

Then $\mu = \mu_f$.

Proof. The premeasure ν_f can be extended by the Caratheodory construction to a σ -field containing $\text{Borel}(\mathbb{R})$. □

Definition 6.7 The measure on $\text{Borel}(\mathbb{R})$ with the distribution function $f(x) = x$ is called the Borel-Lebesgue measure, and denoted λ . Its complete extension is called the Lebesgue measure. In integrals, if the generic variable in \mathbb{R} will be denoted by x , then instead of $d\lambda(x)$ we will usually write dx .

Theorem 6.8 The Borel-Lebesgue measure is the only measure on $\text{Borel}(\mathbb{R})$ invariant wrt translations such that $\mu(]0, 1]) = 1$.

Proof. Let $(\mathbb{R}, \text{Borel}(\mathbb{R}), \mu)$ be translation invariant. This means $\mu(]a, b]) = \mu(]a+x, b+x])$. Using this and $\mu(]0, 1])$ we get $\mu(]k/n, (k+1)/n]) = 1/n$. This easily implies $\mu(]a, b]) = b-a$ for any $a \leq b$. \square

Theorem 6.9 (1) Let μ be a measure on $2^{\mathbb{R}}$ invariant wrt translations (that means if $\tau_t(x) = x-t$, then $\tau_{t*}\mu = \mu$). Suppose that μ is finite on compact sets. Then $\mu = 0$.
(2) Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ denote the 1-dimensional torus. Let μ be a finite measure on $2^{\mathbb{T}}$ invariant wrt translations. Then $\mu = 0$.

Proof. (2) Introduce in \mathbb{T} the equivalence relation

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}.$$

Let \mathcal{I} be the set of equivalence classes. Choose from every class a representant x_i . We have

$$]0, 1[= \bigcup_{y \in \mathbb{Q}} A_y$$

for

$$A_y := \{x_i + y : i \in \mathcal{I}\}.$$

But $\tau_{y_2 - y_1} A_{y_2} = A_{y_1}$. Hence

$$\mu(\mathbb{T}) = \sum_{y \in \mathbb{Q}} \mu(A_y) = \sum_{y \in \mathbb{Q}} \mu(A_0) = \infty \mu(A_0).$$

Thus $\mu(\mathbb{T}) = 0$ or $\mu(\mathbb{T}) = \infty$.

(1) Let μ be a translation invariant measure on $(\mathbb{R}, 2^{\mathbb{R}})$. Consider the canonical projection $\mathbb{R} \rightarrow \mathbb{T}$ restricted to $]0, 1[$ it is a bijection. Denote it by $\pi :]0, 1[\rightarrow \mathbb{T}$. Define $\tilde{\mu}(A) := \mu(\pi^{-1}(A))$, $A \subset \mathbb{T}$.

Let us check that $\tilde{\mu}$ is translation invariant and $\tilde{\mu}(\mathbb{T}) = 1$. By (2) it is zero. \square

6.5 The Cantor set and devil's staircase

Definition 6.10 Let $q \in \mathbb{N}$. To every sequence of numbers $(p_j)_{j \in \mathbb{N}}$ with values in $\{0, 1, \dots, q-1\}$ we assign a number from the interval $[0, 1]$:

$$0.p_1 p_2 \dots := \sum_{j=1}^{\infty} \frac{p_j}{q^j} = x.$$

We say that $0.p_1 p_2 \dots$ denotes the number x in the system based on q

Note that every $x \in [0, 1]$ has such a representation. It is ambiguous only if for some n we have $10^n x \in \mathbb{N}$. Then

$$0.p_1 \dots p_{n-1} p_n 00 \dots = 0.p_1 p_2 \dots p_{n-1} (p_n - 1)(q-1)(q-1) \dots$$

Definition 6.11 The Cantor set C , is the subset of $[0, 1]$ consisting of the numbers that in the tri-nary system have only 0 and 2. It can be defined also as follows: $C_0 = [0, 1]$, $C_1 = C_0 \setminus]\frac{1}{3}, \frac{2}{3}[$, $C_2 = C_1 \setminus]\frac{1}{9}, \frac{2}{9}[\cup]\frac{7}{9}, \frac{8}{9}[$, etc. We set $C = \bigcap_{n=1}^{\infty} C_n$.

It is a closed set with an empty interior, uncountable and has zero Lebesgue measure (because $\lambda(C_n) = \frac{2^n}{3^n}$).

Definition 6.12 Define the transformation, called *devil's staircase*, $F : [0, 1] \rightarrow [0, 1]$ as follows. If $x = 0.p_1p_2 \dots \in C$ in the ternary system, where $p_i \in \{0, 2\}$, then $F(x) = 0.\frac{p_1}{2}\frac{p_2}{2} \dots$ in the binary system. If $x \in [0, 1] \setminus C$, then $x \in]x_-, x_+[$ where $x_- = 0.p_1 \dots p_n 0 2 2 \dots$ and $x_+ = 0.p_1 \dots p_n 2 0 0 \dots$. We see that $F(x_-) = F(x_+)$ and we set $F(x) = F(x_-) = F(x_+)$.

The function F is increasing, continuous, locally constant beyond C , and $F(1) - F(0) = 1$. It defines a Borel measure μ , which is continuous and singular wrt the Lebesgue measure, since $\mu([0, 1] \setminus C) = 0$.

6.6 Transport of the Lebesgue measure in \mathbb{R}

Let $g : [a, b] \rightarrow \mathbb{R}$ be an increasing function. Then there exists a unique increasing function $f : [g(a), g(b)] \rightarrow [a, b]$, which is continuous from the right and $g \circ f(x) = x$, $x \in [g(a), g(b)]$. It is easy to see that $g^*\lambda = \mu_f$. In fact,

$$g^*\lambda([\alpha, \beta]) = \lambda(g^{-1}([\alpha, \beta])) = \lambda([f(\alpha), f(\beta)]) = f(\beta) - f(\alpha).$$

6.7 The Lebesgue measure in \mathbb{R}^n

Let $\text{Borel}(\mathbb{R}^n)$ denote the σ -field of Borel sets in \mathbb{R}^n .

In \mathbb{R}^n we can define the n -dimensional Borel-Lebesgue measure as the measure on $\text{Borel}(\mathbb{R}^n)$ equal $\lambda^n := \lambda \otimes \dots \otimes \lambda$.

Theorem 6.13 (1) *The n -dimensional Borel-Lebesgue measure λ^n is the unique measure on $\text{Borel}(\mathbb{R}^n)$ such that*

$$\lambda^n\left(\prod_{i=1}^n]a_i, b_i]\right) = \prod_{i=1}^n |b_i - a_i|.$$

(2) *It is also the unique translation invariant measure on $\text{Borel}(\mathbb{R}^n)$ such that*

$$\lambda^n\left(\prod_{i=1}^n]0, 1]\right) = 1.$$

We can also consider its completion, called the n -dimensional Lebesgue measure. There are several equivalent ways to construct the Lebesgue measure, described in the following theorem.

Theorem 6.14 *The following measures coincide:*

- (1) $(\lambda^n)^{\text{cp}}$ (the completion of the n -dimensional Borel-Lebesgue measure).
- (2) $(\lambda^{\text{cp}} \otimes \dots \otimes \lambda^{\text{cp}})^{\text{cp}}$.
- (3) Let \mathcal{T}^n be the semiring of sets $\prod_{i=1}^n]a_i, b_i]$. Set

$$\nu^n\left(\prod_{i=1}^n]a_i, b_i]\right) = \prod_{i=1}^n |b_i - a_i|.$$

Then ν^n is a premeasure. We consider the measure obtained by the Caratheodory construction.

6.8 Transport of the Lebesgue measure in \mathbb{R}^n

Theorem 6.15 *Let U be an open subset of \mathbb{R}^n and $\phi : U \rightarrow \mathbb{R}^n$ a C^1 bijection with $\det \phi'(x) \neq 0$, $x \in U$. Let λ be the Lebesgue measure. Then*

$$\frac{d\phi^*\lambda}{d\lambda} = |\det \phi'|. \tag{6.31}$$

Thus if $f \in \mathcal{M}_+(\phi(U))$, then

$$\int_{\phi(U)} f d\lambda = \int_U f \circ \phi |\det \phi'| d\lambda.$$

We will also write the transformation as

$$(y^1, \dots, y^n) \xrightarrow{\phi} (x^1, \dots, x^n).$$

Then we can write

$$\int f(x) dx = \int f(x(y)) \left| \frac{\partial x(y)}{\partial y} \right| dy.$$

Proof. We will say that the transformation ϕ satisfies the change of variables formula iff it satisfies (6.31).

Step 1. If the transformations ϕ, ψ satisfy the change of variables formula, then the transformation $\phi \circ \psi$ satisfies it as well.

Step 2. Transformations of the form $(y^1, \dots, y^n) \mapsto (y^{\pi(1)}, \dots, y^{\pi(n)})$, where π is a permutation, satisfy the change of variables formula.

Step 3. If a transformation ϕ has the form

$$(y^1, \dots, y^n) \mapsto (x^1, \dots, x^n) = (f(y^1, \dots, y^n), y^2, \dots, y^n),$$

then it satisfies the change of variables formula. In fact,

$$\begin{aligned} \int F(x^1, \dots, x^n) dx^1 \cdots dx^n &= \int dx^n \cdots \int dx^1 F(x^1, \dots, x^n) \\ &= \int dx^n \cdots \int dx^2 \int dy^1 \left| \frac{\partial f(y^1, x^2, \dots, x^n)}{\partial y^1} \right| F(f(y^1, x^2, \dots, x^n), x^2, \dots, x^n) \\ &= \int dy^n \cdots \int dy^2 \int dy^1 \left| \frac{\partial f(y^1, y^2, \dots, y^n)}{\partial y^1} \right| F(f(y^1, y^2, \dots, x^n), y^2, \dots, y^n) \\ &= \int F(x(y)) \left| \det \frac{\partial x(y)}{\partial y} \right| dy. \end{aligned}$$

Step 4. We proceed by induction wrt n . We assume that the theorem is true for $n - 1$.

If $F \in \mathcal{M}_+(U)$, then we can find a sequence of functions $F_n \in \mathcal{M}_+(U)$ of compact support with $F_n \nearrow F$.

Therefore, it is sufficient to assume that the support of F is compact.

Let $a = (a^1, \dots, a^n) \in \text{supp} F$. There exist i, j such that $\frac{\partial x^i}{\partial y^j}(a) \neq 0$. We can find $\delta > 0$ such that for $|y^i - a^i| < \delta$, $i = 1, \dots, n$, $\frac{\partial x^i(y^1, \dots, y^n)}{\partial y^j} > 0$, or $\frac{\partial x^i(y^1, \dots, y^n)}{\partial y^j} < 0$. Set $W_a := [a^1 - \delta, a^1 + \delta] \times \cdots \times [a^n - \delta, a^n + \delta]$. Clearly, we can find a finite family of $a_1, \dots, a_n \in \mathbb{R}^n$ such that W_{a_1}, \dots, W_{a_n} covers $\text{supp} F$. Then we can write $F = \sum_{i=1}^n F_i$ with $\text{supp} F_i \subset W_i$.

In what follows we assume that on the support of F , $\frac{\partial x^i(y^1, \dots, y^n)}{\partial y^j} > 0$. By Step 2, we can assume that $i = j = 1$. Define

$$(y^1, \dots, y^n) \xrightarrow{\psi} (z^1, \dots, z^n),$$

where

$$z^1(y^1, \dots, y^n) = x^1(y^1, \dots, y^n), \quad z^2 = y^2, \dots, z^n = y^n.$$

The map ψ is injective. Define $\rho := \phi \psi^{-1}$, that is

$$(z^1, \dots, z^n) \xrightarrow{\rho} (x^1, \dots, x^n).$$

Note that $x^1 = z^1$.

The map ψ is of the type considered in Step 3. Hence it satisfies the change of variables formula. We have $\phi = \rho\psi$. By Step 1, it is thus sufficient to prove that ρ satisfies the change of variables formula.

We have

$$\rho' = \left[\frac{\partial x}{\partial z} \right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\partial x^2}{\partial z^1} & \frac{\partial x^2}{\partial z^2} & \cdots & \frac{\partial x^2}{\partial z^n} \\ \frac{\partial x^n}{\partial z^1} & \frac{\partial x^n}{\partial z^2} & \cdots & \frac{\partial x^n}{\partial z^n} \end{bmatrix},$$

and hence

$$\det \frac{\partial x}{\partial z} = \det \begin{bmatrix} \frac{\partial x^2}{\partial z^2} & \cdots & \frac{\partial x^2}{\partial z^n} \\ \frac{\partial x^n}{\partial z^2} & \cdots & \frac{\partial x^n}{\partial z^n} \end{bmatrix}$$

Thus

$$\begin{aligned} & \int dx^1 \cdots \int dx^n F(x^1, \dots, x^n) \\ &= \int dx^1 \int dx^2 \cdots \int dx^n F(x^1, x^2, \dots, x^n) \\ &= \int dx^1 \int dz^2 \cdots \int dz^n F(x^1, x^2(x^1, z^2, \dots, z^n), \dots, x^n(x^1, z^2, \dots, z^n)) \left| \det \begin{bmatrix} \frac{\partial x^2}{\partial z^2} & \cdots & \frac{\partial x^2}{\partial z^n} \\ \frac{\partial x^n}{\partial z^2} & \cdots & \frac{\partial x^n}{\partial z^n} \end{bmatrix} \right| \\ &= \int dz^1 \int dz^2 \cdots \int dz^n F(x^1(z^1, \dots, z^n), \dots, x^n(z^1, \dots, z^n)) \left| \det \begin{bmatrix} \frac{\partial x^2}{\partial z^2} & \cdots & \frac{\partial x^2}{\partial z^n} \\ \frac{\partial x^n}{\partial z^2} & \cdots & \frac{\partial x^n}{\partial z^n} \end{bmatrix} \right| \\ &= \int F(x(z)) \left| \det \frac{\partial x(z)}{\partial z} \right| dz. \end{aligned}$$

□

7 Charges and the Radon-Nikodym theorem

7.1 Extension of a measure from a σ -ring

Let \mathcal{F}, \mathcal{I} be σ -rings over X and let \mathcal{I} be an ideal in \mathcal{F} . Let (X, \mathcal{I}, μ) be a space with a measure. We can then extend the measure μ from \mathcal{I} to \mathcal{F} . We can do this in many ways.

Theorem 7.1 (1) Define $\mu^{\max} : \mathcal{F} \rightarrow [0, \infty]$ by

$$\mu^{\max}(A) := \inf\{\mu(B) : A \subset B, B \in \mathcal{I}\}.$$

Then μ^{\max} is a measure. We have $\mu^{\max}(A) = \mu(A)$, $A \in \mathcal{I}$, and $\mu^{\max}(A) = \infty$, $A \in \mathcal{F} \setminus \mathcal{I}$. μ^{\max} is the largest measure on \mathcal{F} extending μ onto \mathcal{F} . σ -finite and null sets coincide for μ and μ_{\max} .

(2) Define $\mu^{\min} : \mathcal{F} \rightarrow [0, \infty]$ by

$$\mu^{\min}(A) := \sup\{\mu(B) : B \subset A, B \in \mathcal{I}\}.$$

Then μ^{\min} is a measure. We have $\mu^{\min}(A) = \mu(A)$, $A \in \mathcal{I}$. μ^{\min} is the smallest measure extending μ onto \mathcal{F} .

Proof. (1) is obvious.

(2) Let us prove that μ^{\min} is σ -additive. Let $A_1, A_2, \dots \in \mathcal{F}$ be disjoint, $A = \bigcup_{j=1}^{\infty} A_j$.

Let $B \in \mathcal{I}$, $B \subset A$. Then using $A_j \cap B \in \mathcal{I}$ we get

$$\mu(B) = \sum_{j=1}^{\infty} \mu(A_j \cap B) \leq \sum_{j=1}^{\infty} \mu^{\min}(A_j).$$

Hence

$$\mu^{\min}(A) \leq \sum_{j=1}^{\infty} \mu^{\min}(A_j).$$

Let $B_j \subset A_j$, $B_j \in \mathcal{I}$. Then B_1, B_2, \dots are disjoint and $\cup_{j=1}^{\infty} B_j \subset A$, hence

$$\sum_{j=1}^{\infty} \mu(B_j) = \mu(\bigcup_{j=1}^{\infty} B_j) \leq \mu^{\min}(A).$$

Thus

$$\sum_{j=1}^{\infty} \mu^{\min}(A_j) \leq \mu^{\min}(A).$$

□

7.2 Measures singular and continuous wrt an ideal

Let \mathcal{F} be a σ -ring over X and let \mathcal{I} be a σ -ring—an ideal in \mathcal{F} . Let (X, \mathcal{F}, ν) be a space with a measure.

We say that ν is \mathcal{I} -singular if

$$\nu(A) = \sup\{\nu(B) : B \subset A, B \in \mathcal{I}\}, \quad A \in \mathcal{F}.$$

We say that ν is \mathcal{I} -continuous if

$$A \in \mathcal{I} \Rightarrow \nu(A) = 0. \quad (7.32)$$

(More generally, if ν is a charge, we say it is \mathcal{I} -continuous if (7.32) is true).

In particular, if (X, \mathcal{F}, μ) is also a space with a measure, then

$$\mathcal{F}_{\mu}^0 := \{A \in \mathcal{F} : \mu(A) = 0\}$$

is an ideal in \mathcal{F} . We say that ν is μ -singular if it is \mathcal{F}_{μ}^0 -singular. We say that ν is μ -continuous if it is \mathcal{F}_{μ}^0 -continuous.

Theorem 7.2 *Let (X, \mathcal{F}, ν) be a measure. Let \mathcal{I} be a σ -ring, an ideal in \mathcal{F} .*

(1) *There exists a decomposition*

$$\nu = \nu_{\mathcal{I}s} + \nu_{\mathcal{I}c}, \quad (7.33)$$

where $\nu_{\mathcal{I}s}$ is a \mathcal{I} -singular measure and $\nu_{\mathcal{I}c}$ is a \mathcal{I} -continuous measure. The \mathcal{I} -singular part is uniquely given by

$$\nu_{\mathcal{I}s}(A) := \sup\{\nu(B) : B \subset A, B \in \mathcal{I}\}.$$

The \mathcal{I} -continuous part does not have to be unique, but there is a canonical choice given by

$$\nu_{\mathcal{I}c}(A) := \inf\{\nu(A \setminus B) : B \subset A, B \in \mathcal{I}\}.$$

(2) *If ν is σ -finite, then the decomposition of ν into a \mathcal{I} -singular and a \mathcal{I} -continuous measure is unique.*

(3) If X is $\nu - \sigma$ -finite, then there exists a set $N \in \mathcal{I}$ such that

$$\nu_{\mathcal{I}_s}(A) = \nu(A \cap N), \quad \nu_{\mathcal{I}_c}(A) = \nu(A \setminus N).$$

Proof. The fact that $\nu_{\mathcal{I}_s}$ is a measure follows from Theorem 7.1 applied to $\nu|_{\mathcal{I}}$.

We need to show that $\nu_{\mathcal{I}_c}$ is a measure. Let $A_1, A_2, \dots \in \mathcal{F}$ be disjoint and $A = \bigcup_{j=1}^{\infty} A_j$. Let us prove that

$$\nu_{\mathcal{I}_c}(A) \leq \sum_{j=1}^{\infty} \nu_{\mathcal{I}_c}(A_j). \quad (7.34)$$

It is sufficient to assume that $\nu_{\mathcal{I}_c}(A_j) < \infty$, $j = 1, 2, \dots$. Let $\epsilon > 0$. We will find $B_j \in \mathcal{I}$, $B_j \subset A_j$ with $\nu_{\mathcal{I}_c}(A_j) > \nu_{\mathcal{I}_c}(A_j \setminus B_j) - 2^{-j}\epsilon$. Then

$$\nu_{\mathcal{I}_c}(A) \leq \nu(A \setminus \bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \nu(A_j \setminus B_j) \leq \sum_{j=1}^{\infty} \nu_{\mathcal{I}_c}(A_j) + \epsilon.$$

This proves (7.34).

Let us prove that

$$\sum_{j=1}^{\infty} \nu_{\mathcal{I}_c}(A_j) \leq \nu_{\mathcal{I}_c}(A). \quad (7.35)$$

It is sufficient to assume that $\nu_{\mathcal{I}_c}(A) < \infty$. Let $\epsilon > 0$. We will find $B \in \mathcal{I}$, $B \subset A$ with $\nu_{\mathcal{I}_c}(A) > \nu_{\mathcal{I}_c}(A \setminus B) - \epsilon$. Then

$$\sum_{j=1}^{\infty} \nu_{\mathcal{I}_c}(A_j) \leq \sum_{j=1}^{\infty} \nu(A_j \setminus B) = \nu(A \setminus B) < \nu_{\mathcal{I}_c}(A) + \epsilon.$$

This proves (7.35)

(3) Let us prove the existence of the set N . Assume that X is ν -finite. Let

$$\alpha := \sup \{ \nu(A) : A \in \mathcal{I} \}.$$

Then $\alpha < \infty$. We can find a sequence $(N_j)_{j \in \mathbb{N}}$ in \mathcal{I} such that $\lim_{j \rightarrow \infty} \nu(N_j) = \alpha$. We can assume that the sequence $N_j \nearrow N$. Then $N \in \mathcal{I}$ and $\nu(N) = \alpha$.

It is obvious that $\nu(A \cap N) \leq \nu_{\mathcal{I}_s}(A)$. Suppose that for some $A \in \mathcal{F}$,

$$\nu(A \cap N) < \nu_{\mathcal{I}_s}(A).$$

Then there exists $B \in \mathcal{I}$ with $B \subset A$ and

$$\nu(A \cap N) < \nu(B).$$

Then $B \cup N \in \mathcal{I}$ and

$$\nu(N \cup B) = \nu(N \setminus B) + \nu(B) > \nu(A \cap N) + \nu(N \setminus A) = \nu(N),$$

which is a contradiction

If X is $\nu - \sigma$ -finite, then we can find a sequence $X_n \nearrow X$ such that $\nu(X_n) < \infty$. We will also find sets $N_n \subset X_n$ constructed as above. We easily check that $\nu(A \cap N) = \nu_{\mathcal{I}_s}(A)$.

The decomposition of ν is uniquely determined on σ -finite sets. Hence it is unique. \square

7.3 Pure point and continuous measures

Definition 7.3 Suppose that (X, \mathcal{F}, ν) is a space with measure and

$$\{\{x\} : x \in X\} \subset \mathcal{F}. \quad (7.36)$$

We say that ν is a point (atomic) measure if

$$\nu(A) = \sum_{x \in A} \nu(\{x\}).$$

ν is continuous (diffuse) if

$$\nu(\{x\}) = 0, \quad x \in X.$$

Theorem 7.4 Let (X, \mathcal{F}, ν) be a measure. Assume (7.36).

(1) There exists a decomposition

$$\nu = \nu_p + \nu_c,$$

where ν_p is a point measure and ν_c is a continuous measure. The point part is uniquely given by

$$\nu_p(A) := \sup\{\nu(B) : B \subset A, B \text{ is finite}\},$$

The \mathcal{I} -continuous part does not have to be unique, but there is a canonical choice given by

$$\nu_c(A) := \inf\{\nu(A \setminus B) : B \subset A, B \text{ is finite}\}.$$

(2) If ν is σ -finite, then the decomposition of ν into a point and a continuous measure is unique.

(3) If \mathcal{X} is ν - σ -finite, there exists a countable set $N \in \mathcal{F}$ such that

$$\nu_p(A) = \nu(A \cap N), \quad \nu_c(A) = \nu(A \setminus N).$$

Corollary 7.5 Let (X, \mathcal{F}) be a set with a σ -field Let

$$\{\{x\} : x \in X\} \subset \mathcal{F}$$

Let ν, μ be measures on (X, \mathcal{F}) and let μ be continuous. Then there exists a decomposition

$$\nu = \nu_p + \nu_{sc} + \nu_{ac}$$

such that

ν_p is pure point ,

ν_{sc} is μ -singular and continuous,

ν_{ac} is μ -continuous.

If ν is σ -finite, then the decomposition is unique.

7.4 Charges (signed measures)

Let (X, \mathcal{F}) be a space with a σ -ring. A function $\mu : \mathcal{F} \rightarrow]-\infty, \infty]$ is called a bounded from below charge (or signed measure) if

(1) $\mu(\emptyset) = 0$,

(2) $A_1, A_2 \cdots \in \mathcal{F}$, $A_i \cap A_j = \emptyset$ for $i \neq j \Rightarrow \mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

Proposition 7.6 (1) If $A \subset B$, and $\mu(B) < \infty$, then $\mu(A) < \infty$.

(2) $A_1, A_2, \dots \in \mathcal{F}$, $A_i \cap A_j = \emptyset$, $i \neq j$, and $\mu(\cup_{n=1}^{\infty} A_n) < \infty$, then $\sum_{n=1}^{\infty} \mu(A_n)$ is absolutely convergent.

(3) $A_1, A_2, \dots \in \mathcal{F}$, $A_n \nearrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$.

(4) If $A_1, A_2, \dots \in \mathcal{F}$, $A_n \searrow A$ and for some n , $\mu(A_n) < \infty$, then $\mu(A_n) \rightarrow \mu(A)$.

Proof. (1) $\mu(B) = \mu(A) + \mu(B \setminus A)$ and $\mu(B \setminus A) > -\infty$. Hence $\mu(A) = \mu(B) - \mu(B \setminus A)$ with both summands less than ∞ .

(2) We group the sets A_i into two subfamilies: those with a positive charge and a nonpositive charge. After renumbering we can call the former B_1, B_2, \dots and the latter C_1, C_2, \dots . We have

$$-\sum_{n=1}^{\infty} \mu(B_n) = -\mu(\cup_{n=1}^{\infty} B_n) < \infty$$

and

$$\sum_{n=1}^{\infty} \mu(C_n) = \mu(\cup_{n=1}^{\infty} C_n) = \mu(\cup_{n=1}^{\infty} A_n) - \mu(\cup_{n=1}^{\infty} B_n) < \infty.$$

□

7.5 Hahn and Jordan decompositions of a charge

Let (X, \mathcal{F}, μ) be a space with a bounded from below charge.

We say that $A \in \mathcal{F}$ is positive iff $B \in \mathcal{F}$, $B \subset A$ implies $\mu(B) \geq 0$. We say that $A \in \mathcal{F}$ is negative iff $B \in \mathcal{F}$, $B \subset A$ implies $\mu(B) \leq 0$. Let \mathcal{F}_{μ}^{\pm} denote the family of positive/negative sets.

Theorem 7.7 \mathcal{F}_{μ}^{\pm} are σ -rings and ideals of \mathcal{F} . $\pm\mu$ restricted to \mathcal{F}_{μ}^{\pm} are measures.

For $A \in \mathcal{F}$, we set

$$\mu_+(A) := \sup\{\mu(B) : B \subset A, B \in \mathcal{F}_{\mu}^+\},$$

$$\mu_-(A) := \sup\{-\mu(B) : B \subset A, B \in \mathcal{F}_{\mu}^-\}.$$

Theorem 7.8 (1) μ_-, μ_+ are measures.

(2) On \mathcal{F}_{μ}^{\pm} , μ coincides with $\pm\mu_{\pm}$.

(3) μ_- is finite.

(4) There exists $E \in \mathcal{F}_{\mu}^-$ with $\mu_-(E) = \mu_-(X)$. In what follows we fix such a set E .

(5) If $\mu(A) < \infty$, then $\mu_+(A)$ is finite.

(6) $X \setminus E \in \mathcal{F}_{\mu}^+$.

(7) (**Jordan decomposition**) $\mu = \mu_+ - \mu_-$.

(8) (**Hahn decomposition**) For $A \in \mathcal{F}$,

$$\mu_-(A) = -\mu(A \cap E), \quad \mu_+(A) = \mu(A \setminus E).$$

(9)

$$\mu_+(A) := \sup\{\mu(B) : B \subset A, B \in \mathcal{F}\},$$

$$\mu_-(A) := \sup\{-\mu(B) : B \subset A, B \in \mathcal{F}\}.$$

Proof. (1) and (2) follow immediately from Theorem 7.1 (2).

Let $\beta := \mu_-(X)$. Then there exist negative E_1, E_2, \dots such that $\mu_-(E_n) \rightarrow \beta$. Since negative sets form a σ -ring, $E := \bigcup_{j=1}^{\infty} E_j \in \mathcal{F}_\mu^-$. Clearly, $\mu_-(E) \leq \mu_-(E_n)$. Hence $\beta = \mu_-(E)$. This implies (4) and (3). A similar argument yields (5).

We interrupt the proof. \square

Lemma 7.9 *Suppose that $\mu(X) < \mu_+(X)$. Then there exists $B \in \mathcal{F}_\mu^-$ with $\mu(B) < 0$.*

Proof. If $\mu_+(X) = 0$, then $X \in \mathcal{F}_\mu^-$ and $\mu(X) < 0$. We can thus set $B := X$

The condition $\mu(X) < \infty$ implies that $\mu_+(X) < \infty$.

Let $\mu_+(X) > 0$. We can find $q < 1$ such that

$$\mu(X) - q\mu_+(X) < 0.$$

We can find $E \in \mathcal{F}_\mu^+$ such that $\mu(E) \geq q\mu_+(X)$. Set $X_1 := X \setminus E$. Then

$$\mu_+(X_1) = \mu_+(X) - \mu_+(E) \leq (1 - q)\mu_+(X),$$

$$\mu(X_1) = \mu(X) - \mu(E) \leq \mu(X) - q\mu_+(X) < 0.$$

By induction, we can find a sequence of disjoint sets $E_1, \dots \in \mathcal{F}_\mu^+$ such that for $X_n := X_1 \setminus \bigcup_{j=1}^{n-1} E_j$, then

$$\mu(E_n) \geq q\mu_+(X_n).$$

(Note that $E_j \subset X_j$). Then

$$\begin{aligned} \mu_+(X_{n+1}) &= \mu_+(X_n) - \mu_+(E_n) \\ &= \mu_+(X_n) - \mu(E_n) \leq (1 - q)\mu_+(X_n). \end{aligned}$$

Hence,

$$\mu_+(X_n) \leq (1 - q)^n \mu_+(X_0).$$

Moreover,

$$\mu(X_{n+1}) = \mu(X_1) \setminus \sum_{j=1}^n \mu(E_j) \leq \mu(X_1) < 0.$$

Set $B := \bigcap_j X_j$. Then

$$\mu_+(B) = \lim_{j \rightarrow \infty} \mu_+(X_j) = 0,$$

and hence $B \in \mathcal{F}_\mu^-$, and

$$\mu(B) = \lim_{j \rightarrow \infty} \mu(X_j) \leq \mu(X_1) < 0.$$

\square

Continuation of the proof of Theorem 7.8. Suppose that $X \setminus E$ is not positive. This means that there exists $X_0 \subset X \setminus E$ and $\mu(X_0) < 0$. Then X_0 satisfies the conditions of Lemma 7.9. Hence X_0 contains $B \in \mathcal{F}_\mu^-$ with $\mu(B) < 0$. Hence $E \cup B \in \mathcal{F}_\mu^-$ with $\mu(E \cup B) < \beta$, which is a contradiction. This proves (6).

Now note that for $B \in \mathcal{F}_\mu^+$ we have $\mu(B) = \mu_+(B \setminus E)$. Hence for $A \in \mathcal{F}$,

$$\mu_+(A) = \sup\{\mu_+(B \setminus E) : B \subset A, B \in \mathcal{F}_\mu^+\} = \mu_+(A \setminus E) = \mu(A \setminus E).$$

This proves (8) and (7).

$$\begin{aligned}\mu_+(A) &\leq \sup\{\mu(B) : B \subset A, B \in \mathcal{F}\}, \\ &\leq \sup\{\mu_+(B) : B \subset A, B \in \mathcal{F}\} = \mu_+(A).\end{aligned}$$

This proves (9). \square

7.6 Banach space of finite charges

Let (X, \mathcal{F}) be a set with a σ -field. We define $\text{Ch}(X, \mathcal{F})$ (or $\text{Ch}(X)$) as the ordered linear space of finite charges on (X, \mathcal{F}) . We set

$$\|\mu\| := |\mu|(X).$$

Theorem 7.10 (1) $\text{Ch}(X, \mathcal{F})$ is a Banach space.

(2) $\text{Ch}(X, \mathcal{F})$ is a complete lattice.

(3) $0 \leq \mu \leq \nu$ implies $\|\mu\| \leq \|\nu\|$.

7.7 Measures with a density

Theorem 7.11 Let (X, \mathcal{F}, μ) be a space with a measure. Let $f \in \mathcal{M}_+(X)$. Then

$$\mathcal{F} \ni A \mapsto \nu(A) := \int 1_A f d\mu \tag{7.37}$$

is a measure. If $f \in \mathcal{M}(X)$ and $f_- \in \mathcal{L}^1(\mu)$, then (7.37) is a bounded from below charge.

Definition 7.12 The measure ν is called the measure with the density f wrt the measure μ and is denoted $\nu = f\mu$. We will also write $f := \frac{d\nu}{d\mu}$.

Theorem 7.13 (1) For $f, g \in \mathcal{M}_+(X)$ we have

$$f = g \text{ } \mu\text{-a.e.} \Rightarrow f\mu = g\mu.$$

(2) If $f\mu$ is sum-finite, then the converse implication is also true.

Proof. The implication \Rightarrow is obvious. Let us show the converse statement.

First assume that $f\mu$ is finite, or in other words $f \in \mathcal{L}^1(\mu)$. Let $N := \{f < g\}$ and

$$h := g1_N - f1_N.$$

Clearly, $f1_N \leq f$ and $g1_N \leq g$. Hence, $f1_N \in \mathcal{L}^1(\mu)$, $g1_N \in \mathcal{L}^1(\mu)$. Therefore, $h \in \mathcal{L}^1(\mu)$. Besides, $\int h d\mu = 0$ and $h \geq 0$. Thus $h = 0$ μ -a.e. But $N = \{h > 0\}$. Hence $\mu(N) = 0$.

Assume now that μ is sum-finite. Let X_i be a localizing family. Then $f\mu = g\mu$ restricted to X_i . Hence $f_i = g_i$ on X_i almost everywhere wrt the measure μ restricted to X_i . This implies that $f = g$ μ -a.e. \square

Recall that the charge ν is called continuous wrt μ (or μ -continuous), if

$$\mu(N) = 0 \Rightarrow \nu(N) = 0, \quad N \in \mathcal{F}.$$

Theorem 7.14 (Radon-Nikodym) Let μ be a sum-finite measure on (X, \mathcal{F}) and let ν be a charge. Then the following conditions are equivalent:

- (1) *there exists $f \in \mathcal{M}(X)$ such that $\nu = f\mu$ and $f_- \in \mathcal{L}^1(\mu)$;*
(2) *ν is μ -continuous.*

Proof. The implication \Rightarrow is obvious. Let us show the converse statement.

Step 0. If $\mu = 0$, then $\nu = 0$, and the theorem is obviously true.

Step 1. Assume that $0 < \mu(X) < \infty$, $\nu(X) < \infty$. Let

$$\mathcal{G} := \{g \in \mathcal{M}_+(\mathcal{F}) : g\mu \leq \nu\}.$$

Clearly, \mathcal{G} is non-empty, since $0 \in \mathcal{G}$.

We have

$$g, h \in \mathcal{G} \Rightarrow \sup(g, h) \in \mathcal{G}.$$

In fact, if $A_1 := \{g < h\}$, $A_2 := \{g \geq h\}$ and $A \in \mathcal{F}$, then

$$\int_A \sup(g, h) d\mu = \int_{A \cap A_1} g d\mu + \int_{A \cap A_2} h d\mu \leq \nu(A \cap A_1) + \nu(A \cap A_2) = \nu(A).$$

Let

$$\gamma := \sup\{\int g d\mu : g \in \mathcal{G}\}. \quad (7.38)$$

Then $\gamma \leq \nu(X) < \infty$. We can find $g'_n \in \mathcal{G}$ such that

$$\lim_{n \rightarrow \infty} \int g'_n d\mu = \gamma.$$

Let

$$f := \sup(g'_n)_{n \in \mathbb{N}}.$$

We claim that

$$f \in \mathcal{G} \text{ and } \int f d\mu = \gamma. \quad (7.39)$$

In fact, we have

$$g_n := \sup(g'_1, \dots, g'_n) \in \mathcal{G}, \quad g_n \nearrow f,$$

which immediately implies (7.39).

Suppose now that

$$(f\mu)(X) < \nu(X). \quad (7.40)$$

Using $\mu(X) < \infty$, we can find $\beta > 0$ such that

$$\beta\mu(X) < \nu(X) - (f\mu)(X).$$

Set

$$\rho(A) := \nu(A) - (f\mu)(A) - \beta\mu(A), \quad A \in \mathcal{F}.$$

ρ is a bounded μ -continuous charge satisfying $\rho(X) > 0$. By Lemma 7.9, we can find a ρ -positive set $E \in \mathcal{F}$ such that $\rho(E) > 0$. Recall that ρ -positivity of E means that

$$A \in 2^E \cap \mathcal{F} \Rightarrow \rho(A) \geq 0.$$

Hence $f_0 = f + \beta 1_E \in \mathcal{G}$.

Note that the μ -continuity of the charge ρ and $\rho(E) > 0$ implies $\mu(E) > 0$. Hence

$$\int f_0 d\mu = \int f d\mu + \beta\mu(E) = \gamma + \beta\mu(E) > \gamma,$$

which is a contradiction with (7.38).

Step 2. ν is σ -finite, μ is finite. We decompose X into a disjoint union of sets of finite measure ν and use **Step 1**.

Step 3. ν is arbitrary, μ is finite. Let $\mathcal{F}_\nu^f := \{A \in \mathcal{F} : \nu(A) < \infty\}$ and $\alpha := \sup\{\mu(A) : A \in \mathcal{F}_\nu^f\}$. We can find $A_n \in \mathcal{F}_\nu^f$ such that $\lim_{n \rightarrow \infty} \mu(A_n) = \alpha$. We can assume that $A_n \nearrow X_0$. Then ν on X_0 is σ -finite and on $X_1 := X \setminus X_0$ it has the property

$$\begin{aligned} \text{or } \mu(A) = \nu(A) = 0, \\ \text{or } \mu(A) > 0, \nu(A) = \infty. \end{aligned}$$

In fact, if $A \subset X_1$, $\mu(A) > 0$ and $\nu(A) < \infty$, then $A_n \cup A \in \mathcal{F}_\nu^f$ and $\mu(A_n \cup A) \nearrow \alpha + \mu(A)$, which means $\mu(A) = 0$. We apply **Step 2**. to X_0 and on X_1 we put $\nu = \infty\mu$.

Step 4. ν is arbitrary and μ sum-finite. We decompose X into a union of disjoint sets with a finite measure μ and use **Step 3**. \square

7.8 Dual of $L^p(\mu)$

Theorem 7.15 Let (X, \mathcal{F}, μ) be a space with a measure, $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L^q(\mu)$ set

$$\langle v_g | f \rangle := \int g f d\mu, \quad f \in L^p(\mu). \quad (7.41)$$

Then

(1) $v_g \in (L^p(\mu))^\#$ and $\|v_g\| = \|g\|_q$. Thus

$$L^q(\mu) \ni g \mapsto v_g \in (L^p(\mu))^\# \quad (7.42)$$

is an isometry.

(2)

$$\tilde{f} := \bar{f}|f|^{p-2} / \left(\int |f|^p \right)^{1/q} \quad (7.43)$$

$v_{\tilde{f}}$ is a functional tangent to f .

(3) Let $1 \leq p < \infty$ and let μ be sum-finite. Then for v , a bounded functional on $L^p(\mu)$, there exists a unique $g \in L^q$ such that (7.41) holds. Thus, the map (7.42) is bijective.

Proof. (1) Using the Hölder inequality we check that if $g \in L^q(\mu)$, then $v_g \in (L^p(\mu))^\#$ and $\|v_g\| \leq \|g\|_q$. Setting

$$\tilde{g} := \bar{g}|g|^{q-2} / \left(\int |g|^q \right)^{1/p}, \quad (7.44)$$

we see that $\langle v_g | \tilde{g} \rangle = \|g\|_q$ and $\|\tilde{g}\|_p = 1$. Hence $\|v_g\| = \|g\|_q$.

(3) To prove the existence first assume that the measure is finite and $v \in (L^p(\mu))^\#$. Then

$$\mathcal{F} \ni A \mapsto \langle v | 1_A \rangle$$

is a μ -continuous finite charge. By the Radon-Nikodym Theorem, there exists $g \in \mathcal{M}(X)$ such that

$$\langle v | 1_A \rangle = \int g 1_A d\mu,$$

Assume that $g \notin L^q(\mu)$. Clearly, $g_+ \notin L^q(\mu)$ or $g_- \notin L^q(\mu)$. Hence it is sufficient to assume that $g \geq 0$.

We can find $g_n \in \mathcal{E}_+(X)$ such that $g_n \nearrow g$. Clearly, $\|g_n\|_q \rightarrow \infty$. Set $\tilde{g}_n := g_n^{q-1}/(\int |g_n|^q)^{1/p}$. Clearly, $\tilde{g}_n \in L^p(\mu)$, $\|\tilde{g}_n\| = 1$ and

$$\langle v|\tilde{g}_n \rangle = \int g g_n^{q-1} d\mu / (\int |g_n|^q)^{1/p} \geq \|g_n\|_q \rightarrow \infty.$$

Hence v is not bounded.

Thus $g \in L^q(\mu)$. We already know that v_g is bounded and it coincides with v on $\mathcal{E}(X)$, which is dense in $L^p(\mu)$. Hence (7.41) is true for all $f \in L^p(\mu)$

The uniqueness follows from Theorem (7.13) (2). \square

Theorem 7.16 *Let μ be sum-finite. Then $L^\infty(\mu)$ is a complete lattice.*

8 Measures on topological spaces

8.1 δ -open and σ -closed sets

If X is a topological space, we will write $\text{Open}(X)$, $\text{Open}_0(X)$, $\text{Closed}(X)$ and $\text{Compact}(X)$ for the family of open, open pre-compact, closed and compact subsets of X .

Definition 8.1 *δ -open sets are countable intersection of open sets.*

σ -closed sets are countable unions of closed sets.

The complement of a σ -closed set is a δ -open set and vice versa.

Theorem 8.2 *Let X be a metrizable space. Then every closed set is δ -open.*

Proof. Let C be closed. Define

$$C_n := \{x \in X : d(x, C) < 1/n\}.$$

Then C_n are open and

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Hence C is a δ -open set. \square

8.2 Baire and Borel sets of 1st kind

Theorem 8.3 (1) *Let $f \in C(X, \mathbb{R})$. Then $f^{-1}(]a, \infty[) \in \text{Open} \cap \sigma\text{-Closed}(X)$.*

(2) *Let X be normal and $A \in \text{Open} \cap \sigma\text{-Closed}(X)$. Then there exists $f \in C(X, \mathbb{R})$ such that $A = f^{-1}(]a, \infty[)$*

Proof. We have

$$f^{-1}(]a, \infty[) = \bigcup_{n=1}^{\infty} f^{-1}([a + 1/n, \infty[) \tag{8.45}$$

Clearly, (8.45) are σ -closed.

Let

$$A = \bigcup_{n=1}^{\infty} A_n$$

be open σ -closed and A_n let be closed. We can then find $f_n \in C(X)$ such that $0 \leq f \leq 1$, $f_n = 1$ on A_n and $f_n = 0$ outside A . We define

$$f := \sum_{n=1}^{\infty} 2^{-n} f_n.$$

Then

$$A = f^{-1}(]0, \infty[).$$

□

Definition 8.4 Let X be a topological space. Then the σ -field of Baire sets of 1st kind, denoted $\text{Baire}_1(X)$, is the smallest σ -field such that all elements of $C(X, \mathbb{R})$ are measurable.

Theorem 8.5 Let X be normal, $A \subset B \subset X$, A be closed and B open. Then there exist $A_0 \in \text{Closed} \cap \text{Baire}_1(X)$ and $B_0 \in \text{Open} \cap \text{Baire}_1(X)$ such that $A \subset B_0 \subset A_0 \subset B$.

Proof. We can find $f \in C(X, \mathbb{R})$ such that $f = 0$ on A and $f = 1$ on $X \setminus B$. Then $f^{-1}(] - \infty, \frac{1}{2}[) \in \text{Open} \cap \text{Baire}_1(X)$ and $f^{-1}(] - \infty, \frac{1}{2}[) \in \text{Closed} \cap \text{Baire}_1(X)$. □

Theorem 8.6 (1) In any topological space, $\text{Baire}_1(X)$ is generated e.g. by

$$\begin{aligned} & \{f^{-1}(U) : U \in \text{Open}(\mathbb{R}), f \in C(X, \mathbb{R})\}, \\ & \{f^{-1}(C) : C \in \text{Closed}(\mathbb{R}), f \in C(X, \mathbb{R})\}, \\ & \{f^{-1}(] \alpha, \infty[, \alpha \in \mathbb{R}, f \in C(X, \mathbb{R})\}. \end{aligned}$$

(2) If X is normal, then $\text{Baire}_1(X)$ is generated by

$$\begin{aligned} & \text{Open} \cap \sigma\text{-Closed}(X), \\ & \text{Closed} \cap \delta\text{-Open}(X). \end{aligned}$$

Theorem 8.7 Let X be compact Hausdorff. Then

- (1) $\text{Open} \cap \text{Baire}_1(X) = \text{Open} \cap \sigma\text{-Closed}(X)$,
- (2) $\text{Closed} \cap \text{Baire}_1(X) = \text{Closed} \cap \delta\text{-Open}(X)$.

Proof. It is sufficient to prove (2) \subset .

Let C be closed Baire. By Theorem 1.8, there exists a countable family C_1, C_2, \dots of σ -closed sets such that $C \in \sigma\text{-Ring}(C_1, C_2, \dots)$. We can find functions $f_n \in C(X)$ such that $\{f_n = 0\} = C_n$. Then

$$d(x, y) := \sum_{n=1}^{\infty} 2^{-n} |f_n(x) - f_n(y)|$$

is a semimetric on X .

Let (\tilde{X}, \tilde{d}) be the reduced metric space and $T : X \rightarrow \tilde{X}$ the corresponding reduction. Clearly, $x \in C_n$, $d(x, y) = 0$ imply $y \in C_n$. Therefore, $T^*2^{\tilde{X}}$ contains C_1, C_2, \dots

$T^*2^{\tilde{X}}$ is a σ -ring. Hence

$$T^*2^{\tilde{X}} \supset \sigma\text{-Ring}(C_1, C_2, \dots).$$

Thus there exists $\tilde{C} \in 2^{\tilde{X}}$ with $C = T^{-1}\tilde{C}$. But C is compact and T continuous. Therefore, \tilde{C} is compact as well. Thus \tilde{C} is a closed subset of a metric space, and hence it is σ -open. Hence there exist open $\tilde{U}_1, \tilde{U}_2, \dots$ in \tilde{X} with $\tilde{U}_n \searrow \tilde{C}$. Now $T^{-1}\tilde{U}_n$ are open in X and $T^{-1}\tilde{U}_n \searrow C$. □

Definition 8.8 σ -field of Borel sets of 1st kind, denoted $\text{Borel}_1(X)$, is the σ -field generated by open sets.

Clearly, $\text{Baire}_1(X) \subset \text{Borel}_1(X)$.

Theorem 8.9 If X is metrizable, then $\text{Baire}_1(X) = \text{Borel}_1(X)$.

Proof. In a metrizable space every open set is σ -closed. \square

Example 8.10 Let I be uncountable and X_i be sets of at least two elements. Let $X = \prod_{i \in I} X_i$. Then one-element sets in X are closed (hence Borel) but not δ -open (hence not Baire). In fact, let $x \in Y \subset X$ and Y be δ -open. Then Y contains a subset of the form $\prod_{i \in I} Y_i$ with $Y_i = X_i$ for all but a countable number of $i \in I$.

8.3 Baire and Borel sets of 2nd kind

Theorem 8.11 (1) Let $f \in C_c(X, \mathbb{R})$ and $\alpha \geq 0$. Then $f^{-1}(]0, \infty]) \in \text{Open} \cap \sigma\text{-Compact}(X)$.

(2) Let X be Tikhonov and $A \in \text{Open} \cap \sigma\text{-Compact}(X)$. Then there exists $f \in C_c(X, \mathbb{R})$ such that $A = f^{-1}(]0, \infty])$

Definition 8.12 Let X be a topological space. Then the σ -ring of Baire sets of 2nd kind, denoted $\text{Baire}_2(X)$, is the smallest σ -ring such that all elements of $C_c(X)$ are measurable.

Lemma 8.13 Let X be Tikhonov, $A \subset B \subset X$, A be compact and B open. Then there exist $A_0 \in \text{Compact} \cap \text{Baire}_2(X)$ and $B_0 \in \text{Open} \cap \text{Baire}_2(X)$ such that $A \subset B_0 \subset A_0 \subset B$.

Theorem 8.14 (1) $\text{Baire}_2(X)$ is generated by

$$\begin{aligned} &\{f^{-1}(U) : U \in \text{Open}(\mathbb{R} \setminus \{0\}), f \in C_c(X)\}; \\ &\{f^{-1}(C) : K \in \text{Closed}(\mathbb{R} \setminus \{0\}), f \in C_c(X)\}, \\ &\{f^{-1}(] \alpha, \infty]), f^{-1}(] - \infty, -\alpha]), 0 < \alpha, f \in C_c(X)\}. \end{aligned}$$

(2) The closures of all elements of $\text{Baire}_2(X)$ are σ -compact.

(3) If X is Tikhonov, then $\text{Baire}_2(X)$ is generated by

$$\begin{aligned} &\text{Open} \cap \sigma\text{-Compact}(X), \\ &\text{Compact} \cap \delta\text{-Open}(X). \end{aligned}$$

Theorem 8.15 Let X be Tikhonov. Then

(1) $\text{Compact} \cap \text{Baire}_2(X) = \text{Compact} \cap \delta\text{-Open}(X)$.

(2) $\text{Open} \cap \text{Baire}_2(X) = \text{Open} \cap \sigma\text{-Compact}(X)$

Definition 8.16 σ -ring of Borel sets of 2nd kind, denoted $\text{Borel}_2(X)$, is the σ -ring generated by compact sets.

Clearly, $\text{Baire}_2(X) \subset \text{Borel}_2(X)$.

Theorem 8.17 If X is metrizable, then $\text{Baire}_2(X) = \text{Borel}_2(X)$.

Theorem 8.18 For σ -compact spaces $\text{Baire}_1(X) = \text{Baire}_2(X)$ and $\text{Borel}_1(X) = \text{Borel}_2(X)$.

In what follows, we will consider σ -rings of sets only on locally compact Hausdorff spaces. We will use the σ -rings $\text{Baire}_2(X)$ and $\text{Borel}_2(X)$. We will call them simply Baire and Borel σ -rings and denote by $\text{Baire}(X)$ and $\text{Borel}(X)$ respectively. For σ -compact spaces they are in fact σ -fields and coincide with $\text{Baire}_1(X)$ and $\text{Borel}_1(X)$ respectively.

8.4 Baire measures on compact spaces

Let X be a compact Hausdorff space. A finite measure on $\text{Baire}(X)$ is called a Baire measure on X .

A linear functional $\lambda : C(X) \rightarrow \mathbb{R}$ is called a positive functional (or a Radon measure) if

$$f \in C(X), f \geq 0 \Rightarrow \lambda(f) \geq 0.$$

Theorem 8.19 Let ν be a Baire measure. Then

$$C(X) \ni f \mapsto \int f d\nu \in \mathbb{R} \tag{8.46}$$

is a positive linear functional.

(1) If $C \in \text{Closed} \cap \text{Baire}(X)$, then

$$\nu(C) = \inf\{\int f d\nu : f \in C(X), f = 1 \text{ on } C, 0 \leq f \leq 1\}.$$

(2) If $U \in \text{Open} \cap \text{Baire}(X)$, then

$$\nu(U) = \sup\{\int f d\nu : f \in C(X), \text{supp} f \subset U, 0 \leq f \leq 1\}.$$

Proof. The positivity is obvious.

Let us prove (1). The inequality \leq is obvious.

There exists a sequence $U_1, U_2, \dots \in \text{Open}(X)$ such that $U_n \searrow C$. Let $f_n \in C(X)$, $\text{supp} f_n \subset U_n$, $0 \leq f_n \leq 1$, and $f_n = 1$ on C . Then $f_n \rightarrow 1_C$ pointwise, $f_n \leq 1 \in \mathcal{L}^1(\nu)$. Hence by the Lebesgue theorem

$$\lim_{n \rightarrow \infty} \int f_n d\nu = \nu(C).$$

This shows the inequality \geq . \square

Theorem 8.20 (Riesz-Markov) Let λ be a positive linear functional on $C(X)$. Then there exists a unique Baire measure ν satisfying

$$\lambda(f) = \int f d\nu, \quad f \in C(X). \tag{8.47}$$

The proof of Theorem 8.20 will be split into a number of steps. Let us assume that we are given a positive functional λ .

Lemma 8.21 A Baire measure satisfying (8.47) is uniquely determined.

Proof. By Theorem 8.19 (1), ν is uniquely determined by λ on $\text{Closed} \cap \text{Baire}(X)$. This is a \cap -stable family that generates $\text{Baire}(X)$. Hence ν is uniquely determined. \square

For $U \in \text{Open} \cap \text{Baire}(X)$ we set

$$\nu^*(U) := \sup\{\lambda(f) : f \in C(X), \text{supp}f \subset U, 0 \leq f \leq 1.\} \quad (8.48)$$

For any $A \in 2^X$ we set

$$\nu^*(A) := \inf\{\nu^*(U) : A \subset U, U \in \text{Open} \cap \text{Baire}(X)\}. \quad (8.49)$$

(For $U \in \text{Open} \cap \text{Baire}(X)$, (8.48) agrees with (8.49)).

Lemma 8.22 ν^* is an external measure.

Proof. We need to show that

$$\nu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \nu^*(A_j).$$

If the sum on the right is infinite, the inequality is obvious. Assume it is finite. We will find $U_j \in \text{Open} \cap \text{Baire}(X)$ with $A_j \subset U_j$ and

$$\nu^*(U_j) \leq \nu^*(A_j) + 2^{-j}\epsilon.$$

Let $f \in C(X)$ satisfy $0 \leq f \leq 1$,

$$\text{supp}f \subset \bigcup_{j=1}^{\infty} A_j \text{ i } \nu^*\left(\bigcup_{j=1}^{\infty} A_j\right) < \lambda(f) + \epsilon.$$

By the compactness of $\text{supp}f$, for some n ,

$$\text{supp}f \subset \bigcup_{j=1}^n U_j.$$

We will find $h_1, \dots, h_n \in C(X)$ such that $h_j \geq 0$, $\text{supp}h_j \subset U_j$ and $\sum_{j=1}^n h_j = 1$ on $\text{supp}f$. Hence,

$$\begin{aligned} \nu^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \nu^*\left(\bigcup_{n=1}^{\infty} U_n\right) \\ &\leq \lambda(f) + \epsilon = \sum_{j=1}^n \lambda(fh_j) + \epsilon \\ &\leq \sum_{n=1}^{\infty} \nu^*(U_n) + \epsilon \\ &\leq \sum_{n=1}^{\infty} \nu^*(A_n) + \epsilon \sum_{n=1}^{\infty} 2^{-n} + \epsilon = \sum_{n=1}^{\infty} \nu^*(A_n) + 2\epsilon. \end{aligned}$$

\square

Lemma 8.23 All Baire sets on X are ν^* -measurable.

Proof. It suffices to show that open Baire sets are ν^* -measurable. Let U be open and $Q \in 2^X$.

Let $A \in \text{Open} \cap \text{Baire}(X)$ such that $Q \subset A$ and $\nu^*(A) \leq \nu^*(Q) + \epsilon$. Consider the set $A_1 := A \cap U \in \text{Open} \cap \text{Baire}(X)$. Let $f_1 \in C(A)$ such that $0 \leq f_1 \leq 1$, $\text{supp}f_1 \subset A_1$ and

$$\nu^*(A_1) \leq \lambda(f_1) + \epsilon.$$

Next consider set $A_2 := A \setminus \text{supp} f_1 \in \text{Open}$. We will find $f_2 \in C(X)$ such that $0 \leq f_2 \leq 1$, $\text{supp} f_2 \subset A_2$ and

$$\nu(A_2) \leq \lambda(f_2) + \epsilon.$$

We have $0 \leq f_1 + f_2 \leq 1$ and $\text{supp}(f_1 + f_2) \subset A$. Besides, $Q \cap U \subset A_1$ and $Q \setminus U \subset A_2$. Hence

$$\begin{aligned} & \nu^*(Q \cap U) + \nu^*(Q \setminus U) \\ & \leq \nu^*(A_1) + \nu^*(A_2) \\ & \leq \lambda(f_1) + \lambda(f_2) + 2\epsilon = \lambda(f_1 + f_2) + 2\epsilon \\ & \leq \nu^*(A) + 2\epsilon \leq \nu^*(Q) + 3\epsilon. \end{aligned}$$

Hence,

$$\nu^*(Q \cap U) + \nu^*(Q \setminus U) \leq \nu^*(A).$$

Hence U is ν^* -measurable. \square

Set ν to be equal to ν^* restricted to $\text{Baire}(X)$. By Lemmas 8.22 and 8.23 it is a Baire measure.

Lemma 8.24 *Let $f \in C(X)$. Then*

$$\lambda(f) = \int f d\nu.$$

Proof. We can assume that $0 \leq f \leq 1$. Set

$$U_{n,j} := \{f > j/n\}, \quad C_{n,j} := \{f \geq j/n\}.$$

Let $g_{n,j} \in C(X)$ with $\text{supp} g_{n,j} \subset U_{n,j}$ and $0 \leq g_{n,j} \leq 1$. Set

$$g_n := \frac{1}{n} \sum_{j=1}^{n-1} g_{n,j}.$$

Then $g_n \leq f$. Hence

$$\lambda(f) \geq \lambda(g_n) = \frac{1}{n} \sum_{j=1}^{n-1} \lambda(g_{n,j}).$$

Thus

$$\lambda(f) \geq \frac{1}{n} \sum_{j=1}^{n-1} \nu(U_{n,j}) = \int f_n d\nu,$$

where $f_n := \frac{1}{n} \sum_{j=1}^{n-1} 1_{U_{n,j}}$. But $f_n \rightarrow f$ and $0 \leq f_n \leq 1$. Hence $\int f_n d\nu \rightarrow \int f d\nu$. Thus

$$\lambda(f) \geq \int f d\nu.$$

We will find a sequence of open sets W_n such that $W_n \searrow \text{supp} f$. Choose functions $\tilde{g}_{n,j}$ such that $\text{supp} \tilde{g}_{n,0} \subset W_n$, $\tilde{g}_{n,0} = 1$ on $\text{supp} f$; $\text{supp} \tilde{g}_{n,j} \subset U_{n,j}$, $\tilde{g}_{n,j} = 1$ on $C_{n,j+1}$. Set $\tilde{g}_n := \frac{1}{n} \sum_{j=0}^{n-1} \tilde{g}_{n,j}$. Then $f \leq \tilde{g}_n$. Thus

$$\begin{aligned} \lambda(f) & \leq \lambda(\tilde{g}_n) = \frac{1}{n} \sum_{j=0}^{n-1} \lambda(\tilde{g}_{n,j}) \\ & \leq \frac{1}{n} \nu(W_n) + \frac{1}{n} \sum_{j=0}^{n-1} \nu(U_{n,j}) = \frac{1}{n} \nu(W_n) + \int \tilde{f}_n d\nu, \end{aligned}$$

where $\tilde{f}_n := \frac{1}{n} \sum_{j=1}^{n-1} 1_{U_{n,j}}$. We have $\tilde{f}_n \rightarrow f$ and $0 \leq \tilde{f}_n \leq 1$. Hence, by the Lebesgue theorem, $\int \tilde{f}_n d\nu \rightarrow \int f d\nu$. Hence

$$\lambda(f) \leq \int f d\nu.$$

□

Proof of Theorem 8.20. We define the Baire measure ν as described before Lemma 8.24. By Lemma 8.24, $\lambda(f) = \int f d\nu$. By Lemma 8.21, ν is uniquely defined. □

Theorem 8.25 *Let ν be a Baire measure on X . Then it satisfies the following regularity properties:*

- (1) $\nu(A) = \inf\{\nu(U) : A \subset U, U \in \text{Open} \cap \text{Baire}(X)\}, \quad A \in \text{Baire}(X);$
- (2) $\nu(A) = \sup\{\nu(C) : C \subset A, C \in \text{Closed} \cap \text{Baire}(X)\} \quad A \in \text{Baire}(X).$

Proof. We define λ by the formula (8.46). We construct the corresponding ν^* . By construction, it satisfies

$$\nu^*(A) = \inf\{\nu^*(U) : A \subset U, U \in \text{Open} \cap \text{Baire}(X)\}.$$

But on $\text{Baire}(X)$ ν^* coincides with ν . Hence it satisfies the property (1). □

8.5 Borel measures on compact spaces

Let X be a compact Hausdorff space. A finite measure on $\text{Borel}(X)$ is called a Borel measure on X .

Theorem 8.26 *Let μ be a Borel measure on X . The following conditions are equivalent:*

- (1) $\mu(A) = \inf\{\mu(U) : A \subset U, U \in \text{Open}(X)\}, \quad A \in \text{Borel}(X).$
- (2) $\mu(A) = \sup\{\mu(C) : C \subset A, C \in \text{Closed}(X)\}, \quad A \in \text{Borel}(X).$

If the above conditions are satisfied then μ is called a regular Borel (or Radon) measure on X .

Proof of Theorem 8.26 Using $\mu(X) < \infty$, we get

$$\mu(A) = \mu(X) - \mu(X \setminus A),$$

$$\inf\{\mu(U) : U \in \text{Open}(X)\} = \mu(X) - \sup\{\mu(C) : C \in \text{Closed}(X)\}.$$

□

Theorem 8.27 *Let ν be a Baire measure on X . Then there exists a unique regular Borel measure μ extending ν . It has the following properties:*

- (1) *If $U \in \text{Open}(X)$, then*

$$\mu(U) = \sup\{\nu(C) : C \subset U, C \in \text{Baire} \cap \text{Closed}(X)\};$$

- (2) *If $C \in \text{Closed}(X)$, then*

$$\mu(C) = \inf\{\nu(U) : C \subset U, U \in \text{Baire} \cap \text{Open}(X)\}.$$

Theorems 8.19 and 8.27 imply the following version of the Riesz-Markov theorem:

Theorem 8.28 *Let λ be a positive linear functional on $C(X)$. Then there exists a unique regular Borel measure μ satisfying*

$$\lambda(f) = \int f d\mu, \quad f \in C(X). \quad (8.50)$$

Proof of Theorem 8.27 Let us prove (2).

The inequality \leq is obvious.

For any $U \in \text{Open}(X)$ such that $C \subset U$, There exists an open Baire U_1 such that $C \subset U_1 \subset U$. Therefore,

$$\begin{aligned} \mu(C) &= \inf\{\nu(U) : C \subset U, U \in \text{Open}(X)\} \\ &\geq \inf\{\nu(U) : C \subset U_1, U_1 \in \text{Open} \cap \text{Baire}(X)\}. \end{aligned}$$

This proves the \geq inequality.

It follows from (2) that μ is uniquely determined on the family of closed sets. But this family is \cap -stable and generates $\text{Borel}(X)$. Hence μ is uniquely determined.

Let us now describe the proof of the existence of μ . Define λ as in (). Then for $U \in \text{Open}(X)$ we set

$$\mu^*(U) := \sup\{\lambda(f) : f \in C(X), \text{supp } f \subset U, 0 \leq f \leq 1.\} \quad (8.51)$$

For any $A \in 2^X$ we set

$$\mu^*(A) := \inf\{\nu^*(U) : A \subset U, U \in \text{Open}(X)\}. \quad (8.52)$$

(For $U \in \text{Open}(X)$, (8.51) agrees with (8.52)).

Exactly as in the previous subsection, we show that μ^* is an external measure and Borel sets are μ^* -measurable. We define μ to be the restriction of μ^* to $\text{Borel}(X)$. By (8.52), it is a regular Borel measure.

Clearly, if $U \in \text{Open} \cap \text{Baire}(X)$, then

$$\nu(U) = \nu^*(U) = \mu^*(U) = \mu(U).$$

Thus ν coincides with μ on $\text{Open} \cap \text{Baire}(X)$. But this is a \cap -closed family generating $\text{Baire}(X)$. Hence ν coincides with μ on $\text{Baire}(X)$. \square

8.6 Baire measures on locally compact spaces

Let X be a locally compact space. A measure on $\text{Baire}(X)$ finite on compact sets is called a Baire measure on X .

A linear functional $\lambda : C_c(X) \rightarrow \mathbb{R}$ is called a positive functional if

$$f \in C_c(X), f \geq 0 \Rightarrow \lambda(f) \geq 0.$$

Theorem 8.29 *Let ν be a Baire measure. Then*

$$C_c(X) \ni f \mapsto \int f d\nu \in \mathbb{R} \quad (8.53)$$

is a positive linear functional.

(1) *If $C \in \text{Compact} \cap \text{Baire}(X)$, then*

$$\nu(C) = \inf\{\int f d\nu : f \in C_c(X), f = 1 \text{ on } C, 0 \leq f \leq 1.\}.$$

(2) If $U \in \text{Open}_0 \cap \text{Baire}(X)$, then

$$\nu(U) = \sup\{\int f d\nu : f \in C_c(X), \text{supp}f \subset U, 0 \leq f \leq 1\}.$$

The following theorem is called the Riesz-Markov Theorem.

Theorem 8.30 *Let λ be a positive linear functional on $C_c(X)$. Then there exists a unique Baire measure ν satisfying*

$$\lambda(f) = \int f d\nu, \quad f \in C_c(X). \quad (8.54)$$

Theorem 8.31 *Let ν be a Baire measure on X . Then it satisfies the following regularity properties:*

- (1) $\nu(A) = \inf\{\nu(U) : A \subset U, U \in \text{Open}_0 \cap \text{Baire}(X)\}, \quad A \in \text{Baire}(X);$
- (2) $\nu(A) = \sup\{\nu(C) : C \subset A, C \in \text{Compact} \cap \text{Baire}(X)\} \quad A \in \text{Baire}(X).$

8.7 Borel measures on locally compact spaces

A measure on $\text{Borel}(X)$ finite on compact sets is called a Borel measure on X .

Theorem 8.32 *Let μ be a Borel measure on X . The following conditions are equivalent:*

- (1) $\mu(A) = \inf\{\mu(U) : A \subset U, U \in \text{Open}_0(X)\}, \quad A \in \text{Borel}(X).$
- (2) $\mu(A) = \sup\{\mu(C) : C \subset A, C \in \text{Compact}(X)\}, \quad A \in \text{Borel}(X).$

If the above conditions are satisfied then μ is called a regular Borel (or Radon) measure on X .

Theorem 8.33 *Let ν be a Baire measure on X . Then there exists a unique regular Borel measure μ extending ν . It has the following properties:*

- (1) If $U \in \text{Open}_0(X)$, then

$$\mu(U) = \sup\{\nu(C) : C \subset U, C \in \text{Baire} \cap \text{Compact}(X)\};$$

- (2) If $C \in \text{Compact}(X)$, then

$$\mu(C) = \inf\{\nu(U) : C \subset U, U \in \text{Baire} \cap \text{Open}(X)\}.$$

Theorems 8.29 and 8.33 imply the following version of the Riesz-Markov theorem:

Theorem 8.34 *Let λ be a positive linear functional on $C_c(X)$. Then there exists a unique regular Borel measure μ satisfying*

$$\lambda(f) = \int f d\mu, \quad f \in C_c(X). \quad (8.55)$$

9 Measures on infinite Cartesian products

9.1 Infinite Cartesian products

Let $X_i, i \in I$ be a family of sets. For any $K \subset J \subset I$ we can define the map

$$\pi^{KJ} : \prod_{j \in J} X_j \rightarrow \prod_{k \in K} X_k,$$

where for $x_J = (x_j)_{j \in J} \in \prod_{j \in J} X_j$, $\pi^{KJ} x_J$ is $(x_k)_{k \in K}$. Clearly, $M \subset K \subset J$ implies

$$\pi^{MK} \pi^{KJ} = \pi^{MJ}.$$

If $(X_i, \mathcal{F}_i), i \in I$ is a family of sets with σ -fields, then for $J \subset I$ we set $\bigotimes_{j \in J} \mathcal{F}_j$ to be the family of subsets of $\prod_{j \in J} X_j$ of the form $\prod_{j \in J} A_j$ with $A_j \in \mathcal{F}_j$ and $A_j = X_j$ for all but a finite number of $j \in J$. We set $\bigotimes_{j \in J} \mathcal{F}_j := \sigma\text{-Field}(\bigotimes_{j \in J} \mathcal{F}_j)$.

Clearly, the maps π^{KJ} for $K \subset J \subset I$ are measurable.

9.2 Compatible measures

Let $(X_i, \mathcal{F}_i), i \in I$ is a family of sets with σ -fields. Let $K \subset J \subset I$ and μ_J, μ_K are probabilistic measures on $(\prod_{j \in J} X_j, \bigotimes_{j \in J} \mathcal{F}_j)$ and $(\prod_{k \in K} X_k, \bigotimes_{k \in K} \mathcal{F}_k)$ respectively. We say that they are compatible iff $\pi_{KJ}^* \mu_J = \mu_K$, that means

$$\int f(x_K) d\mu_K(x_K) = \int f(x_K) d\mu_J(x_K, x_{J \setminus K}).$$

Theorem 9.1 *If μ_I is a measure on $(\prod_{i \in I} X_i, \bigotimes_{i \in I} \mathcal{F}_i)$, then for any $K \subset J \subset I$, the measures $\pi_{KI}^* \mu_I$ and $\pi_{JI}^* \mu_I$ are compatible.*

9.3 Infinite tensor product of measures

Theorem 9.2 *$(X_i, \mathcal{F}_i, \mu_i), i \in I$, be a family of spaces with probabilistic measures. Then there exists a unique measure $\bigotimes_{i \in I} \mu_i$ on $(\prod_{i \in I} X_i, \bigotimes_{i \in I} \mathcal{F}_i)$ such that for any $A_i \in \mathcal{F}_i$, where all but a finite number of $A_i = X_i$,*

$$\bigotimes_{i \in I} \mu_i \left(\prod_{i \in I} A_i \right) = \prod_{i \in I} \mu_i(A_i).$$

$K \subset J \subset I$, the measures $\bigotimes_{j \in J} \mu_j$ and $\bigotimes_{k \in K} \mu_k$ are compatible.

9.4 The Kolmogorov theorem

Suppose that $X_i, i \in I$, is a family of compact sets.

Theorem 9.3 *Suppose that for any finite set $J \in 2^I$ we are given a Baire measure ν_J on $\prod_{j \in J} X_j$.*

Assume that for any finite $K, J \in 2^I$ with $K \subset J$, ν_K is compatible with ν_J . Then there exists a unique Baire measure ν_I on $\prod_{i \in I} X_i$ compatible with all ν_J for finite J .

Proof. It is easy to see that the family of measures μ_J defines a regular content on $\text{Ring}(\ast_{i \in I} \text{Baire}(X_i))$. By Theorem 6.1, it is a premeasure. Hence it admits a unique extension to

$$\sigma\text{-Ring}\left(\text{Ring}(\ast_{i \in I} \text{Baire}(X_i))\right) = \text{Baire}(\times_{i \in I} X_i).$$

□

Theorem 9.4 *Let $X_i, i \in I$ be a family of topological spaces. Then $\text{Baire}_1(\prod_{i \in I} X_i) = \otimes_{i \in I} \text{Baire}_1(X_i)$.*

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