Unbounded linear operators

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1 Unbounded operators

1.1 Relations

Let X, Y be sets. R is called a relation iff $R \subset Y \times X$. We will also write $R : X \to Y$. (Note the inversion of the direction). An example of a relation is the identity $1_X := \{(x, x) \ : \ x \in X\} \subset X \times X$.

Introduce the "projections"

$$Y \times X \ni (y, x) \mapsto \pi_Y(y, x) := y \in Y,$$

$$Y \times X \ni (y, x) \mapsto \pi_X(y, x) := x \in X,$$

and the "flip"

$$Y \times X \ni (y, x) \mapsto \tau(y, x) := (x, y) \in X \times Y.$$

The domain of R is defined as $\mathrm{Dom} R := \pi_X R$, its range is $\mathrm{Ran}\, R = \pi_Y R$, the inverse of R is defined as $R^{-1} := \tau R \subset X \times Y$. If $S \subset Z \times Y$, then the superposition of S and R is defined as $S \circ R \subset Z \times X$, $S \circ R := \{(z, x) \in Z \times X \ : \ \underset{y \in Y}{\vee} \ (z, y) \in S, \ (y, x) \in R\}$.

If $X_0 \subset X$, then the restriction of R to X_0 is defined as

$$R\Big|_{X_0} := R \cap Y \times X_0.$$

If, moreover, $Y_0 \subset Y$, then
$$R\Big|_{X_0 \to Y_0} := R \cap Y_0 \times X_0.$$

We say that a relation R is right-unique, if for any $x \in X$ $\pi_Y(R \cap Y \times \{x\})$ is one-element. We say that R has a maximal domain if $\mathrm{Dom} R = X$.

Proposition 1.1 *a) If R, S are right-unique, then so is $S \circ R$.*
b) If R, S have a maximal domain, then so does $S \circ R$.

A right unique relation will be also called a pseudo-transformation (-operator, etc). Instead of writing $(y, x) \in R$, we will then write $y = R(x)$ or, in some contexts, $y = Rx$. We also introduce the graph of R:

$$\mathrm{Gr}\, R := \{(y, x) \in Y \times X \ : \ y = R(x), \ x \in \mathrm{Dom} R\}.$$

Note that strictly speaking $\mathrm{Gr}\, R = R$. The difference of $\mathrm{Gr}\, R$ and R lies only in their syntactic role.

Note that a superposition of pseudotransformations is a pseudotransformation.

We say that a pseudotransformation is injective if it is left-unique. The inverse of a pseudotransformation is a pseudotransformation iff it is injective.

A transformation is a pseudotransformation with a maximal domain. The composition of transformations is a transformation.

We say that a transformation R is bijective iff it is left-unique and $\mathrm{Ran}\, R = Y$. The inverse of a transformation is a transformation iff it is bijective.

Proposition 1.2 *Let $R \subset X \times Y$ and $S \subset Y \times X$ be transformations such that $R \circ S = 1_Y$ and $S \circ R = 1_X$. Then S and R are bijections and $S = R^{-1}$.*

1.2 Linear pseudooperators

Let $\mathcal{X}, \mathcal{Y}$ be vector spaces.

Proposition 1.3 *1) A linear subspace $\mathcal{V} \subset \mathcal{Y} \oplus \mathcal{X}$ is a graph of a certain pseudooperator iff $(y, 0) \in \mathcal{V}$ implies $y = 0$.*
2) A pseudooperator A is injective iff $(0, x) \in \mathrm{Gr}\, A$ implies $x = 0$.

From now on by an "operator" we will mean a "pseudooperator". To say that A is a true operator we will write $\mathrm{Dom} A = \mathcal{X}$.

1.3 Closed operators

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces.

Theorem 1.4 *Let $A : \mathcal{X} \to \mathcal{Y}$ be an operator. The following conditions are equivalent:*

(1) *$\mathrm{Gr}\, A$ is closed in $\mathcal{Y} \times \mathcal{X}$.*

(2) *If $x_n \to x$, $x_n \in \mathrm{Dom} A$ and $Ax_n \to y$, then $x \in \mathrm{Dom} A$ and $y = Ax$.*

(3) *For some $p \in [1, \infty]$, $\mathrm{Dom} A$ with the norm*

$$\|x\|_{A,p} := (\|Ax\|^p + \|x\|^p)^{\frac{1}{p}}.$$

is a Banach space.

Proof. The equivalence of (1), (2) and (3) is obvious, if we note that

$$\mathrm{Dom}A \ni x \mapsto (Ax, x) \in \mathrm{Gr}\, A$$

is a bijection. $\Box$

Definition 1.5 *An operator satisfying the above conditions is called closed.*

We will say that $A : \mathcal{X} \to \mathcal{Y}$ is bounded iff

$$\|Ax\| \leq c\|x\|, \quad x \in \mathrm{Dom}A. \tag{1.1}$$

$B(\mathcal{X}, \mathcal{Y})$ will denote all bounded operators from $\mathcal{X}$ to $\mathcal{Y}$ with the domain equal to $\mathcal{X}$.

Proposition 1.6 *A bounded operator A is closed iff* $\mathrm{Dom}A$ *is closed.*

Theorem 1.7 *Let $B \in B(\mathcal{X}, \mathcal{Y})$ be invertible and $A : \mathcal{X} \to \mathcal{Y}$ closed. Then BA is closed on* $\mathrm{Dom}A$ *and AB is closed on $B^{-1}\mathrm{Dom}A$.*

Proof. We check that

$$\begin{aligned} \|x\|_A &\leq \max(1, \|B^{-1}\|)\|x\|_{BA}, \\ \|x\|_{BA} &\leq \max(1, \|B\|)\|x\|_A, \\ \|Bx\|_A &\leq \max(1, \|B\|)\|x\|_{AB}, \\ \|x\|_{AB} &\leq \max(1, \|B^{-1}\|)\|Bx\|_A. \end{aligned}$$

$\Box$

Theorem 1.8 *If A is closed and injective, then so is A^{-1}.*

Proof. The flip $\tau : \mathcal{Y} \times \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ is continuous. $\Box$

1.4 Closable operators

Theorem 1.9 *Let $A : \mathcal{X} \to \mathcal{Y}$ be an operator. The following conditions are equivalent:*

(1) *There exists a closed operator B such that $B \supset A$.*

(2) $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ *is the graph of an operator.*

(3) $(y, 0) \in (\mathrm{Gr}\, A)^{\mathrm{cl}} \;\Rightarrow\; y = 0$.

(4) $(x_n) \subset \mathrm{Dom}A$, $x_n \to 0$, $Ax_n \to y$ *implies* $y = 0$.

Definition 1.10 *An operator A satisfying the conditions of Theorem 1.9 is called closable. If the conditions of Theorem 1.9 hold, then the operator whose graph equals* $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ *is denoted by* A^{cl} *and called the closure of A.*

Proof of Theorem 1.9 To show (2)$\Rightarrow$(1) it suffices to take as B the operator A^{cl}. Let us show (1)$\Rightarrow$(2). Let B be a closed operator such that $A \subset B$. Then $(\mathrm{Gr}\, A)^{\mathrm{cl}} \subset (\mathrm{Gr}\, B)^{\mathrm{cl}} = \mathrm{Gr}\, B$. But $(y, 0) \in \mathrm{Gr}\, B \;\Rightarrow\; y = 0$, hence $(y, 0) \in (\mathrm{Gr}\, A)^{\mathrm{cl}} \;\Rightarrow\; y = 0$. Thus $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ is the graph of an operator. $\Box$

As a by-product of the above proof, we obtain

Proposition 1.11 *If A is closable, B closed and $A \subset B$, then $A^{\mathrm{cl}} \subset B$.*

Proposition 1.12 *Let A be bounded. Then A is closable,* $\mathrm{Dom}A^{\mathrm{cl}} = (\mathrm{Dom}A)^{\mathrm{cl}}$ *and* $A^{\mathrm{cl}}x = \lim_n Ax_n$ *for* $x_n \underset{n\to\infty}{\to} x$, $x_n \in \mathrm{Dom}A$. *Besides, A^{cl} satisfies (1.1).*

Let A be a closed operator. We say that a linear subspace $\mathcal{D}$ is an essential domain for A iff $\mathcal{D}$ is dense in $\mathrm{Dom}A$ in the graph topology. In other words, $\mathcal{D}$ is an essential domain for A, if

$$\left(A\Big|_{\mathcal{D}}\right)^{\mathrm{cl}} = A.$$

Theorem 1.13 (1) *If $A \in B(\mathcal{X}, \mathcal{Y})$, then a linear subspace $\mathcal{D} \subset \mathcal{X}$ is an essential domain for A iff it is dense in $\mathcal{X}$ (in the usual topology).*

(2) *If A is closed, has a dense domain and $\mathcal{D}$ is its essential domain, then $\mathcal{D}$ is dense in $\mathcal{X}$.*

1.5 Perturbations of closed operators

Definition 1.14 *Let B, $A : \mathcal{X} \to \mathcal{Y}$. We say that B is bounded relatively to A iff* $\mathrm{Dom}A \subset \mathrm{Dom}B$ *and there exist constants a, b such that*

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \quad x \in \mathrm{Dom}A. \tag{1.2}$$

The infimum of a satisfying (1.2) is called the A-bound of B. In other words: the A-bound of B equals

$$\inf_{c>0} \sup_{x\in\mathrm{Dom}A\backslash\{0\}} \frac{\|Bx\|}{\|Ax\| + c\|x\|}.$$

In particular, if B is bounded, then its A-bound equals 0.

If A is unbounded, then its A-bound equals 1.

In the case of Hilbert spaces it is more convenient to use the following condition to define the relative boundedness:

Theorem 1.15 *B is bounded relatively to A with the A-bound a iff* $\mathrm{Dom}A \subset \mathrm{Dom}B$ *and*

$$\inf_{c>0} \sup_{x\in\mathrm{Dom}A\backslash\{0\}} \left(\frac{\|Bx\|^2}{\|Ax\|^2 + c\|x\|^2}\right)^{1/2} < \infty. \tag{1.3}$$

If this is the case, then (1.3) equals the A-bound of B

Proof. For any $\epsilon > 0$ we have

$$\begin{aligned}
&\left(\|Ax\|^2 + c^2\|x\|^2\right)^{\frac{1}{2}} \\
&\leq \|Ax\| + c\|x\| \\
&\leq \left((1+\epsilon^2)\|Ax\|^2 + c^2(1+\epsilon^{-2})\|x\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

□

Theorem 1.16 *Let A be closed and let B be bounded relatively to A with the A-bound less than 1. Then $A+B$ with the domain* $\mathrm{Dom}A$ *is closed. All essential domains of A are essential domains of $A+B$*

Proof. We know that

$$\|Bx\| \leq a\|Ax\| + b\|x\|$$

for some $a < 1$ and b. Hence

$$\|(A+B)x\| + \|x\| \leq (1+a)\|Ax\| + (1+b)\|x\|$$

and

$$(1-a)\|Ax\| + \|x\| \leq \|Ax\| - \|Bx\| + (1+b)\|x\| \leq \|(A+B)x\| + (1+b)\|x\|.$$

Hence the norms $\|Ax\| + \|x\|$ and $\|(A+B)x\| + \|x\|$ are equivalent on $\mathrm{Dom}A$. □

Theorem 1.17 *Suppose that A, C are two operators with the same domain $\mathrm{Dom}A = \mathrm{Dom}C = \mathcal{D}$ satisfying*

$$\|(A-C)x\| \leq a(\|Ax\| + \|Cx\|) + b\|x\|$$

for some $a < 1$. Then

(1) *A is closed on $\mathcal{D}$ iff C is closed on $\mathcal{D}$.*

(2) *A is closable on $\mathcal{D}$ iff C is closable on $\mathcal{D}$ and then the domains of A^{cl} and C^{cl} coincide.*

Proof. Define $B := C - A$ and $F(t) := A + tB$ with the domain $\mathcal{D}$. For $0 \leq t \leq 1$, we have

$$\begin{aligned}
\|Bx\| &\leq a(\|Ax\| + \|Cx\|) + b\|x\| \\
&= a\left(\|(F(t) - tB)x\| + \|(F(t) + (1-t)B)x\|\right) + b\|x\| \\
&\leq 2a\|F(t)x\| + a\|Bx\| + b\|x\|
\end{aligned}$$

Hence

$$\|Bx\| \leq \frac{2a}{1-a}\|F(t)x\| + \frac{b}{1-a}\|x\|.$$

Therefore, if $|s| < \frac{1-a}{2a}$ and $t, t+s \in [0,1]$, then $F(t+s)$ is closed iff $F(t)$ is closed. □

1.6 Invertible unbounded operators

Definition 1.18 *We say that an operator A is invertible iff $A^{-1} \in B(\mathcal{Y}, \mathcal{X})$.*

Theorem 1.19 *Let A be closed. Suppose that for some $c > 0$*

$$\|Ax\| \geq c\|x\|, \quad x \in \mathrm{Dom}A. \tag{1.4}$$

Then $\mathrm{Ran}\,A$ is closed. If $\mathrm{Ran}\,A = Y$, then A is invertible and

$$\|A^{-1}\| \leq c^{-1}$$

Proof. Let $y_n \in \mathrm{Ran}\,A$ and $y_n \to y$. Let $Ax_n = y_n$. Then x_n is a Cauchy sequence. Hence there exists $\lim_{n\to\infty} x_n := x$. But A is closed, hence $Ax = y$. Therefore, $\mathrm{Ran}\,A$ is closed. □

Corollary 1.20 *Let A be closed. Suppose that for some $c > 0$*

$$\|Ax\| \geq c\|x\|,$$

and $\mathrm{Ran}\,A$ is dense in $\mathcal{Y}$. Then A is invertible

Theorem 1.21 *Let A be closable and for some c (1.4) holds. Then (1.4) holds for A^{cl} as well.*

Theorem 1.22 (1) *Let A be injective and* $\mathrm{Dom}B \supset \mathrm{Dom}A$*. Then B has the A-bound less than $\|BA^{-1}\|$.*

(2) *If, moreover, $\|BA^{-1}\| < 1$, then $A + B$ with the domain* $\mathrm{Dom}A$ *is closed, invertible and*

$$(A+B)^{-1} = \sum_{j=0}^{\infty} (-1)^j A^{-1} (BA^{-1})^j.$$

Proof. Let $a := \|BA^{-1}\|$. By the estimate

$$\|Bx\| \leq a\|Ax\|, \quad x \in \mathrm{Dom}A,$$

we see that B has the A-bound less than or equal to a. This proves (1).

Assume now that $a < 1$. Let

$$C_n := \sum_{j=0}^{n} (-1)^j A^{-1} (BA^{-1})^j.$$

Then $\lim_{n\to\infty} C_n =: C$ exists.

Let $y \in \mathcal{Y}$. Clearly, $\lim_{n\to\infty} C_n y = Cy$.

$$(A+B)C_n y = y + (-1)^n (BA^{-1})^{n+1} y \to y.$$

But $A + B$ is closed, hence $Cy \in \mathrm{Dom}(A+B)$ and $(A+B)Cy = y$.

If $x \in \mathrm{Dom}(A+B)$, then

$$C_n (A+B)x = x + (-1)^n A^{-1} (BA^{-1})^n Ax \to x.$$

Hence $C(A+B)y = y$. □

Proposition 1.23 *Let A and B be invertible and* $\mathrm{Dom}B \supset \mathrm{Dom}A$*. Then*

$$B^{-1} - A^{-1} = B^{-1}(A-B)A^{-1}.$$

1.7 Spectrum of unbounded operators

Let A be an operator on $\mathcal{X}$. We define the resolvent set of A as

$$\mathrm{rs}A := \{z \in \mathbb{C} \ : \ z - A \ \text{ is invertible } \}.$$

We define the spectrum of A as $\mathrm{sp}A := \mathbb{C}\backslash \mathrm{rs}A$.

We say that $x \in \mathcal{X}$ is an eigenvector of A with the eigenvalue $\lambda \in \mathbb{C}$ iff $x \in \mathrm{Dom}A$, $x \neq 0$ and $Ax = \lambda x$. The set of eigenvalues is called the point spectrum of A and denoted $\mathrm{sp_p}A$. Clearly, $\mathrm{sp_p}A \subset \mathrm{sp}A$.

Let $\mathbb{C}^{\mathrm{comp}}$ denote the Riemann sphere (the one-point compactification of $\mathbb{C}$). In the case of unbounded operators it is sometimes convenient to use the "extended spectrum", which is a subset of $\mathbb{C}^{\mathrm{comp}}$, instead of the usual spectrum—a subset of $\mathbb{C}$.

The extended resolvent set is defined as $\mathrm{rs}^{\mathrm{ext}}A := \mathrm{rs}A \cup \{\infty\}$ if $A \in B(\mathcal{X})$ and $\mathrm{rs}^{\mathrm{ext}}A := \mathrm{rs}A$, if A is unbounded. The extended spectrum is defined as

$$\mathrm{sp}^{\mathrm{ext}}A = \mathbb{C}^{\mathrm{comp}} \backslash \mathrm{rs}^{\mathrm{ext}}A.$$

If $A \in B(\mathcal{X})$, we set $(\infty - A)^{-1} = 0$.

Theorem 1.24 (1) *If* $\lambda, \mu \in \mathrm{rs}A$, *then*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}.$$

(2) *If* $\lambda \in \mathrm{rs}A$ *and* $\|(\lambda - A)^{-1}\| = c$, *then* $\{z : \ |z - \lambda| < c^{-1}\} \subset \mathrm{rs}A$.

(3) $\|(z - A)^{-1}\| \geq (\mathrm{dist}(z, \mathrm{sp}A))^{-1}$.

(4) *If* $\mathrm{rs}A$ *is nonempty, then* A *is closed.*

(5) $\mathrm{sp}^{\mathrm{ext}}A$ *is a compact subset of* $\mathbb{C}^{\mathrm{comp}}$.

(6) $(z - A)^{-1}$ *is analytic on* $\mathrm{rs}^{\mathrm{ext}}A$.

(7) $(z - A)^{-1}$ *cannot be analytically extended to a larger subset of* $\mathbb{C}^{\mathrm{comp}}$ *than* $\mathrm{rs}^{\mathrm{ext}}(A)$.

(8) $\mathrm{sp}^{\mathrm{ext}}(A) \neq \emptyset$

(9) $\mathrm{Ran}\,(z - A)^{-1}$ *does not depend on* $z \in \mathrm{rs}A$ *and equals* $\mathrm{Dom}A$.

(10) $\mathrm{Ker}(z - A)^{-1} = \{0\}$.

Proof. Let us show (4). If $\lambda \in \mathrm{rs}(A)$, then $\lambda - A$ is invertible, hence closed. □

It is easy to define the functional calculus for unbounded operators with a non-empty resolvent set. The definition **??** can be repeated verbatim, replacing $\mathrm{sp}A$ with $\mathrm{sp}^{\mathrm{ext}}(A)$. Theorem **??** remains valid except that the point about convergent power series should be dropped

Proposition 1.25 *Suppose that* $\mathrm{rs}A$ *is non-empty and* $\mathrm{Dom}A$ *is dense. Then* $\mathrm{Dom}A^2$ *is dense.*

Proof. Let $z \in \mathrm{rs}A$. Then $(z - A)^{-1}\mathrm{Dom}A$ is dense in $\mathcal{X}$ by Theorem **??** and $A(z - A)^{-1}\mathrm{Dom}A \subset \mathrm{Dom}A + z(z - A)^{-1}\mathrm{Dom}A = \mathrm{Dom}A$. Hence $(z - A)^{-1}\mathrm{Dom}A \subset \mathrm{Dom}A^2$. □

Theorem 1.26 *Let* A *and* B *be operators on* $\mathcal{X}$ *with* $A \subset B$, $A \neq B$. *Then* $\mathrm{rs}A \subset \mathrm{sp}B$, *and hence* $\mathrm{rs}B \subset \mathrm{sp}A$.

Proof. Let $\lambda \in \mathrm{rs}A$. Let $x \in \mathrm{Dom}B \backslash \mathrm{Dom}A$. We have $\mathrm{Ran}\,(\lambda - A) = \mathcal{X}$, hence there exists $y \in \mathrm{Dom}A$ such that $(\lambda - A)y = (\lambda - B)x$. Hence $(\lambda - B)y = (\lambda - B)x$. Hence $\lambda \notin \mathrm{rs}B$. □

1.8 Examples of unbounded operators

Example 1.27 *Let* I *be an infinite set and* $(a_i)_{i\in I}$ *be an unbounded complex sequence. Let* $C_0(I)$ *be the space of sequences with a finite number of non-zero elements. For* $1 \leq p < \infty$ *we define the operator*

$$L^p(I) \supset C_0(I) \ni x \mapsto Ax \in L^p(I)$$

by the formula

$$(Ax)_i = a_i x_i.$$

(We can use $C_\infty(I)$ *instead of* $L^p(I)$, *then* $p = \infty$ *in the formulas below). Then the operator* A *is unbounded and non-closed. Besides,*

$$\mathrm{sp}_{\mathrm{p}}(A) = \{a_i \ : \ i \in I\},$$
$$\mathrm{sp}A = \mathbb{C}.$$

The closure of A *has the domain*

$$\mathrm{Dom}A^{\mathrm{cl}} := \{(x_i)_{i\in I} \in L^p(I) \ : \ \textstyle\sum_{i\in I} |a_i x_i|^p < \infty\} \tag{1.5}$$

We then have

$$\mathrm{sp_p}(A^{\mathrm{cl}}) = \{a_i \ : \ i \in I\},$$
$$\mathrm{sp}A^{\mathrm{cl}} = \{a_i \ : \ i \in I\}^{\mathrm{cl}}.$$

To prove this let $\mathcal{D}$ be the rhs of (1.5) and $x \in \mathcal{D}$. Then there exists a countable set I_1 such that $i \notin I_1$ implies $x_i = 0$. We enumerate the elements of I_1: $i_1, i_2, \ldots$. Define $x^n \in C_0(I)$ setting $x^n_{i_j} = x_{i_j}$ for $j \leq n$ and $x^n_i = 0$ for the remaining indices. Then $\lim_{n\to\infty} x^n = x$ and $Ax^n \to Ax$. Hence, $\{(x, Ax) \ : \ x \in \mathcal{D}\} \subset (\mathrm{Gr}\, A)^{\mathrm{cl}}$.

If x^n belongs to (1.5) and $(x^n, Ax^n) \to (x, y)$, then $x^n_i \to x_i$ and $a_i x^n_i = (Ax^n)_i \to y_i$. Hence $y_i = a_i x_i$. Using that $y \in L^p(I)$ we see that x belongs to (1.5).

Example 1.28 *Let $p^{-1} + q^{-1} = 1$, $1 < p \leq \infty$ and let $(w_i)_{i\in I}$ be a sequence that does not belong to $L^q(I)$. Let $C_0(I)$ be as above. Define*

$$L^p(I) \supset C_0(I) \ni x \mapsto \langle w|x\rangle := \sum_{i\in I} x_i w_i \in \mathbb{C}.$$

Then $\langle w|$ is non-closable.

It is sufficient to assume that $I = \mathbb{N}$ and define $v^n_i := \frac{|w_i|^q}{w_i(\sum_{i=1}^n |w_i|^q)}$, $i \leq n$, $v^n_i = 0$, $i > n$. Then $\langle w|v^n\rangle = 1$ and $\|v^n\|_p = (\sum_{i=1}^n |w_i|^q)^{-\frac{1}{q}} \to 0$. Hence $(0, 1)$ belongs to the closure of the graph of the operator.

1.9 Pseudoresolvents

Definition 1.29 *Let $\Omega \subset \mathbb{C}$ be open. Then the continuous function*

$$\Omega \ni z \mapsto R(z) \in B(\mathcal{X})$$

is called a pseudoresolvent if

$$R(z_1) - R(z_2) = (z_2 - z_1)R(z_1)R(z_2). \tag{1.6}$$

Evidently, every resolvent of a closed operator is a pseudoresolvent.

Proposition 1.30 *Let $\Omega \ni z \mapsto R_n(z) \in B(\mathcal{X})$ be a sequence of pseudoresolvents and $R(z) := \mathrm{s-}\lim_{n\to\infty} R_n(z)$. Then $R(z)$ is a pseudoresolvent.*

Theorem 1.31 *Let $\Omega \ni z \mapsto R(z) \in B(\mathcal{X})$ be a pseudoresolvent. Then*

(1) *$\mathcal{R} := \mathrm{Ran}\, R(z)$ does not depend on $z \in \Omega$.*

(2) *$\mathcal{N} := \mathrm{Ker} R(z)$ does not depend on $z \in \Omega$.*

(3) *$R(z)$ is an analytic function and*
$$\frac{\mathrm{d}}{\mathrm{d}z} R(z) = -R(z)^2.$$

(4) *$R(z)$ is a resolvent of a certain operator iff $\mathcal{N} = \{0\}$. The domain of this operator equals $\mathcal{R}$.*

Proof. Let us prove (4)$\Leftarrow$. Fix $z_1 \in \Omega$. If $\mathcal{N} = \{0\}$, then every element of $\mathcal{R}$ can be uniquely represented as $R(z_1)x$, $x \in \mathcal{X}$. Define $HR(z_1)x := -x + z_1 R(z_1)x$. By formula (1.6) we check that the definition does not depend on z_1. □

2 One-parameter semigroups in Banach spaces

2.1 (M,β)-type semigroups

Let $\mathcal{X}$ be a Banach space.

Definition 2.1 $[0,\infty[\ni t \mapsto W(t) \in B(\mathcal{X})$ *is called a strongly continuous one-parameter semigroup iff*

(1) $W(0) = 1$*;*

(2) $W(t_1)W(t_2) = W(t_1 + t_2)$*,* $t_1, t_2 \in [0,\infty[$*;*

(3) $\lim_{t\downarrow 0} W(t)x = x$*,* $x \in \mathcal{X}$*;*

(4) *for some* $t_0 > 0$*,* $\|W(t)\| < M$*,* $0 \leq t \leq t_0$*.*

Remark 2.2 *Using the Banach-Steinhaus Theorem one can show that (4) follows from the remaining points.*

Theorem 2.3 *If* $W(t)$ *is a strongly continuous semigroup, then*

$$[0,\infty[\times X \ni (t,x) \mapsto W(t)x \in \mathcal{X}$$

is a continuous function. Besides, there exist constants M*,* β *such that*

$$\|W(t)\| \leq M\mathrm{e}^{\beta t}. \tag{2.7}$$

Proof. By (4), for $t \leq nt_0$ we have $\|W(t)\| \leq M^n$. Hence, $\|W(t)\| \leq M\exp(\frac{t}{t_0}\log M)$. Therefore, (2.7) is satisfied.

Let $t_n \to t$ and $x_n \to x$. Then

$$\begin{aligned} \|W(t_n)x_n - W(t)x\| &\leq \|W(t_n)x_n - W(t_n)x\| + \|W(t_n)x - W(t)x\| \\ &\leq M\mathrm{e}^{\beta t_n}\|x_n - x\| + M\mathrm{e}^{\beta\min(t_n,t)}\|W(|t-t_n|)x - x\|. \end{aligned}$$

□

We say that the semigroup $W(t)$ is (M,β)-type, if the condition (2.7) is satisfied.

Clearly, if $W(t)$ is (M,β)-type, then $W(t)\mathrm{e}^{-\beta t}$ is $(M,0)$-type. Since $W(0) = 1$, no semigroups (M,β) exist for $M < 1$.

2.2 Generator of a semigroup

If $W(t)$ is a strongly continuous one-parameter semigroup, we define

$$\mathrm{Dom}A := \{x \in \mathcal{X} \ : \ \text{there exists } \lim_{t\to 0} t^{-1}(W(t)x - x)\}.$$

The operator A with the domain $\mathrm{Dom}A$ is defined by the formula

$$Ax := \lim_{t\to 0} t^{-1}(W(t)x - x).$$

A will be called the generator of $W(t)$.

Theorem 2.4 (1) *A is a closed densely defined operator;*

(2) $W(t)\mathrm{Dom}A \subset \mathrm{Dom}A$ *and* $W(t)A = AW(t)$*;*

(3) *If* $W_1(t)$ *and* $W_2(t)$ *are two different semigroups, then their generators are different.*

By (3), if A is an operator, which is a generator of a semigroup $W(t)$, then such $W(t)$ is unique. We will write $W(t) =: \mathrm{e}^{tA}$.

Proof of Theorem 2.4 (2). Let $x \in \mathrm{Dom}A$. Then

$$\lim_{s\downarrow 0} s^{-1}(W(s)-1)W(t)x = W(t)\lim_{s\downarrow 0} s^{-1}(W(s)-1)x = W(t)Ax. \tag{2.8}$$

Hence the limit of the left hand side of (2.8) exists. Hence $W(t)x \in \mathrm{Dom}A$ and $AW(t)x = W(t)Ax$. □

Lemma 2.5 *For $x \in \mathcal{X}$ put*

$$B_t x := t^{-1}\int_0^t W(s)x\mathrm{d}s.$$

Then

(1) $\mathrm{s-}\lim_{t\to 0} B_t = 1$.

(2) $B_t W(s) = W(s)B_t$.

(3) *If $x \in \mathcal{X}$, then $B_t x \in \mathrm{Dom}A$,*

$$AB_t x = t^{-1}(W(t)x - x), \tag{2.9}$$

(4) *For $x \in \mathrm{Dom}A$, $AB_t x = B_t Ax$.*

Proof. (1) follows by

$$B_t x - x = t^{-1}\int_0^t (W(s)x - x)\mathrm{d}s \underset{t\downarrow 0}{\to} 0.$$

(2) is obvious. (4) is proven as Theorem 2.4 (2). To prove (3) we note that

$$u^{-1}(W(u)-1)B_t x = t^{-1}(W(t)-1)B_u x \underset{u\downarrow 0}{\to} t^{-1}(W(t)x - x).$$

□

Proof of Theorem 2.4 (1), (3) The density of $\mathrm{Dom}A$ follows by Lemma 2.5 (1) and (3).

Let us show that A is closed. Let $x_n \underset{n\to\infty}{\to} x$ and $Ax_n \underset{n\to\infty}{\to} y$. Using the boundedness of $B_t A = AB_t$ we get

$$B_t y = \lim_{n\to\infty} B_t A x_n = B_t A x.$$

Hence

$$y = \lim_{t\downarrow 0} B_t y = \lim_{t\downarrow 0} B_t A x = Ax.$$

□

Proposition 2.6 *Let $W(t)$ be a semigroup and A its generator. Then, for any $x \in \mathrm{Dom}A$ there exists a unique solution of*

$$[0,t_0] \ni t \mapsto x(t) \in \mathrm{Dom}A, \quad \frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t), \tag{2.10}$$

(for $t = 0$ the derivative is right-sided). The solution is given by $x(t) = W(t)x$.

Proof. Let us show that $x(t) := W(t)x$ solves (2.10). We already know that the right-sided derivative equals $Ax(t)$. It suffices to prove the same about the left-sided derivative. For $0 \le u \le t$ we have

$$(-u)^{-1}(W(t-u)x - W(t)x) = W(t-u)u^{-1}(W(u)-1)x \to_{u\to 0} W(t)Ax = AW(t)x.$$

Let us show now the uniqueness. Let $x(t)$ solve (2.10) . Let $y(s) := W(t-s)x(s)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}y(s) = W(t-s)Ax(s) - AW(t-s)x(s) = 0$$

Hence $y(s)$ does not depend on s. At $s = t$ it equals $x(t)$, and at $s = 0$ it equals $W(t)x$. $\Box$

Proof of Theorem 2.4 (3) By Theorem 2.6 (2), $W(t)$ is uniquely determined by A on Dom$A$. But $W(t)$ is bounded and Dom$A$ is dense, hence $W(t)$ is uniquely determined. $\Box$

2.3 Norm continuous semigroups

Theorem 2.7 (1) *If $A \in B(\mathcal{X})$, then $\mathbb{C} \ni z \mapsto \mathrm{e}^{zA}$ is a norm continuous group and A is its generator.*

(2) *If a one-parameter semigroup $W(t)$ is norm continuous, then its generator is bounded.*

Proof. (1) follows by the functional calculus.

Let us show (2). $W(t)$ is norm continuous, hence $\lim_{t\to 0} B_t = 1$. Therefore, for $0 < t < t_0$

$$\|B_t - 1\| < 1.$$

Hence B_t is then invertible.

We know that for $x \in \mathrm{Dom}A$

$$t^{-1}(W(t) - 1)x = B_t Ax.$$

For $0 \leq t < t_0$ we can write this as

$$Ax = t^{-1}B_t^{-1}(W(t) - 1)x.$$

Hence $\|Ax\| \leq c\|x\|$. $\Box$

2.4 Essential domains of generators

Theorem 2.8 *Let $W(t)$ be a strongly continuous one-parameter semigroup and let A be its generator. Let $\mathcal{D} \subset$ Dom$A$ be dense in $\mathcal{X}$ and $W(t)\mathcal{D} \subset \mathcal{D}$, $t > 0$. Then $\mathcal{D}$ is dense in* Dom$A$ *in the graph topology—in other words, $\mathcal{D}$ is an essential domain of A.*

Lemma 2.9 (1) *For $x \in \mathcal{X}$, $\|B_t x\|_{\mathrm{Dom}A} \leq (Ct^{-1} + 1)\|x\|$;*

(2) *For $x \in \mathrm{Dom}A$, $\lim_{t\downarrow 0} \|B_t x - x\|_{\mathrm{Dom}A} = 0$;*

(3) *$W(t)$ is a strongly continuous semi-group on* Dom$A$ *equipped with the graph norm.*

(4) *If $\tilde{\mathcal{D}}$ is a closed subspace in* Dom$A$ *invariant wrt $W(t)$, then it is invariant also wrt B_t.*

Proof. (1) follows by Lemma 2.5 (3).

(2) follows by Lemma 2.5 (1) and because $B(t)$ commutes with A.

(3) follows from the fact that $W(t)$ is a strongly continuous semigroup on $\mathcal{X}$, preserves Dom$A$ and commutes with A.

To show (4), note that $B_t x$ is defined using an integral involving $W(s)x$. $W(s)x$ depends continuously on s in the topology of Dom$A$, as follows by (3). Hence this integral (as Riemann's integral) is well defined. Besides, $B_t x$ belongs to the closure of the space spanned by $W(s)x$, $0 \leq s \leq t$. $\Box$

Proof of Theorem 2.8. Let $x \in \mathrm{Dom}A$, $x_n \in \mathcal{D}$ and $x_n \underset{n\to\infty}{\to} x$ in $\mathcal{X}$. Let $\tilde{\mathcal{D}}$ be he closure of $\mathcal{D}$ in Dom$A$. Then $B_t x_n \in \tilde{\mathcal{D}}$, by Lemma 2.9 (4). By Lemma 2.9 (1) we have

$$\|B_t x_n - B_t x\|_{\mathrm{Dom}A} \leq C_t\|x_n - x\|.$$

Hence $B_t x \in \tilde{\mathcal{D}}$. By Lemma 2.9 (2)

$$\|B_t x - x\|_{\mathrm{Dom} A} \underset{t\downarrow 0}{\rightarrow} 0.$$

Hence, $x \in \tilde{\mathcal{D}}$. □

2.5 Operators of (M,β)-type

Theorem 2.10 *Let A be a densely defned operator. Then the following conditions are equivalent:*
(1) $[\beta,\infty[\subset \mathrm{rs}(A)$ *and*

$$\|(x-A)^{-m}\| \leq M|x-\beta|^{-m}, \quad m=1,2,\dots, \quad x\in[\beta,\infty[$$

(2) $\{z\in\mathbb{C} \ : \ \mathrm{Re} z>\beta\}\subset \mathrm{rs}(A)$ *and*

$$\|(z-A)^{-m}\| \leq M|\mathrm{Re} z-\beta|^{-m}, \quad m=1,2,\dots, \quad z\in\{z\in\mathbb{C} \ : \ \mathrm{Re} z>\beta\}.$$

Proof. It suffices to prove (1)⇒(2). Let (1) be satisfied. It suffices to assume that $\beta=0$. Let $z=x+\mathrm{i}y$. Then for $t>0$

$$\begin{aligned}(z-A)^{-m} &= (x+t-A)^{m}(1+(\mathrm{i}y-t)(x+t-A)^{-1})^{-m} \\ &= \sum_{j=0}^{\infty}(x+t-A)^{-m-j}(iy-t)^{j}\begin{pmatrix} -m \\ j\end{pmatrix}.\end{aligned}$$

Using the fact that $\begin{pmatrix} -m \\ j\end{pmatrix}$ has an alternating sign we get

$$\begin{aligned}\|(z-A)^{-m}\| &\leq \textstyle\sum_{j=0}^{\infty}|x+t|^{-m-j}(-1)^{j}|iy-t|^{j}\begin{pmatrix} -m \\ j\end{pmatrix} \\ &= M|x+t|^{m}\Big(1-\tfrac{|iy-t|}{x+t}\Big)^{-m} \\ &= M(x+t-|iy-t|)^{-m} \underset{t\to\infty}{\rightarrow} Mx^{-m}.\end{aligned}$$

□

Definition 2.11 *We say that an operator A is (M,β)-type, iff the conditions of Theorem 2.10 are satisfied.*

Obviously, if A is of (M,β)-type, then $A-\beta$ is of $(M,0)$-type.

2.6 The Hille-Philips-Yosida theorem

Theorem 2.12 *If $W(t)$ is a semigroup of (M,β)-type, then its generator A is also of (M,β)-type. Besides,*

$$(z-A)^{-1}=\int_0^\infty \mathrm{e}^{-tz}W(t)\mathrm{d}t, \quad \mathrm{Re} z>\beta.$$

Proof. Set
$$R(z)x := \int_0^\infty \mathrm{e}^{-zt}W(t)x\mathrm{d}t.$$
Let $y = R(z)x$. Then
$$\begin{aligned}&u^{-1}(W(u)-1)y\\ &= -u^{-1}\mathrm{e}^{zu}\int_0^u \mathrm{e}^{-zt}W(t)x\mathrm{d}t + u^{-1}(\mathrm{e}^{zu}-1)\int_0^\infty \mathrm{e}^{-zt}W(t)x\mathrm{d}t \underset{u\downarrow 0}{\to} -x+zy.\end{aligned}$$
Hence $y \in \mathrm{Dom}A$ and $(z-A)R(z)x = x$.

Suppose now that $x \in \mathrm{Ker}(z-A)$. Then $x_t := \mathrm{e}^{zt}x \in \mathrm{Dom}A$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}x_t = Ax_t$. Hence $x_t = W(t)x$. But $\|x_t\| = \mathrm{e}^{\mathrm{Re}zt}\|x\|$, which is impossible.

By the formula
$$(z-A)^{-m} = \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-z(t_1+\cdots+t_m)}W(t_1+\cdots+t_m)\mathrm{d}t_1\cdots\mathrm{d}t_m$$
we get the estimate
$$\|(z-A)^{-m}\| \le \int_0^\infty \cdots \int_0^\infty M\mathrm{e}^{-(z-\beta)(t_1+\cdots+t_m)}\mathrm{d}t_1\cdots\mathrm{d}t_m = M|z-\beta|^{-m}.$$
□

Theorem 2.13 *If A is an operator of (M,β)-type, then it is the generator of a semigroup $W(t)$. This semigroup is of (M,β)-type.*

To simplify, let us assume that $\beta = 0$ (which does not restrict the generality). Then we have the formula
$$W(t) = \mathrm{s-}\lim_{n\to\infty}\left(1-\frac{t}{n}A\right)^{-n},$$
$$\left\|W(t)x - \left(1-\frac{t}{n}A\right)^{-n}x\right\| \le M\frac{t^2}{2}\|A^2x\|, \quad x \in \mathrm{Dom}A^2.$$

Proof. Set
$$V_n(t) := \left(1-\frac{t}{n}A\right)^{-n}.$$
Let us first show that
$$\mathrm{s-}\lim_{t\downarrow 0} V_n(t) = 1. \tag{2.11}$$
To prove (2.11) it suffices to prove that
$$\mathrm{s-}\lim_{s\downarrow 0}(1-sA)^{-1} = 1. \tag{2.12}$$
We have $(1-sA)^{-1} - 1 = (s^{-1}-A)^{-1}A$. Hence for $x \in \mathrm{Dom}A$
$$\|(1-sA)^{-1}x - x\| \le Ms^{-1}\|Ax\|,$$
which proves (2.12).

Let us list some other properties of $V_n(t)$: for $\mathrm{Re}t > 0$, $V_n(t)$ is holomorphic, $\|V_n(t)\| \leq M$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}V_n(t) = A\left(1 - \frac{t}{n}A\right)^{-n-1}.$$

To show that $V_n(t)x$ is a Cauchy sequence for $x \in \mathrm{Dom}(A^2)$, we compute

$$\begin{aligned} V_n(t)x - V_m(t)x &= \lim_{s\downarrow 0} V_n(t-s)V_m(s)x - \lim_{s\uparrow t} V_n(t-s)V_m(s)x \\ &= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \tfrac{\mathrm{d}}{\mathrm{d}s} V_n(t-s)V_m(s)x \\ &= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \Big(- V_n'(t-s)V_m(s) + V_n(t-s)V_m'(s)\Big)x \\ &= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \left(\tfrac{s}{n} - \tfrac{t-s}{m}\right)\left(1 - \tfrac{t-s}{n}A\right)^{-n-1}\left(1 - \tfrac{s}{n}A\right)^{-m-1} A^2x. \end{aligned}$$

Hence for $x \in \mathrm{Dom}(A^2)$

$$\begin{aligned} \|V_n(t)x - V_m(t)x\| &\leq \|A^2x\| \int_0^t |\tfrac{s}{m} - \tfrac{t-s}{n}| M^2 \mathrm{d}s \\ &= M^2(\tfrac{1}{n} + \tfrac{1}{m})\tfrac{t^2}{2}. \end{aligned}$$

By the Proposition 1.25, $\mathrm{Dom}(A^2)$ is dense in $\mathcal{X}$. Therefore, there exists a limit uniform on $[0, t_0]$

$$\mathrm{s-}\lim_{n\to\infty} V_n(t) =: W(t),$$

which depends strongly continuously on t.

Finally, let us show that $W(t)$ is a semigroup with the generator A. To this end it suffices to show that for $x \in \mathrm{Dom}A$

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t)x = AW(t)x. \tag{2.13}$$

But $x \in \mathrm{Dom}A$

$$V_n(t+u)x = V_n(t)x + \int_t^{t+u} A\Big(1 - \frac{s}{n}A\Big)^{-1} V_n(s)x\mathrm{d}s$$

Hence passing to the limit we get

$$W(t+u)x = W(t)x + \int_t^{t+u} AW(s)x\mathrm{d}s.$$

This implies (2.13). □

2.7 Semigroups of contractions and dissipative operators

Theorem 2.14 *Let A be a closed operator on $\mathcal{X}$. Then the following conditions are eqivalent:*

(1) *A is a generator of a semigroup of contractions, eg.* $\|\mathrm{e}^{tA}\| \leq 1$

(2) *The operator A is of $(1,0)$-type.*

(3) $]0,\infty[\subset \mathrm{rs}(A)$ *and*

$$\|(\mu - A)^{-1}\| \leq \mu^{-1}, \quad \mu \in \mathbb{R},\ \mu > 0,$$

(4) $\{z \in \mathbb{C} \ :\ \mathrm{Re}z > 0\} \subset \mathrm{rs}(A)$ *and*

$$\|(z - A)^{-1}\| \leq |\mathrm{Re}z|^{-1}, \quad z \in \mathbb{C},\ \mathrm{Re}z > 0.$$

(5) $\{z \in \mathbb{C} \ : \ \mathrm{Re} z > 0\} \subset \mathrm{rs} A$*; besides, if* $x \in \mathrm{Dom} A$, $v \in \mathcal{X}^{\#}$, $\|v\| = 1$ *and* $\langle v|x\rangle = \|x\|$, *then*

$$\mathrm{Re}\langle v|Ax\rangle \leq 0.$$

(6) *There exists* $z \in \mathbb{C}$ *with* $\mathrm{Re} z > 0$ *such that* $\mathrm{Ran}\,(z - A) = X$*; besides if* $x \in \mathrm{Dom} A$, *then there* $\vee\, v \in X^{\#}$ *such that* $\|v\| = 1$, $\langle v|x\rangle = \|x\|$, *and*

$$\mathrm{Re}\langle v|Ax\rangle \leq 0.$$

Proof. The equivalence of (1) and (2) is a special case of Theorems 2.12 and 2.13. The implications (2)⇒(3) and (2)⇒(4) are obvious, the converse implications are easy.

Let us show (1),(3)⇒(5). We have

$$\mathrm{Re}\langle v|x\rangle = \langle v|x\rangle = \|x\|,$$

$$\mathrm{Re}\,\langle v|\mathrm{e}^{tA}x\rangle \leq |\langle v|\mathrm{e}^{tA}x\rangle| \leq \|x\|.$$

Hence

$$\mathrm{Re}\langle v|Ax\rangle = \lim_{t\downarrow 0} \mathrm{Re} t^{-1}(\langle v|\mathrm{e}^{tA}x\rangle - \langle v|x\rangle) \leq 0.$$

We know that if $\mathrm{Re} z > 0$, then $z \in \mathrm{rs}(A)$. Hence $\mathrm{Ran}\,(z - A) = \mathcal{X}$.

The implication (5)⇒(6) is obvious.

Let us prove (6)⇒(3).

$$\begin{aligned} \|(z-A)x\| &\geq |\langle v|(z-A)x\rangle| \\ &\geq \mathrm{Re}\langle v|(z-A)x\rangle \quad \geq \mathrm{Re} z\langle v|x\rangle = \|x\|\mathrm{Re} z. \end{aligned}$$

Using $\mathrm{Ran}\,(z - A)^{-1} = \mathcal{X}$, we conclude that $(z - A)^{-1}$ exists and $\|(z - A)^{-1}\| \leq |\mathrm{Re} z|^{-1}$. □

3 Unbounded operators in Hilbert spaces

3.1 Graph scalar product

Let $\mathcal{V}$, $\mathcal{W}$ be Hilbert spaces. Let $A : \mathcal{V} \to \mathcal{W}$ be an operator with domain $\mathrm{Dom} A$. It is natural to treat $\mathrm{Dom} A$ as a space with a scalar product

$$(v_1|v_2)_A := (v_1|v_2) + (Av_1|Av_2).$$

Clearly, $\mathrm{Dom} A$ is a Hilbert space with this product iff A is closed.

3.2 The adjoint of an operator

Definition 3.1 *Let* $A : \mathcal{V} \to \mathcal{W}$ *have a dense domain. Then* $w \in \mathrm{Dom} A^*$, *iff the functional*

$$\mathrm{Dom} A \ni v \mapsto (w|Av)$$

is bounded (in the topology of $\mathcal{V}$*). Hence there exists a unique* $y \in \mathcal{V}$ *such that*

$$(w|Av) = (y|v), \quad v \in \mathcal{V}.$$

We set then

$$A^* w = y.$$

If $C_{\mathcal{V}}$, $C_{\mathcal{W}}$ are the Riesz antiisomorphisms, then we have the relationship between the (Banach space)-conjugate $A^{\#}$ and the (Hilbert space)-adjoint A^*:

$$A^* = C_{\mathcal{V}}^{-1} A^{\#} C_{\mathcal{W}}.$$

Theorem 3.2 *Let $A : \mathcal{V} \to \mathcal{W}$ have a dense domain. Then*

(1) *A^* is closed.*

(2) *$\mathrm{Dom} A^*$ is dense in $\mathcal{W}$ iff A is closable.*

(3) *$(\mathrm{Ran}\, A)^{\perp} = \mathrm{Ker} A^*$.*

(4) *$\mathrm{Dom} A \cap (\mathrm{Ran}\, A^*)^{\perp} \supset \mathrm{Ker} A$.*

Proof. Let $j : \mathcal{V} \oplus \mathcal{W} \to \mathcal{W} \oplus \mathcal{V}$, $j(v, w) := (-w, v)$. Note that j is unitary. We have

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^{\perp}.$$

Hence $\mathrm{Gr}\, A^*$ is closed. This proves (1).

Let us prove (2).

$$\begin{aligned} w \in (\mathrm{Dom} A^*)^{\perp} \quad &\Leftrightarrow (0, w) \in (\mathrm{Gr}\, A^*)^{\perp} \\ &\Leftrightarrow (w,) \in (\mathrm{Gr}\, A)^{\perp\perp} = (\mathrm{Gr}\, A)^{\mathrm{cl}}. \end{aligned}$$

Proof of (3):

$$\begin{aligned} w \in \mathrm{Ker} A^* \quad &\Leftrightarrow (A^* w | v) = 0, \quad v \in \mathcal{V} \\ &\Leftrightarrow (A^* w | v) = 0, \quad v \in \mathrm{Dom} A \\ &\Leftrightarrow (w | A v) = 0, \quad v \in \mathrm{Dom} A \\ &\Leftrightarrow w \in (\mathrm{Ran}\, A)^{\perp}. \end{aligned}$$

Proof of (4)

$$\begin{aligned} v \in \mathrm{Ker} A \quad &\Leftrightarrow (w | A v) = 0, \quad w \in \mathcal{W} \\ &\Rightarrow (w | A v) = 0, \quad w \in \mathrm{Dom} A^* \\ &\Leftrightarrow (A^* w | v) = 0, \quad w \in \mathrm{Dom} A^* \\ &\Leftrightarrow v \in (\mathrm{Ran}\, A^*)^{\perp}. \end{aligned}$$

Theorem 3.3 *Let $A : \mathcal{V} \to \mathcal{W}$ be closable with a dense domain. Then*

(1) *A^* is closed with a dense domain.*

(2) *$A^* = (A^{\mathrm{cl}})^*$.*

(3) *$(A^*)^* = A^{\mathrm{cl}}$*

(4) *$(\mathrm{Ran}\, A)^{\perp} = \mathrm{Ker} A^*$. Hence A^* is injective iff $\mathrm{Ran}\, A$ is dense.*

(5) *$(\mathrm{Ran}\, A^*)^{\perp} = \mathrm{Ker} A$. Hence A is injective iff $\mathrm{Ran}\, A^*$ is dense.*

Proof. (1) was proven in Theorem 3.2.

To see (2) note that

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^{\perp} = j((\mathrm{Gr}\, A)^{\mathrm{cl}})^{\perp} = \mathrm{Gr}\, A^{\mathrm{cl}*}.$$

To see (3) we use

$$\mathrm{Gr}\ (A^{**}) = j^{-1} \left(j(\mathrm{Gr}\, A)^{\perp} \right)^{\perp} = (\mathrm{Gr}\, A)^{\perp\perp} = (\mathrm{Gr}\, A)^{\mathrm{cl}}.$$

(4) is proven in Theorem 3.2.

To prove (5) note that in the second line of the proof of Theorem 3.2 (4) we can use the fact that $\mathrm{Dom}A^*$ is dense in $\mathcal{W}$ to replace $\Rightarrow$ with $\Leftrightarrow$. □

3.3 Inverse of the adjoint operator

Theorem 3.4 *Let A be densely defined, closed, injective and with a dense range. Then*

(1) *A^{-1} is densely defined, closed, injective and with a dense range.*

(2) *A^* is densely defined, closed, injective and with a dense range.*

(3) $(A^*)^{-1} = (A^{-1})^*$.

Proof. (1) and (2) sum up previously proven facts.

To prove (3), recall the maps $\tau, j : \mathcal{V} \oplus \mathcal{W} \to \mathcal{W} \oplus \mathcal{V}$. We have

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^\perp, \quad \mathrm{Gr}\, A^{-1} = \tau(\mathrm{Gr}\, A).$$

Hence

$$\mathrm{Gr}\, A^{-1*} = j(\tau(\mathrm{Gr}\, A))^\perp = \tau^{-1}(j(\mathrm{Gr}\, A)^\perp) = \mathrm{Gr}\, A^{*-1}.$$

□

Theorem 3.5 *Let $A : \mathcal{V} \to \mathcal{W}$ be densely defined and closed. Then the following conditions are equivalent:*

(1) *A is invertible.*

(2) *A^* is invertible.*

(3) *For some $c > 0$, $\|Av\| \geq c\|v\|$, $v \in \mathcal{V}$ and $\|A^*w\| \geq c\|v\|$, $w \in \mathcal{W}$.*

Moreover, $\mathrm{sp}^{\mathrm{ext}}(A) = \overline{\mathrm{sp}^{\mathrm{ext}}(A^*)}$.

Proof. (1)$\Rightarrow$(2). Let A be invertible. Then $A^{-1} \in B(\mathcal{W}, \mathcal{V})$. Hence, $A^{-1*} \in B(\mathcal{V}, \mathcal{W})$.

Clearly, the assumptions of Theorem 3.4 are satisfied, and hence $A^{*-1} = A^{-1*}$. Therefore, $A^{*-1} \in B(\mathcal{V}, \mathcal{W})$.

(1)$\Leftarrow$(2). A^* is also densely defined and closed. Hence the same arguments as above apply.

It is obvious that (1) and (2) imply (3). Let us prove that (3)$\Rightarrow$(1). $\|A^*v\| \geq c\|v\|$ implies that $\mathrm{Ker}A^* = \{0\}$. Hence $(\mathrm{Ran}\, A)^\perp$ is dense. This together with $\|Av\| \geq c\|v\|$ implies that $\mathrm{Ran}\, A = \mathcal{W}$, and consequently, A is invertible. □

3.4 Maximal operators

The numerical range of the operator T is defined as

$$\mathrm{Num}T = \{(v|Tv) \ : \ v \in \mathrm{Dom}T, \quad \|v\| = 1\}.$$

Theorem 3.6 (1) $\|(z - T)v\| \geq \mathrm{dist}(z, \mathrm{Num}T)\|v\|, \quad v \in \mathrm{Dom}T.$

(2) *If T is a closed operator and $z \in \mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$, then $z - T$ has a closed range.*

(3) *If $z \in \mathrm{rs}T\backslash\mathrm{Num}T$, then $\|(z - T)^{-1}\| \leq |\mathrm{dist}(z, \mathrm{Num}T)|^{-1}$.*

(4) *Let Δ be a connected component of $\mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$. Then either $\Delta \subset \mathrm{sp}T$ or $\Delta \subset \mathrm{rs}T$.*

Proof. To prove (1), take $(z_0 \notin \mathrm{Num}T)^{\mathrm{cl}}$. Recall that $\mathrm{Num}T$ is convex. Hence, replacing T wih $\alpha T+\beta$ we can assume that $z_0 = \mathrm{i}\nu$ and $0 \in \mathrm{Num}T \subset \{\mathrm{Im}z \leq 0\}$. Thus $\nu = \mathrm{dist}(\mathrm{i}\nu, \mathrm{Num}T)$ and

$$\begin{aligned}\|(\mathrm{i}\nu - T)v\|^2 &= (Tv|Tv) - i\nu(v|Tv) + \mathrm{i}\nu(Tv|v) + |\nu|^2\|v\|^2 \\ &= (Tv|Tv) - 2\nu\mathrm{Im}(v|Tv) + |\nu|^2\|v\|^2 \\ &\geq |\nu|^2\|v\|^2.\end{aligned}$$

(1) implies (2) and (3).

Let $z_0 \in \mathrm{rs}T\backslash\mathrm{Num}T$. By (3), if $r = \mathrm{dist}(z_0, \mathrm{Num}T)$, then $\{|z - z_0| < r\} \subset \mathrm{rs}T$. This proves (4). □

Definition 3.7 *An operator T is called maximal, if* $\mathrm{sp}T \subset (\mathrm{Num}T)^{\mathrm{cl}}$.

Clearly, if T is a maximal operator, and $z \notin (\mathrm{Num}T)^{\mathrm{cl}}$, then

$$\|(z-T)^{-1}\| \leq (\mathrm{dist}(z, \mathrm{Num}T)))^{-1}.$$

If T is bounded, then T is maximal.

Theorem 3.8 *Suppose that T is an operator and for any connected component Δ_i of $\mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$ we choose $\lambda_i \in \Delta_i$. Then the following conditions are necessary and sufficient for T to be maximal*

(1) *For all i, $\lambda_i \notin \mathrm{sp}T$;*

(2) *T is closable and for all i,* $\mathrm{Ran}\,(\lambda_i - T) = \mathcal{V}$.

(3) *T is closed and for all i,* $\mathrm{Ran}\,(\lambda_i - T)$ *is dense in $\mathcal{V}$.*

(4) *T is closed and for all i,* $\mathrm{Ker}(\overline{\lambda}_i - T^*) = \{0\}$.

3.5 Dissipative operators

We say that an operator A is dissipative iff

$$\mathrm{Im}(v|Av) \leq 0, \quad v \in \mathrm{Dom}A.$$

Equivalently, A is dissipative iff $\mathrm{Num}A \subset \{\mathrm{Im}z \leq 0\}$.

A is maximally dissipative iff A is dissipative and $\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}$.

Theorem 3.9 *Let A be a densely defined operator. Then the following conditions are equivalent:*

(1) *$-\mathrm{i}A$ is the generator of a strongly continuous semigroup of contractions.*

(2) *A is maximally dissipative.*

Proof. (1) ⇒(2) We have

$$\mathrm{Re}(v|\mathrm{e}^{-\mathrm{i}tA}v) \leq |(v|\mathrm{e}^{-\mathrm{i}tA}v)| \leq \|v\|^2.$$

Hence

$$\begin{aligned}\mathrm{Im}(v|Av) &= -\mathrm{Re}\mathrm{i}(v|Av) \\ &= \mathrm{Re}\lim_{t\nearrow 0} t^{-1}\left((v|\mathrm{e}^{-\mathrm{i}tA}v) - \|v\|^2\right) \leq 0.\end{aligned}$$

Hence A is dissipative.

We know that the generators of contractions satisfy $\{\mathrm{Re}z > 0\} \subset \mathrm{rs}(-\mathrm{i}A)$.

(2)⇒(1) Let $\mathrm{Re}z > 0$. We have

$$\begin{aligned}\|v\|\|(z+\mathrm{i}A)v\| &\geq |(v|(z+\mathrm{i}A)v)| \\ &\geq \mathrm{Re}(v|(z+\mathrm{i}A)v) \geq \mathrm{Re}z\|v\|^2.\end{aligned}$$

Hence, noting that $z \in \mathrm{rs}A$, we obtain $\|(z+\mathrm{i}A)^{-1}\| \leq \mathrm{Re}z^{-1}$. Therefore, A is an operator of the type $(1,0)$. □

Theorem 3.10 *Let A be dissipative. Then the following conditions are equivalent:*

(1) *A is maximally dissipative.*

(2) *A is closable and there exists z_0 with $\mathrm{Im}z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A) = \mathcal{V}$.*

(3) *A is closed and there exists z_0 with $\mathrm{Im}z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A)$ dense in $\mathcal{V}$.*

(4) *A is closed and there exists z_0 with $\mathrm{Im}z_0 > 0$ and $\mathrm{Ker}(\overline{z}_0 - A^*) = \{0\}$.*

3.6 Symmetric operators I

An operator $A : \mathcal{V} \to \mathcal{V}$ is symmetric iff

$$(Aw|v) = (w|Av), \;\; w, v \in \mathrm{Dom}A.$$

Clearly, A is symmetric iff $\mathrm{Num}A \subset \mathbb{R}$.

Theorem 3.11 *Let A be densely defined and symmetric. Then A is closable. Besides, one of the following possibilities is true:*

(1) $\mathrm{sp}A \subset \mathbb{R}$.

(2) $\mathrm{sp}A = \{\mathrm{Im}z \geq 0\}$.

(3) $\mathrm{sp}A = \{\mathrm{Im}z \leq 0\}$.

(4) $\mathrm{sp}A = \mathbb{C}$.

Proof. $A \subset A^*$ and A^* is closed. Hence A is closable. □

Theorem 3.12 *Let A be a densely defined operator. Then the following conditions are equivalent:*

(1) *$-\mathrm{i}A$ is the generator of a strongly continuous semigroup of isometries.*

(2) *A is symmetric and $\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}$.*

Proof. (1)⇒(2) For $v \in \mathrm{Dom}A$,

$$0 = \partial_t(\mathrm{e}^{-\mathrm{i}tA}v|\mathrm{e}^{-\mathrm{i}tA}v)\Big|_{t=0} = -\mathrm{i}(Av|v) + \mathrm{i}(v|Av).$$

(2)⇒(1) We know that $\mathrm{e}^{-\mathrm{i}tA}$ is the generator of a strongly continuous contractive semigroup. For $v \in \mathrm{Dom}A$,

$$0 = \partial_t(\mathrm{e}^{-\mathrm{i}tA}v|\mathrm{e}^{-\mathrm{i}tA}v)$$

Hence, for $v \in \mathrm{Dom}A$, $\|\mathrm{e}^{-\mathrm{i}tA}v\|^2 = \|v\|^2$. □

Theorem 3.13 *Let A be symmetric. Then the following conditions are equivalent:*

(1) $\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}$.

(2) *There exists z_0 with $\mathrm{Im}z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A) = \mathcal{V}$.*

(3) *A is closed and there esxists z_0 with $\mathrm{Im}z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A)$ dense in $\mathcal{V}$.*

(4) *A is closed and there exists z_0 with $\mathrm{Im}z_0 > 0$ and $\mathrm{Ker}(\overline{z}_0 - A^*) = \{0\}$.*

3.7 Self-adjoint operators

Let T be a densely defined operator on $\mathcal{V}$. T is self-adjoint iff $T^* = T$, that means if for $w \in \mathcal{W}$ there exists $y \in \mathcal{V}$ such that

$$(y|v) = (w|Tv), \ v \in \mathrm{Dom}T,$$

then $w \in \mathrm{Dom}T$ and $Tw = y$.

Theorem 3.14 *Every self-adjoint operator is symmetric and closed. If A is bounded, then it is self-adjoint iff it is symmetric.*

Theorem 3.15 *Fix $z_\pm$ with $\pm\mathrm{Im}z_\pm > 0$. Let A be symmetric. Then the following conditions are necessary and sufficient for A to be self-adjoint:*

(1) $\mathrm{sp}A \subset \mathbb{R}$.

(2) $z_\pm \notin \mathrm{sp}A$.

(3) $\mathrm{Ran}\,(z_\pm - A) = \mathcal{V}$.

(4) A *is closed and* $\mathrm{Ran}\,(z_\pm - A)$ *is dense in* $\mathcal{V}$.

(5) A *is closed and* $\mathrm{Ker}(\overline{z}_\pm - A^*) = \{0\}$.

Theorem 3.16 *Let $\lambda_0 \in \mathbb{R}$. Let A be symmetric. Then the following conditions are sufficient for A to be self-adjoint:*

(1) $\lambda_0 \notin \mathrm{sp}A$.

(2) $\mathrm{Ran}\,(\lambda_0 - A) = \mathcal{V}$.

(3) A *is closed and* $\mathrm{Ran}\,(\lambda_0 - A)$ *is dense in* $\mathcal{V}$.

(4) A *is closed and* $\mathrm{Ker}(\lambda_0 - A^*) = \{0\}$.

Theorem 3.17 *Let A be self-adjoint. Then $U := (A + \mathrm{i})(A - \mathrm{i})^{-1}$ is a unitary operator with* $\mathrm{sp}U = (\mathrm{sp}^{\mathrm{ext}}A + \mathrm{i})(\mathrm{sp}^{\mathrm{ext}}A - \mathrm{i})^{-1}$.

If $f \in C(\mathrm{sp}^{\mathrm{ext}}A)$, we can define

$$f(A) := f(\mathrm{i}(U + \mathrm{i})(U - 1)^{-1}).$$

Of course, we can also apply the functional calculus for measurable functions. In particular, the function $\mathrm{sp}A \ni x \mapsto \mathrm{id}(x) := x$ is a measurable function on $\mathrm{sp}A$. We have $\mathrm{id}A = A$.

Theorem 3.18 (Stone Theorem) *Let A be an operator. Then the following conditions are equivalent:*

(1) $\mathrm{i}A$ *is the generator of a strongly continuous group of unitary operators.*

(2) A *is self-adjoint.*

Proof. To prove (1)⇒(2), suppose that $\mathbb{R} \mapsto U(t)$ is a strongly continuous unitary group. Let $-\mathrm{i}A$ be its generator. Then $[0,\infty[\ni U(t), U(-t)$ are semigroups of contractions with the generators $\mathrm{i}A$ and $-\mathrm{i}A$. By Theorem 3.18, A is symmetric and $\mathrm{sp}A \subset \mathbb{R}$. Hence A is self-adjoint.

(2)⇒(1) follows by the spectral theorem. □

3.8 Essentially self-adjoint operators

Definition 3.19 *An operator $A : \mathcal{V} \to \mathcal{V}$ is essentially self-adjoint iff A^{cl} is self-adjoint.*

Theorem 3.20 *Every essentially self-adjoint operator is symmetric and closable.*

Theorem 3.21 *Fix $z_\pm$ with $\pm \mathrm{Im} z_\pm > 0$ Let A be symmetric. Then the following conditions are necessary and sufficient for A to be essentially self-adjoint:*

(1) *A^* is self-adjoint.*

(2) $\mathrm{Ran}\,(z_\pm - A)$ *is dense in $\mathcal{V}$.*

(3) $\mathrm{Ker}(\overline{z}_\pm - A^*) = \{0\}$.

Theorem 3.22 *Let $\lambda_0 \in \mathbb{R}$. Let A be symmetric. Then the following conditions are sufficient for A to be self-adjoint:*

(1) $\mathrm{Ran}\,(\lambda_0 - A)$ *is dense in $\mathcal{V}$.*

(2) $\mathrm{Ker}(\lambda_0 - A^*) = \{0\}$.

3.9 Scale of Hilbert spaces

Let B be a positive operator on $\mathcal{V}$ with $B \geq 1$. We define the family of Hilbert spaces $\mathcal{V}_\alpha$, $\alpha \in \mathbb{R}$ as follows. For $\alpha \geq 0$, we set $\mathcal{V}_\alpha := \mathrm{Ran}\, B^{-\alpha/2}$ with the scalar product

$$(v|w)_\alpha := (v|B^\alpha w),$$

and $\mathcal{V}_{-\alpha} := \mathcal{V}_\alpha^*$, (where $\mathcal{V}_\alpha^*$ denotes the space of bounded antilinear functionals on $\mathcal{V}_\alpha$). Note that we have the identification $\mathcal{V} = \mathcal{V}^*$, hence both definitions give $\mathcal{V}_0 = \mathcal{V}$.

It is clear, that for $0 \leq \alpha \leq \beta$, $\mathcal{V} \supset \mathcal{V}_\alpha \supset \mathcal{V}_\beta$. Hence $\mathcal{V}_{-\alpha} = \mathcal{V}_\alpha^*$ can be identified with a subspace of $\mathcal{V}_{-\beta} = \mathcal{V}_\beta^*$. Thus we obtain $\mathcal{V}_\alpha \supset \mathcal{V}_\beta$ for any $\alpha \leq \beta$.

Note that for $\alpha > 0$ $\mathcal{V}$ is embedded in $\mathcal{V}_{-\alpha}$ and for $v, w \in \mathcal{V}$

$$(v|w)_{-\alpha} = \left(B^{-\alpha/2} v | B^{-\alpha/2} w\right).$$

Moreover, $\mathcal{V}$ is dense in $\mathcal{V}_{-\alpha}$.

It is easy to see that, for $\alpha \geq 0$, $B_0^{-\alpha} := B^{-\alpha}$ is a unitary operator from $\mathcal{V}_0$ to $\mathcal{V}_{2\alpha}$. Moreover, the operator B^α, defined by the functional calculus (or as the inverse of the bounded operator $B^{-\alpha}$, with the domain $\mathcal{V}_{2\alpha}$ extends to a unitary operator from $\mathcal{V}_0$ to $\mathcal{V}_{-\alpha}$, which we will denote by B_0^α. For any α, β, setting $B_\alpha^{\alpha-\beta} := B_0^{-\beta}(B_0^{-\alpha})^{-1}$ we obtain a unitary operator from $\mathcal{V}_{2\alpha}$ to $\mathcal{V}_{2\beta}$. These operators satisfy the chain rule:

$$B_{\alpha+2\beta}^{-\gamma} B_\alpha^{-\beta} = B_\alpha^{-\gamma-\beta}.$$

Sometimes we will use a different notation: $B^{-\alpha}\mathcal{V} = \mathcal{V}_{2\alpha}$. If A is a self-adjoint operator, then we will use the notation $\langle A \rangle := (1 + A^2)^{1/2}$.

Lemma 3.23 *Let $0 \leq \alpha \leq 1$. Then* $\mathrm{Dom} B = \{v \in \mathcal{V}_\alpha \ : \ B_\alpha v \in \mathcal{V}\}$.

3.10 Sesquilinear forms

Let $\mathcal{V}, \mathcal{W}$ be complex spaces. We say that $\mathfrak{t}$ is a sesquilinear quasiform on $\mathcal{W} \times \mathcal{V}$ iff there exist subspaces $\mathrm{Dom}_{\mathrm{L}}\mathfrak{t} \subset \mathcal{W}$ and $\mathrm{Dom}_{\mathrm{R}}\mathfrak{t} \subset \mathcal{V}$ such that

$$\mathrm{Dom}_{\mathrm{L}}\mathfrak{t} \times \mathrm{Dom}_{\mathrm{R}}\mathfrak{t} \ni (w, v) \mapsto \mathfrak{t}(w, v) \in \mathbb{C}$$

is a sesquilinear map. From now on by a sesquilinear form we will mean a sesquilinear quasiform.

We define a form $\mathfrak{t}^*$ with the domains $\mathrm{Dom}_{\mathrm{L}}\mathfrak{t}^* := \mathrm{Dom}_{\mathrm{R}}\mathfrak{t}$, $\mathrm{Dom}_{\mathrm{R}}\mathfrak{t}^* := \mathrm{Dom}_{\mathrm{L}}\mathfrak{t}$, by the formula $\mathfrak{t}^*(v, w) := \overline{\mathfrak{t}(w, v)}$. If $\mathfrak{t}_1$ are $\mathfrak{t}_2$ forms, then we define $\mathfrak{t}_1 + \mathfrak{t}_2$ with the domain $\mathrm{Dom}_{\mathrm{L}}\mathfrak{t}_1 + \mathfrak{t}_2 := \mathrm{Dom}_{\mathrm{L}}\mathfrak{t}_1 \cap \mathrm{Dom}_{\mathrm{L}}\mathfrak{t}_1$, $\mathrm{Dom}_{\mathrm{R}}\mathfrak{t}_1 + \mathfrak{t}_2 := \mathrm{Dom}_{\mathrm{R}}\mathfrak{t}_1 \cap \mathrm{Dom}_{\mathrm{R}}\mathfrak{t}_1$ by $(\mathfrak{t}_1 + \mathfrak{t}_2)(w, v) := \mathfrak{t}_1(w, v) + \mathfrak{t}_2(w, v)$. We write $\mathfrak{t}_1 \subset \mathfrak{t}_2$ if $\mathrm{Dom}_{\mathrm{L}}\mathfrak{t}_1 \subset \mathrm{Dom}_{\mathrm{L}}\mathfrak{t}_2$, $\mathrm{Dom}_{\mathrm{R}}\mathfrak{t}_1 \subset \mathrm{Dom}_{\mathrm{R}}\mathfrak{t}_2$, and $\mathfrak{t}_1(w, v) = \mathfrak{t}_2(w, v)$ $w \in \mathrm{Dom}_{\mathrm{L}}\mathfrak{t}_1$, $v \in \mathrm{Dom}_{\mathrm{R}}\mathfrak{t}_1$.

$\mathfrak{t}$ is bounded iff

$$|\mathfrak{t}(w, v)| \leq c\|w\|\|v\|, \quad w \in \mathrm{Dom}_{\mathrm{L}}\mathfrak{t}, \quad v \in \mathrm{Dom}_{\mathrm{R}}\mathfrak{t}.$$

From now on, we will usually assume that $\mathcal{W} = \mathcal{V}$ and $\mathrm{Dom}_{\mathrm{L}}\mathfrak{t} = \mathrm{Dom}_{\mathrm{R}}\mathfrak{t}$ and the latter subspace will be simply denoted by $\mathrm{Dom}_{\mathrm{F}}\mathfrak{t}$.

Recall that the numerical range of the form $\mathfrak{t}$ is defined as

$$\mathrm{Num}\mathfrak{t} := \{\mathfrak{t}(v) \ : \ v \in \mathrm{Dom}_{\mathrm{F}}\mathfrak{t}, \ \|v\| = 1\}.$$

We proved that $\mathrm{Num}\mathfrak{t}$ is a convex set.

The form $\mathfrak{t}$ is bounded iff $\mathrm{Num}\mathfrak{t}$ is bounded. Equivalently, $|\mathfrak{t}(v)| \leq c\|v\|^2$.

$\mathfrak{t}$ is bounded from below, if there exists c such that

$$\mathrm{Num}\mathfrak{t} \subset \{z \ : \ \mathrm{Re} z > c\}.$$

$\mathfrak{t}$ is hermitian iff $\mathrm{Num}\mathfrak{t} \subset \mathbb{R}$. The equivalent condition: $\mathfrak{t}(w, v) = \overline{\mathfrak{t}(v, w)}$.

If T is an operator on $\mathcal{V}$, then $\mathfrak{t}(w, v) := (w, Tv)$ with the domain $\mathrm{Dom}T$ is a form called the form associated with the operator T. Clearly, $\mathrm{Num}\mathfrak{t} = \mathrm{Num}T$.

3.11 Closed positive forms

Let $\mathfrak{s}$ be a positive form.

Definition 3.24 *We say that $\mathfrak{s}$ is a closed form iff $\mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$ with the scalar product*

$$(w, v)_{\mathfrak{s}} := (\mathfrak{s} + 1)(w, v), \quad w, v \in \mathrm{Dom}_{\mathrm{F}}\mathfrak{s},$$

is a Hilbert space.

Theorem 3.25 *The form $\mathfrak{s}$ is closed iff for any sequence (x_n) in $\mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$, if $x_n \to x$ and $\mathfrak{s}(x_n - x_m) \to 0$, then $x \in \mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$ and $\mathfrak{s}(x_n - x) \to 0$.*

Example 3.26 *Let A be an operator. Then $(Aw|Av)$ with the domain $\mathrm{Dom}A$ is a closed form iff A is closed.*

3.12 Closable positive forms

Let $\mathfrak{s}$ be a positive form.

Definition 3.27 *We say that $\mathfrak{s}$ is a closable form iff there exists a closed form $\mathfrak{s}_1$ such that $\mathfrak{s} \subset \mathfrak{s}_1$.*

Theorem 3.28 (1) *The form* $\mathfrak{t}$ *is closable* $\Leftrightarrow$ *for any sequence* $(x_n) \subset \mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$, *if* $x_n \to 0$ *and* $\mathfrak{s}(x_n - x_m) \to 0$, *then* $\mathfrak{s}(x_n) \to 0$.

(2) *If* $\mathfrak{s}$ *is closable, then there exists the smallest closed form* $\mathfrak{s}_1$ *such that* $\mathfrak{s} \subset \mathfrak{s}_1$. *We will denote it by* $\mathfrak{s}^{\mathrm{cl}}$.

(3) $\mathrm{Num}\mathfrak{s}$ *is dense in* $\mathrm{Num}\mathfrak{s}^{\mathrm{cl}}$

Proof. (1) $\Rightarrow$ follows immediately from Theorem 3.25.

To prove (1) $\Leftarrow$, define $\mathfrak{s}_1$ as follows: $v \in \mathrm{Dom}_{\mathrm{F}}\mathfrak{s}_1$, iff there exists a sequence $(v_n) \subset \mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$ such that $v_n \to v$ and $\mathfrak{s}(v_n - v_m) \to 0$. We set then $\mathfrak{s}_1(v) := \lim_{n\to\infty}\mathfrak{s}(v_n)$ (the limit exists, because $\mathfrak{s}(v_n)$ is a Cauchy sequence).

To show that the definition is correct, suppose that $w_n \in \mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$, $w_n \to v$ and $\mathfrak{s}(w_n - w_m) \to 0$. Then $v_n - w_n - (v_m - w_m) \to 0$ and $v_n - w_n \to 0$. By the hypothesis we get $\mathfrak{s}(v_n - w_n) \to 0$. Hence, $\lim_{n\to\infty}\mathfrak{s}(v_n) = \lim_{n\to\infty}\mathfrak{s}(w_n)$. Thus the definition of $\mathfrak{s}_1$ does not depend on the choice of the sequence v_n. It is clear that $\mathfrak{s}_1$ is a closed form containing $\mathfrak{s}$. Hence $\mathfrak{s}$ is closable.

To prove (2) note that the form $\mathfrak{s}_1$ constructed above is the smallest closed form containg $\mathfrak{s}$. □

Example 3.29 *Let* A *be an operator. Then* $(Aw|Av)$ *with the domain* $\mathrm{Dom}A$ *is closable iff* A *is a closable operator. Then* $(A^{\mathrm{cl}}w|A^{\mathrm{cl}}v)$ *with the domain* $\mathrm{Dom}A^{\mathrm{cl}}$ *is its closure.*

3.13 Operators associated with positive forms

Let S be a positive self-adjoint operator. We define the form $\mathfrak{s}$ as follows: $\mathrm{Dom}_{\mathrm{F}}\mathfrak{s} := \mathrm{Dom}S^{1/2}$ and $\mathfrak{s}(v,w) := (S^{1/2}v|S^{1/2}w)$. We will say that $\mathfrak{s}$ is the form associated with S.

Theorem 3.30 (1) $\mathfrak{s}$ *is a closed form.*

(2) $\mathrm{Dom}S$ *is an essential domain for* $\mathfrak{s}$.

(3) *NumS is dense in* $\mathrm{Num}\mathfrak{s}$.

The next theorem describes the converse construction.

Theorem 3.31 *Let* $\mathfrak{s}$ *be a densely defined closed positive form. Then there exists a unique positive self-adjoint operator* S *such that* $\mathrm{Dom}_{\mathrm{F}}\mathfrak{s} = \mathrm{Dom}S^{1/2}$ *and* $\mathfrak{s}(v,w) := (S^{1/2}v|S^{1/2}w)$. *We will say that* S *is the operator associated with the form* $\mathfrak{s}$.

Proof. For $w \in \mathcal{V}$, $v \in \mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$, we have

$$|(v|w)| \leq \|v\|\|w\| \leq \|v\|_{\mathfrak{s}}\|w\|.$$

By the Riesz lemma, there exists $A : \mathcal{V} \to \mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$ such that

$$(v|w) = (v|Aw)_{\mathfrak{s}}, \tag{3.14}$$

$$\|Aw\| \leq \|Aw\|_{\mathfrak{s}} \leq \|w\|.$$

$\mathrm{Ker}A = \{0\}$, because $Aw = 0$ implies $(v|w) = 0$ for $v \in \mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$, and $\mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$ s dense in $\mathcal{V}$. Besides, A is self-adjoint. Putting

$$S := A^{-1} - 1$$

we define a positive self-adjoint operator.

$$\mathfrak{s}(v,y) = (v|Sy), \quad v \in \mathrm{Dom}_{\mathrm{F}}\mathfrak{s}, \quad y \in \mathrm{Dom}S = \mathrm{Ran}\,A.$$

Let us show that $\mathrm{Dom}_{\mathbf{F}}\mathfrak{s}$ is an essential domain for $\mathfrak{s}$. Let $v \in \mathrm{Dom}_{\mathbf{F}}\mathfrak{s}$ is $\mathfrak{s}$-orthogonal to $\mathrm{Ran}\, A = \mathrm{Dom}S$. Then v is orthogonal to $\mathrm{Dom}_{\mathbf{F}}\mathfrak{s}$—see (3.14). Hence $v = 0$.

Define $\mathfrak{s}_1$ by $\mathrm{Dom}_{\mathbf{F}}\mathfrak{s}_1 = \mathrm{Dom}S^{\frac{1}{2}}$ and $\mathfrak{s}_1(w,v) = (S^{\frac{1}{2}}w|S^{1/2}v)$. The form $\mathfrak{s}$ and $\mathfrak{s}_1$ coincide on $\mathrm{Dom}S \subset \mathrm{Dom}S^{\frac{1}{2}} \cap \mathrm{Dom}_{\mathbf{F}}\mathfrak{s}$. We proved above that $\mathrm{Dom}S$ is an essential domain for $\mathfrak{s}$. The form $\mathfrak{s}_1$ is obviously closed and $\mathrm{Dom}_{\mathbf{F}}\mathfrak{s}_1$ is an essential domain for $\mathfrak{s}_1$. Hence, $\mathfrak{s}_1 = \mathfrak{s}$. □

3.14 Polar decomposition

Theorem 3.32 *Let A be a densely defined closed operator. Let B be the operator associated to the form $(Aw|Av)$. Let $\mathcal{V}_\alpha$ be the scale of spaces $(B+1)^{-\alpha}\mathcal{V}$, so that $\mathcal{V}_1 = \mathrm{Dom}A$ and $\mathcal{V}_{-1} = \mathcal{V}_1^*$. Then*

(1) *A, treated as an operator from $\mathcal{V}_1$ to $\mathcal{V}_0$, and denoted A_1, is bounded.*

(2) *A^* extends by density to a bounded operator, denoted A_0^* from $\mathcal{V}_0$ to $\mathcal{V}_{-1}$. We have $A_0^* = (A_1)^*$.*

(3) $\mathrm{Dom}A^* = \{v \in \mathcal{V} \ : \ A_0^*v \in \mathcal{V}\}$.

(4) $\mathrm{Dom}B = \{v \in \mathrm{Dom}A \ : \ Av \in \mathrm{Dom}A^*\}$ *and for* $v \in \mathrm{Dom}B$, $Bv = A^*Av$.

Proof. (1) is obvious.

To see (2), note that for $v \in \mathrm{Dom}A^*$, $w \in \mathrm{Dom}A$ we have

$$|(A^*v|w)| = (v|Aw)| \leq \|v\|\|w\|_1.$$

Hence $\|A^*v\|_{-1} \leq \|v\|$, and so $A^* : \mathcal{V} \to \mathcal{V}_{-1}$ is bounded. $\mathrm{Dom}A^*$ is dense in $\mathcal{V}$. Hence A^* extends to a bounded operator $A_0^* : \mathcal{V} \to \mathcal{V}_{-1}$.

To prove (3), let $v \in \mathcal{V}$, $A_0^*v = w \in \mathcal{V}$. Then there exists $(v_n) \subset \mathrm{Dom}A^*$ such that $v_n \to v$ in the norm of $\mathcal{V}$ and $A^*v_n \to w$ in the norm of $\mathcal{V}_{-1}$. Hence for $x \in \mathrm{Dom}A$,

$$(v_n|Ax) = (A^*v_n|x) \to (w|x).$$

Therefore, $(w|x) = (v|Ax)$. Hence, $v \in \mathrm{Dom}A^*$ and $A^*v = w$.

To prove (4), denote B_1 as the extension of B to the operator from $\mathcal{V}_1$ to $\mathcal{V}_{-1}$. Note that B_1 is bounded.

We have $B_1 = A_0^*A_1$. In fact, for $v, w \in \mathcal{V}_1$

$$(w|B_1v) = (B^{1/2}w|B^{1/2}v) = (Aw|Av) = (w|A_0^*A_1v).$$

Now

$$\begin{aligned}\mathrm{Dom}B &= \{v \in \mathcal{V}_1 \ : \ B_1v \in \mathcal{V}\}\\ &= \{v \in \mathcal{V}_1 \ : \ A_0^*A_1v \in \mathcal{V}\}\\ &= \{v_1 \ : \ Av \in \mathrm{Dom}A^*\}.\end{aligned}$$

□

Motivated by the above theorem we will write A^*A for B.

Theorem 3.33 *Let A be closed. Then there exist a unique positive operator $|A|$ and a unique partial isometry U such that* $\mathrm{Ker}U = \mathrm{Ker}A$ *and* $A = U|A|$. *We have then* $\mathrm{Ran}\, U = \mathrm{Ran}\, A^{\mathrm{cl}}$.

Proof. The operator A^*A is positive. By the spectral theorem, we can then define

$$|A| := \sqrt{A^*A}.$$

On Ran $|A|$ the operator U is defined by

$$U\ |A|v := Av.$$

It is isometric, because

$$\||A|v\|^2 = (v||A|^2v) = (v|A^*Av) = \|Av\|^2,$$

and correctly defined. We can extend it to $(\mathrm{Ran}\,|A|)^{\mathrm{cl}}$ by continuity. On $\mathrm{Ker}|A| = (\mathrm{Ran}\,|A|)^{\mathrm{cl}}$, we extend it by putting $Uv = 0$. □

3.15 Sectorial forms

A subset of $\mathbb{C}$ of the form $\mathrm{Sec}(a,\theta) := a + \{z\ :\ |\arg z| < \theta\}$ with $\theta < \frac{\pi}{2}$ will be called a sector. a will be called its tip and θ its angle. We say that a form $\mathfrak{t}$ is sectorial iff there exists $a \in \mathbb{C}$ and $\theta < \frac{\pi}{2}$ such that $\mathrm{Num}\mathfrak{t} \subset \mathrm{Sec}(a,\theta)$.

Lemma 3.34 *Let* $\mathfrak{t}$ *be a sectorial form with the sector given by* a, θ. *Then*

$$|(\mathfrak{t}-a)(w,v)| \leq (1+\tan\theta)\mathrm{Re}(\mathfrak{t}-a)(w)^{\frac{1}{2}}\mathrm{Re}(\mathfrak{t}-a)(v)^{\frac{1}{2}}.$$

Clearly, if $\mathfrak{t}$ is sectorial and a is the tip of a sector containing $\mathrm{Num}\mathfrak{t}$, then $\mathrm{Re}\mathfrak{t} + \mathrm{Re}a$ is a positive form. We say that a sectorial form $\mathfrak{t}$ is closed iff $\mathrm{Re}\mathfrak{t}$ is closed. We say that a sectorial form $\mathfrak{t}$ is closable iff $\mathrm{Re}\mathfrak{t}$ is closable.

It is easy to see, using Lemma 3.34, that Theorems 3.25 and 3.28 remain true if the form $\mathfrak{s}$ is assumed to be sectorial.

3.16 Operators associated with sectorial forms

Definition 3.35 *An operator* T *is called sectorial iff the associated form is sectorial, that means, if* $\mathrm{Num}T \subset \mathrm{Sec}(a,\theta)$. *A maximal sectorial operator is called shortly m-sectorial.*

Theorem 3.36 (1) *Let* $\mathfrak{t}$ *be a sectorial form. Then there exists a unique m-sectorial operator* T *such that* $\mathrm{Dom}T \subset \mathrm{Dom}\mathfrak{t}$ *and*

$$\mathfrak{t}(w,v) = (w|Tv), \quad v \in \mathrm{Dom}T,\ w \in \mathrm{Dom}_{\mathrm{F}}\mathfrak{t}.$$

T is called the operator associated with the form $\mathfrak{t}$ *and denoted by* $T_{\mathfrak{t}}$.

(2) $\mathrm{Dom}T$ *is an essential domain for* $\mathfrak{t}$.

(3) $\mathrm{Num}T$ *is dense in* $\mathrm{Num}\mathfrak{t}$.

(4) *If* $\mathfrak{t}$ *is bounded, then so is* T.

We will assume that the sector of $\mathfrak{t}$ has the tip at 0. We will write $\mathfrak{s} := \mathrm{Re}\mathfrak{t}$.

Lemma 3.37 *There exists an invertible operator* $B \in B(\mathrm{Dom}_{\mathrm{F}}\mathfrak{s})$ *such hat*

$$(\mathfrak{t}+1)(w,v) = \mathfrak{s}(w,Bv) + (w|Bv).$$

Proof. By Lemma 3.34, the form $\mathfrak{t}+1$ is bounded in the Hilbert space $\mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$. Hence there exists $B \in B(\mathrm{Dom}_{\mathrm{F}}\mathfrak{s})$ such that

$$(\mathfrak{t}+1)(w,v) = (w|Bv)_{\mathfrak{s}} = (w|Bv) + \mathfrak{s}(w,Bv).$$

We have

$$\|v\|_{\mathfrak{s}}^2 = \mathrm{Re}(\mathfrak{t}+1)(v) = \mathrm{Re}(v|Bv)_{\mathfrak{s}} \leq \|Bv\|_{\mathfrak{s}}\|v\|_{\mathfrak{s}}.$$

Hence $\|v\|_{\mathfrak{s}} \leq \|Bv\|_{\mathfrak{s}}$. Therefore, $\mathrm{Ran}\, B$ is closed.

If w is orthogonal in $\mathrm{Dom}_{\mathrm{F}}\mathfrak{s}$ to $\mathrm{Ran}\, B$, then

$$\|w\|_{\mathfrak{s}}^2 = \mathrm{Re}(w|Bw)_{\mathfrak{s}} = 0.$$

Hence, $w = 0$. Therefore, B is invertible. □

Proof of Theorem 3.36. Let S denote the operator associated with $\mathfrak{s}$. Then $T := S(B+1) - 1$ with the domain $\mathrm{Dom}T = \mathrm{Dom}S$ satisfies the conditions of the theorem. □

3.17 Perturbations of sectorial forms

Theorem 3.38 *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be sectorial forms.*

(1) *$\mathfrak{t}_1 + \mathfrak{t}_2$ is also a sectorial form.*

(2) *If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closed, then $\mathfrak{t}_1 + \mathfrak{t}_2$ is closed as well.*

(3) *If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closable, then $\mathfrak{t}_1 + \mathfrak{t}_2$ is closable as well and $(\mathfrak{t}_1 + \mathfrak{t}_2)^{\mathrm{cl}} \subset \mathfrak{t}_1^{\mathrm{cl}} + \mathfrak{t}_2^{\mathrm{cl}}$.*

Definition 3.39 *Let $\mathfrak{p}$, $\mathfrak{s}$ be forms and let $\mathfrak{s}$ be positive. We say that $\mathfrak{p}$ is $\mathfrak{s}$-bounded iff* $\mathrm{Dom}_{\mathrm{F}}\mathfrak{s} \subset \mathrm{Dom}_{\mathrm{F}}\mathfrak{p}$ *and*

$$b := \inf_{c>0} \sup_{v\in \mathrm{Dom}_{\mathrm{F}} s} \frac{|\mathfrak{p}(v)|}{\mathfrak{s}(v) + c\|v\|^2} < \infty.$$

The number b is called the $\mathfrak{s}$-bound of $\mathfrak{p}$.

Theorem 3.40 *Let $\mathfrak{t}$ be sectorial and let $\mathfrak{p}$ be* $\mathrm{Re}\mathfrak{t}$*-bounded with the* $\mathrm{Re}\mathfrak{t}$*-bound* < 1. *Then*

(1) *The form $\mathfrak{t}+\mathfrak{p}$ (with the domain* $\mathrm{Dom}_{\mathrm{F}}\mathfrak{t}$*) is sectorial as well.*

(2) *$\mathfrak{t}$ is closed $\Leftrightarrow$ $\mathfrak{t}+\mathfrak{p}$ is closed.*

(3) *$\mathfrak{t}$ is closable $\Leftrightarrow$ $\mathfrak{t}+\mathfrak{p}$ is closable, and then* $\mathrm{Dom}_{\mathrm{F}}(\mathfrak{t}+\mathfrak{p})^{\mathrm{cl}} = \mathrm{Dom}_{\mathrm{F}}\mathfrak{t}^{\mathrm{cl}}$.

Proof. Let us prove (1). For some $b < 1$, we have

$$|\mathfrak{p}(v)| \leq b\mathrm{Re}\mathfrak{t}(v) + c\|v\|^2.$$

Hence

$$\begin{aligned} \mathrm{Im}(\mathfrak{t}+\mathfrak{p})(v) &\leq |\mathrm{Im}\mathfrak{t}(v)| + |\mathrm{Im}\mathfrak{p}(v)| \leq (\tan\theta + b)\mathrm{Re}\mathfrak{t}(v) + c\|v\|^2, \\ \mathrm{Re}(\mathfrak{t}+\mathfrak{p})(v) &\geq \mathrm{Re}\mathfrak{t}(v) - |\mathrm{Im}\mathfrak{p}(v)| \geq (1-b)\mathrm{Re}\mathfrak{t}(v) - c\|v\|^2. \end{aligned} \tag{3.15}$$

Hence,

$$|\mathrm{Im}(\mathfrak{t}+\mathfrak{p})(v)| \leq (1-b)^{-1}(\tan\theta + b)\left(\mathrm{Re}(\mathfrak{t}+\mathfrak{p})(v) + c\|v\|^2\right) + c\|v\|^2.$$

This proves that $\mathfrak{t}+\mathfrak{p}$ is sectorial.

To see (2) and (3), note that (3.15) and

$$(1+b)\mathrm{Re}\mathfrak{t}(v) + c\|v\|^2 \geq \mathrm{Re}(\mathfrak{t}+\mathfrak{p})(v)$$

prove that the norms $\|\cdot\|_{\mathfrak{t}}$ and $\|\cdot\|_{\mathfrak{t}+\mathfrak{p}}$ are equivalent. □

3.18 Friedrichs extensions

Theorem 3.41 *Let T be a sectorial operator. Then the form $\mathfrak{t}(w,v) := (w|Tv)$ is closable.*

Proof. It suffices to assume that the tip of the sector of $\mathfrak{t}$ is 0. Suppose that $w_n \in \mathrm{Dom}T = \mathrm{Dom}_{\mathrm{F}}\mathfrak{t}$, $w_n \to 0$, $\lim_{n,m\to\infty} \mathfrak{t}(w_n - w_m) = 0$. Then

$$\begin{aligned} |\mathfrak{t}(w_n)| &\leq |\mathfrak{t}(w_n - w_m, w_n)| + |\mathfrak{t}(w_m, w_n)| \\ &\leq (1+\tan\theta)(\mathrm{Re}\mathfrak{t}(w_n))^{\frac{1}{2}}(\mathrm{Re}\mathfrak{t}(w_n - w_m))^{\frac{1}{2}} + (w_m|Tw_n). \end{aligned}$$

For any $\epsilon > 0$ there exists N such that for $n, m > N$ we have $\mathrm{Re}\mathfrak{t}(w_n - w_m) \leq \epsilon^2$. Besides, $\lim_{m\to\infty}(w_m|Tw_n) = 0$. Therefore,

$$|\mathfrak{t}(w_n)| \leq \epsilon(1+\tan\theta)|\mathfrak{t}(w_n)|^{1/2}.$$

Hence $\mathfrak{t}(w_n) \to 0$. □

Thus there exists a unique m-sectorial operator T_{Fr} associated with the form $\mathfrak{t}^{\mathrm{cl}}$. The operator T_{Fr} is called the Friedrichs extension of T.

3.19 Relative operator boundedness

Theorem 3.42 *Let A be a closed operator and B an operator with $\mathrm{Dom}B \supset \mathrm{Dom}A$. Then the following statements are equivalent:*

(1) *B has the A-bound equal to a_1, that is*

$$\inf_{\nu>0}\sup_{v\neq 0}\left(\frac{\|Bv\|^2}{\|Av\|^2 + \nu^2\|v\|}\right)^{\frac{1}{2}} = a_1.$$

(2)

$$\inf_{\nu>0}\|B(A^*A+\nu^2)^{-1/2}\| = a_1.$$

If, moreover, A is self-adjoint, then the above statements are equivalent to

(3) $\inf_{\nu>0}\|B(\mathrm{i}\nu - A)^{-1}\| = a_1$

Proof. The equivalence of (1) and (2) is evident.

The equivalence of (2) and (3) for a self-adjoint A is the consequence of the unitarity of

$$(A^2+\nu^2)^{-1/2}(\mathrm{i}\nu - A).$$

□

Theorem 3.43 (Kato-Rellich) *Let A be self-adjoint, B symmetric. Let B be A-bounded with the $A-$bound < 1. Then*

(1) *$A+B$ is self-adjoint on* $\mathrm{Dom}A$.

(2) *If A is essentally self-adjoint on $\mathcal{D}$, then $A+B$ is essentially self-adjoint on $\mathcal{D}$.*

Proof. Clearly, $A+B$ is symmetric on $\mathrm{Dom}A$. Moreover, for some ν, $\|B(\mathrm{i}\nu - A)^{-1}\| < 1$ and (which is equivalent by the unitarity of $(A-\mathrm{i}\nu)(A+\mathrm{i}\nu)^{-1}$), $\|B(-\mathrm{i}\nu - A)^{-1}\| < 1$. Hence, $\mathrm{i}\nu - A - B$ and $-\mathrm{i}\nu - A - B$ are invertible. □

Let us note an improved version of the notion of the operator boundedness:

Theorem 3.44 *Let A be a closed operator and B an operator with* $\mathrm{Dom}B \supset \mathrm{Dom}A$*. Then the following statements are equivalent:*
(1)

$$a_2 = \inf_{\mu,\nu>0} \sup_{v\neq 0} \left(\frac{\|Bv\|^2}{\|(A-\mu)v\|^2 + \nu^2\|v\|} \right)^{\frac{1}{2}}.$$

(2)

$$\inf_{\mu,\nu>0} \|B((A-\mu)^*(A-\mu)+\nu^2)^{-1/2}\| = a_2.$$

If, moreover, A is self-adjoint, then the above statements are equivalent to

(3) $\inf_{\mu,\nu>0} \|B(\mu + \mathrm{i}\nu - A)^{-1}\| = a_2$

Note hat the analog of Theorem 3.43 is true with a_1 replaced with a_2.

3.20 Relative form boundedness

Theorem 3.45 *Let A be a self-adjoint operator. Let $\mathfrak{b}$ be a hermitian form with* $\mathrm{Dom}_{\mathrm{F}}\mathfrak{b} \supset \mathrm{Dom}|A|^{\frac{1}{2}}$*. Then the following statements are equivalent:*
(1)

$$a_3 = \inf_{\mu,\nu>0} \sup_{v\neq 0} \frac{\mathfrak{b}(v)}{(v|((A-\mu)^2+\nu^2)^{\frac{1}{2}}v)}.$$

(2)

$$\inf_{\mu,\nu>0} \sup_{\|v\|=1} \|\mathfrak{b}\big(((A-\mu)^2+\nu^2)^{-\frac{1}{4}}v, ((A-\mu)^2+\nu^2)^{-\frac{1}{4}}v\big)\| = a_3.$$

(3)

$$\inf_{\mu,\nu>0} \sup_{\|v\|=1} \|\mathfrak{b}\big((\mu+\mathrm{i}\nu-A)^{-\frac{1}{2}}v, (\mu+\mathrm{i}\nu-A)^{-\frac{1}{2}}v\big)\| = a_3.$$

If the conditions of the above theorem are satified, then we say that the A-form-bound of $\mathfrak{b}$ equals a_3. If $a_3 < \infty$, then there exists a bounded operator from $\langle A\rangle^{-1/2}\mathcal{V}$ to $\langle A\rangle^{1/2}\mathcal{V}$ such that $b(v) = (v|B\phi)$.

Theorem 3.46 *Let A be a self-adjoint operator. Let $\mathfrak{b}$ be a form with the A-form-bound less than1. Let B be the corresponding operator. Then there exists a open subsets in the upper and lower complex half-plane such that the series*

$$R(z) := \sum_{j=0}^{\infty} (z-A)^{-1}(B(z-A)^{-1})^j$$

is convergent. Moreover, $R(z)$ is a resolvent of a self-adjoint operator, which will be called the form sum of A and B. If A is bounded from below, then so is $A+B$ and $\mathrm{Dom}|A+B|^{\frac{1}{2}} = \mathrm{Dom}|A|^{\frac{1}{2}}$.

3.21 Non-maximal operators

Theorem 3.47 $\dim \mathrm{Ran}\,(z-A)^{\perp} = \dim \mathrm{Ker}(\overline{z}-A^*)$ *is a constant function on connected components of* $\mathbb{C}\backslash(\mathrm{Num}\,A)^{\mathrm{cl}}$.

Proof. Let us show that if $|\alpha| < \lambda$, then

$$\mathrm{Ran}\,(\mathrm{i}\lambda - A) \cap \mathrm{Ran}\,(\mathrm{i}\lambda + \alpha - A)^{\perp} = \{0\}. \tag{3.16}$$

Let $w \in \mathrm{Ran}\,(\mathrm{i}\lambda - A)$. Then there exists $v \in \mathrm{Dom}A$ such that

$$w = (\mathrm{i}\lambda - A)v$$

and $\|v\| \leq \lambda^{-1}\|w\|$. If moreover, $w \in \mathrm{Ran}\,(\mathrm{i}\lambda + \alpha - A)^{\perp} = \mathrm{Ker}(-\mathrm{i}\lambda - \overline{\alpha} - A^*)$, then

$$\begin{aligned} 0 &= ((-\mathrm{i}\lambda + \overline{\alpha} - A^*)w|v) \\ &= (w|(\mathrm{i}\lambda - A)v) + \alpha(w|v) \\ &= \|w\|^2 + \alpha(w|v). \end{aligned}$$

But

$$|\|w\|^2 + \alpha(w|v)| \geq (1 - |\alpha|/|\lambda|)\|w\|^2 > 0,$$

which is a contradiction and completes the proof of (3.16).

Now (3.16) implies that $\dim \mathrm{Ran}\,(\mathrm{i}\lambda - A) \leq \dim \mathrm{Ran}\,(\mathrm{i}\lambda + \alpha - A)$. □

3.22 Symmetric operators II

Let A be closed symmetric.

Theorem 3.48 *The so-called defect indices*

$$n_{\pm} := \dim \mathrm{Ker}(\lambda - A^*),\ \ \lambda \in \mathbb{C}_{\pm}$$

does not depend on λ. *One of the following possibilities is true:*

$$\begin{array}{lll} 1)\ \mathrm{sp}A \subset \mathbb{R}, & n_{\pm} = 0, & A \textit{ is self-adjoint;} \\ 2)\ \mathrm{sp}A = \{\mathrm{Im}z \geq 0\}, & n_+ \neq 0, n_- = 0, & A \textit{ is not self-adjoint}; \\ 3)\ \mathrm{sp}A = \{\mathrm{Im}z \leq 0\}, & n_+ = 0, n_- \neq 0, & A \textit{ is not self-adjoint}; \\ 4)\ \mathrm{sp}A = \mathbb{C} & n_+ \neq 0, n_- \neq 0, & A \textit{ is not self-adjoint.} \end{array}$$

Definition 3.49 *Let* A *be symmetric and closed. Define on* $\mathrm{Dom}A^*$ *the following scalar product:*

$$(v|w)_{A^*} := (v|w) + (A^*v|A^*w)$$

and the following antisymmetric form:

$$[v|w]_{A^*} := (A^*v|w) - (v|A^*w).$$

The A^*-*completeness and the* A^*-*orthogonality is defined using the scalar product* $(\cdot|\cdot)_{A^*}$. *A space is* A^*-*symmetric iff* $[\cdot|\cdot]_{A^*}$ *vanishes on this subspace.*

Theorem 3.50 (1) *Every closed extension of* A *is a restriction of* A^* *to an* A^*-*closed subspace in* $\mathrm{Dom}A^*$ *containing* $\mathrm{Dom}A$.

(2)

$$\mathrm{Dom}A^* = \mathrm{Dom}A \oplus \mathrm{Ker}(A^* + \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i})$$

and the components in the above direct sum are A^*-closed, A^*-orthogonal and

$$(w \oplus w_+ \oplus w_- | v \oplus v_+ \oplus v_-)_{A^*} = (w|v) + (Aw|Av) + 2(w_+|v_+) + 2(w_-|v_-),$$

$$[w \oplus w_+ \oplus w_- | v \oplus v_+ \oplus v_-]_{A^*} = 2i(w_+|v_+) - 2i(w_-|v_-).$$

Proof. (1) is obvious. In (2) the A^*-orthogonality and the A^*-closedness are easy.

Let $w \in \mathrm{Dom}A^*$ and

$$w \perp \mathrm{Dom}A \oplus \mathrm{Ker}(A^* + \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i})$$

in the sense of the product $(\cdot|\cdot)_{A^*}$. In particular, for $v \in \mathrm{Dom}A$ we have

$$0 = (A^*w|A^*v) + (w|v) = (A^*w|Av) + (w|v).$$

Hence $A^*w \in \mathrm{Dom}A^*$ and

$$A^*A^*w = -w.$$

Therefore,

$$(A^* + \mathrm{i})(A^* - \mathrm{i})w = 0.$$

Thus

$$(A^* - \mathrm{i})w \in \mathrm{Ker}(A^* + \mathrm{i}). \tag{3.17}$$

If $y \in \mathrm{Ker}(A^* + \mathrm{i})$, then

$$\mathrm{i}(y|(A^* - \mathrm{i})w) = (A^*\eta|A^*w) + (\eta|w) = (\eta|w)_{A^*} = 0$$

In particular, setting $y = (A^* - \mathrm{i})w$ and using (3.17), we get

$$w \in \mathrm{Ker}(A^* - \mathrm{i}).$$

But $w \perp \mathrm{Ker}(A^* - \mathrm{i})$, hence $w = 0$. $\Box$

Every A^*-closed subspace containing $\mathrm{Dom}A$ is of the form $\mathrm{Dom}A \oplus \mathcal{Z}$, where

$$\mathcal{Z} \subset \mathrm{Ker}(A^* + \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i}).$$

If

$$A \subset B \subset A^*,$$

then the subspace $\mathcal{Z}$ corresponding to B will be denoted by $\mathcal{Z}_B$. We will write

$$\mathcal{Z}^{\mathrm{per}} := \{v \in \mathrm{Ker}(A^* - \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i}) \ : \ [z, v]_{A^*} = 0, \ z \in \mathcal{Z}\}.$$

The subspace $\mathcal{Z}$ is A^*-symmetric iff

$$\mathcal{Z}^{\mathrm{per}} \supset \mathcal{Z}.$$

Theorem 3.51 *We have*

$$\mathcal{Z}_{B^*} = (\mathcal{Z}_B)^{\mathrm{per}}.$$

In particular, B is symmetric iff $\mathcal{Z}$ is A^*-symmetric. Every A^*-symmetric subspace corresponds to a partial isometry $U : \mathrm{Ker}(A^* + \mathrm{i}) \to \mathrm{Ker}(A^* - \mathrm{i})$. Then

$$\mathcal{Z} := \{z \oplus Uz \ : \ z \in \mathrm{Ran}\, U^*U\}.$$

B is self-adjoint iff U is unitary.

4 Discrete and essential spectrum

4.1 Extended discrete and essential spectrum

Let $\mathcal{X}$ be a Banach space and A a closed operator on $\mathcal{X}$. Recall that we defined the discrete and essential spectra of A denoted by $\mathrm{sp}_{\mathrm{d}}A$ and $\mathrm{sp}_{\mathrm{ess}}A$.

We say that ∞ belongs to the extended discrete spectrum iff there is a decomposition of $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1$ into the direct sum of two closed subspaces such that $\mathcal{X}_1$ is nonzero finite dimensional, $\mathrm{Dom}A = \mathcal{X}_0$ and A maps $\mathcal{X}_0$ into itself and A restricted to $\mathcal{X}_0$ is bounded. Equivalently, ∞ is a discrete point in $\mathrm{sp}^{\mathrm{ext}}A$ and $1_{\{\infty\}}(A)$ is finite dimensional. The discrete extended spectrum is denoted by $\mathrm{sp}_{\mathrm{d}}^{\mathrm{ext}}(A)$. The essential extended spectrum is defined as

$$\mathrm{sp}_{\mathrm{ess}}^{\mathrm{ext}}A := \mathrm{sp}^{\mathrm{ext}}A \backslash \mathrm{sp}_{\mathrm{d}}^{\mathrm{ext}}A.$$

Theorem 4.1 *Let $z_0 \in \mathrm{rs}A$. Then*

$$\mathrm{sp}_{\mathrm{ess}}^{\mathrm{ext}}(z_0 - A)^{-1} = (z_0 - \mathrm{sp}_{\mathrm{ess}}^{\mathrm{ext}}A)^{-1}, \quad \mathrm{sp}_{\mathrm{d}}^{\mathrm{ext}}(z_0 - A)^{-1} = (z_0 - \mathrm{sp}_{\mathrm{d}}^{\mathrm{ext}}A)^{-1}.$$

4.2 Operators with a compact resolvent

Theorem 4.2 *Let A be an operator with a non-empty resolvent set. Then the following conditions are equivalent:*

(1) *$(z_0 - A)^{-1}$ is compact for some $z_0 \in \mathrm{rs}A$.*

(2) *$(z - A)^{-1}$ is compact for all $z \in \mathrm{rs}A$.*

Proof. We use the resolvent equation

$$(z - A)^{-1} = (z_0 - A)^{-1}\left(1 - (z - z_0)(z - A)^{-1}\right).$$

□

When the conditions of Theorem 4.2 are satisfied, then we say that the operator A has a compact resolvent.

Theorem 4.3 (1) *Let A be normal. Then A has a compact resolvent iff $\mathrm{sp}A = \mathrm{sp}_{\mathrm{d}}A$.*

(2) *Let A be bounded from below and self-adjoint. Then A has a compact resolvent iff $\mu_n(A) \to \infty$.*

Theorem 4.4 *Let $f, g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$, $\lim_{|x|\to\infty} f(x) = \infty$ and $\lim_{|x|\to\infty} g(x) = \infty$. Then*

$$H := f(x) + g(D)$$

has a compact resolvent.

Proof. Clearly, the functions f, g are bounded from below. Fix $\epsilon > 0$. For $r > 0$,let

$$H_r := f(x) + \min(g(D), r).$$

Then for $z \in \mathbb{C}\backslash\mathbb{R}$ the operator $(\min(g(D), r) - r)(z - f(x) - r)^{-1}$ is compact. Hence also the operator

$$(z - H_r)^{-1} - (z - f(x) - r)^{-1} = (z - H_r)^{-1}(\min(g(D), r) - r)(z - f(x) - r)^{-1}$$

is compact. Thus

$$\mathrm{sp}_{\mathrm{ess}}(H_r) = \mathrm{sp}_{\mathrm{ess}}(f(x) + r) \subset [r + \inf f, \infty[.$$

Therefore, there exists N such that for $n > N$

$$\mu_n(H_r) \geq r - \epsilon + \inf f.$$

But $H_r \leq H$. Hence

$$\mu_n(H_r) \leq \mu_n(H).$$

Therefore, for $n > N$, we have $\mu_n(H) \geq R - \epsilon + \inf f$. Thus $\mu_n(H) \to \infty$. □

4.3 Stability of essential spectrum

Theorem 4.5 *Let $\mathcal{V}$ be a Hilbert space and $A \in B(\mathcal{V})$ be normal. Then $\lambda \in \mathrm{sp}_{\mathrm{ess}} A$ iff there exits a sequence of vectors v_n such that*

$$\|v_n\| = 1, \quad \mathrm{w}-\lim_{n\to\infty} v_n = 0, \quad \lim_{n\to\infty} (A-\lambda)v_n = 0. \tag{4.18}$$

Proof. $\Rightarrow$ We know that for any n $\dim 1_{B(\lambda,\frac{1}{n})}(A) = \infty$. Therefore, we can find an orthonormal system $v_1, v_2, \ldots$ such that $v_n \in \mathrm{Ran}\, 1_{B(\lambda,\frac{1}{n})}(A)$. The sequence $v_1, v_2, \ldots$ satisfies (4.18).

$\Leftarrow$ Let $\epsilon > 0$. We have

$$(1 - 1_{B(\lambda,\epsilon)}(A))v_n = (1 - 1_{B(\lambda,\epsilon)}(A))(A-\lambda)^{-1}(A-\lambda)v_n \to 0.$$

Hence

$$c_n := \|1_{B(\lambda,\epsilon)}(A)v_n\| \to 1.$$

Let

$$\tilde{v}_n := \frac{1}{c_n} 1_{B(\lambda,\epsilon)}(A)v_n.$$

Then $\|\tilde{v}_n\| = 1$, $\mathrm{w}-\lim_{n\to\infty} \tilde{v}_n = 0$ and $\tilde{v}_n \in \mathrm{Ran}\, 1_{B(\lambda,\epsilon)}(A)$. Hence $\mathrm{Span}\{\tilde{v}_1, \tilde{v}_2, \ldots\}$ is infinite dimensional. Thus, $1_{B(\lambda,\epsilon)}(A)$ is infinite dimensional. □

Definition 4.6 *A sequence of vectors v_n satisfying the conditions of the above theorem will be called a Weyl sequence for λ and the operator A.*

Theorem 4.7 (Weyl) *Let $A, B \in B(\mathcal{V})$ be normal and let $B - A$ be compact. Then $\mathrm{sp}_{\mathrm{ess}} A = \mathrm{sp}_{\mathrm{ess}} B$.*

Proof. Assume that $\lambda \in \mathrm{sp}_{\mathrm{ess}} A$. Then there exists a Weyl sequence $v_1, v_2, \ldots$ for λ and the operator A. We have $\lim_{n\to\infty}(B-A)v_n = 0$. Hence $v_1, v_2, \ldots$ is a Weyl sequence for λ and the operator B. □

Theorem 4.8 *Assume that A, B are normal operators such that for some $z_0 \notin \mathrm{sp}A \cup \mathrm{sp}B$ the operator $(z_0 - A)^{-1} - (z_0 - B)^{-1}$ is compact. Then*

$$\mathrm{sp}_{\mathrm{ess}} A = \mathrm{sp}_{\mathrm{ess}} B.$$

Proof. By the Weyl theorem $\mathrm{sp}_{\mathrm{ess}}(z_0 - A)^{-1} = \mathrm{sp}_{\mathrm{ess}}(z_0 - B)^{-1}$. Then we use Theorem 4.1 to normal operators A and B. □

Theorem 4.9 *Let $f \in L^\infty_{\rm loc}(\mathbb{R}^d)$, $\lim_{|x|\to\infty} f(x) = \infty$ and $g \in L^\infty(\mathbb{R}^d)$, $\lim_{|x|\to\infty} g(x) = 0$. Then*

$$\mathrm{sp}_{\mathrm{ess}}(f(D) + g(x)) = \mathrm{sp}_{\mathrm{ess}}(f(D)).$$

Proof. The operator $g(x)(z_0 - f(D))^{-1}$ is compact. Hence, the operator

$$(z_0 - f(D) - g(x))^{-1} - (z_0 - f(D))^{-1} = (z_0 - f(D) - g(x))^{-1} g(x)(z_0 - f(D))^{-1}$$

is compact as well. We can thus use Theorem 4.8. □

5 Momentum and Laplacian in one dimension

5.1 Momentum on the line

The equation

$$U(t)f(x) := f(x-t), \quad f \in L^2(\mathbb{R}), \quad t \in \mathbb{R},$$

defines a unitary strongly continuous group on $L^2(\mathbb{R})$. Let the momentum operator D be defined by

$$U(t) = \mathrm{e}^{-\mathrm{i}tD}.$$

Theorem 5.1 (1) *D is a self-adjoint operator.*

(2) *The integral kernel of $(z-D)^{-1}$ equals*

$$R(z,x,y) = \left\{ \begin{array}{ll} -\mathrm{i}\theta(x-y)\mathrm{e}^{iz(x-y)}, & \mathrm{Im}z > 0, \\ +\mathrm{i}\theta(y-x)\mathrm{e}^{iz(x-y)}, & \mathrm{Im}z < 0. \end{array} \right. ,$$

where θ is the Heavyside function

(3) $\mathrm{Dom}D \subset C_\infty(\mathbb{R})$ *and* $\mathrm{Dom}D \ni f \mapsto f(x) \in \mathbb{C}$ *is a continuous functional.*

(4) $\{f \in L^2(\mathbb{R}) \cap C^1(\mathbb{R}) \ : \ f' \in L^2(\mathbb{R})\} \subset \mathrm{Dom}D$ *and for f in this space*

$$Df(x) := \frac{1}{i}\partial_x f(x). \tag{5.19}$$

(5) *If $f \in \mathrm{Dom}D$ and $Df \in C(\mathbb{R})$, then $f \in C^1(\mathbb{R})$ and (5.19) is true.*

(6) *$C_0^\infty(\mathbb{R})$ is an essential domain of D.*

(7) $\mathrm{sp}D = \mathbb{R}$.

(8) $\mathrm{sp_p}D = \emptyset$.

(9) *If $f \in \mathrm{Dom}D$ and $f = 0$ on $]a,b[$, then $Df = 0$ on $]a,b[$.*

Proof. (2) For $\mathrm{Im}z > 0$

$$(z-D)^{-1} = -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}U(t)\mathrm{d}t.$$

Hence

$$(z-D)^{-1}f(x) = -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}f(x-t)\mathrm{d}t = -\mathrm{i}\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}(x-y)z}\theta(x-y)f(y)\mathrm{d}y.$$

For $\mathrm{Im}z < 0$ we can use

$$(z-D)^{-1*} = (\overline{z} - D)^{-1}.$$

(3) $\mathrm{Dom}D = \mathrm{Ran}\,(\mathrm{i}-D)^{-1}$. Now $(\mathrm{i}-D)^{-1}$ is the convolution with $-\mathrm{i}\theta(x)\mathrm{e}^{-|x|}$, which belongs to $L^2(\mathbb{R})$. The convolution of two $L^2(\mathbb{R})$ functions belongs to $C_\infty(\mathbb{R})$.

(4) First let $f \in C_0^1(\mathbb{R})$. Then

$$t^{-1}(f(x+t)-f(x)) = t^{-1}\int_x^{x+t} f'(y)\mathrm{d}y.$$

f' is uniformy continuous. Hence we will find $t_0 > 0$ such that for $|y_1-y_2| < t_0$, we have $|f'(y_1)-f'(y_2)| < \epsilon$. Therefore, for $|t| < t_0$

$$|t^{-1}(f(x+t)-f(x)) - f'(x)| < \epsilon.$$

Using the compactness of the support of f we obtain that $t^{-1}(f(x+t)-f(x)) - f'(x) \to 0$ in $L^2(\mathbb{R})$. Thus $C_0^1(\mathbb{R}) \subset \mathrm{Dom}D$ and (5.19) is true on this subspace.

If $f \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$, then choose $j \in C_0^\infty(\mathbb{R})$ such that $j = 1$ on a neighborhood of zero. Set $j_r(x) := j(x/r)$. Then $j_r f \in C_0^1$, hence $Dj_r f = -\mathrm{i}\partial_x j_r f$. We easily check that $j_r f \to f$ and $Dj_r f \to -\mathrm{i}\partial_x f$ in $L^2(\mathbb{R})$. Hence, by the closedness of D we get $Df = -\mathrm{i}\partial_x f$.

(5) Let $f \in \mathrm{Dom}D$, $g \in C(\mathbb{R})$ and $Df = g$. Let $x \in \mathbb{R}$, $r > 0$. Set $h := 1_{[x,x+r]}$. Then

$$\begin{aligned} t^{-1}(h|U(t)f-f) &= t^{-1}\int_{x-t}^{x+r-t} f(y)\mathrm{d}y - t^{-1}\int_x^{x+r} f(y)\mathrm{d}y \\ &= -t^{-1}\int_{x+r-t}^{x+r} f(y)\mathrm{d}y + t^{-1}\int_{x-t}^{x} f(y)\mathrm{d}y \to -f(x+r)+f(x). \end{aligned}$$

Therefore

$$\mathrm{i}(h|g) = \mathrm{i}\int_x^{x+r} g(y)\mathrm{d}y = -f(x+r)+f(x).$$

Hence, using the continuity of g,

$$\lim_{r\to 0}\frac{f(x+r)-f(x)}{r} = -\mathrm{i}g(x).$$

(7) Let $k \in \mathbb{R}$. Consider $f_{\epsilon,k} = \sqrt{\pi\epsilon}\mathrm{e}^{-\epsilon x^2+\mathrm{i}kx}$. Then $\|f_{\epsilon,k}\| = 1$, $f_{\epsilon,k} \in \mathrm{Dom}D$ and $(k-D)f_{\epsilon,k} \to 0$ as $\epsilon \to 0$. Hence $k \in \mathrm{sp}D$.

(8) Suppose that $f \in \mathrm{Dom}D$ and $Df = kf$. Clearly, $f \in \mathrm{Dom}D^2$. Hence, by (), $f \in C^1(\mathbb{R})$ and $Df = -\mathrm{i}\partial_x f = kf$. It is well know that the only solution is $f = c\mathrm{e}^{\mathrm{i}kx}$, which does not belong to $L^2(\mathbb{R})$.

(9) is obvious for $f \in C_0^1(\mathbb{R})$. It extends by density.□

5.2 Laplacian on the line

The operator D^2 from the previous subsection will be denoted $-\Delta$.

Theorem 5.2 (1) *$-\Delta$ is a positive self-adjoint operator.*

(2) $\mathrm{sp}(-\Delta) = [0,\infty[$.

(3) *The integral kernel of* $(k^2-\Delta)^{-1}$, *for* $\mathrm{Re}k > 0$, *equals*

$$R(k,x,y) = \frac{1}{2k}\mathrm{e}^{-k|x-y|}.$$

(4) *The integral kernel of* $\mathrm{e}^{t\Delta}$ *equals*

$$K(t,x,y) = (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}}.$$

(5) $\mathrm{sp_p}(-\Delta)=\emptyset$.

(6) $\{f\in C^2(\mathbb{R})\cap L^2(\mathbb{R})\ :\ f',f''\in L^2(\mathbb{R})\}$ *is contained in* $\mathrm{Dom}(-\Delta)$ *and on this set*

$$-\Delta f(x)=-\partial_x^2 f(x).$$

(7) $C_0^\infty(\mathbb{R})$ *is an essential domain of* $-\Delta$.

Proof. (3) Let $\mathrm{Re}k>0$. Then

$$(\mathrm{i}k-D)^{-1}(x,y)=-\mathrm{i}\theta(x-y)\mathrm{e}^{-k|x-y|},\quad (-\mathrm{i}k-D)^{-1}(x,y)=\mathrm{i}\theta(y-x)\mathrm{e}^{-k|x-y|}.$$

Now

$$\begin{aligned}(k^2-\Delta)^{-1}\ &=(\mathrm{i}k-D)^{-1}(-\mathrm{i}k-D)^{-1}\\ &=(-2\mathrm{i}k)^{-1}\left((\mathrm{i}k-D)^{-1}-(-\mathrm{i}k-D)^{-1}\right).\end{aligned}\tag{5.20}$$

The integral kernel of (5.20) equals $(2k)^{-1}\mathrm{e}^{-k|x-y|}$.

(4) We have

$$\mathrm{e}^{t\Delta}=(2\pi\mathrm{i})^{-1}\int_\gamma(z-\Delta)^{-1}\mathrm{e}^{tz}\mathrm{d}z,$$

where γ is a contour of the form $\mathrm{e}^{-\mathrm{i}\alpha}]0,\infty[\cup\mathrm{e}^{\mathrm{i}\alpha}[0,\infty[$ and $\pi/2<\alpha<\pi$. Hence

$$\mathrm{e}^{t\Delta}(x,y)=(2\pi\mathrm{i})^{-1}\int_{\tilde\gamma}\mathrm{e}^{-k|x-y|+tk^2}\mathrm{d}k$$

where $\tilde\gamma$ is a contour of the form $\mathrm{e}^{-\mathrm{i}\alpha/2}[0,\infty[\cup\mathrm{e}^{\mathrm{i}\alpha/2}[0,\infty[$. We put $k=\mathrm{i}u$ and obtain

$$\mathrm{e}^{t\Delta}(x,y)=(2\pi\mathrm{i})^{-1}\int_{-\infty}^{\infty}\mathrm{e}^{-iu|x-y|-tu^2}\mathrm{i}\mathrm{d}u$$

□

5.3 Sobolev spaces in one dimension

Let $L^2_\alpha(\mathbb{R})$ be the scale of spaces associated with D. This means in particular, that $L^2_1(\mathbb{R})=\mathrm{Dom}D$ and $L^2_2(\mathbb{R})=\mathrm{Dom}(-\Delta)$.

Theorem 5.3 $L^2_{n+1}(\mathbb{R})\subset C^n(\mathbb{R})$ *and* $L^2_{n+1}(\mathbb{R})\ni f\mapsto f^{(j)}(x)$ *for* $j=0,\dots,n-1$ *are continuous functionals depending continuously on* $x\in\mathbb{R}$.

Proof. We use induction. The step $n=0$ was proven. Suppose that we know that $L^2_{n+1}(\mathbb{R})\subset C^n(\mathbb{R})$. Let $f\in L^2_{n+2}(\mathbb{R})$. Then $(\mathrm{i}-D)f=g\in L^2_{n+1}(\mathbb{R})$. Clearly, $L^2_{n+2}\subset L^2_{n+1}(\mathbb{R})$ hence $f\in C^n(\mathbb{R})$. Likewise, $g\in C^n(\mathbb{R})$, by the induction assumption. Now $Df=-g+\mathrm{i}f\in C^n(\mathbb{R})$. Hence, $f\in C^{n+1}(\mathbb{R})$. □

Define

$$L^2_{n,\min}([0,\infty[):=\{f\in L^2_n(\mathbb{R})\ :\ f(x)=0,\ x<0\}.$$

Define $L^2_{n,\min}(]-\infty,0])$ in a similar way.

Theorem 5.4 (1) $L^2_{n,\min}([0,\infty[)$ *is orthogonal to* $L^2_{n,\min}(]-\infty,0])$.

(2) *The codimension of*

$$L^2_{n,\min}(]-\infty,0])\oplus L^2_{n,\min}([0,\infty[)\tag{5.21}$$

equals n.

(3) *(5.21) equals*
$$\{f \in L^2_n(\mathbb{R}) \ : \ f^{(j)}(0) = 0, \ j = 0, \dots, n-1\}.$$

(4) *D maps $L^2_{n,\min}([0,\infty[)$ into $L^2_{n-1,\min}([0,\infty[)$.*

(5) $L_{n+1,\min}([0,\infty[) \subset L_{n,\min}([0,\infty[)$

(6) $L^2_{0,\min}([0,\infty[) = L^2([0,\infty[)$.

We define
$$L^2_{n,\max}([0,\infty[) := L^2_n(\mathbb{R}) \ominus L^2_{n,\min}(]-\infty,0]).$$

Theorem 5.5 (1) *$L^2_{n,\min}([0,\infty[)$ is a subspace of $L^2_{n,\max}([0,\infty[)$ of codimension n.*

(2) *$L^2_{n,\min}([0,\infty[)$ equals*

$$\{f \in L^2_{n,\max}([0,\infty[) \ : \ f^{(j)}(0) = 0, \ j = 0, \dots, n-1\}.$$

(3) *D maps $L^2_{n,\max}([0,\infty[)$ into $L^2_{n-1,\max}([0,\infty[)$.*

(4) $L_{n+1,\max}([0,\infty[) \subset L_{n,\max}([0,\infty[)$

(5) $L^2_{0,\max}([0,\infty[) = L^2([0,\infty[)$.

5.4 Momentum on the half-line

Define $D_{\max}$ as an operator on $L^2([0,\infty[)$ equal to the restriction of D to $L^2_{1,\max}([0,\infty[)$. Likewise, define $D_{\min}$ as an operator on $L^2([0,\infty[)$ equal to the restriction of D to $L^2_{1,\min}([0,\infty[)$.

Theorem 5.6 (1) $D_{\min} \subset D_{\max}$, $D^*_{\min} = D_{\max}$, $D^*_{\max} = D_{\min}$

(2) *The operators $D_{\min}$ and $-D_{\max}$ are m-dissipative (in particular, they are closed); the operator $D_{\min}$ is symmetric.*

(3) $\mathrm{sp_p} D_{\max} = \{\mathrm{Im} z > 0\}$, $\quad \mathrm{sp_p} D_{\min} = \emptyset$*;*

$$D_{\max} \mathrm{e}^{\mathrm{i}zx} = z\mathrm{e}^{\mathrm{i}zx}, \quad \mathrm{e}^{\mathrm{i}zx} \in \mathrm{Dom} D_{\max}, \quad \mathrm{Im} z > 0, \tag{5.22}$$

(4) $\mathrm{sp} D_{\max} = \{\mathrm{Im} z \geq 0\}$, $\mathrm{sp} D_{\min} = \{\mathrm{Im} z \leq 0\}$*;*

(5) *The integral kernels of $(z - D_{\max})^{-1}$ and $(z - D_{\min})^{-1}$ are equal*

$$R_{\max}(z, x, y) = \mathrm{i}\theta(y - x)\mathrm{e}^{\mathrm{i}z(x-y)}, \ \mathrm{Im} z < 0.$$

$$R_{\min}(z, x, y) = -\mathrm{i}\theta(x - y)\mathrm{e}^{\mathrm{i}z(x-y)}, \ \mathrm{Im} z > 0.$$

(6) *The semigroups generated by these operators:*

$$\mathrm{e}^{\mathrm{i}tD_{\max}} f(x) = f(x + t), \quad t \geq 0.$$

$$\mathrm{e}^{-\mathrm{i}tD_{\min}} f(x) = \begin{cases} f(x - t), & x \geq t \geq 0. \\ 0, & t > x, \end{cases}$$

5.5 Laplacian on the halfline I

Define $-\Delta_{\max}$ as an operator on $L^2([0,\infty[)$ equal to the restriction of $-\Delta$ to $L^2_{2,\max}([0,\infty[)$. Likewise, define $-\Delta_{\min}$ as an operator on $L^2([0,\infty[)$ equal to the restriction of $-\Delta$ to $L^2_{2,\max}([0,\infty])$.

Theorem 5.7 (1) $-\Delta_{\min}^* = -\Delta_{\max}, \quad -\Delta_{\min} \subset -\Delta_{\max}$.

(2) *The operators* $-\Delta_{\min}$ *and* $-\Delta_{\max}$ *are closed and* $-\Delta_{\min}$ *is symmetric.*

(3) $\mathrm{sp_p}(-\Delta_{\max}) = \mathbb{C}\backslash[0,\infty[, \quad \mathrm{sp_p}(-\Delta_{\min}) = \emptyset$

$$-\Delta_{\max}\mathrm{e}^{ikx} = k^2\mathrm{e}^{ikx},\ \mathrm{Im}k > 0, \qquad \mathrm{e}^{ikx} \in \mathrm{Dom}(-\Delta_{\max}).$$

(4) $\mathrm{sp}(-\Delta_{\max}) = \mathbb{C}, \ \mathrm{sp}(-\Delta_{\min}) = \mathbb{C}.$

(5) $-\Delta_{\min} = D_{\min}^2, \quad -\Delta_{\max} = D_{\max}^2$.

5.6 Laplacian on the halfline II

Let $\mu \in \mathbb{C} \cap \{\infty\}$,

$$\mathrm{Dom}(-\Delta_\mu) = \{f \in L^2_{2,\max}([0,\infty[) \ : \ \mu f(0) = f'(0)\}. \tag{5.23}$$

(If $\mu = \infty$, these are the Dirichlet boundary conditions, that means $f(0) = 0$, if $\mu = 0$, these are the Neumann boundary conditions, that means $f'(0) = 0$). Let $-\Delta_\mu$ be the restriction of $-\Delta_{\max}$ to (5.23).

Define also the form δ_μ as follows. If $\mu \in \mathbb{R}$, then $\mathrm{Dom}\delta_\mu = L^2_{1,\max}([0,\infty[)$ and

$$\delta_\mu(f,g) := -\mu\overline{f(0)}g(0) + \int \overline{f'(x)}g'(x)\mathrm{d}x.$$

For $\mu = \infty$, we set $\mathrm{Dom}\delta_\infty := L^2_{1,\min}([0,\infty[)$ and

$$\delta_\infty(f,g) := \int \overline{f'(x)}g'(x)\mathrm{d}x.$$

Theorem 5.8 (1) $-\Delta_{\min} \subset -\Delta_\mu \subset -\Delta_{\max}$.

(2) $-\Delta_\mu^* = -\Delta_{\overline{\mu}}$.

(3) *The operator* $-\Delta_\mu$ *are generators of groups. For* $\mu \in \mathbb{R} \cup \{\infty\}$ *it is self-adjoint.*

(4) $\mathrm{sp_p}(-\Delta_\mu) = \begin{cases} \{\mu^2\}, & \mathrm{Im}\mu > 0 \\ \emptyset, & \text{otherwise,} \end{cases}$;

$$-\Delta_\mu \mathrm{e}^{\mathrm{i}\mu x} = \mu^2\mathrm{e}^{\mathrm{i}\mu x},\ \mathrm{Im}\mu > 0, \qquad \mathrm{e}^{\mathrm{i}\mu x} \in \mathrm{Dom}(-\Delta_\mu).$$

(5) $\mathrm{sp}(-\Delta_\mu) = \begin{cases} \{\mu^2\} \cup [0,\infty[, & \mathrm{Im}\mu > 0, \\ [0,\infty[, & \text{otherwise,} \end{cases}$;

(6)

$$-\Delta_0 = D_{\max}^* D_{\max}, \quad -\Delta_\infty = D_{\min}^* D_{\min}.$$

(7) *The forms* δ_μ *are closed and associated with the operator* $-\Delta_\mu$.

(8) *The integral kernel of* $(k^2-\Delta_\mu)^{-1}$ *is equal*

$$R_\mu(z,x,y)=\frac{1}{2k}\mathrm{e}^{-k|x-y|}-\frac{1}{2k}\frac{(\mu-k)}{(\mu+k)}\mathrm{e}^{-k(x+y)},$$

in particular, for the Dirichlet boundary conditions,

$$R_\infty(z,x,y)=\frac{1}{2k}\mathrm{e}^{-k|x-y|}-\frac{1}{2k}\mathrm{e}^{-k(x+y)},$$

and for the Neumann boundary conditions

$$R_0(z,x,y)=\frac{1}{2k}\mathrm{e}^{-k|x-y|}+\frac{1}{2k}\mathrm{e}^{-k(x+y)}.$$

(9) *The semigroups* $\mathrm{e}^{t\Delta_\mu}$ *have the integral kernel*

$$K_\mu(t,x,y)=(4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}}+(2\pi)^{-1}\int_{-\infty}^{\infty}\frac{\mu-\mathrm{i}u}{\mu+\mathrm{i}u}\mathrm{e}^{-\mathrm{i}u(x+y)-tu^2}\mathrm{d}u,$$

In particular, in the Dirichlet case

$$K_\infty(t,x,y)=(4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}}-(4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x+y)^2}{4t}},$$

and in the Neumann case

$$K_\infty(t,x,y)=(4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}}+(4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x+y)^2}{4t}}.$$

The semigroup $K_\mu(t)$ for $\mu\in\mathbb{R}$ can be used to describe the diffusion with a sink or source at the end of the halfline. Note that $K_\mu(t)$ preserves the pointwise positivity. If $p_t=K_\mu(t)p_0$, $0<a<b$, then

$$\partial_t\int_a^b p_t(x)\mathrm{d}x=p'(b)-p'(a).$$

$$\partial_t\int_0^a p_t(x)\mathrm{d}x=p'(a)-\mu p(0).$$

Thus at 0 there is a sink of p with the rate μ.

5.7 Momentum on an interval I

We define $L^2_{n,\max}([-\pi,\pi])$ and $L^2_{n,\min}([-\pi,\pi])$ as above.

Define $D_{\max}$ as an operator on $L^2([-\pi,\pi])$ equal to the restriction of D to $L^2_{1,\max}([-\pi,\pi])$. Likewise, define $D_{\min}$ as an operator on $L^2([-\pi,\pi])$ equal to the restriction of D to $L^2_{1,\min}([-\pi,\pi])$.

Theorem 5.9 (1) $D_{\min}\subset D_{\max}$, $D^*_{\min}=D_{\max}$, $D^*_{\max}=D_{\min}$

(2) *The operators* $D_{\min}$ *and* $D_{\max}$ *are closed; the operator* $D_{\min}$ *is symmetric.*

(3) $\mathrm{sp_p}D_{\max}=\mathbb{C}$, $\mathrm{sp_p}D_{\min}=\emptyset$;

$$D_{\max}\mathrm{e}^{\mathrm{i}zx}=z\mathrm{e}^{\mathrm{i}zx},\quad \mathrm{e}^{\mathrm{i}zx}\in\mathrm{Dom}D_{\max},\quad z\in\mathbb{C},\tag{5.24}$$

(4) $\mathrm{sp}D_{\max}=\mathbb{C}$, $\mathrm{sp}D_{\min}=\mathbb{C}$;

5.8 Momentum on an interval II

Let $\kappa \in \mathbb{C}$. Let the operator D_κ on $L^2([-\pi,\pi])$ be defined as the restriction of $D_{\max}$ to

$$\mathrm{Dom} D_\kappa = \{f \in L^2_{1,\max}([-\pi,\pi]) \ : \ \mathrm{e}^{\mathrm{i}2\pi\kappa} f(-\pi) = f(\pi)\}.$$

Theorem 5.10 (1) $D_\kappa^* = D_{\overline{\kappa}}, \qquad D_\kappa = D_{\kappa+1}.$

(2) $D_{\min} \subset D_\kappa \subset D_{\max}$.

(3) *Operators D_κ are closed and for $\kappa \in \mathbb{R}$ self-adjoint.*

(4) $\mathrm{sp} D_\kappa = \mathrm{sp}_{\mathrm{p}} D_\kappa = \mathbb{Z} + \kappa$,

$$D_\kappa \mathrm{e}^{\mathrm{i}(n+\kappa)x} = (n+\kappa)\mathrm{e}^{\mathrm{i}(n+\kappa)x}, \ n \in \mathbb{Z}.$$

(5) *The integral kernel of $(z - D_\kappa)^{-1}$ equals*

$$R_\kappa(z,x,y) = \frac{1}{2\sin\pi(z-\kappa)} \left(\mathrm{e}^{-\mathrm{i}(z-\kappa)\pi} \mathrm{e}^{\mathrm{i}z(x-y)} \theta(x-y) + \mathrm{e}^{\mathrm{i}(z-\kappa)\pi} \mathrm{e}^{\mathrm{i}z(x-y)} \theta(y-x) \right).$$

(6) *The group generated by $\mathrm{i}D_\kappa$ equals*

$$\mathrm{e}^{\mathrm{i}tD_\kappa}\phi(x) = \mathrm{e}^{\mathrm{i}\pi n\kappa}\phi(x+t), \quad (2n-1)\pi < x+t < (2n+1)\pi.$$

(7) *The operators D_κ are similar to one another up to an additive constant:*

$$\mathrm{Dom} D_\kappa = \mathrm{e}^{\mathrm{i}\kappa x} \mathrm{Dom} D_0, \qquad D_\kappa = \mathrm{e}^{\mathrm{i}\kappa x} D_0 \mathrm{e}^{-\mathrm{i}\kappa x} + \kappa. \tag{5.25}$$

5.9 Momentum on an interval III

Let the operator $D_{\pm\mathrm{i}\infty}$ on $L^2([-\pi,\pi])$ be defined as the restriction of $D_{\max}$ to

$$\mathrm{Dom} D_{\pm\mathrm{i}\infty} = \{f \in L^2_{1,\max}([-\pi,\pi]) \ : \ f(\pm\pi) = 0\}.$$

Theorem 5.11 (1) $D_{\pm\mathrm{i}\infty}^* = D_{\mp\mathrm{i}\infty}$.

(2) $D_{\min} \subset D_{\pm\mathrm{i}\infty} \subset D_{\max}$.

(3) *The operators $D_{\pm\mathrm{i}\infty}$ are closed.*

(4) $\mathrm{sp} D_{\pm\mathrm{i}\infty} = \emptyset$.

(5) *The integral kernel of $(z - D_{\pm\mathrm{i}\infty})^{-1}$ equals*

$$R_{\pm\mathrm{i}\infty}(z,x,y) = \pm\mathrm{i}\mathrm{e}^{\mathrm{i}z(x-y\pm\pi)}\theta(\pm y \mp x), \quad z \in \mathbb{C}.$$

(6) $\pm\mathrm{i}D_{\pm\mathrm{i}\infty}$ *generate the semigroups of contractions for $t \geq 0$:*

$$\mathrm{e}^{\pm\mathrm{i}tD_{\mathrm{i}\infty}} f(x) = \begin{cases} f(x\pm t), & |x\pm t| \leq \pi, \\ 0 & |x \pm t| > \pi. \end{cases}$$

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