

ASYMPTOTIC COMPLETENESS
OF N-BODY SCATTERING

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In my opinion, scattering theory for N -body Schrödinger operators is one of the greatest successes of 20th century mathematical physics.

On the physical side, we have a rigorous framework that explains why nonrelativistic matter is built out of well defined clusters of nuclei and electrons, such as atoms, ions, molecules.

On the mathematical side, we have a deep analysis of a large family of nontrivial operators with continuous spectrum, combining ideas from classical and quantum mechanics.

A single quantum particle in an external potential is described by the Hilbert space $L^2(\mathbb{R}^d)$ and the Schrödinger Hamiltonian

$$H = H_0 + V(x),$$

where

$$H_0 = \frac{p^2}{2m}, \quad p = \frac{1}{i}\partial_x.$$

A typical example of a potential is

$$V(x) = \frac{c}{|x|}.$$

THEOREM. Assume that $V(x)$ is **short range**, that is,

$$|V(x)| \leq c\langle x \rangle^{-\mu_s}, \quad \mu_s > 1.$$

Then there exist **wave (Møller) operators**

$$\Omega^\pm := s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0},$$

they are **isometric**, they **intertwine the free and full Hamiltonian**:

$$\Omega^\pm H_0 = H \Omega^\pm,$$

and they are **complete**:

$$\Omega^\pm \Omega^{\pm*} = \mathbb{1}_c(H).$$

THEOREM. Assume that $V(x)$ is **long range**, that is,

$$V(x) = V_1(x) + V_s(x),$$

where $V_s(x)$ is short range and

$$|\partial_x^\alpha V_1(x)| \leq c_\alpha \langle x \rangle^{-|\alpha| - \mu_1}, \quad \mu_1 > 0, \quad \alpha \in \mathbb{N}^d.$$

Then there exists a function $(t, \xi) \mapsto S_t(\xi)$ and **modified Møller operators**

$$\Omega^\pm := s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} e^{-iS_t(p)},$$

which satisfy the same properties as those stated for the short-range case.

Thus the Hilbert space is the direct sum of **bound states** and of **scattering states** – states which evolve for large times as free waves. One can define the **scattering operator**,

$$S := \Omega^+ \Omega^{-*},$$

which is unitary. The integral kernel of S defines **scattering amplitudes**. The square of the absolute value of a scattering amplitude is the **scattering cross-section** describing the probability of a scattering process.

The most difficult part of the above theorems is to prove that the range of (modified) wave operators fills the whole continuous spectral space of H . This is called **asymptotic completeness (AC)**.

2 interacting quantum particles are described by the Hilbert space $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \simeq L^2(\mathbb{R}^{2d})$ and the Hamiltonian

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x_1 - x_2).$$

Introduce the **center-of-mass coordinate** $x_{12} := \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ and the **relative coordinate** $x^{12} := x_2 - x_1$. The Hilbert space factorizes

$$L^2(\mathbb{R}^{2d}) = L^2(X_{12}) \otimes L^2(X^{12}).$$

Let $m_{12} := m_1 + m_2$ be the **total mass** and $m^{12} := (m_1^{-1} + m_2^{-1})^{-1}$ be the **reduced mass**. Then we can write

$$H = \frac{p_{12}^2}{2m_{12}} + H^{12},$$

where

$$H^{12} := \frac{(p^{12})^2}{2m^{12}} + V(x^{12}).$$

Thus the problem of two interacting particles is reduced to a single particle in an external potential.

N interacting quantum particles are described by the Hilbert space

$$\bigotimes_{i=1}^N L^2(\mathbb{R}^d) \simeq L^2(X),$$

where $X := \mathbb{R}^{Nd}$, and the Hamiltonian is

$$H := \sum_{j=1}^N \frac{p_j^2}{2m_j} + \sum_{1 \leq i < j \leq N} V_{ij}(x_i - x_j).$$

A typical potential is

$$V_{ij}(x_i - x_j) = \frac{Z_i Z_j e^2}{4\pi |x_i - x_j|}.$$

A **cluster decomposition** is a partition of $\{1, \dots, N\}$ into **clusters**:

$$a = \{c_1, \dots, c_k\}.$$

The **Hamiltonian of a cluster c** is

$$H_c := \sum_{j \in c} \frac{p_j^2}{2m_j} + \sum_{i, j \in c} V_{ij}(x_i - x_j).$$

The **Hamiltonian of a cluster decomposition a** is

$$H_a = H_{c_1} + \dots + H_{c_k}.$$

Note that cluster decompositions have a natural order. In particular, there is a minimal cluster decomposition, where all clusters are 1-element. Every pair determines a cluster decomposition.

Define the **collision plane of a** as

$$X_a := \{(x_1, \dots, x_N) \in \mathbb{R}^{Nd} : (ij) \leq a \Rightarrow x_i = x_j\}.$$

Consider the quadratic form on X

$$\sum \frac{m_i}{2} x_i^2.$$

Let X^a denote the **internal plane of a** , defined as the orthogonal complement of X_a wrt this form. We will write $x \mapsto x_a$ and $x \mapsto x^a$ for the orthogonal projections onto X_a and X^a .

We have

$$X = X_a \oplus X^a, \quad X^a = X^{c_1} \oplus \dots \oplus X^{c_k}.$$

Therefore,

$$L^2(X) = L^2(X_a) \otimes L^2(X^a), \quad L^2(X^a) = L^2(X^{c_1}) \otimes \dots \otimes L^2(X^{c_k}).$$

The cluster Hamiltonian decomposes:

$$H_a = T_a + H^a, \quad H^a = H^{c_1} + \dots + H^{c_k}.$$

Introduce

$$\mathcal{H}^a := \text{Ran} \mathbb{1}_p(H^a) \simeq \text{Ran} \mathbb{1}_p(H^{c_1}) \otimes \cdots \otimes \text{Ran} \mathbb{1}_p(H^{c_k}).$$

Let

$$E^a := H^a \Big|_{\mathcal{H}^a} = H^{c_1} \Big|_{\mathcal{H}^{c_1}} + \cdots + H^{c_k} \Big|_{\mathcal{H}^{c_k}}$$

be the operator describing the **bound state energies of clusters**. Let

$$J^a : L^2(X_a) \otimes \mathcal{H}^a \rightarrow L^2(X)$$

be the embedding of **bound states of clusters** into the full Hilbert space.

THEOREM. Assume that the potentials V_{ij} are short range. Then for any cluster decomposition a there exists the corresponding **partial wave operator**

$$\Omega_a^\pm := s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} J_a e^{-it(T_a + E^a)}.$$

Ω_a^\pm are isometric, they intertwine the cluster and the full Hamiltonian:

$$\Omega_a^\pm (T_a + E^a) = H \Omega_a^\pm$$

and are complete:

$$\bigoplus_a \text{Ran} \Omega_a^\pm = L^2(X).$$

THEOREM. Assume that the potentials V_{ij} are long range with

$$\mu_1 > \sqrt{3} - 1.$$

Then for any cluster decomposition a there exists a function $(t, \xi_a) \mapsto S_{a,t}(\xi_a)$, the corresponding **partial modified wave operator**

$$\Omega_a^\pm := s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} J_a e^{-i(S_{a,t}(p_a) + tE^a)},$$

which satisfy the same properties as those stated in the short range case.

AC means that all states in $L^2(X)$ can be decomposed into states with a clear physical/chemical interpretation such as **atoms, ions and molecules**.

We can introduce **partial scattering operators**

$$S_{ab} := \Omega_a^{+*} \Omega_b^-$$

describing various processes, such as **elastic and inelastic scattering, ionization, capture of an electron, chemical reactions**.

The partial wave operators Ω_a^\pm can be organized into

$$\bigoplus_a L^2(X_a) \otimes \mathcal{H}^a \ni (\psi_a) \mapsto \sum_a \Omega_a^\pm \psi_a \in L^2(X),$$

which is unitary. The partial scattering operators S_{ab} arranged in the matrix $[S_{ab}]$ also describe a unitary operator.

2-body scattering theory, including AC in both short- and long-range case, was understood already in the 60's.

Existence of N -body wave operators and the orthogonality of their ranges was established about the same time. What was missing for a long time was **Asymptotic Completeness** – the fact that the ranges of wave operators span the whole Hilbert space.

Below I review the various methods that were used, more or less successfully, to prove this.

The **stationary approach** to scattering theory is based on resolvent identities. For example, if $H = H_0 + V$, then the identity

$$(z - H)^{-1} = (z - H_0)^{-1} + (z - H_0)^{-1}V^{1/2} \left(1 - |V|^{1/2}(z - H_0)^{-1}V^{1/2} \right)^{-1} |V|^{1/2}(z - H_0)^{-1}$$

can be used to prove AC in the 2-body case.

L.Faddeev found a resolvent identity that can be used to study 3-body scattering. A number of other resolvent identities were used (eg. **G.Hagedorn's** for 4 bodies). The results about AC with $N \geq 3$ proven using the stationary approach involve implicit assumptions on invertibility of certain complicated operators and on properties of bound and almost-bound states. They also require a very fast decay of potentials and $d \geq 3$.

However, in principle, the stationary approach leads to explicit formulas for scattering amplitudes.

V.Enss introduced time-dependent methods into proofs of AC. In his approach an important tool was the **RAGE Theorem** saying that for K compact and $\psi \in \text{Ran} \mathbb{1}^c(H)$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|K e^{itH} \psi\|^2 dt = 0.$$

Enss started with proving the 2-body AC (late 70's), and managed to prove 3-body AC including the long-range case with $\mu_1 > \sqrt{3} - 1$ (late 80's).

Let us describe an idea that turned out to be important: One needs to look for observables A such that $i[H, A]$ is in some sense positive. Here is an important example of this idea:

E.Mourre (1981). Suppose that E is not a **threshold** (it is not an eigenvalue of H_a for any a). Then there exists an interval I around E and $c_0 > 0$ such that

$$\mathbb{1}_I(H)i[H, A]\mathbb{1}_I(H) \geq c_0\mathbb{1}_I(H),$$

where $A = \sum_i \frac{1}{2}(p_i x_i + x_i p_i)$ is the generator of dilations.

The **Mourre estimate** has important implications both in the stationary and time-dependent approach.

I.M.Sigal devoted a large part of his research career to N -body AC. After working with the stationary approach he switched to the time-dependent approach. Together with **A.Soffer** he obtained the first proof of the N -body AC in the short range case (announced 1985, published 1987). They first used heavily **propagation estimates**. Below we summarize abstractly the time-dependent version of this technique:

If $\Phi(t)$ is a uniformly bounded observable on a Hilbert space \mathcal{H} and

$$\frac{d}{dt}\Phi(t) + i[H, \Phi(t)] \geq \Psi^*(t)\Psi(t),$$

then

$$\int_1^\infty \|\Psi(t)e^{-itH}v\|^2 dt < \infty, \quad v \in \mathcal{H}.$$

A new and elegant proof of the N -body AC in the short range case was given by **G.M.Graf** (1989). Just as Sigal-Soffer's, it was also time-dependent, used propagation estimates and Mourre estimate. It introduced a clever observable, the **Graf vector field**, whose commutator with H is positive.

First proof of AC in the long range case for any N with $\mu_1 > \sqrt{3} - 1$ (which includes the physical **Coulomb potentials**) was given by **J.D** (announced 1991, published 1993). There exists a monograph **J.D and C.Gérard** in Springer Tracts and Monographs in Physics, 1997¹ about this subject.

In what follows I describe the main steps of the proof. My presentation will stress some additional features of N -body scattering, which I find interesting.

¹<http://www.fuw.edu.pl/~derezins/bookn.pdf>

First assume the long-range condition on the potentials with

$$\mu_1 > 0.$$

Following the ideas of the proof of Graf for the short-range case one can show the existence of the so-called **asymptotic velocity**:

THEOREM For any function $f \in C_c^\infty(X)$ there exists limits

$$\text{s-} \lim_{t \rightarrow \pm\infty} e^{itH} f\left(\frac{x}{t}\right) e^{-itH}. \quad (*)$$

There exists a family of commuting self-adjoint operators P^\pm such that $(*)$ equals $f(P^\pm)$.

Of course, we can replace H with H^a obtaining P^{a+} , the asymptotic velocity corresponding to a . The following fact follows by arguments involving the Mourre estimate, and is also essentially due to Graf:

THEOREM For any a

$$\mathbb{1}_{\{0\}}(P^{a+}) = \mathbb{1}^{\text{P}}(H^a).$$

For any a introduce

$$Z_a := X_a \setminus \bigcup_{b \not\leq a} X_b.$$

Then the family Z_a is a partition of X . In particular,

$$\mathbb{1} = \sum_a \mathbb{1}_{Z_a}(P^+).$$

Now in the short-range case AC follows easily by proving that

$$\lim_{t \rightarrow \pm\infty} e^{itH_a} e^{-itH} \mathbb{1}_{Z_a}(P^+)$$

exists and coincides with $\Omega_a^{\pm*}$.

In the long-range case one needs an additional step.

THEOREM Let $\phi = \mathbb{1}_{Z^a}(P^\pm)\phi$ and $\delta = \frac{2}{2+\mu}$. Then there exists c such that

$$\lim_{t \rightarrow \pm\infty} \mathbb{1}(t^{-\delta}|x^a| > c) e^{\mp itH} \phi = 0.$$

To see that this bound is natural note that Newton's equation in the potential $V(x) = -|x|^{-\mu}$ at zero energy has trajectories of the form

$$x(t) = ct^{\frac{2}{2+\mu}}.$$

To prove the existence of the modified wave operator we need to show that the variation of the potential that comes from outside of the given cluster decomposition within a wave packet is integrable in time. The variation of the potential can be estimated by

$$\begin{aligned} & (\text{spread of wave packet}) \times (\text{derivative of potential}) \\ & \sim t^{\frac{2}{2+\mu}} \times t^{-1-\mu}. \end{aligned}$$

The integrability condition gives

$$\frac{2}{2+\mu} - 1 - \mu < -1,$$

which is solved by $\mu > \sqrt{3} - 1$.