Ćwiczenia do Teorii Operatorów.
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0.1 Banach spaces

Hölder inequality:

$$\|fg\|_p \leq \|f\|_q \|g\|_r, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}, \quad 1 \leq p, q, r \leq \infty. \tag{0.1}$$

Minkowski inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p, \quad 1 \leq p \leq \infty. \tag{0.2}$$

Zadanie 1 Jeśli $p \leq r$, $x \in \mathbb{C}^n$, to

$$\|x\|_p \leq n^{\frac{1}{p} - \frac{1}{r}} \|x\|_r.$$

Zadanie 2 Jeśli $1 \leq q \leq r \leq \infty$, $x \in l^q$, to $\|x\|_q \geq \|x\|_r$. Zatem $l^q \subset l^r$.

Solution. Najpierw pokażemy, że $1 \leq p$, to

$$\|x\|_p \leq \|x\|_1. \tag{0.3}$$

Niech $y_i = [0, \ldots, x_i, \ldots 0]$. Wtedy $x = y_1 + \cdots + y_n$. Stosujemy nierówność Minkowskiego

$$\|x\|_p \leq \|y_1\|_p + \cdots + \|y_n\|_p = |x_1| + \cdots |x_n| = \|x\|_1.$$

Następnie stosujemy (0.3) do $y = [x_1^q, \cdots, x_n^q]$:

$$\left(\sum |x_i|^r \right)^{\frac{q}{r}} = \|y\|_{\frac{r}{q}} \leq \|y\|_1 = \sum |x_i|^q.$$

Zadanie 3 Jeśli $1 \leq p \leq r \leq \infty$, $f \in L^r[0,1]$, to

$$\|f\|_p \leq \|f\|_r.$$

Zatem $L^p[0,1] \supset L^r[0,1]$.

Zadanie 4 A linear operator from $\mathbb{C}^m$ to $\mathbb{C}^n$ can be defined by a matrix $[a_{ij}]$.

(1) Jeśli $\mathbb{C}^m$ jest wyposażone w normę $\|\cdot\|_1$ a $\mathbb{C}^n$ w normę $\|\cdot\|_\infty$, wtedy $\|A\| = \max\{|a_{ij}|\}$.
(2) Jeśli $\mathbb{C}^m$ jest wyposażone w normę $\|\cdot\|_\infty$ a $\mathbb{C}^n$ w normę $\|\cdot\|_1$, wtedy $\|A\| \leq \sum_{i,j} |a_{ij}|$.
(3) Jeśli $\mathbb{C}^m$ jest wyposażone w normę $\|\cdot\|_1$ a $\mathbb{C}^n$ w normę $\|\cdot\|_1$, wtedy $\|A\| = \max_j \{\sum_i |a_{ij}|\}$.
(4) Jeśli $\mathbb{C}^m$ jest wyposażone w normę $\|\cdot\|_\infty$ a$\mathbb{C}^n$ w normę $\|\cdot\|_\infty$, wtedy $\|A\| = \max_i \{\sum_j |a_{ij}|\}$.

0.2 Przestrzenie Hilberta

Niech $\mathcal{H}$ będzie przestrzenią Hilberta. Będziemy stosować notację dla iloczynu skalarnego podobną do notacji Diraca:

$$(v|w), \quad v, w \in \mathcal{H}.$$

Jedną z jej zalet jest możliwość "oderwania" $(v|$, $|w)$, traktując je jako operatory

$$\mathbb{C} \ni z \mapsto |w)z := wz \in \mathcal{H}, \tag{0.4}$$

$$\mathcal{H} \ni h \mapsto (v|h := (v|h) \in \mathbb{C}. \tag{0.5}$$

Na przykład

$$|w)(v|h = w(v|h).$$

Jeśli $(v|w) = 1$, jest to rzut na w wzdłuż $\mathrm{Ker}(v|$. Jeśli $\|v\| = 1$, to $|v)(v|$ jest rzutem ortogonalnym na v. Jeśli $e_1, \ldots, e_n$ jest bazą ortonormalną, to

$$A = \sum_{i,j=1}^{n} A_{ij} |e_i)(e_j|.$$

Zadanie 5 In l^2 we define the spaces

$$W := \{(x_n) \in l^2 \ : x_{2k} = 0, \ k \in \mathbb{N}\}, \tag{0.6}$$

$$Z := \{(x_n) \in l^2 \ : \ x_{2k-1} + \sqrt{k} x_{2k} = 0, \ k \in \mathbb{N}\}. \tag{0.7}$$

Obviously, W and Z are closed. Show that $W + Z$ is dense in l^2 but not closed.

Solution. Let $x \perp W + Z$. Because $x \perp V$, $x_{2k-1} = 0$. Because $x \perp Z$, $x_{2k-1} - \frac{1}{\sqrt{k}} x_{2k} = 0$. Hence $x = 0$. Therefore, $W + Z$ is dense in l^2.

Consider $x \in l^2$, $x_i = \frac{1}{n}$. Let $x = v + z$, $v \in V$, $z \in Z$. Then

$$z_{2k} = \frac{1}{2k}, \quad z_{2k-1} = -\sqrt{k} z_{2k} = -\frac{1}{2\sqrt{k}}.$$

But $z \notin l^2$.

Zadanie 6 Let A be a self-adjoint operator on $\mathbb{C}^n$. Show

$$\|A\| = \max\{|\lambda| \ : \ \lambda \in \mathrm{sp}(A)\} = \sup\{(v|Av) \mid \|v\| = 1\}.$$

Solution.

$$A = \sum_{\lambda \in \mathrm{sp}(A)} \lambda P_\lambda.$$

$$\|Av\|^2 = \sum \lambda^2 \|P_\lambda v\| \leq \max\{|\lambda| \ : \ \lambda \in \mathrm{sp}(A)\} \sum \|P_\lambda v\|^2 \tag{0.8}$$

$$= \max\{|\lambda| \ : \ \lambda \in \mathrm{sp}(A)\} \|v\|^2. \tag{0.9}$$

Zadanie 7 $\|B\|^2 = \|B^*B\|$. **Solution.** Clearly, B^*B is self-adjoint.

$$\|B\| = \sup\{\|Bv\| \mid \|v\| = 1\} = \sup\{(v|B^*Bv) \mid \|v\| = 1\} = \|B^*B\|.$$

Zadanie 8 Find the norm of $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. **Solution.**

$$B^*B = \begin{bmatrix} \overline{B_{11}}B_{11} + \overline{B_{21}}B_{21} & \overline{B_{11}}B_{12} + \overline{B_{21}}B_{22} \\ \overline{B_{12}}B_{11} + \overline{B_{22}}B_{21} & \overline{B_{12}}B_{12} + \overline{B_{22}}B_{21} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$\det(C - \lambda \mathbb{1}) = (C_{11} - \lambda)(C_{22} - \lambda) - C_{12}C_{21} = 0.$$

$$\|B\|^2 = \frac{C_{11} + C_{22} + \sqrt{(C_{11} - C_{22})^2 + 4C_{12}C_{21}}}{2}.$$

Zadanie 9 Let $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ be a matrix with determinant 1. Prove that the transformation on $L^2(\mathbb{R})$ given by

$$Uf(x) := \frac{1}{|cx + d|} f\left(\frac{ax + b}{cx + d}\right), \quad x \in \mathbb{R}$$

is unitary.

Zadanie 10 Let A be an invertible operator. Then there exists a unique positive operatori B and unitary U such that

$$A = UB.$$

Solution. We have $A^* = BU^*$, $A^*A = B^2$. Hence $B = \sqrt{A^*A}$. B^2 is invertible. Hence so is B. Therefore, $U = AB^{-1}$. Then we check that U is unitary.

Zadanie 11 Let S^1 be the unit circle parametrized with angle $\phi \in [0, 2\pi[$. Let $S^1 \ni \phi \mapsto \psi(\phi) \in S^1$ be the bijection of class C^1 such that $\frac{\mathrm{d}\phi}{\mathrm{d}\psi}$ is bounded. Define the operator W on functions on S^1 by

$$Wf(\phi) = f(\psi(\phi))).$$

(i) Find W^*.
(ii) Find the unitary operator U and the positive operator A such that $W = BU$.
(iii) Show that że W is bounded on $L^2(S^1)$ and find $\|W\|$.

Solution.

$$W^*g(\psi) = \left|\frac{\mathrm{d}\phi}{\mathrm{d}\psi}\right| g\big(\phi(\psi)\big).$$

Hence,

$$W^*Wf(\psi) = \left|\frac{\mathrm{d}\phi}{\mathrm{d}\psi}\right| f(\psi), \tag{0.10}$$

$$\sqrt{W^*W}f(\psi) = \left|\frac{\mathrm{d}\phi}{\mathrm{d}\psi}\right|^{\frac{1}{2}} f(\psi), \tag{0.11}$$

$$Uf(\phi) = \left|\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\right|^{\frac{1}{2}} f(\psi(\phi)). \tag{0.12}$$

0.3 Fourier series

Zadanie 12 Consider $\mathbb{C}^n$ with the canonical basis $(\delta_j \ : \ j = 0, 1, \dots, n-1)$. Define the operators

$$U := \sum_{j=0}^{n-2} |\delta_{j-1})(\delta_j| + |\delta_{n-1})(\delta_0|, \quad R = |\delta_0)(\delta_0| + \sum_{j=1}^{n-1} |\delta_{n-j})(\delta_j|.$$

(i) Show that U and R are unitary.
(ii) Show that $UR = RU^*$ and $(U + U^*)R = R(U + U^*)$.
(iii) Find an orthonormal basis that diagonalizes U.
(iv) Find an orthonormal basis that diagonalizes $U + U^*$ and R.

Solution. (i) is obvious, because both U and R permute an orthonormal basis. The basis

$$e_j = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \mathrm{e}^{\frac{\mathrm{i}jk2\pi}{n}} \delta_k$$

diagonalizes U:

$$Ue_j = \mathrm{e}^{\frac{\mathrm{i}j2\pi}{n}} e_j, \quad U = \sum_{j=0}^{n-1} \mathrm{e}^{\frac{\mathrm{i}j2\pi}{n}} |e_j)(e_j|,$$

We have $Re_j = e_{-j}$. The basis

$$e_0^+ = e_0, \quad e_{\frac{n}{2}}^+ = e_{\frac{n}{2}}^+ \text{ if } n \text{ is even} \tag{0.13}$$

$$e_j^+ = \frac{1}{\sqrt{2}}(e_j + e_{-j}), \quad 0 < j < \left[\frac{n}{2}\right], \tag{0.14}$$

$$e_j^- = \frac{1}{\mathrm{i}\sqrt{2}}(e_j - e_{-j}), \quad 0 < j < \left[\frac{n}{2}\right]. \tag{0.15}$$

diagonalizes simultaneously $U + U^*$ and R:

$$(U + U^*)e_j^+ = 2\cos\frac{j2\pi}{n} e_j^+, \qquad Re_j^+ = e_j^+ \tag{0.16}$$

$$(U + U^*)e_j^- = 2\cos\frac{j2\pi}{n} e_j^-, \qquad Re_j^- = -e_j^-. \tag{0.17}$$

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Set $\mathcal{F}e_j = \delta_j$, or

$$\mathcal{F} = \sum |\delta_j)(e_j| = \sum_{j,k=0}^{n-1} \frac{1}{\sqrt{n}} \mathrm{e}^{-\frac{\mathrm{i}jk2\pi}{n}} |\delta_j)(\delta_k|.$$

Then

$$\mathcal{F}U\mathcal{F}^* = \sum_{j=0}^{n-1} \mathrm{e}^{\frac{\mathrm{i}j2\pi}{n}} |\delta_j)(\delta_j|.$$

Zadanie 13 Consider $L^2[-\pi,\pi]$, where $[-\pi,\pi]$ is treated as the circle. Define the operators

$$U(t)f(\phi) := f(\phi - t), \qquad Rf(\phi) := f(-\phi).$$

(i) Show that $U(t)$ and R are unitary.
(ii) Show that $U(t)U(s) = U(t+s)$ and $U(t)R = RU(-t)$.
(iii) Find an orthonormal basis that diagonalizes $U(t)$.
(iv) Find an orthonormal basis that diagonalizes $U(t) + U(-t)$ and R.

Solution. The on. basis

$$e_j(\phi) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}j\phi}$$

diagonalizes $U(t)$:

$$U(t)e_j = \mathrm{e}^{\mathrm{i}jt} e_j, \quad U(t) = \sum_{j=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}jt} |e_j)(e_j|,$$

We have $Re_j = e_{-j}$. The basis

$$e_0^+ = e_0^+, \quad e_0^+(\phi) = \frac{1}{\sqrt{2\pi}}; \tag{0.18}$$

$$e_j^+ = \frac{1}{\sqrt{2}}(e_j + e_{-j}), \quad e_j^+(\phi) = \frac{1}{\sqrt{\pi}} \cos(j\phi), \quad j = 1, \dots, \tag{0.19}$$

$$e_j^- = \frac{1}{\mathrm{i}\sqrt{2}}(e_j - e_{-j}), \quad e_j^-(\phi) = \frac{1}{\sqrt{\pi}} \sin(j\phi), \quad j = 0, 1, 2, \dots. \tag{0.20}$$

diagonalizes simultaneously $U(t) + U(-t)$ and R:

$$(U(t) + U(-t))e_j^+ = 2\cos(jt)e_j^+, \qquad Re_j^+ = e_j^+ \tag{0.21}$$

$$(U(t) + U(-t))e_j^- = 2\cos(jt)e_j^-, \qquad Re_j = -e_j^-. \tag{0.22}$$

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Let $\{\delta_j \ : \ j \in \mathbb{Z}\}$ denote the canonical basis in $l^2(\mathbb{Z})$. Define the unitary Fourier transformation $\mathcal{F} : L^2[-\pi,\pi] \to l^2(\mathbb{Z})$ as

$$\mathcal{F} = \sum_{j=-\infty}^{\infty} |\delta_j)(e_j|,$$

or

$$(\mathcal{F}f)_j = \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-\mathrm{i}j\phi} f(\phi) \mathrm{d}\phi.$$

The Fourier transformation diagonalizes translations:

$$\mathcal{F}U(t)\mathcal{F}^* = \sum_{j=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}jt} |\delta_j)(\delta_j|.$$

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Zadanie 14 Define $L^2_\pm[-\pi,\pi] := \{f \in L^2[-\pi,\pi] \mid f(\phi) = \pm f(-\phi)$. Then $L^2[-\pi,\pi] = L^2_+[-\pi,\pi] \oplus L^2_-[-\pi,\pi]$. Besides, e_n^+, $n = 0,1,2,\dots$ is an orthonormal basis of $L^2_+[-\pi,\pi]$ and e_n^-, $n = 1,2,\dots$ of $L^2_-[-\pi,\pi]$.

Zadanie 15 Prove that $\sqrt{\frac{2}{\pi}}\cos n\phi$ $n = 1,2,\dots,$ $\frac{1}{\sqrt{\pi}}$, is an orthogonal basis of $L^2([0,\pi])$.

Prove that $\sqrt{\frac{2}{\pi}}\sin n\phi$, $n = 1,2,\dots,$ is an orthogonal basis of $L^2([0,\pi])$.

Solution Note that

$$L^2_\pm[-\pi,\pi] \ni f \mapsto U_\pm f := \sqrt{2} f\Big|_{[0,\pi]} \in L^2[0,\pi]$$

is a unitary operator and

$$\begin{aligned} U_+ e_0^+ &= \frac{1}{\sqrt{\pi}}, \\ U_+ e_n^+ &= \sqrt{\frac{2}{\pi}}\cos(n\phi), \\ U_- e_n^- &= \sqrt{\frac{2}{\pi}}\sin(n\phi). \end{aligned}$$

Niech I będzie zbiorem. Definiujemy

$$l^2(I) := \{(f_i)_{i\in I} \ : \ \sum |f_i|^2 =: \|f\|^2 < \infty\}.$$

Jeśli $\mathcal{H}$ jest przestrzenią Hilberta z bazą ortonormalną $\{e_i \ : \ i \in I\}$, to

$$(\mathcal{F}f)_i := (e_i|f), \quad f \in \mathcal{H}$$

definiuje operator unitarny $\mathcal{F} : \mathcal{H} \to l^2(I)$. Na przykład, transformata Fouriera

$$L^2[-\pi,\pi] \ni f \mapsto \frac{1}{\sqrt{2\pi}}\hat{f} \in l^2(\mathbb{Z})$$

jest takim operatorem, gdzie

$$\hat{f}_n := \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}n\phi} f(\phi)\mathrm{d}\phi.$$

We will write $c_n = \cos(n\phi)$, $s_n = \sin(n\phi)$.

Zadanie 16 Jedne funkcje lepiej jest rozwijać w szereg kosinusów a inne w szereg sinusów:

$$\begin{aligned} 1 &= c_0 \\ &= \frac{1}{\pi}\sum_{m=0}^{\infty} \frac{2}{2m+1} s_{2m+1}, \\ \sin\phi &= s_1 \\ &= \frac{1}{\pi}\sum_{m=1}^{\infty} \left(\frac{1}{2m-1} - \frac{1}{2m+1}\right) c_{2m}. \end{aligned}$$

Można wykorzystać

$$\sin\phi\cos(n\phi) = \frac{1}{2}\Big(\sin(n+1)\phi - \sin(n-1)\phi\Big).$$

Zadanie 17 $h(\phi) := (a - \mathrm{e}^{\mathrm{i}\phi})^{-1}$, $a > 1$. Wtedy

$$\hat{h}_n = \begin{cases} 2\pi a^{-n-1}, & n = 0, 1, \ldots; \\ 0, & n = -1, -2, \ldots. \end{cases}$$

Zadanie 18 $h(\phi) := (\mathrm{e}^{\mathrm{i}\phi} - a)^{-1}$, $a < 1$. Wtedy

$$\hat{h}_n = \begin{cases} 0, & n = 0, 1, 2, \ldots; \\ 2\pi a^{-n-1}, & n = -1, -2, \ldots. \end{cases}$$

Zadanie 19 $h(\phi) := \phi$. Wtedy

$$\hat{h}_n = \begin{cases} \frac{\mathrm{i}2\pi(-1)^n}{n}, & n \neq 0 \\ 0. & n = 0. \end{cases}$$

Aby to otrzymać można zauważyć, że możemy napisać

$$\log(1 + \mathrm{e}^{\pm\mathrm{i}\phi}) = \log\left(\mathrm{e}^{\pm\frac{\mathrm{i}}{2}\phi}\cos\frac{\phi}{2}\right) = \pm\mathrm{i}\phi + \log\left(\cos\frac{\phi}{2}\right). \tag{0.23}$$

(Używamy gałęzi główniej logarytmu). Dlatego,

$$h(\phi) = -\mathrm{i}\log(1 + a\mathrm{e}^{\mathrm{i}\phi}) + \mathrm{i}\log(1 + a\mathrm{e}^{-\mathrm{i}\phi}).$$

$$\log(1 + \mathrm{e}^{\pm\mathrm{i}\phi}) = \lim_{a\searrow 0}\log\left(1 + a\mathrm{e}^{\pm\mathrm{i}\phi}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\mathrm{e}^{\pm\mathrm{i}\phi n}}{n}.$$

Z tego wynika (0.23).

Częściową suma Fouriera

$$h_{(n)}(\phi) := \sum_{|j|\leq n}\frac{\hat{h}_j\mathrm{e}^{\mathrm{i}n\phi}}{2\pi},$$

jest zbieżna punktowo do ϕ na $]-\pi,\pi[$. Ale w otoczeniu $\phi = \pm\pi$ obserwujemy tzw. zjawisko Gibbsa: funkcja $h_{(n)}$ "przestrzeliwuje" wartość funkcji h. Mamy bowiem

$$h_{(n)}(-\pi + \epsilon) = -2\sum_{j=1}^{n}\frac{\sin\epsilon j}{j}.$$

W otoczeniu nieciągłości funkcji h obserwujemy "zafalowanie" funkcji $h_{(n)}$, które w miarę wzrostu n zwęża się, ale nie zmniejsza swej wysokości zachowując swoją wysokość. To zafalowanie ma w granicy ściśle określony kształt (z dokładnością do zwężania), mamy bowiem

$$\lim_{n\to\infty} h_{(n)}\left(-\pi + \frac{y}{n}\right) = -2\int_0^y\frac{\sin x}{x}\mathrm{d}x =: -2F(y).$$

Funkcja F jest nieparzysta, $\lim\limits_{x\to\infty} F(x) = \frac{\pi}{2}$ i ma maksimum dla $y = \pi$ równe

$$G := \int_0^\pi \frac{\sin x}{x} \mathrm{d}x \approx 1,81,$$

zwane stałą Wilbrahama-Gibbsa.

Ta własność sumy częściowej szeregu Fouriera występuje zawsze, kiedy mamy do czynienia z nieciągłą funkcją. Prowadzi ono do tego, że dla funkcji nieciągłej o skoku $a2\pi$ w sumie częściowej szeregu Fouriera będzie skok $4aG > a2\pi$. Mamy $(4G - 2\pi) \approx 0.18$.

Zadanie 20 Rozważmy $l^2(\mathbb{Z})$ z bazą kanoniczną $(\delta_j \ : \ j \in \mathbb{Z})$. Zdefiniujmy operatory

$$U := \sum_{j=-\infty}^{\infty} |\delta_{j+1})(\delta_j|, \quad R = \sum_{j=-\infty}^{\infty} |\delta_{-j})(\delta_j|.$$

(i) Pokazać, że U i R są unitarne.

(ii) Czy istnieje baza ortonormalna w której U jest diagonalny?

(iii) Odwrotna transformata Fouriera $\mathcal{F}^* : l^2(\mathbb{Z}) \to L^2(S^1)$ diagonalizuje U:

$$\frac{1}{2\pi}\mathcal{F}^* U \mathcal{F} = B, \quad (Bf)(\phi) = \mathrm{e}^{\mathrm{i}\phi} f(\phi), \quad f \in L^2(S^1).$$

(iv) Podać operator unitarny $V : l^2(\mathbb{Z}) \to L^2(0,\pi) \oplus L^2(0,\pi)$ taki, że

$$VRV^* = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \quad V(U + U^*)V^* = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix},$$

gdzie

$$(Cg)(\phi) = 2\cos\phi g(\phi), \quad g \in L^2(0,\pi).$$

Wskazówka. Najpierw rozwiązać zadanie 13.

Zadanie 21 Pokazać, że jeśli $f^{(n)}$ istnieje, to

$$|\hat{f}_k| \le |k|^{-n} \int_{-\pi}^{\pi} \left| f^{(n)}(x) \mathrm{d}x \right|.$$

Solution.

$$k^n \hat{f}_k = \int_{-\pi}^{\pi} f(x) \mathrm{i}^n \partial_x^n \mathrm{e}^{\mathrm{i}kx} \mathrm{d}x \tag{0.24}$$

$$= \int_{-\pi}^{\pi} (-\mathrm{i})^n \big(\partial_x^n f(x)\big) \mathrm{e}^{\mathrm{i}kx} .\mathrm{d}x \tag{0.25}$$

Zadanie 22 Pokazać, że jeśli dla $\epsilon > 0$,

$$|\hat{f}_k| \leq \frac{C}{(|k|+1)^{n+1+\epsilon}},$$

to f jest n-krotnie różniczkowalne.

Solution.

$$\partial_x^n f(x) = \frac{1}{2\pi} \sum k^n \hat{f}_k e^{ikx}.$$

Zatem

$$|\partial_x^n f(x)| \leq \frac{1}{2\pi} \sum |k^n \hat{f}_k|.$$

0.4 Falki Haara.

Zdefiniujmy

$$\psi_{k,n}(x) \quad := \quad \begin{cases} 2^{k/2}, & 2^{-k}n \leq x < 2^{-k}n + 2^{-k-1}, \\ -2^{k/2}, & 2^{-k}n + 2^{-k-1} \leq x < 2^{-k}(n+1), \\ 0, & x \notin [2^{-k}n, 2^{-k}(n+1)[; \end{cases}$$

$$\phi_{k,n}(x) \quad := \quad \begin{cases} 2^{k/2}, & 2^{-k}n \leq x < 2^{-k}(n+1), \\ 0, & x \notin [2^{-k}n, 2^{-k}(n+1)[. \end{cases}$$

Czasami nazywa się ψ_{00} "falką matką" a ϕ_{00} "falką ojcem".

Zadanie 23 Wprowadźmy operatory unitarne translacji i skalowania

$$\begin{aligned} (U_t f)(x) &:= f(x-t), \\ (W_s f)(x) &:= s^{-\frac{1}{2}} f(s^{-1}x). \end{aligned}$$

Zauważmy, że możemy napisać

$$\psi_{k,n} \quad = \quad W_{2^{-k}} U_n \psi_{00}, \quad \psi_{k,n}(x) = 2^{k/2} \psi_{00}(2^k x - n).$$

Pokazać, że $\{\psi_{k,n} \mid k = 0, 1, 2, \ldots, \quad n = 0, 1, \ldots, 2^k - 1\}$ oraz funkcja ϕ_{00} stanowią bazę ortonormalną $L^2[0,1]$.

Solution. Sprawdzamy najpierw ortonormalność. Oczywiste jest, że $\mathrm{Span}\{\phi_{kn} \mid k \geq 0\}$ jest gęste i zawiera $\{\psi_{k,n} \mid k = 0, 1, 2, \ldots, \quad n = 0, 1, \ldots, 2^k - 1\}$ oraz ϕ_{00}. Przeciwna inkluzja też jest łatwa.

Zadanie 24 (1) Niech $m \in \mathbb{Z}$. Wtedy

$$\mathcal{V}_m := (\mathrm{Span}\{\psi_{k,n} \ : \ k \leq m, \ n =\})^{\mathrm{cl}} = (\mathrm{Span}\{\phi_{k,n} \ : \ k \geq m+1, \ n \in \mathbb{Z}\})^{\mathrm{cl}}. \quad (0.26)$$

(2) $\{\psi_{k,n} \mid k,n \in \mathbb{Z}\}$ stanowią bazę ortonormalną $L^2(\mathbb{R})$.

(3) $\psi_{m,n}$, $n \in \mathbb{Z}$ stanowią bazę ortonormalną w $\mathcal{V}_m \ominus \mathcal{V}_{m+1}$ (w dopełnieniu ortogonalnym do $\mathcal{V}_{m+1}$ wewnątrz $\mathcal{V}_m$).

Solution. (1): $\subset$ jest oczywiste. Mamy

$$\sum_{j=0}^{\infty} 2^{-\frac{j}{2}} \psi_{-j0} = \sqrt{2}\phi_{10}.$$

To pokazuje $\supset$. (2) Najpierw sprawdzamy ortonormalność ψ_{kn}. Oczywiste jest, że $\mathrm{Span}\{\phi_{kn} \mid k,n \in \mathbb{Z}\}$ jest gęste w $L^2(\mathbb{R})$.

0.5 Operatory

Zadanie 25 Pokazać, że $\|A\| = \|A^*\|$ **Solution**.

$$\|A\| = \sup_{\|v\|\le 1} \|Av\| = \sup_{\|v\|,\|w\|\le 1} |(w|Av)| = \sup_{\|v\|,\|w\|\le 1} |(A^*w|v)| = \sup_{\|w\|\le 1} \|A^*w\| = \|A^*\|.$$

Zadanie 26 Rozważmy $L^2(S^1)$ z bazą ortonormalną $e_n(\phi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}n\phi}$, $n \in \mathbb{Z}$. Niech $(c_n \ : \ n \in \mathbb{Z})$ będzie ciągiem ograniczonym. Zdefinujmy operator

$$C = \sum_{n=-\infty}^{\infty} c_n |e_n)(e_n|.$$

(i) Pokazać, że C jest ograniczony i ma normę $\sup\{|c_n| \ : \ n \in \mathbb{Z}\}$.

(ii) Pokazać, że jeśli $\sum\limits_{n=-\infty}^{\infty} |c_n| < \infty$, to C posiada jądro całkowe równe

$$C(\phi,\psi) = \frac{\hat{c}(-\phi+\psi)}{2\pi},$$

gdzie $\hat{c}(\psi) = \sum\limits_{n=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\psi n} c_n.$.

Zadanie 27 Na $L^2(S^1)$ rozważyć operator P_ϵ z jądrem całkowym

$$P_\epsilon(\phi,\psi) = \frac{\sinh\epsilon}{\cosh\epsilon - \cos(\phi-\psi)}.$$

Pokazać, że $\mathrm{s}-\lim\limits_{\epsilon\searrow 0} P_\epsilon = 1\!\!1$ i $\|P_\epsilon\| = 1$.

Wskazówka. Pokazać, używając bazy ortonormalnej z poprzedniego zadanie, że

$$P_\epsilon = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\epsilon|n|} |e_n)(e_n|.$$

Zadanie 28 Niech $f : [0,\infty[\to [0,1]$ będzie funkcją ciągłą i malejącą, taką, że $f(0) = 1$ i $\lim_{t\to\infty} f(t) = 0$. Rozważmy przestrzeń Hilberta l^2 z bazą kanoniczną $(\delta_j \ : \ j \in \mathbb{N}\}$. Zdefiniujmy roddzinę operatorów

$$C_\epsilon := \sum_{j=1}^{\infty} f(\epsilon j)|\delta_j)(\delta_j|.$$

(i) Pokazać, że funkcja $[0,\infty[\ni \epsilon \mapsto C_\epsilon$ jest normowo ciągła na $]0,\infty[$ lecz normowo nieciągła w $\epsilon = 0$.

(ii) Pokazać, że funkcja $[0,\infty[\ni \epsilon \mapsto C_\epsilon$ jest silnie ciągła.

Zadanie 29 Mówimy, że P jest rzutem, gdy $P^2 = P$.

(i) Pokazać, że jeśli P jest rzutem niezerowym, to $\|P\| \geq 1$.

(ii) Pokazać, że dla każdego $c \geq 1$ istnieje rzut na przestrzeni Hilberta taki, że $\|P\| = c$.
Wskazówka. Wystarczy rozważać 2-wymiarowe przestrzenie Hilberta.

(iii) Pokazać, że jeśli P jest rzutem na przestrzeni Hilberta takim, że $\|P\| = 1$, to jest to rzut ortogonalny.

Zadanie 30 Niech $(U_n \ : \ n = 1, 2, \dots)$ będzie ciągiem operatorów unitarnych.

(i) Pokazać, że jeśli $\lim_{n\to\infty} U_n = U$, to U jest unitarny.

(ii) Pokazać, że jeśli s$-\lim_{n\to\infty} U_n = U$ i s$-\lim_{n\to\infty} U_n^* = U^*$, to U jest unitarny.

(iii) Pokazać, że jeśli s$-\lim_{n\to\infty} U_n = U$, to U jest izometrią. Podać przykład ciągu operatorów unitarnych, którego silna granica nie jest unitarna.

(iv) Pokazać, że jeśli w$-\lim_{n\to\infty} U_n = U$, to $\|U\| \leq 1$. Podać przykład ciągu operatorów unitarnych, którego słaba granica jest zerem.

Zadanie 31 Niech $(P_n \ : \ n = 1, 2, \dots)$ będzie ciągiem rzutów. Niech s$-\lim_{n\to\infty} P_n = P$. Pokazać, że P jest rzutem.
Wskazówka. Można założyć, że $\sup \|P_n\| < \infty$. (Wynika to z Tw. Banacha-Steinhausa i silnej zbieżności ciągu (P_n)).

Zadanie 32 Niech A_n będzie ciągiem samosprzężonych operatorów ograniczonych na przestrzeni Hilberta takich, że w$-\lim_{n\to\infty} A_n = A$. Pokazać, że A jest samosprzężony.

Zadanie 33 Niech $(P_n \ : \ n = 1, 2, \dots)$ będzie ciągiem rzutów ortogonalnych.

(i) Niech s$-\lim_{n\to\infty} P_n = P$. Pokazać, że P jest rzutem ortogonalnym.

(ii) Podać przykład ciągu $(P_n \ : \ n = 1, 2, \dot{)}$ rzutów ortogonalnych takich, że w$-\lim_{n\to\infty} P_n = \frac{1}{2}\mathbb{1}$.

Zadanie 34 Dla $f \in L^2([0,\infty[)$ definiujemy

$$(Tf)(x) := x^{-1} f(x^{-1}).$$

Czy T jest operatorem

(i) ograniczonym,
(ii) unitarnym,
(iii) samosprzężonym.

Zadanie 35 Niech $\alpha \in \mathbb{C}$, $\mathrm{Re}\alpha \leq 0$. Niech $(\delta_n : n \in \mathbb{N})$ oznacza bazę kanoniczną w $l^2(\mathbb{N})$. Zdefiniujmy następujący operator na $l^2(\mathbb{N})$:

$$T := \sum_{n=1}^{\infty} \mathrm{e}^{n\alpha} |\delta_{n+1})(\delta_n|.$$

(i) Pokazać, że T jest operatorem ograniczonym i znaleźć jego normę.
(ii) Dla jakich α operator T jest izometrią?
(iii) Policzyć T^*T.
(iv) Policzyć T^2.
(v) Dla jakich α istnieje $\mathrm{s} - \lim_{n\to\infty} T^n$?
(vi) Dla jakich α istnieje $\lim_{n\to\infty} T^n$?

Zadanie 36 Rozważmy $l^2(\mathbb{Z})$ z bazą kanoniczną $(\delta_n : n \in \mathbb{Z})$. Niech $\theta \in \mathbb{R}$. Zdefiniujmy wektory

$$b_n := \cos(\theta n)\delta_n + \sin(\theta n)\delta_{-n}.$$

Pokazać, że $(b_n : n \in \mathbb{Z})$ jest bazą ortnormalną.

Zadanie 37 Policzyć tranformatę Fouriera funkcji

$$f(x) = \mathrm{e}^{-\frac{3}{4}x^2} \cos x^2.$$

Zadanie 38 Niech $0 < \alpha < \pi$. Dla $f \in L^2(S^1)$ zdefiniujmy

$$(Tf)(\phi) = \int_{-\alpha}^{\alpha} f(\phi - \psi)\mathrm{d}\psi.$$

(i) Policzyć Te_n, gdzie $e_n(\phi) := \mathrm{e}^{\mathrm{i}n\phi}$.
(ii) Pokazać, że T jest operatorem ograniczonym i znaleźć jego normę.

Zadanie 39 Niech $1 < p < \infty$ i $f \in L^p(\mathbb{R}^3)$. Niech $|x|$ oznacza normę euklidesową wektora $x \in \mathbb{R}^3$. Dla jakiego m funkcja $(1 + |x|^2)^{-m} f$ należy do $L^1(\mathbb{R}^d)$?

Zadanie 40 Niech d będzie liczbą naturalną. Dla jakiego m następująca funkcja należy do $L^2(\mathbb{R}^d)$:

(i) $|x|^{-m}$,
(ii) $(1 + |x|)^{-m}$,
(iii) $\prod_{i=1}^{d}(1 + |x_i|)^{-m}$.

Zadanie 41 Dla $t > 0$ kładziemy $g_t(x) := (2\pi t)^{-1/2} \mathrm{e}^{-x^2/2t}$. Znaleźć $g_t * g_s$.
Wskazówka. $\hat{g}_t(\xi) = \mathrm{e}^{-t\xi^2/2}$.

Zadanie 42 Dla $t \neq 0$ kładziemy $g_t(x) = (\mathrm{i}x + t)^{-1}$. Znaleźć $g_t * g_s$.
Wskazówka. $\hat{g}_t(\xi) = 2\pi(\mathrm{sgn} t)\theta(\xi \mathrm{sgn} t)\mathrm{e}^{-t\xi}$.

Zadanie 43 Pokazać, że $\mathrm{Span}\{(x + \alpha)^{-1} \ : \ \mathrm{Im}\alpha > 0\}$ nie jest podprzestrzenią gęstą w $L^2(\mathbb{R})$.

Zadanie 44 Niech $g \in L^1(\mathbb{R})$. Pokazać, że operator $Tf := f * g$ jest dobrze zdefiniowany dla $f \in L^2(\mathbb{R})$, jest ograniczony i $\|T\| \leq \|g\|_1$. Czy zawsze $\|T\| = \|g\|_1$?
Wskazówka. Zastosować transformację Fouriera.

Zadanie 45 Niech $1 \leq p \leq r \leq q \leq \infty$. Pokazać, że

$$L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^r(\mathbb{R}) \subset L^p(\mathbb{R}) + L^q(\mathbb{R}).$$

Zadanie 46 Dla $m \in \mathbb{R}$, $1 \leq p \leq \infty$ definiujemy

$$L^p_m(\mathbb{R}) := \{f \ : \ \|(1 + |x|)^m f\|_p < \infty\}.$$

Pokazać, że jeśli $r \leq q$, $m > \frac{1}{r} - \frac{1}{q} + k$, to $L^r_k(\mathbb{R}) \supset L^q_m(\mathbb{R})$.
Wskazówka. Wykorzystać uogólnioną nierówność Höldera

$$\|fg\|_r \leq \|f\|_p \|g\|_q, \quad 1 \leq p \leq r \leq q \leq \infty, \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

0.6 Dystrybucje

Odwzorowania z $\mathcal{D}(\mathbb{R}^d) \to \mathbb{C}$, zwane dystrybucjami, bywają zapisywne w różny sposób, np.:

$$\mathcal{D}(\mathbb{R}^d) \ni \phi \mapsto T(\phi) = \langle T|\phi\rangle = \int T(x)\phi(x)\mathrm{d}x.$$

Spełniają one następujący warunek: dla każdego zwartego $K \subset \mathbb{R}^d$ istnieje N i C takie, że dla $\phi \in \mathcal{D}(\mathbb{R}^d)$ spełniających $\mathrm{supp}\phi \subset K$,

$$|\langle T|\phi\rangle| \leq C \max_{n \leq N} \sup_x |\partial_x^n \phi(x)|.$$

Przykład: jeśli $T \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, to dystrybucją regularną związaną z F, nazywamy

$$\langle T_F|\phi\rangle = \int T(x)f(x)\mathrm{d}x.$$

A oto delta Diraca w $a \in \mathbb{R}^d$:

$$\langle \delta_a|\phi\rangle = \int \delta(x - a)\phi(x)\mathrm{d}x = \phi(a).$$

Zadanie 47 Pokazać, że

$$\mathcal{P}\int \frac{1}{x}\phi(x)\mathrm{d}x := \lim_{\epsilon\searrow 0}\Big(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\Big)\frac{\phi(x)}{x}\mathrm{d}x$$

jest dystrybucją.

Solution. Niech supp$\phi \subset K$.

$$\begin{aligned}&\mathcal{P}\int \frac{1}{x}\phi(x)\mathrm{d}x\\ =&\Big(\int_{-\infty}^{-1} + \int_{1}^{\infty}\Big)\frac{\phi(x)}{x}\mathrm{d}x + \lim_{\epsilon\searrow 0}\Big(\int_{-1}^{-\epsilon} + \int_{\epsilon}^{1}\Big)\frac{\phi(x)}{x}\mathrm{d}x \ = \ I + II\end{aligned}$$

$$|II| \leq 2\sup|\phi'|, \quad |I| \leq |K|\sup|\phi|.$$

Zadanie 48 Pokazać, że

$$\mathcal{P}\int_0^{\infty} \frac{1}{x}\phi(x)\mathrm{d}x := \lim_{\epsilon\searrow 0}\Big(\int_{\epsilon}^{\infty}\frac{\phi(x)}{x}\mathrm{d}x + \phi(0)\log\epsilon\Big)$$

jest dystrybucją.

Solution.

$$\begin{aligned}&\mathcal{P}\int_0^{\infty} \frac{1}{x}\phi(x)\mathrm{d}x\\ =&\int_1^{\infty}\frac{\phi(x)}{x}\mathrm{d}x + \int_0^1\frac{\phi(x)-\phi(0)}{x}\mathrm{d}x.\end{aligned}$$

Następnie korzystamy z tego, że funkcja

$$x \mapsto \begin{cases}\frac{\phi(x)-\phi(0)}{x}, & x \in]0,1],\\ \phi'(0), & x = 0\end{cases}$$

jest ciągła.

Zadanie 49 Zróżniczkować n-krotnie $\frac{1}{2}\theta(x)x^2$

Zadanie 50 Niech δ_a będzie deltą Diraca w punkcie $a \in \mathbb{R}$. Pokazać, że operator $\mathcal{S}(\mathbb{R}) \ni f \mapsto T_a f := \delta_a * f \in \mathcal{S}(\mathbb{R})$ rozszerza się do operatora unitarnego na $L^2(\mathbb{R})$. Czy T_a dla $a \to \infty$ jest zbieżny normowo, silnie lub słabo? Ewentualnie policzyć granicę.

0.7 Zbieżność dystrybucji

Mówimy, że ciąg dystrybucji T_n jest zbieżny (w sensie dystrybucyjnym) do dystrybucji T, gdy

$$\langle T_n|\phi\rangle \to \langle T|\phi\rangle, \quad \phi \in \mathcal{D}(\mathbb{R}^d).$$

Zadanie 51 Niech $f \in L^1(\mathbb{R})$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-1} f(x\epsilon^{-1})$. Wtedy

$$\lim_{\epsilon \searrow 0} = \delta.$$

Solution. Niech $\delta > 0$.

$$\begin{aligned}
\int f_\epsilon(x)\phi(x)\mathrm{d}x - \phi(0) &= \int f_\epsilon(x)\big(\phi(x) - \phi(0)\big)\mathrm{d}x \\
&= \int_{|x|<\delta} f_\epsilon(x)\big(\phi(x) - \phi(0)\big)\mathrm{d}x + \int_{|x|>\delta} f_\epsilon(x)\big(\phi(x) - \phi(0)\big)\mathrm{d}x \quad = \quad I + II; \\
|I| &\leq \sup_{|x|<\delta} |\phi(x) - \phi(0)| \int |f(x)|\mathrm{d}x \leq \delta \sup_x |\phi'(x)| \int |f(x)|\mathrm{d}x; \\
|II| &\leq 2 \sup |\phi(x)| \int_{|x|>\delta/\epsilon} |f(x)|\mathrm{d}x.
\end{aligned}$$

Ale $\lim_{\epsilon \searrow 0} \int_{|x|>\delta/\epsilon} |f(x)|\mathrm{d}x = 0$. Więc

$$\Big| \int f_\epsilon(x)\phi(x)\mathrm{d}x - \phi(0) \Big| \leq A\delta.$$

Ale $\delta > 0$ było dowolne.

Zadanie 52 Pokazać wzór Sochockiego.

$$\lim_{\epsilon \searrow 0} \frac{1}{x + \mathrm{i}\epsilon} = \mathcal{P}\frac{1}{x} - \mathrm{i}\pi\delta(x).$$

Solution. Mamy

$$\frac{1}{x + \mathrm{i}\epsilon} = \frac{x}{x^2 + \epsilon^2} - \frac{\mathrm{i}\epsilon}{x^2 + \epsilon^2}.$$

Mamy

$$\int \frac{1}{\pi} \int \frac{\epsilon}{x^2 + \epsilon^2}\mathrm{d}x = 1.$$

Więc, z poprzedniego zadania mamy

$$\lim_{\epsilon \searrow 0} \frac{\mathrm{i}\epsilon}{x^2 + \epsilon^2} = \mathrm{i}\pi\delta(x).$$

Podobnie pokazujemy

$$\lim_{\epsilon \searrow 0} \frac{x}{x^2 + \epsilon^2} = \mathcal{P}\frac{1}{x}.$$

0.8 Równania dystrybucyjne

Zadanie 53 Znaleźć wszystkie dystrybucje spełniające

$$k^m T = 0. \tag{0.27}$$

Solution. T musi mieć nośnik $\{0\}$. Zatem musi mieć postać

$$\sum_{j=0}^{n} c_j \delta^{(j)}(k).$$

Wtedy

$$\langle k^m T|\phi\rangle = \sum_{j=m}^{n} c_j (-1)^j \big(k^m \phi(k)\big)^{(j)}\Big|_{k=0} = \sum_{j=m}^{n} c_j (-1)^j j(j-1)\cdots(j-m+1)\phi^{(j-m)}(0).$$

Czyli rozwiązaniem są

$$T = \sum_{j=0}^{m-1} c_j \delta^{(j)}(k).$$

Zadanie 54 Znaleźć wszystkie dystrybucje spełniające

$$kT = 1. \tag{0.28}$$

Solution.

$$T = \mathcal{P}\frac{1}{k} + c\delta(k).$$

Zadanie 55 Znaleźć wszystkie dystrybucje spełniające

$$(k^2 - 1)T = 1. \tag{0.29}$$

Solution.

$$T = \mathcal{P}\frac{1}{k^2-1} + c_+\delta(k-1) + c_-\delta(k+1).$$

Zadanie 56 Znaleźć wszystkie dystrybucje spełniające

$$k^2 T = 1. \tag{0.30}$$

Solution. Zdefiniujmy dystrybucję $\mathcal{P}\frac{1}{k^2}$ wzorem

$$\mathcal{P}\int \frac{\phi(k)}{k^2}\mathrm{d}k := \int_{|k|<1} \frac{\phi(k) - \phi(0) - k\phi'(0)}{k^2}\mathrm{d}k + \int_{|k|>1} \frac{\phi(k)}{k^2}\mathrm{d}k.$$

Solutionm jest

$$T = \mathcal{P}\frac{1}{k^2} + c_0\delta(k) + c_1\delta^{(1)}(k).$$

0.9 Transformata Fouriera

Zadanie 57 Niiech $m > 0$. Pokazać, że

$$\begin{aligned} \int \frac{\mathrm{e}^{-\mathrm{i}\xi s}}{(s+\mathrm{i}m)}\mathrm{d}\ s &= -2\pi\mathrm{i}\theta(\xi)\mathrm{e}^{-m\xi}, && (0.31)\\ \int \frac{\mathrm{e}^{-\mathrm{i}\xi s}}{(s-\mathrm{i}m)}\mathrm{d}\ s &= 2\pi\mathrm{i}\theta(-\xi)\mathrm{e}^{-m|\xi|}, && (0.32)\\ \int \frac{\mathrm{e}^{-\mathrm{i}\xi s}s}{(s^2+m^2)}\mathrm{d}\ s &= \pi\mathrm{i}\mathrm{sgn}(\xi)\mathrm{e}^{-m|\xi|}, && (0.33)\\ \int \frac{\mathrm{e}^{-\mathrm{i}\xi s}m}{(s^2+m^2)}\mathrm{d}\ s &= \pi\mathrm{e}^{-m|\xi|}. && (0.34) \end{aligned}$$

Zadanie 58 Przechodząc do granicy z m do zera w poprzednim zadaniu, pokazać, że

$$\begin{aligned} \int \theta(\pm x)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x &= \frac{\mp\mathrm{i}}{k\mp\mathrm{i}0}, && (0.35)\\ \int \frac{\mathrm{e}^{-\mathrm{i}kx}}{x\pm\mathrm{i}0}\mathrm{d}x &= \mp 2\pi\mathrm{i}\theta(\pm x), && (0.36)\\ \int \mathrm{sgn}(x)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x &= -2\mathrm{i}\mathcal{P}\left(\frac{1}{k}\right), && (0.37)\\ \int \mathcal{P}\frac{\mathrm{e}^{-\mathrm{i}kx}}{x}\mathrm{d}x &= \pi\mathrm{i}\mathrm{sgn}(k), && (0.38) \end{aligned}$$

Zadanie 59 Pokazać, że dla $\epsilon \geq 0$, $\lambda > -1$, transformata Fouriera funkcji $\theta(x)x^\lambda \mathrm{e}^{-\epsilon x}$ jest równa $\mathrm{e}^{-\mathrm{i}(1+\lambda)\frac{\pi}{2}}\Gamma(\lambda+1)(\xi-\mathrm{i}\epsilon)^{-1-\lambda}$.

Zadanie 60 Pokazać, że dla $f \in \mathcal{D}(\mathbb{R})$

$$Tf(x) := \mathcal{P}\int \frac{f(y)}{x-y}\mathrm{d}y$$

należy do $L^2(\mathbb{R})$ i że T rozszerza się do operatora ograniczonego na $L^2(\mathbb{R})$. Policzyć T^2.
Wskazówka. Warto zastosować transformatę Fouriera.

Zadanie 61 Pokazać, że

$$\mathbb{1}_{[-1,1]} * \mathbb{1}_{[-1,1]} = (2-|x|)\mathbb{1}_{[-1,1]}. \qquad (0.39)$$

Wiedząc, że transformata Fouriera $\mathbb{1}_{[-1,1]}$ jest równa $\frac{2\sin(\xi)}{\xi}$, policzyć transformatę Fouriera (0.39).

Solution. $\frac{4\sin^2(\xi)}{\xi^2}$.

0.10 Funkcje Greena

Zadanie 62 Znaleźć dystrybucje G spełniające

$$(\partial_x + 1)G(x) = \delta(x). \tag{0.40}$$

Pokazać, że

$$f := G * h$$

spełnia

$$(\partial_x + 1)f(x) = h(x). \tag{0.41}$$

Solution. Metoda 1. Solutionm równania jednorodnego

$$(\partial_x + 1)G_0(x) = 0. \tag{0.42}$$

jest $G_0(x) := c\mathrm{e}^{-x}$. Uzmienniając stałą c dostajemy równanie

$$\partial_x c(x) = \delta(x).$$

Stąd

$$G(x) = c\mathrm{e}^{-x} + \theta(x)\mathrm{e}^{-x}.$$

W szczególności, mamy funkcję Greena retardowaną $G_+(x) := \theta(x)\mathrm{e}^{-x}$ i adwansowaną $G_- := -\theta(-x)\mathrm{e}^{-x}$. Retardowana jest jedyną dystrybucją temperowaną spośród funkcji Greena.

Metoda 2.

$$\hat{G}(k) = \frac{1}{\mathrm{i}k + 1}.$$

Więc

$$G(x) = \int \frac{\mathrm{e}^{-\mathrm{i}kx}}{\mathrm{i}k + 1}\mathrm{d}k = \theta\mathrm{e}^{-x}.$$

Zadanie 63 Znaleźć dystrybucje G spełniające

$$(\partial_x + x)G(x, y) = \delta(x - y). \tag{0.43}$$

Pokazać, że

$$f(x) := \int G(x, y)h(y)\mathrm{d}x$$

spełnia

$$(\partial_x + x)f(x) = h(x). \tag{0.44}$$

Solution. Solutionm równania jednorodnego

$$(\partial_x + x)G_0(x, y) = 0. \tag{0.45}$$

jest $G_0(x) := c(y)\mathrm{e}^{-\frac{x^2}{2}}$. Uzmienniając stałą $c(y)$ dostajemy równanie

$$\partial_x c(x,y) = \mathrm{e}^{\frac{x^2}{2}}\delta(x-y) = \mathrm{e}^{\frac{y^2}{2}}\delta(x-y).$$

Stąd

$$G(x,y) = \mathrm{e}^{-\frac{x^2}{2}+\frac{y^2}{2}}\theta(x-y) + g(y)\mathrm{e}^{-\frac{x^2}{2}}\mathrm{e}^{-x}.$$

W szczególności, mamy funkcję Greena retardowaną G_+ i adwansowaną G_-:

$$G_+(x) := \mathrm{e}^{-\frac{x^2}{2}+\frac{y^2}{2}}\theta(x-y),$$
$$G_-(x) := -\mathrm{e}^{-\frac{x^2}{2}+\frac{y^2}{2}}\theta(y-x).$$

Zadanie 64 Rozważmy $\mathbb{R}^3$. Pokazać, że $G(x) := \frac{\mathrm{e}^{-m|x|}}{4\pi|x|}$ jest rozwiązaniem równania

$$(-\Delta + m^2)G(x) = \delta(x).$$

Solution. Metoda 1. Niech

$$G(x) = \frac{1}{(2\pi)^3}\int \hat{G}(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k.$$

Wtedy

$$\begin{aligned}
\hat{G}(k) &= \frac{1}{(k^2+m^2)},\\
G(x) &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{e}^{\mathrm{i}kx}}{(k^2+m^2)}\mathrm{d}k\\
&= \frac{1}{(2\pi)^2}\int_0^\infty |k|^2\mathrm{d}|k|\int_0^\pi \sin\theta\mathrm{d}\theta\frac{\mathrm{e}^{\mathrm{i}|k||x|\cos\theta}}{(k^2+m^2)}\\
&= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}|k|\frac{\mathrm{e}^{\mathrm{i}|k||x|}-\mathrm{e}^{-\mathrm{i}|k||x|}}{|k||x|\mathrm{i}(k^2+m^2)}\\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^\infty \mathrm{d}s\frac{s\mathrm{e}^{\mathrm{i}s|x|}}{|x|\mathrm{i}(s^2+m^2)} = \frac{\mathrm{e}^{-m|x|}}{4\pi|x|}.
\end{aligned}$$

Metoda 2. Zastosujemy wzór Greena:

$$\int_\Omega(\Delta fg - f\Delta g) = \int_{\partial\Omega}(\nabla fg - f\nabla g)\mathrm{d}s.$$

Kładziemy $f=\phi$, $g=\frac{\mathrm{e}^{-m|x|}}{4\pi|x|}$, $\Omega = \mathbb{R}^3\backslash K(r)$. Mamy na $\partial K(r) = r^2S^2$ z miarą $r^2\mathrm{d}\omega$, gdzie $\omega\in S^2$.

Sprawdzamy, że poza zerem

$$(-\Delta + m^2)\frac{\mathrm{e}^{-m|x|}}{4\pi|x|} = 0.$$

$$\int_{\mathbb{R}^3\backslash K(r)} (-\Delta + m^2)\phi(x)\frac{\mathrm{e}^{-m|x|}}{4\pi|x|}\mathrm{d}x = \int_{S^2} \Big(\partial_r\phi(r,\omega)\frac{\mathrm{e}^{-mr}}{4\pi r} - \phi(r,\omega)\partial_r\frac{\mathrm{e}^{-mr}}{4\pi r}\Big)r^2\mathrm{d}\omega$$

Wreszcie, korzystamy z

$$\partial_r\frac{\mathrm{e}^{-mr}}{r} = -\frac{\mathrm{e}^{-mr}}{r^2} - \frac{m\mathrm{e}^{-mr}}{r}, \qquad \int_{S^2}\mathrm{d}\omega = 4\pi.$$

Zadanie 65 Niech P będzie wielomianem takim, że $P(\xi) \neq 0$, $\xi \in \mathbb{R}$. Niech $g \in \mathcal{S}(\mathbb{R})$. Pokazać, że istnieje $f \in \mathcal{S}(\mathbb{R})$ taka, że

$$P(\mathrm{i}\partial_x)f = g.$$

Pokazać, że istnieje G zależne tylko od P takie, że to rozwiązanie może być zapisane jako

$$f = G * g.$$

Znaleźć G dla $P(\xi) = \xi^2 + m^2$

0.11 Finite dimensional matrices

Let A be a linear operator on a finite dimensional space $\mathcal{V}$. Let $\lambda \in \mathbb{C}$. TFAE:

(1) $\det(A - \lambda) = 0$,

(2) There exists $v \neq 0$ s.t. $Av = \lambda v$

(3) $(\lambda - A)$ is not invertible.

The set of such λ is called the spectrum of A and denoted sp$A$.

Problem. Let $\{\lambda_1, \dots, \lambda_n\} = \mathrm{sp}A$ Show that if $v_i \in \mathrm{Ker}(\lambda_i - A)$, and $v_1 + \cdots + v_n = 0$, then $v_1 = \cdots = v_n = 0$.

Solution. Suppose that this is not true. We can assume that p is the smallest possible number of nonzero $v_i \in \mathrm{Ker}(\lambda_i - A)$ such that $v_1 + \cdots + v_p = 0$. Then

$$0 = \lambda_p(v_1 + \cdots v_p) - A(v_1 + \cdots v_p) = (\lambda_p - \lambda_1)v_1 + \cdots (\lambda_p - \lambda_{p-1})v_{p-1} = 0,$$

which is a contradiction. □

We say that A is diagonalizable if $\sum\limits_{i=1}^{n} \mathrm{Ker}(\lambda_i - A) = \mathcal{V}$. In other words, $\mathcal{V} = \bigoplus\limits_{i=1}^{n} (\mathrm{Ker}(\lambda_i - A)$. We can then define the projection P_i onto $\mathcal{V}_i$ along $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_{i-1} \oplus \mathcal{V}_{i+1} \oplus \cdots \oplus \mathcal{V}_n$. We have then

$$A = \sum_{i=1}^{n} \lambda_i P_i.$$

Problem. Let A be an arbitrary matrix. Let $\lambda_i \in \mathrm{sp}(A)$. Show that

$$\mathrm{Ker}(\lambda_i - A) \subset \cdots \subset \mathrm{Ker}(\lambda_i - A)^m.$$

Show that this sequence stablizes. Suppose this happens for $m_i, m_i+1, \dots$. Set

$$\mathcal{V}_i := \mathrm{Ker}(\lambda_i - A)^{m_i}.$$

Show that $\mathcal{V}_i$ is an invariant subspace of A.
****

We have

$$\mathcal{V} = \overset{n}{\underset{i=1}{\oplus}} \mathcal{V}_i, \quad A = \overset{n}{\underset{i=1}{\oplus}} (\lambda_i + N_i),$$

where N_i is nilpotent on $\mathcal{V}_i$. Set

$$D := \overset{n}{\underset{i=1}{\oplus}} \lambda_i, \quad N := \overset{n}{\underset{i=1}{\oplus}} N_i.$$

We have

$$A = D + N, \tag{0.46}$$

where D is diagonalisable, N is nilpotent and $DN = ND$. (0.46) is called *Jordan-Chevalley decomposition*.

Problem. Let $f(z) = \sum_{k=0}^{n} f_k z^k$ be a polynomial Show that

$$f(A) := \sum_{k=0}^{n} f_k A^k = \sum \frac{N^j}{j!} f^{(j)}(D). \tag{0.47}$$

Solution. It is enough to assume that $A = \lambda + N$ where N is nilpotent.

$$f(A) = \sum f_k \sum N^j \lambda^{k-j} \frac{k!}{(k-j)!j!} = \sum_{j=0} \frac{N!}{j!} \sum_{j=0}^{m} k(k-1)\cdots(k-j+1) f_k \lambda^{k-j}. \tag{0.48}$$

Note that in order to compute $f(A)$ it is enough to know

$$f(\lambda_i), \cdots f^{(m_i-1)}(\lambda_i),$$

Spectral Theorem. Let $AA^* = A^*A$ (A is normal). Then A is diagonalizable and the spaces $\mathrm{Ker}(\lambda_i - A)$ are orthogonal. Therefore, P_i are orthogonal projections.

If A is Hermitian, then $\mathrm{sp}A \subset \mathbb{R}$.

If A is unitary, then $\mathrm{sp}A \subset \{|z| = 1\}$.

Problem. Let A be normal. Then

$$\|A\| = \sup\{|\lambda| \ : \ \lambda \in \mathrm{sp}A\}.$$

In particular

$$\|(z-A)^{-1}\| = \Big(\min\{|z-\lambda| \ : \ \lambda \in \mathrm{sp}A\}\Big)^{-1}.$$

Solution.

$$\|Au\|^2 = \sum |\lambda_i|^2 \|P_i u\|\|^2$$
$$\leq \sup|\lambda_i|^2 \sum \|P_i u\|^2 = (\sup|\lambda_i|)^2 \|u\|^2.$$

Problem. Let A be any operator with spectrum $\{\lambda_1, \dots, \lambda_n\}$, and degrees of nilpotency $m_1, \dots, m_n$. Then

$$\|(z-A)^{-1}\| \leq c \sum_{i=1}^{n} |z-\lambda_i|^{m_i}.$$

Solution.

$$\|(z-A)^{-1}u\| \leq \sum_{i=1}^{n} \|(z-\lambda_i - N_i)^{-1} P_i u\|.$$

$$(z-\lambda_i - N_i)^{-1} = \sum_{j=0}^{m_j-1} (z-\lambda_i)^{-1-j} N_j.$$

Problem. Let $A, B \in B(\mathcal{H})$. Prove that $\mathrm{sp}AB \cup \{0\} = \mathrm{sp}BA \cup \{0\}$.
Solution. Let $z \in \mathrm{rs}AB \backslash \{0\}$.Then

$$z^{-1}(1 + B(z-AB)^{-1}A)(z-BA) = z^{-1}(z-BA) + z^{-1}B(z-AB)^{-1}(1-AB)A = 1.$$

Hence $z^{-1}(1 + B(z-AB)^{-1}A) = (z-BA)^{-1}$ and $z \in \mathrm{rs}(BA)$.

0.12 Spectrum

We say that A is involution if $A^2 = \mathbb{1}$.
Problem. Show that the following are equivalent:

(1) A is self-adjoint and A is an involution.

(2) A is an involution and A is unitary

(3) A is unitary and A is self-adjoint.

Problem. Let A be an involution. Find its spectrum and its spectral projections.
Solution. We guess: set $P_\pm := \frac{1}{2}(\mathbb{1} \mp A)$. Then $AP_\pm = \pm P_\pm$ and $A = P_+ - P_-$. Therefore, $\mathrm{sp}(A) = \{-1, 1\}$ and $\mathbb{1}_{\{\pm 1\}}(A) = P_\pm$.
Problem. Let $U^n = 1$. Find spectral projections of U. **Solution.** Similarly, as in the previous problem, we guess:

$$P_k = \frac{1}{n} \sum_{j=0}^{n-1} U^j \mathrm{e}^{\frac{-\mathrm{i}jk2\pi}{n}} = \mathbb{1}_{\{\mathrm{e}^{\frac{\mathrm{i}2\pi k}{n}}\}}(U), \tag{0.49}$$

$$U = \sum_{k=0}^{n=1} \mathrm{e}^{\frac{\mathrm{i}k2\pi}{n}} P_k. \quad \mathrm{sp}(U) = \{\mathrm{e}^{\frac{\mathrm{i}2\pi k}{n}} \ : \ k = 0, \dots, n-1\}. \tag{0.50}$$

Problem. Find the spectrum of the Fourier transformation $\mathcal{F}$. **Solution.** $\mathcal{F}^4 = \mathbb{1}$. Hence $\mathrm{sp}\mathcal{F} \subset \{1, \mathrm{i}, -1, -\mathrm{i}\}$. Let $\Omega(x) = \mathrm{e}^{-\frac{x^2}{2}}$, $a^* = x - \partial_x$. Then $\mathcal{F}\Omega = \Omega$ and $\mathcal{F}a^* = \mathrm{i}a^*\mathcal{F}$. Hence $\mathcal{F}^n a^{*n}\Omega = \mathrm{i}^n a^{*n}\Omega$. But $a^{*n}\Omega$ is a complete set of eigenvectors of the harmonic oscillator.

0.13 Operator inequalities

We say that $A \in B(\mathcal{H})$ satisfies $A \geq 0$ if

$$(v|Av) \geq 0, \quad v \in \mathcal{H}.$$

Equivalent condition: A is self-adjoint and $\mathrm{sp}A \subset [0, \infty[$.

Let A be self-adjoint. Set $\inf \mathrm{sp}A = a_-$, $\sup \mathrm{sp}A = a_+$. Then

$$a_- \leq A \leq a_+.$$

For any A, $A^*A \geq 0$.

Problem. Show that

$$AA^* \leq 1 \quad \Leftrightarrow \quad A^*A \leq 1. \tag{0.51}$$

Solution. We have

$$\|Av\|^2 = (v|A^*Av) \leq (v|v) = \|v\|^2 \Leftrightarrow \|A\| \leq 1,$$
$$\|A^*v\|^2 = (v|AA^*v) \leq (v|v) = \|v\|^2 \Leftrightarrow \|A^*\| \leq 1.$$

But $\|A\| = \|A^*\|$.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Let $A, B \in B(\mathcal{H})$. We write $A \leq B$ if

$$(v|Av) \leq (v|Bv), \quad v \in \mathcal{H}.$$

$$A_1 \leq B_1, \quad A_2 \leq B_2 \text{ implies } A_1 + A_2 \leq B_1 + B_2. \tag{0.52}$$

$$A \leq B \text{ implies } CAC^* \leq CBC^*. \tag{0.53}$$

Problem. Let $0 \leq A \leq B$. Show that for $t \geq 0$,

$$(t + B)^{-1} \leq (t + A)^{-1}. \tag{0.54}$$

Solution. Using (0.51), we obtain the following implications:

$$t + A \leq t + B, \tag{0.55}$$
$$(t + B)^{-\frac{1}{2}}(t + A)(t + B)^{-\frac{1}{2}} \leq 1, \tag{0.56}$$
$$(t + A)^{\frac{1}{2}}(t + B)^{-1}(t + A)^{\frac{1}{2}} \leq 1, \tag{0.57}$$
$$(t + B)^{-1} \leq (t + A)^{-1}. \tag{0.58}$$

Problem. Let $A \geq 0$ and $\mathrm{Ker}A = \{0\}$. Let $0 < \alpha, \beta < 1$. Then

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{t^{-\alpha}\mathrm{d}t}{(A + t)}, \tag{0.59}$$
$$A^{\beta} = \frac{\sin \pi\beta}{\pi} \int_0^\infty \Big(\frac{1}{t} - \frac{1}{(A + t)}\Big) t^{\beta}\mathrm{d}t. \tag{0.60}$$

Solution. We start from identity

$$\frac{\pi}{\sin \pi\alpha} = \int_0^\infty \frac{s^{-\alpha}\mathrm{d}s}{(1+s)}. \tag{0.61}$$

We substitute $s = \frac{t}{A}$ to get (0.59). Next we multiply (0.59) by A, use

$$\frac{A}{(t+A)} = \Big(\frac{1}{t} - \frac{1}{(A+t)}\Big)t$$

and set $\beta = 1 - \alpha$, to obtain (0.60).
Problem. Let $0 \leq A \leq B$ and $\mathrm{Ker}A = \{0\}$. Let $0 < \alpha < 1$. Then

$$B^{-\alpha} \leq A^{-\alpha}, \tag{0.62}$$
$$A^{\alpha} \leq B^{\alpha}. \tag{0.63}$$

Solution. We have $(t+B)^{-1} \leq (t+A)^{-1}$. Therefore (0.62) follows from (0.59).
We have $\frac{1}{t} - \frac{1}{(A+t)} \leq \frac{1}{t} - \frac{1}{(B+t)}$. Therefore, (0.63) follows from (0.60).
Problem. Find an example of $A \leq B$ such that $A^2 \leq B^2$ is not true.
Solution. We use the following criterion for positivity:

$$\begin{bmatrix} a & b \\ \bar{b} & d \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad a \geq 0 \text{ and } ad - |b|^2 \geq 0.$$

Set

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_\epsilon := \begin{bmatrix} 1+\epsilon^2 & \epsilon \\ \epsilon & 1 \end{bmatrix}.$$

Clearly, $A \leq B$. Now

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_\epsilon^2 = \begin{bmatrix} 1+3\epsilon^2+\epsilon^4 & 2\epsilon+\epsilon^3 \\ 2\epsilon+\epsilon^3 & 1+\epsilon^2 \end{bmatrix}.$$

Then

$$B^2 - A^2 = \begin{bmatrix} 3\epsilon^2+\epsilon^4 & 2\epsilon+\epsilon^3 \\ 2\epsilon+\epsilon^3 & 1+\epsilon^2 \end{bmatrix}, \quad \det(B^2 - A^2) = 3\epsilon^2 - 4\epsilon^2 + O(\epsilon^4).$$

Hence $\det(B^2 - A^2) < 0$ for small ϵ.

0.14 Polar decomposition

Let A be an operator such that $\mathrm{Ker}A = \{0\}$ and $\mathrm{Ker}A^* = \{0\}$. Then there exists a unique positive operator, denoted $|A|$ and a unitary operator U such that

$$A = |A|U. \tag{0.64}$$

Besides,

$$|A| = \sqrt{AA^*}, \quad U = |A|^{-1}A.$$

(0.64) is called the *polar decomposition* of A.

The above definition has a generalization to an arbitrary operator. More precisely, if A is arbitrary, then there exists a unique positive operator $|A|$ and a unique partial isometry U such that $\mathrm{Ker}U = \mathrm{Ker}|A|$ and

$$A = U|A|. \tag{0.65}$$

Then (0.65) is called *polar decomposition* of A and $|A| = \sqrt{A^*A}$.

Let B be the inverse of $|A|$ restricted to $\mathrm{Ran}Q$, extended by 0 on $\mathrm{Ker}A$. Then

$$A^+ = BU^*, \tag{0.66}$$
$$|A|^+ = B, \tag{0.67}$$
$$U^+ = U^*. \tag{0.68}$$

Uwaga 0.1 *Let us denote the orthogonal projection onto the closure of* $\mathrm{Ran}A$ *by* P *and onto* $(\mathrm{Ker}A)^\perp$ *by* Q. *The* Moore-Penrose pseudoinverse *is defined as the unique operator* A^+ *such that*

$$AA^+ = P, \quad A^+A = Q. \tag{0.69}$$

Problem. Find the polar decomposition of A on $l^2(\mathbb{Z})$ given by

$$Ae_n = a_{n+1}e_{n+1},$$

where $a_n \neq 0$, $n \in \mathbb{Z}$.

Solution.

$$|A|e_n = |a_n|e_n, \quad Ue_n = \frac{a_{n+1}}{|a_{n+1}|}e_{n+1}.$$

Problem. Find the polar decomposition of A_z on $L^2(\mathbb{R})$ given by the integral kernel

$$A_z(x,y) = \mathrm{e}^{-z\frac{(x-y)^2}{2}}, \quad \mathrm{Re}z > 0.$$

Solution. First we compute

$$A_zA_w = \sqrt{\frac{zw2\pi}{z+w}}A_{z+w}.$$

Therefore,

$$|A_z|(x,y) = \sqrt{2\pi\mathrm{Im}z}\mathrm{e}^{-\mathrm{Re}z\frac{(x-y)^2}{2}}, \quad U_z(x,y) = \frac{1}{\sqrt{2\pi\mathrm{Im}z}}\mathrm{e}^{-\mathrm{Im}z\frac{(x-y)^2}{2}}.$$

Problem. Let $\mathbb{R} \ni x \mapsto y(x) \in \mathbb{R}$ be an increasing bijection. Find the polar decomposition of A on $L^2(\mathbb{R})$

$$Af(x) := f(y(x)).$$

Solution. First we compute

$$A^*g(y) = g(x(y))\Big|\frac{\mathrm{d}x}{\mathrm{d}y}(y)\Big|.$$

Therefore,

$$|A|f(y) = \left|\frac{\mathrm{d}x}{\mathrm{d}y}(y)\right|^{\frac{1}{2}} f(y),$$
$$Uf(x) = \left|\frac{\mathrm{d}y}{\mathrm{d}x}(x)\right|^{\frac{1}{2}} f(y(x)).$$

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Problem.

(1) Let U be a unitary operator and P an orthogonal projection. Then $W := UP$ is a partial isometry.

(2) Let W be a partial isometry on a finite dimensional Hilbert space. Then there exists a unitary operator U and an orthogonal projection P such that $W = UP$.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Problem. Find the polar decomposition of $|v)(w|$, where v, w are arbitrary vectors.
Solution.

$$|v)(w| = \frac{|v)(w|}{\|v\|\|w\|} \cdot \frac{|w)(w|\|v\|}{\|w\|}.$$

0.15 Rank one perturbations

Let us start with a physical example. Consider $l^2(\mathbb{Z})$ with the canonical basis e_n, $n \in \mathbb{Z}$. Consider the Hamiltonian

$$H_0 e_n = e_{n-1} + e_{n+1}, \quad \text{or} \quad (H_0 f)_n = f_{n+1} + f_{n-1},$$

perturbed by λV, where

$$V e_n = \delta_{0,n} e_0, \quad \text{or} \quad (Vf)_n = \delta_{0,n} f_0.$$

We would like to find the spectrum of $H = H_0 + \lambda V$.

Introduce the Fourier transformation $\mathcal{F} : l^2(\mathbb{Z}) \to L^2[-\pi, \pi]$

$$(\mathcal{F} e_n)(k) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}nk}.$$

Then

$$\mathcal{F} H_0 \mathcal{F}^{-1} f(k) = 2\cos k f(k), \quad \mathcal{F} V \mathcal{F}^{-1} = |v)(v|$$

$v(k) = \frac{1}{\sqrt{2\pi}}$. Thus $\sigma(H_0) = [-2, 2]$.

In the sequel we will consider an abstract version of this problem. We assume that H_0 is an operator of multiplication

$$Hf(x) = xf(x),$$

on $L^2[a, b]$ and $v \in L^2[a, b]$. Let

$$Hf = \beta f.$$

Then

$$xf(x) + v(x)\lambda \int \overline{v}(y) f(y) \mathrm{d}y = \beta f(x).$$

Hence

$$f(x) = \frac{\lambda v(x)}{\beta - x} \int \overline{v(y)} f(y) \mathrm{d}y.$$

$$1 = \lambda \int \frac{|v(x)|^2}{\beta - x} \mathrm{d}x.$$

Assume that v is continuous and nonzero on $]a, b[$. Then $\int \frac{|v(x)|^2}{\beta - x} \mathrm{d}x = \infty$ for $\beta \in]a, b[$. We have

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \int \frac{|v(x)|^2}{\beta - x} \mathrm{d}x = - \int \frac{|v(x)|^2}{(\beta - x)^2} \mathrm{d}x < 0, \tag{0.70}$$

$$\lim_{\beta \to \pm\infty} \int \frac{|v(x)|^2}{\beta - x} \mathrm{d}x = 0. \tag{0.71}$$

Set

$$A := \int \frac{|v(x)|^2}{a - x} \mathrm{d}x, \qquad B := \int \frac{|v(x)|^2}{b - x} \mathrm{d}x.$$

Hence on $]-\infty, a[$ we have exactly one eigenvalue for $\lambda \in]-\infty, A^{-1}[$ and on $]b, \infty[$ for $\lambda \in]B^{-1}, \infty[$. We have

$$\lim_{\lambda \to \pm\infty} \frac{\beta(\lambda)}{\lambda} = 1.$$

The eigenvector is

$$\Psi_\lambda(x) = \Big(\int \frac{|v(x)|^2}{(\beta - x)^2} \mathrm{d}x \Big)^{-1} \frac{v(x)}{(\beta - x)}.$$

Let us compute the resolvent:

$$(z - H)^{-1} = (z - H_0)^{-1} + \Big(\lambda^{-1} - (v|(z - H_0)^{-1} v) \Big)^{-1} (z - H_0)^{-1} |v)(v| (z - H_0)^{-1}$$

Hence, by computing the residue of the resolvent at β, we get

$$1\!\!1_{\{\beta\}}(H) = (\beta - H_0)^{-1} |v)(v| (\beta - H_0)^{-1} \frac{1}{(v|(\beta - H_0)^{-2} v)}.$$

0.16 Resonances

For $a \in \mathbb{R}$, using z as a real variable, let us first define the distribution on $\mathbb{R}$

$$\frac{1}{(z - a + \mathrm{i}0)} := \lim_{\epsilon \searrow 0} \frac{1}{(z - a + \mathrm{i}\epsilon)}. \tag{0.72}$$

Note that it is a tempered distribution and we can compute its Fourier transform:

$$\frac{1}{2\pi \mathrm{i}} \int \frac{\mathrm{e}^{-\mathrm{i}tz}}{z - a + \mathrm{i}0} \mathrm{d}z = -\mathrm{e}^{-\mathrm{i}ta} \theta(t). \tag{0.73}$$

Indeed,

$$\int \theta(t)\mathrm{e}^{-\mathrm{i}ta}\mathrm{e}^{\mathrm{i}tz}\mathrm{d}t = \lim_{\epsilon\searrow 0}\int_0^\infty \mathrm{e}^{\mathrm{i}t(z-a+\mathrm{i}\epsilon)}\mathrm{d}t \tag{0.74}$$

$$= \lim_{\epsilon\searrow 0}\frac{\mathrm{i}}{z-a+\mathrm{i}\epsilon} =\frac{\mathrm{i}}{z-a+\mathrm{i}0}. \tag{0.75}$$

Let H be a self-adjoint operator. Clearly,

$$\mathbb{C}\backslash\mathrm{sp}(H)\ni z\mapsto (z-H)^{-1}$$

is an analytic function that has poles at points of the discrete spectrum and the residues are the corresponding spectral projections. We cannot extend this function to a larger domain. However, sometimes we can extend

$$z\mapsto (\Phi|(z-H)^{-1}\Psi) \tag{0.76}$$

for some vectors Ψ. The additional domain arising from this extension is sometimes called the "non-physical sheet of the complex plane".

Suppose that and $\mathcal{D}$ a distinguished subspace of $\mathcal{H}$. Suppose that for all $\Psi\in\mathcal{D}$ (0.76) can be extended to some common region Ξ We say that $E\in\Xi$ is a *resonance* if (0.76) has a singularity at E.

Note that for $\epsilon>0$

$$\int_0^\infty \mathrm{e}^{\mathrm{i}t(z-H+\mathrm{i}\epsilon)}\mathrm{d}t = \frac{\mathrm{i}}{(z-H+\mathrm{i}\epsilon)}. \tag{0.77}$$

The limit of (0.77) for $\epsilon\searrow 0$ does not exist in terms of operators, but for appropriate Φ,Ψ there may exist the limit of matrix elements:

$$\Big(\Phi|\int_0^\infty \mathrm{e}^{\mathrm{i}t(z-H)}\mathrm{d}t\Psi\Big) = \lim_{\epsilon\searrow 0}\Big(\Phi|\int_0^\infty \mathrm{e}^{\mathrm{i}t(z-H+\mathrm{i}\epsilon)}\mathrm{d}t\Psi\Big) \tag{0.78}$$

$$= \lim_{\epsilon\searrow 0}\Big(\Phi|\frac{\mathrm{i}}{(z-H+\mathrm{i}\epsilon)}\Psi\Big) =: \Big(\Phi|\frac{\mathrm{i}}{(z-H+\mathrm{i}0)}\Psi\Big). \tag{0.79}$$

Therefore, (omitting Φ,Ψ) for $t>0$, applying the inverse Fourier transformation

$$\mathrm{e}^{-\mathrm{i}tH} = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^\infty (z-H+\mathrm{i}0)^{-1}\mathrm{e}^{-\mathrm{i}tz}\mathrm{d}z.$$

By deforming the contour, pushing it down and picking up the residue at E, we obtain for $\Phi,\Psi\in\mathcal{D}$

$$(\Phi|\mathrm{e}^{-\mathrm{i}tH}\Psi) = \frac{1}{2\pi\mathrm{i}}\int_\gamma (\Phi|(z-H)^{-1}\Psi)\mathrm{e}^{-\mathrm{i}tz} + (\Phi|R\Psi)\mathrm{e}^{-\mathrm{i}tE}.$$

0.17 The Friedrichs Hamiltonian

On the Hilbert space $L^2]a,b[\oplus\mathbb{C}$ consider

$$G = \begin{bmatrix} H_0 & |v) \\ (v| & \varepsilon \end{bmatrix}$$

$$H_0 f(k) = k f(k).$$

Let us look for an eigenvector of the form (f, g).

$$kf(k) + v(k)g = zf(k) \tag{0.80}$$

$$\int v(k)f(k) + \epsilon g = zg. \tag{0.81}$$

If $g \neq 0$, this yields

$$z = \varepsilon + \int_a^b \frac{|v(k)|^2}{z-k} \mathrm{d}k, \quad f = \frac{v(k)g}{z-k}. \tag{0.82}$$

If

$$\varepsilon \leq (v|H_0^{-1}v) = \int_a^b \frac{|v(k)|^2}{k-a} \mathrm{d}k, \tag{0.83}$$

then there exists a unique solution less than a of (0.82). Let us call this solution E. Then we have an eigenvector with the eigenprojection

$$\left(1 + \int_a^b \frac{|v(k)|^2}{(E-k)^2} \mathrm{d}k\right)^{-1} \begin{bmatrix} \frac{v(k)}{E-k} \\ 1 \end{bmatrix} \begin{bmatrix} \overline{\frac{v(k)}{E-k}} & 1 \end{bmatrix}. \tag{0.84}$$

(0.84) can be also obtained from the resolvent:

$$\begin{aligned}
&(z-G)^{-1} \\
=& \begin{bmatrix} 1\!\!1 & \frac{v(k)}{(z-k)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{z-k} & 0 \\ 0 & \left(z - \varepsilon - \int \frac{|v(k)|^2 \mathrm{d}k}{(z-k)}\right)^{-1} \end{bmatrix} \begin{bmatrix} 1\!\!1 & 0 \\ \frac{v(k)}{(z-k)} & 1 \end{bmatrix} \\
=& \begin{bmatrix} \frac{1}{z-k} & 0 \\ 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} \frac{v(k)}{(z-k)} \\ 1 \end{bmatrix} \left(z - \varepsilon - \int \frac{|v(k)|^2 \mathrm{d}k}{(z-k)}\right)^{-1} \begin{bmatrix} \frac{v(k)}{(z-k)} & 1 \end{bmatrix}
\end{aligned}$$

Suppose now that (0.83) is not satisfied and $v(k) \neq 0$ for all $k \geq 0$. The vector $\begin{bmatrix} \frac{v(k)}{E-k} \\ 1 \end{bmatrix}$ is formally an eigenvector of G, but it is nonnormalizable, because of the singularity $\frac{v(E)}{(E-k)}$ Therefore, there is no eigenvalue. However, there may be a resonance. If we replace v with λv, where v is small, then

$$E_\lambda \simeq \varepsilon + \lambda^2 \mathcal{P} \int_a^b \frac{|v(k)|^2}{(E-k)} \mathrm{d}k - \mathrm{i}\pi\lambda^2 |v(E)|^2. \tag{0.85}$$

This is the Fermi Golden Rule I.

This (at least on the heuristic level) implies

$$(\Phi_0|\mathrm{e}^{-\mathrm{i}tH}\Phi_0) \approx \mathrm{e}^{-\mathrm{i}Et},$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\left(\Phi_0|\mathrm{e}^{-\mathrm{i}tH}\Phi_0\right)\right|^2 = 2\pi|v(E)|\left|\left(\Phi_0|\mathrm{e}^{-\mathrm{i}tH}\Phi_0\right)\right|^2,$$

which is called the Fermi Golden Rule II.

Here is an alternative, differential derivation of this rule. Set

$$\begin{bmatrix}\Psi_t\\ \Phi_t\end{bmatrix} := \mathrm{e}^{-\mathrm{i}tG}\begin{bmatrix}0\\ \Phi_0\end{bmatrix}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(k) = -\mathrm{i}k\Psi_t(k) - \mathrm{i}\lambda v(k)\Phi_t, \tag{0.86}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t = -\mathrm{i}\lambda\int_a^b \overline{v(k)}\Psi_t(k)\mathrm{d}k - \mathrm{i}\varepsilon\Phi_t. \tag{0.87}$$

Set

$$\tilde{\Psi}_t = \mathrm{e}^{\mathrm{i}tk}\Psi_t, \quad \tilde{\Phi}_t := \mathrm{e}^{\mathrm{i}t\varepsilon}\Phi_t.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\Psi}_t(k) = -\mathrm{i}\lambda v(k)\mathrm{e}^{\mathrm{i}t(k-\varepsilon)}\tilde{\Phi}_t, \tag{0.88}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\Phi}_t = -\mathrm{i}\lambda\int_a^b \overline{v(k)}\mathrm{e}^{\mathrm{i}(\varepsilon-k)t}\tilde{\Psi}_t(k)\mathrm{d}k. \tag{0.89}$$

Using the first approximation $\tilde{\Psi}_t(k) = \tilde{\Psi}_0(k) = 0$, $\Phi_t = \Phi_0 = 1$ we obtain after one iteration

$$\tilde{\Psi}_t(k) = -\lambda v(k)\frac{\mathrm{e}^{\mathrm{i}t(k-\varepsilon)} - 1}{k-\varepsilon}\tilde{\Phi}_0.$$

Thus at $t = 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\Phi}_t = \mathrm{i}\lambda^2\int_a^b |v(k)|^2\frac{(1-\mathrm{e}^{\mathrm{i}t(\varepsilon-k)})}{(k-\varepsilon)}\mathrm{d}k\tilde{\Phi}_t \tag{0.90}$$

$$= \mathrm{i}\lambda^2\int_a^b |v(\varepsilon - t^{-1}y)|^2\frac{(\mathrm{e}^{\mathrm{i}y}-1)}{y}\mathrm{d}y\tilde{\Phi}_t \tag{0.91}$$

$$\approx -\pi\lambda^2|v(\varepsilon)|^2\tilde{\Phi}_t. \tag{0.92}$$

where we used

$$\int\frac{\mathrm{e}^{\mathrm{i}y}-1}{y}\mathrm{d}y = \mathrm{i}\pi.$$

Can we have exact exponential decay? Assume $a = -\infty$, $b = \infty$ and $v(k) = \lambda$. Note that formally

$$\mathcal{P}\int\frac{1}{E-k}\mathrm{d}k = -\mathrm{i}\pi|\lambda|^2. \tag{0.93}$$

We have

$$
\begin{aligned}
&(z-G)^{-1}\\
&=\begin{bmatrix}\frac{1}{z-k} & 0\\ 0 & 0\end{bmatrix}\\
&\quad+\begin{bmatrix}\frac{\lambda}{(z-k)}\\ 1\end{bmatrix}\left(z-\varepsilon+\mathrm{i}\pi|\lambda|^2\right)^{-1}\begin{bmatrix}\frac{\lambda}{(z-k)} & 1\end{bmatrix}
\end{aligned}
$$

Thus

$$
\left(\begin{bmatrix}0\\1\end{bmatrix}|\mathrm{e}^{-\mathrm{i}tG}\begin{bmatrix}0\\1\end{bmatrix}\right)=\mathrm{e}^{-\mathrm{i}t\epsilon-\pi|\lambda|^2t}. \tag{0.94}
$$

0.18 Feshbach-Schur formula

Suppose that the space is $\mathcal{V}=\mathcal{V}_{\mathrm{S}}\oplus\mathcal{V}_{\mathrm{R}}$. (S stands for a „small system" and R for a „reservoir"). An operator on $\mathcal{V}$ can be written as

$$
H=\begin{bmatrix}H_{\mathrm{SS}} & H_{\mathrm{SR}}\\ H_{\mathrm{RS}} & H_{\mathrm{RR}}\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}.
$$

We also introduce the imbeddings J_{R} and J_{S} of $\mathcal{V}_{\mathrm{R}}$, resp. $\mathcal{V}_{\mathrm{S}}$ into $\mathcal{V}$.
Problem. Write H as

$$
H=\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}\begin{bmatrix}1 & 0\\ x & 1\end{bmatrix}.
$$

Solution

$$
H=\begin{bmatrix}1 & bd^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}a-bd^{-1}c & 0\\ 0 & d\end{bmatrix}\begin{bmatrix}1 & 0\\ d^{-1}c & 1\end{bmatrix}. \tag{0.95}
$$

Problem. Compute $J_{\mathrm{S}}^*H^{-1}J_{\mathrm{S}}$.

Solution Note that $\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1 & -y\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}^{-1}=\begin{bmatrix}\alpha^{-1} & 0\\ 0 & \beta^{-1}\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ x & 1\end{bmatrix}^{-1}=\begin{bmatrix}1 & 0\\ -x & 1\end{bmatrix}$.
Therefore, application of (0.95) yields the inverse of H:

$$
H^{-1}=\begin{bmatrix}1 & 0\\ -d^{-1}c & 1\end{bmatrix}\begin{bmatrix}(a-bd^{-1}c)^{-1} & 0\\ 0 & d^{-1}\end{bmatrix}\begin{bmatrix}1 & -bd^{-1}\\ 0 & 1\end{bmatrix}
$$

Now $J_{\mathrm{S}}^*=\begin{bmatrix}1 & 0\end{bmatrix}$ and $J_{\mathrm{S}}=\begin{bmatrix}1\\0\end{bmatrix}$. Hence

$$
J_{\mathrm{S}}^*H^{-1}J_{\mathrm{S}}=(a-bd^{-1}c)^{-1}.
$$

Applied to $z-H$ instead of H it is sometimes called the *Feshbach-Schur formula*

$$
J_{\mathrm{S}}^*(z-H)^{-1}J_{\mathrm{S}}=\left(z-H_{\mathrm{SS}}-H_{\mathrm{SR}}(z-H_{\mathrm{RR}})^{-1}H_{\mathrm{RS}}\right)^{-1}. \tag{0.96}
$$

0.19 Perturbation theory

Assume now that H_0 and V are self-adjoint operators on $\mathcal{V}$ and $H_\lambda := H_0 + \lambda V$. Set $R_\lambda(z) = (z - H_\lambda)^{-1}$.

Let $\mathcal{V}_{\rm S}$ be the spectral subspace of H_0 onto the eigenvalue E_0 and $\mathcal{V}_{\rm R}$ its orthogonal complement. We can write the Feshbach-Schur formula as

$$J_{\rm S}^* R_\lambda(E) J_{\rm S} = \Big(E - E_0 - \lambda V_{\rm SS} - \lambda^2 V_{\rm SR}(z - H_{0\rm RR} - \lambda V_{\rm RR})^{-1} V_{\rm RS}\Big)^{-1}. \tag{0.97}$$

Therefore,

$$\{E \in \mathbb{R} \ : \quad E - E_0 - \lambda V_{\rm SS} - \lambda^2 V_{\rm SR}(z - H_{0\rm RR} - \lambda V_{\rm RR})^{-1} V_{\rm RS} \quad \text{is non-invertible}\} \tag{0.98}$$

is contained in the spectrum of H.

Problem. Assume that $\dim \mathcal{V}_{\rm S}$ is finite and E_0 is a discrete eigenvalue of H_0. Find an equation for eigenvalues of H_λ, which for small λ is close to E_0

Solution We can expect that these eigenvalues coincide with (0.98). A finite matrix is non-invertible iff its determinant is zero. Therefore, the condition for these eigenvalues is

$$\det\Big(E - E_0 - \lambda V_{\rm SS} - \lambda^2 V_{\rm SR}(E - H_{0\rm RR} - \lambda V_{\rm RR})^{-1} V_{\rm RS}\Big) = 0. \tag{0.99}$$

Let us remark that we obtain a polynomial in E of degree $\dim \mathcal{V}_{\rm S}$. In general it has $\dim \mathcal{V}_{\rm S}$ solutions $\lambda \to E_j(\lambda)$

%%%%%%%%%%%%%%%%%%%%%%%

Let $\dim \mathcal{V}_{\rm S} = 1$, so that $J_{\rm S} = |\Phi_0)$. Then we expect that close to E_0 there is only one eigenvalue of H_λ. We introduce

$$F_\lambda(E) := E_0 + (\Phi_0|V\Phi_0) + \lambda^2(\Phi_0|V(E - H_{0\rm RR} - \lambda V_{\rm RR})^{-1} V\Phi_0).$$

The eigenvalue E_λ is the solution of

$$E_\lambda = F_\lambda(E_\lambda). \tag{0.100}$$

We can try to solve it by iterations:

$$E_\lambda^{(j)} = F_\lambda^j(E_0). \tag{0.101}$$

The first iteration is

$$E_\lambda^{(1)} = F_\lambda(E_0) = E_0 + (\Phi_0|V\Phi_0) + \lambda^2(\Phi_0|V(E_0 - H_{0\rm RR} - \lambda V_{\rm RR})^{-1} V\Phi_0) \tag{0.102}$$

$$\simeq E_0 + (\Phi_0|V\Phi_0) + \lambda^2(\Phi_0|V(E_0 - H_0)_{\rm RR}^{-1} V\Phi_0) + O(\lambda^3). \tag{0.103}$$

This method of finding eigenvalues is called the *Brillouin-Wigner perturbation theory.*

There is an alternative method, called the *Rayleigh-Schrödinger perturbation theory.* Recall that we have $H = H_0 + \lambda V$, $H_0\Psi_0 = E_0\Psi_0$ and E_0 is a nondegenerate eigenvalue.

We make an ansatz

$$\Psi_\lambda = \sum_{n=0}^{\infty} \lambda^n \Psi_n, \qquad E_\lambda = \sum_{n=0}^{\infty} \lambda^n E_n. \tag{0.104}$$

We assume in addition that

$$(\Psi_0|\Psi_n) = 0, \qquad n = 1, 2, \dots. \tag{0.105}$$

We insert (0.104) into

$$(H_0 + \lambda V)\Psi_\lambda = E_\lambda \Psi_\lambda. \tag{0.106}$$

We obtain a formal series in the powers of λ. At λ^n we have

$$H_0\Psi_n + V\Psi_{n-1} = \sum_{j=0}^{n} E_j \Psi_{n-j}. \tag{0.107}$$

We take the scalar product with Ψ_0:

$$(\Psi_0|H_0\Psi_n) + (\Psi_0|V\Psi_{n-1}) = \sum_{j=0}^{n} E_j(\Psi_0|\Psi_{n-j}). \tag{0.108}$$

With help of (0.105) we simplify (0.108) obtaining

$$(\Psi_0|V\Psi_{n-1}) = E_n. \tag{0.109}$$

(0.106) can be rewritten

$$(E_0 - H_0)\Psi_n = V\Psi_{n-1} - \sum_{j=0}^{n-1} E_j \Psi_{n-j}. \tag{0.110}$$

We multiply (0.110) by $P_{\mathrm{R}} := \mathbb{1} - |\Psi_0)(\Psi_0|$, which does not affect the lhs. Setting

$$R_0' := (E_0 - H_0)^{-1} P_{\mathrm{R}},$$

we obtain

$$\Psi_n = R_0'\Big(V\Psi_{n-1} - \sum_{j=0}^{n-1} E_j \Psi_{n-j}\Big). \tag{0.111}$$

Here are the first iterations:

$$E_1 = (\Psi_0|V\Psi_0), \tag{0.112}$$

$$\Psi_1 = R_0' V\Psi_0, \tag{0.113}$$

$$E_2 = (\Psi_0|VR_0'V\Psi_0), \tag{0.114}$$

$$\Psi_2 = R_0'VR_0'V\Psi_0 - (\Psi_0|V\Psi_0)R_0'V\Psi_0. \tag{0.115}$$