Operators on Hilbert spaces

Jan Dereziński
Department of Mathematical Methods in Physics
Faculty of Physics
University of Warsaw
ul. Pasteura 5, 02-093, Warszawa, Poland

January 19, 2026

Contents

Chapter 1

Introduction

One can argue that in practical applications most operators are unbounded. Unfortunately, *unbounded operators* is a relatively technical and complicated subject, and for that reason this is a topic avoided in many presentations of the theory of operators, or postponed to its later parts. To my knowledge, in most mathematics departments of the world it does not belong to the standard curriculum, except maybe for some rudimentary elements. Most courses of functional analysis limit themselves to bounded operators, which are much cleaner and easier to discuss.

Of course, in physics departments unbounded operators do not belong to the standard curriculum either. However, implicitly, they appear very often in physics courses.

These lecture notes grew out of a course *"Mathematics of quantum theory"* given at Faculty of Physics, University of Warsaw. The aim of the course was not only to give a general theory of unbounded operators, but also to illustrate it with many interesting examples. These examples often allow us to compute exactly various quantities of interest. Often, they are related to special functions, group symmetries, etc.

Hilbert spaces constitute the most useful class of topological vector spaces, and also the most regular one. Therefore, the setting of most of this text is that of Hilbert spaces. Only a small part of the material is presented in the more general setting of Banach spaces. In particular, we try to avoid speaking about duals of Banach spaces, Banach space adjoints, etc. This is motivated by our desire to reduce the amount of "abstract nonsense", which many students do not like, and those who do like, do not have time to study seriously applications.

Chapter 2

Banach spaces

2.1 Vector spaces

Let $\mathbb{K}$ denote the field $\mathbb{C}$ or $\mathbb{R}$.

If the vector space $\mathcal{X}$ over $\mathbb{K}$ is isomorphic to $\mathbb{K}^n$, we say that $\mathcal{X}$ is of a finite dimension and its dimension is n.

If $A \subset \mathcal{X}$, then $\mathrm{Span}A$ denotes the set of finite linear combinations of elements of A. Clearly, $\mathrm{Span}A$ is a subspace of $\mathcal{X}$.

Let $L(\mathcal{X}, \mathcal{Y})$ denote the set of linear transformations from $\mathcal{X}$ to $\mathcal{Y}$ and $L(\mathcal{X}) := L(\mathcal{X}, \mathcal{X})$. For $A \in L(\mathcal{X}, \mathcal{Y})$, $\mathrm{Ker}A$ denotes the kernel of A and $\mathrm{Ran}\,A$ the range of A. A is injective iff $\mathrm{Ker}A = \{0\}$.

If A is bijective, then $A^{-1} \in L(\mathcal{Y}, \mathcal{X})$.

2.2 Norms and seminorms

Definition 2.1 *Let $\mathcal{X}$ be a vector space over $\mathbb{K}$. $\mathcal{X} \ni x \mapsto \|x\| \in \mathbb{R}$ is called a seminorm iff*

$$1)\ \|x\| \geq 0$$

$$2)\ \|\lambda x\| = |\lambda|\|x\|,$$

$$3)\ \|x + y\| \leq \|x\| + \|y\|.$$

If in addition

$$4)\ \|x\| = 0 \iff x = 0,$$

then it is called a norm.

If $\mathcal{X}$ is a space with a seminorm, then $\mathcal{N} := \{x \in \mathcal{X}\ :\ \|x\| = 0\}$ is a linear subspace. Then on $\mathcal{X}/\mathcal{N}$ we define

$$\|x + \mathcal{N}\| := \|x\|,$$

which is a norm on $\mathcal{X}/\mathcal{N}$.

If $\|\cdot\|$ is a norm, then

$$\mathrm{d}(x,y) := \|x-y\|$$

defines a metric.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathcal{X}$. They are equivalent iff there exist $c_1, c_2 > 0$ such that

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1.$$

The equivalence of norms is an equivalence relation. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, then the corresponding metrics are equivalent.

Theorem 2.2 (1) *All norms on a finite dimensional vector space are equivalent.*

(2) *Finite dimensional vector spaces are complete.*

(3) *Every finite dimensional subspace of a normed space is closed.*

For $r > 0$, $(\mathcal{X})_r$ denotes the closed ball in $\mathcal{X}$ of radius r, that is $(\mathcal{X})_r := \{x \in \mathcal{X} \ : \ \|x\| \leq r\}$.

If $\mathcal{V} \subset \mathcal{X}$, then $\mathcal{V}^{\mathrm{cl}}$ will denote the closure of $\mathcal{V}$, $\mathcal{V}^o$ its interior.

2.3 Banach spaces

Definition 2.3 *$\mathcal{X}$ is a Banach space if it has a norm and is complete.*

Definition 2.4 *Let x_i, $i \in I$, be a family of vectors in a normed space. Then*

$$\sum_{i\in I} x_i = x \iff \underset{\epsilon>0}{\forall} \ \underset{I_0\in 2^I_{\mathrm{fin}}}{\exists} \ \underset{I_0\subset I_1\in 2^I_{\mathrm{fin}}}{\forall} \left\|\sum_{i\in I_1} x_i - x\right\| < \epsilon.$$

We say then that $\sum\limits_{i\in I} x_i$ is convergent to x.

Clearly,

$$\left\|\sum_{i\in I} x_i\right\| \leq \sum_{i\in I} \|x_i\|.$$

If $c_n \in \mathbb{R}$ and $\sum\limits_{i\in I} c_i$ is convergent, then only a countable number of terms $c_n \neq 0$.

Theorem 2.5 *1) Let $\mathcal{X}$ be a Banach space, $x_i \in \mathcal{X}$ and*

$$\sum_{i\in I} \|x_i\| < \infty.$$

Then there exists

$$\sum_{i\in I} x_i.$$

2) Conversely, if $\mathcal{X}$ is a normed space such that

$$\sum_{n=1}^{\infty} \|x_n\| < \infty$$

implies the convergence of

$$\sum_{n=1}^{\infty} x_n,$$

then $\mathcal{X}$ is a Banach space.

Proof. 1) Since only a countable number of terms x_n is different from zero, the nonzero terms can be treated as a usual sequence indexed by integers. Let

$$y_N := \sum_{n=1}^{N} x_n.$$

For $n \leq m$

$$\|y_n - y_m\| = \left\| \sum_{i=n+1}^{m} x_i \right\| \leq \sum_{i=n+1}^{m} \|x_i\| \to_{n,m\to\infty} 0.$$

Hence (y_N) is Cauchy and therefore convergent.

2) Let (x_n) be a Cauchy sequence in $\mathcal{X}$. By induction we can find a subsequence (x_{n_j}) of the sequence (x_n) such that

$$\|x_{n_{j+1}} - x_{n_j}\| < 2^{-n}.$$

By assumption,

$$\sum_{j=1}^{\infty} (x_{n_{j+1}} - x_{n_j})$$

is convergent. The mth partial sum equals $x_{n_{m+1}} - x_{n_1}$. Hence x_{n_j} is convergent to some $x \in \mathcal{X}$. Since (x_n) was Cauchy, it also has to be convergent to x. □

Theorem 2.6 *Let $\mathcal{X}_0$ be a normed space. Then there exists a unique up to an isometry Banach space $\mathcal{X}$, such that $\mathcal{X}_0 \subset \mathcal{X}$ and $\mathcal{X}_0$ is dense in $\mathcal{X}$. $\mathcal{X}$ is called the completion of $\mathcal{X}_0$ and is denoted $\mathcal{X}_0^{\mathrm{cpl}}$.*

2.4 Bounded operators

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. An operator $A : \mathcal{X} \to \mathcal{Y}$ is called bounded iff there exists a number C such that

$$\|Ax\| \leq C\|x\|, \quad x \in \mathcal{X}. \tag{2.1}$$

We define the norm of A:

$$\|A\| := \inf\{C \ : \ \|Ax\| \leq C\|x\|, \ x \in \mathcal{X}\},$$

or

$$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Ax\|.$$

The set of operators such that $\|A\| < \infty$ is denoted $B(\mathcal{X}, \mathcal{Y})$. We write $B(\mathcal{X}) := B(\mathcal{X}, \mathcal{X})$.

Theorem 2.7 *The following conditions are equivalent:*
1. A is bounded;
2. A is uniformly continuous;
3. A is continuous;
4. A is continuous in one point.

Proof. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ is obvious. Clearly, 4. holds $\iff$ A is continuous at 0. Let us show that it implies the boundedness of A.

Suppose A is not bounded. Then there exists a sequence (x_n) such that $\|x_n\| = 1$ and

$$\|Ax_n\| \geq n.$$

Then

$$\lim_{n \to \infty} \frac{x_n}{\sqrt{n}} = 0, \quad \lim_{n \to \infty} \left\| A \frac{x_n}{\sqrt{n}} \right\| = \infty.$$

Thus A is not continuous at 0. $\square$

Example 2.8 *A linear operator from $\mathbb{C}^m$ to $\mathbb{C}^n$ can be defined by a matrix $[a_{ij}]$.*

(1) *If $\mathbb{C}^m$ is equipped with the norm $\|\cdot\|_1$ and $\mathbb{C}^n$ with the norm $\|\cdot\|_\infty$, then $\|A\| = \max\{|a_{ij}|\}$.*

(2) *If $\mathbb{C}^m$ is equipped with the norm $\|\cdot\|_\infty$ and $\mathbb{C}^n$ with the norm $\|\cdot\|_1$, then $\|A\| \leq \sum_{i,j} |a_{ij}|$.*

(3) *If $\mathbb{C}^m$ is equipped with the norm $\|\cdot\|_1$ and $\mathbb{C}^n$ with the norm $\|\cdot\|_1$, then $\|A\| = \max_j\{\sum_i |a_{ij}|\}$.*

(4) *If $\mathbb{C}^m$ is equipped with the norm $\|\cdot\|_\infty$ and $\mathbb{C}^n$ with the norm $\|\cdot\|_\infty$, then $\|A\| = \max_i\{\sum_j |a_{ij}|\}$.*

Proposition 2.9 *All linear operators on a finite dimensional space are bounded.*

Theorem 2.10 *If $\mathcal{Y}$ is a Banach space, then $B(\mathcal{X}, \mathcal{Y})$ is a Banach space. Besides, if $A \in B(\mathcal{X}, \mathcal{Y})$ and $B \in B(\mathcal{Y}, \mathcal{Z})$, then*

$$\|BA\| \leq \|B\|\|A\|.$$

Proof. Clearly, $B(\mathcal{X},\mathcal{Y})$ is a normed space. Let us show that it is complete. Let (A_n) be a Cauchy sequence in $B(\mathcal{X},\mathcal{Y})$. Then (A_nx) is a Cauchy sequence in $\mathcal{Y}$. Define

$$Ax := \lim_{n\to\infty} A_nx.$$

Obviously, A is linear.

Fix n. Clearly,

$$(A - A_n)x = \lim_{m\to\infty}(A_m - A_n)x.$$

Hence

$$\begin{aligned}&\|(A - A_n)x\|\\ &= \lim_{m\to\infty}\|(A_m - A_n)x\| \leq \|x\| \lim_{m\to\infty}\|(A_m - A_n)\|.\end{aligned}$$

Thus,

$$\|A - A_n\| \leq \lim_{m\to\infty}\|A_m - A_n\|.$$

Therefore, by the Cauchy condition,

$$\lim_{n\to\infty}\|A - A_n\| = 0.$$

Thus the sequence A_n is convergent to A. □

Theorem 2.11 *Let $\mathcal{X},\mathcal{Y}$ be Banach spaces and $\mathcal{X}_0$ a dense subspace of $\mathcal{X}$. Let $A_0 \in B(\mathcal{X}_0,\mathcal{Y})$. Then there exists a unique $A \in B(\mathcal{X},\mathcal{Y})$ such that $A\Big|_{\mathcal{X}_0} = A_0$. Moreover, $\|A\| = \|A_0\|$.*

Theorem 2.12 *Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces. Let $A : \mathcal{X} \to \mathcal{Y}$ be bounded, $\mathcal{X}_0$ dense in $\mathcal{X}$ and $\operatorname{Ran} A$ dense in $\mathcal{Y}$. Then $A\mathcal{X}_0$ is dense in $\mathcal{Y}$.*

Proof. Let $y \in \mathcal{Y}$ and $\epsilon > 0$. There exists $y_1 \in \operatorname{Ran} A$ such that $\|y - y_1\| < \epsilon/2$. Let $x_1 \in \mathcal{X}$ such that $Ax_1 = y_1$. Then there exists $x_0 \in \mathcal{X}_0$ such that $\|x - x_0\| < \|A\|^{-1}\epsilon/2$. Hence

$$\|y - Ax_0\| \leq \|y - y_0\| + \|A(x_1 - x_0)\| < \epsilon.$$

□

2.5 Continuous embedding

Let $\mathcal{Y},\mathcal{X}$ be Banach spaces. Suppose that $\mathcal{Y} \subset \mathcal{X}$. (We do not assume that the norms agree on $\mathcal{Y}$). We say that $\mathcal{Y}$ is continuously embedded in $\mathcal{X}$ iff the embedding is continuous. Equivalently, for some C,

$$\|y\|_{\mathcal{X}} \leq C\|y\|_{\mathcal{Y}}, \quad y \in \mathcal{Y}.$$

Proposition 2.13 *Let $\mathcal{Y},\mathcal{X}$ be Banach spaces with $\mathcal{Y}$ continuously embedded in $\mathcal{X}$. Let $\mathcal{V}$ be dense in $\mathcal{Y}$, and let $\mathcal{Y}$ be dense in $\mathcal{X}$. Then $\mathcal{V}$ is dense in $\mathcal{X}$.*

2.6 Direct sum of Banach spaces

If $\mathcal{X}, \mathcal{Y}$ are Banach spaces and π is an arbitrary norm in $\mathbb{R}^2$, then $\mathcal{X} \oplus \mathcal{Y}$ becomes a Banach space if we equip it with the norm

$$\|(x, y)\|_\pi = \pi(\|x\|, \|y\|).$$

All these norms in $\mathcal{X} \oplus \mathcal{Y}$ are equivalent and generate the product topology. Thus $(x_n, y_n) \to (x, y)$ is equivalent to $x_n \to x$, $y_n \to y$.

For instance, we can take

$$\|x, y\|_1 := \|x\| + \|y\|.$$

If $\mathcal{X}$, $\mathcal{Y}$ are Hilbert spaces, we will usually prefer

$$\|x, y\|_2 := \sqrt{\|x\|^2 + \|y\|^2}.$$

2.7 Vector valued functions

For continuous $]a, b[\ni t \mapsto v(t) \in \mathcal{X}$ we can define the Riemann integral. It has all the usual properties, for instance,

$$\left\| \int_a^b v(t)\mathrm{d}t \right\| \leq \int_a^b \|v(t)\|\mathrm{d}t,$$

if $A \in B(\mathcal{X}, \mathcal{Y})$, then

$$A \int_a^b v(t)\mathrm{d}t = \int_a^b Av(t)\mathrm{d}t.$$

Let $]a, b[\ni t \mapsto v(t) \in \mathcal{X}$. The (norm) derivative of $v(t)$ is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t_0) := \lim_{h \to 0} \frac{v(t_0 + h) - v(t_0)}{h}.$$

It has all the usual properties, for instance,

$$\frac{\mathrm{d}}{\mathrm{d}t} Av(t_0) := A \frac{\mathrm{d}}{\mathrm{d}t} v(t_0),$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^t v(s)\mathrm{d}s = v(t).$$

We assume now that $\mathbb{K} = \mathbb{C}$. Let Ω be an open subset of $\mathbb{C}$. We say that $\Omega \ni z \mapsto v(z) \in \mathcal{X}$ is analytic iff for any $z_0 \in \Omega$ there exists

$$\frac{\mathrm{d}}{\mathrm{d}z} v(z_0) := \lim_{h \to 0} \frac{v(z_0 + h) - v(z_0)}{h}.$$

Theorem 2.14 (1) *Let* $x_0, x_1, \dots \in \mathcal{X}$ *and* $r^{-1} := \limsup_{n\to\infty} \|x_n\|^{\frac{1}{n}}$. *Then*

$$v(z) := \sum_{n=0}^{\infty} x_n z^n, \ z \in \mathbb{C}$$

is absolutely uniformly convergent for $|z| < r_1 < r$ *and divergent for* $|z| > r$. *In* $B(0,r)$ *it is analytic*

(2) $\Omega \ni z \mapsto v(z) \in \mathcal{X}$ *is analytic iff around any* $z_0 \in \Omega$ *we can develop it into a power series. Its radius of convergence equals*

$$\left(\limsup_{n\to\infty} \left\| \frac{v^{(n)}(z_0)}{n!} \right\|^{\frac{1}{n}}\right)^{-1}.$$

(3) *If* v *is analytic on* Ω, *continuous on* Ω^{cl} *and* $z_0 \in \Omega$, *then*

$$v(z_0) = \frac{1}{2\pi\mathrm{i}} \int_{\partial\Omega} v(z)\mathrm{d}z.$$

Chapter 3

Partial operators on Banach spaces

3.1 Relations

One of the problems with unbounded operators is confusing terminology used in their theory. In particular, they are not true *operators*, which is usually used as one of synonyms of the word *function*–they are a special kind of *relations* sometimes called *partial operators*. Therefore, in order to be precise and clear, before starting to discuss unbounded operators, it is helpful to reexamine the concepts of a function and relation.

Let X, Y be sets. R is called a *relation* iff $R \subset Y \times X$. We will also say that R is a relation from X to Y. We will sometimes write $R : X \to Y$. Note that there is a problem with the order of X and Y. We chose the order $Y \times X$ to have a more transparent picture for the composition of relations. However, the usual order in the literature is $X \times Y$. To be consistent with the literature, we introduce also the *graph* of R:

$$\mathrm{Gr}\, R := \{(x, y) \in X \times Y \ : \ (y, x) \in R\}.$$

An example of a relation is the *identity*

$$\mathbb{1}_X := \{(x, x) \ : \ x \in X\} \subset X \times X.$$

Introduce the *projections*

$$Y \times X \ni (y, x) \mapsto \pi_Y(y, x) := y \in Y,$$

$$Y \times X \ni (y, x) \mapsto \pi_X(y, x) := x \in X.$$

and the *flip*

$$Y \times X \ni (y, x) \mapsto \tau(y, x) := (x, y) \in X \times Y.$$

The *domain* of R is defined as $\operatorname{Dom} R := \pi_X R$, its *range* is $\operatorname{Ran} R = \pi_Y R$, its *inverse* is $R^{-1} := \tau R \subset X \times Y$. If $S \subset Z \times Y$, then the *superposition* of S and R is defined as

$$S \circ R := \{(z, x) \in Z \times X \ : \ \exists_{y \in Y} \ (z, y) \in S, \ (y, x) \in R\}. \tag{3.1}$$

If $X_0 \subset X$, then the *restriction* of R to X_0 is defined as

$$R\Big|_{X_0} := R \cap Y \times X_0.$$

If, moreover, $Y_0 \subset Y$, then

$$R\Big|_{X_0 \to Y_0} := R \cap Y_0 \times X_0.$$

We say that a relation R is *injective*, if $\pi_X(R \cap \{y\} \times X)$ is one-element for any $y \in \operatorname{Ran} R$. We say that R is *surjective* if $\operatorname{Ran} R = Y$.

We say that a relation R is *coinjective*, if $\pi_Y(R \cap Y \times \{x\})$ is one-element for any $x \in \operatorname{Dom} R$. We say that R is *cosurjective* if $\operatorname{Dom} R = X$.

Proposition 3.1 *a) If R, S are coinjective, then so is $S \circ R$.*
b) If R, S are cosurjective, then is $S \circ R$.

In a basic course of set theory we learn that a coinjective cosurjective relation is called a *function*. One also introduces many synonyms of this word, such as a *transformation*, an *operator*, a *map*, etc.

The composition of transformations is a transformation. We say that a transformation R is *bijective* iff it is injective and surjective. The inverse of a transformation is a transformation iff it is bijective.

Proposition 3.2 *Let $R \subset Y \times X$ and $S \subset X \times Y$ be transformations such that $R \circ S = 1\!\!1_Y$ and $S \circ R = 1\!\!1_X$. Then S and R are bijections and $S = R^{-1}$.*

In what follows we will need a weaker concept than a function: A coinjective relation will be called a *partial function* (or a *partial transformation, operator*, etc).

In the sequel, if R is a partial function, instead of writing $(y, x) \in R$ we will write $y = R(x)$, or perhaps $(x, y) \in \operatorname{Gr} R$.

A superposition of partial transformations is a partial transformation. The inverse of a partial transformation is a partial transformation iff it is injective.

3.2 Linear partial operators

Let $\mathcal{X}, \mathcal{Y}$ be vector spaces. We say that $R : \mathcal{X} \to \mathcal{Y}$ is a *linear partial operator* if $\operatorname{Dom} R$ is a linear subspace of $\mathcal{X}$ and $R : \operatorname{Dom} R \to \mathcal{Y}$ is a linear operator in the usual sense.

Proposition 3.3 (1) *$\mathcal{V} \subset \mathcal{X} \oplus \mathcal{Y}$ is a graph of a certain linear partial operator iff $\mathcal{V}$ is a linear subspace and $(0, y) \in \mathcal{V}$ implies $y = 0$.*

(2) *A linear partial operator A is injective iff $(x, 0) \in \mathrm{Gr}\, A$ implies $x = 0$.*

From now on by an "operator" we will mean a "linear partial operator". To say that $A : \mathcal{X} \to \mathcal{Y}$ is a true operator we will write $\mathrm{Dom}\, A = \mathcal{X}$ or that it is *everywhere defined*. Note however that by writing $A \in L(\mathcal{X}, \mathcal{Y})$ or $A \in B(\mathcal{X}, \mathcal{Y})$ we will still imply that $\mathrm{Dom}\, A = \mathcal{X}$.

As before, for operators we will write Ax instead of $A(x)$ and AB instead of $A \circ B$. We define the *kernel* of an operator A:

$$\mathrm{Ker} A := \{x \in \mathrm{Dom}\, A \ : \ Ax = 0\}.$$

Suppose that A, B are two operators $\mathcal{X} \to \mathcal{Y}$. Then by $A + B$ we will mean the obvious operator with domain $\mathrm{Dom}\, A \cap \mathrm{Dom}\, B$.

3.3 Closed operators

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Recall that $\mathcal{X} \oplus \mathcal{Y}$ can viewed as a Banach space equipped eg. with a norm

$$\|(x, y)\|_1 := \|x\| + \|y\|.$$

Theorem 3.4 *Let $A : \mathcal{X} \to \mathcal{Y}$ be an operator. The following conditions are equivalent:*

(1) $\mathrm{Gr}\, A$ *is closed in $\mathcal{X} \oplus \mathcal{Y}$.*

(2) *If $x_n \to x$, $x_n \in \mathrm{Dom}\, A$ and $Ax_n \to y$, then $x \in \mathrm{Dom}\, A$ and $y = Ax$.*

(3) $\mathrm{Dom}\, A$ *with the norm*

$$\|x\|_A := \|x\| + \|Ax\|.$$

is a Banach space.

Proof. The equivalence of (1), (2) and (3) is obvious, if we note that

$$\mathrm{Dom}\, A \ni x \mapsto (x, Ax) \in \mathrm{Gr}\, A$$

is a bijection. □

Definition 3.5 *An operator satisfying the above conditions is called closed.*

Theorem 3.6 *If A is closed and injective, then so is A^{-1}.*

Proof. The flip $\tau : \mathcal{X} \oplus \mathcal{Y} \to \mathcal{Y} \oplus \mathcal{X}$ is continuous. □

Proposition 3.7 *If A is a closed operator, then* $\mathrm{Ker} A$ *is closed.*

3.4 Bounded operators as closed operators

For any operator A from $\mathcal{X}$ to $\mathcal{Y}$ we can define its norm

$$\|A\| := \sup_{\|x\|=1,\ x\in \mathrm{Dom}\, A} \|Ax\|. \tag{3.2}$$

We say that A is bounded if $\|A\| < \infty$. As already defined before, $B(\mathcal{X}, \mathcal{Y})$ denotes all bounded everywhere defined operators from $\mathcal{X}$ to $\mathcal{Y}$.

Proposition 3.8 *A bounded operator A is closed iff* $\mathrm{Dom}\, A$ *is closed.*

If $A : \mathcal{X} \to \mathcal{Y}$ is closed, then $A \in B(\mathrm{Dom}\, A, \mathcal{Y})$.
Let us quote without a proof a well known theorem:

Theorem 3.9 (Closed graph theorem) *Let $A : \mathcal{X} \to \mathcal{Y}$ be a closed operator with* $\mathrm{Dom}\, A = \mathcal{X}$. *Then A is bounded.*

Proposition 3.10 *Let ξ be a densely defined linear form. The following conditions are equivalent:*

(1) *ξ is closed.*

(2) *ξ is everywhere defined and bounded.*

(3) *ξ is everywhere defined and* $\mathrm{Ker}\xi$ *is closed.*

3.5 Closable operators

Theorem 3.11 *Let $A : \mathcal{X} \to \mathcal{Y}$ be an operator. The following conditions are equivalent:*

(1) *There exists a closed operator B such that $B \supset A$.*

(2) $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ *is the graph of an operator.*

(3) $(0, y) \in (\mathrm{Gr}\, A)^{\mathrm{cl}} \;\Rightarrow\; y = 0$.

(4) $(x_n) \subset \mathrm{Dom}\, A$, $x_n \to 0$, $Ax_n \to y$ *implies* $y = 0$.

Definition 3.12 *An operator A satisfying the conditions of Theorem 3.11 is called* closable. *If the conditions of Theorem 3.11 hold, then the operator whose graph equals* $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ *is denoted by* A^{cl} *and called the* closure *of A.*

Proof of Theorem 3.11 To show (2)⇒(1) it suffices to take as B the operator A^{cl}. Let us show (1)⇒(2). Let B be a closed operator such that $A \subset B$. Then $(\mathrm{Gr}\, A)^{\mathrm{cl}} \subset (\mathrm{Gr}\, B)^{\mathrm{cl}} = \mathrm{Gr}\, B$. But $(0, y) \in \mathrm{Gr}\, B \;\Rightarrow\; y = 0$, hence $(0, y) \in (\mathrm{Gr}\, A)^{\mathrm{cl}} \;\Rightarrow\; y = 0$. Thus $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ is the graph of an operator. □

As a by-product of the above proof, we obtain

Proposition 3.13 *If A is closable, B closed and $A \subset B$, then $A^{\mathrm{cl}} \subset B$.*

Proposition 3.14 *Let A be bounded. Then A is closable,* $\mathrm{Dom}\, A^{\mathrm{cl}} = (\mathrm{Dom}\, A)^{\mathrm{cl}}$ *and* $\|A^{\mathrm{cl}}\| = \|A\|$.

Proposition 3.15 *If A is a closable operator, then* $(\mathrm{Ker} A)^{\mathrm{cl}} \subset \mathrm{Ker} A^{\mathrm{cl}}$

Example 3.16 *Let $\mathcal{V}$ be a subspace in $\mathcal{X}$ and $x_0 \in \mathcal{X} \backslash \mathcal{V}$. Define the linear functional w such that* $\mathrm{Dom}\, w = \mathcal{V} + \mathbb{C}x_0$, $\mathrm{Ker} w = \mathcal{V}$ *and* $\langle w|x_0\rangle = 1$. *Then w is closable iff $x_0 \notin \mathcal{V}^{\mathrm{cl}}$. In particular, if $\mathcal{V}$ is dense, then w is nonclosable.*

3.6 Essential domains

Let A be a closed operator. We say that a linear subspace $\mathcal{D}$ is an *essential domain* of A iff $\mathcal{D}$ is dense in $\mathrm{Dom}\, A$ in the graph topology. In other words, $\mathcal{D}$ is an essential domain for A, if

$$\left(A\Big|_{\mathcal{D}}\right)^{\mathrm{cl}} = A.$$

Theorem 3.17 (1) *If $A \in B(\mathcal{X}, \mathcal{Y})$, then a linear subspace $\mathcal{D} \subset \mathcal{X}$ is an essential domain for A iff it is dense in $\mathcal{X}$ (in the usual topology).*

(2) *If A is closed, has a dense domain and $\mathcal{D}$ is its essential domain, then $\mathcal{D}$ is dense in $\mathcal{X}$.*

(2) follows from the following fact:

Proposition 3.18 *Let $\mathcal{V} \subset \mathcal{X}$ be Banach spaces with $\|x\|_{\mathcal{X}} \le \|x\|_{\mathcal{V}}$. Then a dense subspace in $\mathcal{V}$ is dense in $\mathcal{X}$.*

3.7 Perturbations of closed operators

Definition 3.19 *Let B, $A : \mathcal{X} \to \mathcal{Y}$. We say that B is* bounded relatively *to A iff* $\mathrm{Dom}\, A \subset \mathrm{Dom}\, B$ *and there exist constants a, b such that*

$$\|Bx\| \le a\|Ax\| + b\|x\|, \quad x \in \mathrm{Dom}\, A. \tag{3.3}$$

The infimum of a satisfying (3.3) is called the A-bound *of B. If* $\mathrm{Dom}\, A \not\subset \mathrm{Dom}\, B$ *the A-bound of B is set $+\infty$.*

In other words: the A-bound of B equals

$$a_1 := \inf_{c>0} \sup_{x \in \mathrm{Dom}\, A \backslash \{0\}} \frac{\|Bx\|}{\|Ax\| + c\|x\|}.$$

In particular, if B is bounded, then its A-bound equals 0.

If A is unbounded, then its A-bound equals 1.

In the case of Hilbert spaces it is more convenient to use the following condition to define the relative boundedness:

Theorem 3.20 *The A-bound of B equals*

$$a_1 = \inf_{c>0} \sup_{x \in \mathrm{Dom}\, A \setminus \{0\}} \left(\frac{\|Bx\|^2}{\|Ax\|^2 + c\|x\|^2} \right)^{1/2}. \tag{3.4}$$

Proof. For any $\epsilon > 0$ we have

$$\begin{aligned} \left(\|Ax\|^2 + c^2\|x\|^2\right)^{\frac{1}{2}} &\leq \|Ax\| + c\|x\| \\ &\leq \left((1+\epsilon^2)\|Ax\|^2 + c^2(1+\epsilon^{-2})\|x\|^2\right)^{\frac{1}{2}}. \end{aligned}$$

□

Theorem 3.21 *Let A be closed and let B be bounded relatively to A with the A-bound less than 1. Then $A+B$ with the domain* $\mathrm{Dom}\, A$ *is closed. All essential domains of A are essential domains of $A+B$.*

Proof. We know that

$$\|Bx\| \leq a\|Ax\| + b\|x\|$$

for some $a < 1$ and b. Hence

$$\|(A+B)x\| + \|x\| \leq (1+a)\|Ax\| + (1+b)\|x\|$$

and

$$(1-a)\|Ax\| + \|x\| \leq \|Ax\| - \|Bx\| + (1+b)\|x\| \leq \|(A+B)x\| + (1+b)\|x\|.$$

Hence the norms $\|Ax\| + \|x\|$ and $\|(A+B)x\| + \|x\|$ are equivalent on $\mathrm{Dom}\, A$.
□

In particular, every bounded operator with domain containing $\mathrm{Dom}\, A$ is bounded relatively to A.

Proposition 3.22 *Suppose that $\mathcal{X} = \mathcal{Y}$. Then we have the following seemingly different definition of the A-bound of B:*

$$a_1 := \inf_{\mu \in \mathbb{C}} \inf_{c>0} \sup_{x \in \mathrm{Dom}\, A \setminus \{0\}} \frac{\|Bx\|}{\|(A-\mu)x\| + c\|x\|}. \tag{3.5}$$

Proof. It is obvious that (3.5)$\leq$(3.4). To see the converse inequality, it suffices to note that

$$\|Ax\| + c\|x\| \leq \|(A-\mu)x\| + (|\mu| + c)\|x\|.$$

□

Theorem 3.23 *Suppose that A, C are two operators with the same domain* $\mathrm{Dom}\, A = \mathrm{Dom}\, C = \mathcal{D}$ *satisfying*

$$\|(A-C)x\| \leq a(\|Ax\| + \|Cx\|) + b\|x\|$$

for some $a < 1$. Then

(1) *A is closed on $\mathcal{D}$ iff C is closed on $\mathcal{D}$.*

(2) *$\mathcal{D}$ is an essential domain of A^{cl} iff it is an essential domain of C^{cl}.*

Proof. Define $B := C - A$ and $F(t) := A + tB$ with the domain $\mathcal{D}$. For $0 \leq t \leq 1$, we have

$$\begin{aligned} \|Bx\| &\leq a(\|Ax\| + \|Cx\|) + b\|x\| \\ &= a\left(\|(F(t) - tB)x\| + \|(F(t) + (1-t)B)x\|\right) + b\|x\| \\ &\leq 2a\|F(t)x\| + a\|Bx\| + b\|x\| \end{aligned}$$

Hence

$$\|Bx\| \leq \frac{2a}{1-a}\|F(t)x\| + \frac{b}{1-a}\|x\|.$$

Therefore, if $|s| < \frac{1-a}{2a}$ and $t, t+s \in [0,1]$, then $F(t+s)$ is closed iff $F(t)$ is closed. □

3.8 Invertible operators

Let $\mathcal{X}$, $\mathcal{Y}$ be vector spaces and $A \in L(\mathcal{X}, \mathcal{Y})$. We say that A is *invertible* if A is bijective. Then clearly $A^{-1} \in L(\mathcal{Y}, \mathcal{X})$.

If $\mathcal{X}, \mathcal{Y}$ are finite dimensional, then $A \in L(\mathcal{X}, \mathcal{Y})$ can be invertible only if $\mathcal{X}$ and $\mathcal{Y}$ are of the same dimension. We can thus assume that $\mathcal{X} \simeq \mathcal{Y} \simeq \mathbb{K}^n$ and assume that A is given by a square matrix $[A_{ij}]$. Then one of facts of basic linear algebra says that A is invertible iff $\det[A_{ij}] \neq 0$ iff $\mathrm{Ker} A = \{0\}$.

Suppose now that $\mathcal{X}, \mathcal{Y}$ be Banach spaces.

Theorem 3.24 *Let $A \in B(\mathcal{X}, \mathcal{Y})$. If A is invertible, then $A^{-1} \in B(\mathcal{Y}, \mathcal{X})$.*

Proof. $A \in B(\mathcal{X}, \mathcal{Y})$ implies that A is closed. Hence so is A^{-1}. Therefore, by Thm 3.9, A^{-1} is bounded. □

Let A be an operator from $\mathcal{X}$ to $\mathcal{Y}$.

Theorem 3.25 (Closed range theorem) *Let A be closed. Then the following conditions are equivalent:*

(1) *For some $c > 0$*

$$\|Ax\| \geq c\|x\|, \quad x \in \mathrm{Dom}\, A. \tag{3.6}$$

(2) *A is injective and* $\mathrm{Ran}\, A$ *is closed.*

Proof. (1)$\Rightarrow$(2): The injectivity is obvious. Let $y_n \in \operatorname{Ran} A$ and $y_n \to y$. Let $Ax_n = y_n$. Then x_n is a Cauchy sequence. Hence there exists $\lim_{n\to\infty} x_n := x$. But A is closed, hence $Ax = y$. Therefore, $\operatorname{Ran} A$ is closed.

(1)$\Leftarrow$(2): By Thm 3.9, A^{-1} is a bounded operator from $\operatorname{Ran} A$ to $\mathcal{X}$. □

Proposition 3.26 *Let A be closable and suppose that for some $c > 0$ (3.6) holds. Then (3.6) holds for A^{cl} as well.*

Definition 3.27 *We say that an operator A is* invertible *(or* boundedly invertible*) iff $A^{-1} \in B(\mathcal{Y}, \mathcal{X})$.*

Note that we do not demand that A be densely defined. Note also that Definition 3.27 is consistent with the definition of invertibilty for bounded operators.

Theorem 3.28 *Let A be an operator. The following conditions are equivalent:*

(1) *A is invertible.*

(2) *A is closed, injective and $\operatorname{Ran} A = \mathcal{Y}$.*

(3) *A is closable, for some $c > 0$, $\|Ax\| \geq c\|x\|$ and $\operatorname{Ran} A = \mathcal{Y}$.*

(4) *A is closed, for some $c > 0$, $\|Ax\| \geq c\|x\|$ and $\operatorname{Ran} A$ is dense in $\mathcal{Y}$.*

Moreover, if these conditions are true then

$$\|A^{-1}\| = \Big(\max\{c \;:\; \|Ax\| \geq c\|x\|\}\Big)^{-1}. \tag{3.7}$$

The following criterion for the invertibility is obvious:

Proposition 3.29 *Let $C \in B(\mathcal{Y}, \mathcal{X})$ be such that $\operatorname{Ran} C \subset \operatorname{Dom} A$ and*

$$AC = \mathbb{1}, \quad CA = \mathbb{1}\Big|_{\operatorname{Dom} A}.$$

Then A is invertible and $C = A^{-1}$.

Theorem 3.30 *Let A be invertible and $\operatorname{Dom} B \supset \operatorname{Dom} A$.*

(1) *B has the A-bound $\leq \|BA^{-1}\|$.*

(2) *If $\|BA^{-1}\| < 1$, then $A + B$ with the domain $\operatorname{Dom} A$ is closed, invertible and*

$$(A + B)^{-1} = \sum_{j=0}^{\infty} (-1)^j A^{-1}(BA^{-1})^j.$$

(3) *$\|(A + B)^{-1}\| \leq \|A^{-1}\|(1 - \|BA^{-1}\|)^{-1}$.*

(4) *$\|A^{-1} - (A + B)^{-1}\| \leq \|A^{-1}BA^{-1}\|(1 - \|BA^{-1}\|)^{-1}$.*

Proof. By the estimate

$$\|Bx\| \leq \|BA^{-1}\|\|Ax\|, \quad x \in \operatorname{Dom} A,$$

we see that B has the A-bound $\leq \|BA^{-1}\|$. This proves (1).

Assume now that $\|BA^{-1}\| < 1$. Let

$$C_n := \sum_{j=0}^{n} (-1)^j A^{-1} (BA^{-1})^j.$$

Then $\lim_{n\to\infty} C_n =: C$ exists.

Let $y \in \mathcal{Y}$. Clearly, $\lim_{n\to\infty} C_n y = Cy$.

$$(A+B)C_n y = y + (-1)^n (BA^{-1})^{n+1} y \to y.$$

But $A+B$ is closed, hence $Cy \in \operatorname{Dom}(A+B)$ and $(A+B)Cy = y$.

Let $x \in \operatorname{Dom} A$. Then

$$C_n(A+B)x = x + (-1)^n A^{-1}(BA^{-1})^n Bx \to x.$$

Hence $C(A+B)x = x$.

By Prop. 3.29, $A+B$ is invertible and $C = (A+B)^{-1}$, which proves (2). □

As a corollary of Thm 3.30 we note that invertible elements form an open subset of $B(\mathcal{X}, \mathcal{Y})$ on which the inverse is a continuous function.

Theorem 3.31 *Let A and C be invertible and $\operatorname{Dom} C \supset \operatorname{Dom} A$. Then*

$$C^{-1} - A^{-1} = C^{-1}(A - C)A^{-1}.$$

3.9 Product of operators

Let B be an operator from $\mathcal{X}$ to $\mathcal{Y}$ and A an operator from $\mathcal{Y}$ to $\mathcal{Z}$. Then we define its product as an operator from $\mathcal{X}$ to $\mathcal{Z}$ with the domain

$$\operatorname{Dom} AB := \{x \in \operatorname{Dom} B \ : \ Bx \in \operatorname{Dom} A\},$$

and for $x \in \operatorname{Dom} AB$, $ABx := A(Bx)$. (Note that this is a special case of (3.1)).

Proposition 3.32 1. *Let A be closed and S bounded. Then AS is closed.*

2. *Suppose in addition that S is invertible. Let a subspace $\mathcal{D} \subset \operatorname{Dom} A$ be dense in the norm $\|\cdot\|_A$. Then $S^{-1}\mathcal{D}$ is dense in $\operatorname{Dom} AS$ in the norm $\|\cdot\|_{AS}$.*

Proof. (1): Let $(u_n) \subset \operatorname{Dom} AS$ and $ASu_n \to v$, $u_n \to u$. Set $w_n := Su_n$. Then $(w_n) \subset \operatorname{Dom} A$, $Aw_n \to v$, $w_n \to Su$. Hence, $Su \in \operatorname{Dom} A$ and $ASu = \lim_{n\to\infty} Aw_n = v$. Therefore, AS is closed.

(2): Let $u \in \operatorname{Dom} AS$. Then $Su \in \operatorname{Dom} A$. Hence there exists $(v_n) \subset \mathcal{D}$ with $v_n \to Su$ and $Av_n \to ASu$. Set $u_n := S^{-1}v_n \in S^{-1}\mathcal{D}$. Then $ASu_n \to ASu$ and $u_n \to u$. Hence $u_n \to u$ in $\|\cdot\|_{AS}$. □

Proposition 3.33 *1. Let A be closed and T be invertible. Then TA is closed.*

2. Suppose in addition that T is bounded. Let a subspace $\mathcal{D} \subset \operatorname{Dom} A$ be dense in the norm $\|\cdot\|_A$. Then $\mathcal{D}$ is dense in $\operatorname{Dom} TA$ in the norm $\|\cdot\|_{TA}$.

Proof. (1): Let $(u_n) \subset \operatorname{Dom} TA$ and $TAu_n \to v$, $u_n \to u$. Then $Au_n \to T^{-1}v$. Hence $u \in \operatorname{Dom} A$ and $Au = T^{-1}v$. Hence $u \in \operatorname{Dom} TA$ and $TAu = v$.Therefore, TA is closed.

(2): Let $u \in \operatorname{Dom} TA$. Let $(u_n) \subset \mathcal{D}$ with $Au_n \to u$, $u_n \to u$. Then $TAu_n \to Tu$. Hence $u_n \to u$ in $\|\cdot\|_{TA}$. □

Chapter 4

Spectral theory of operators on Banach spaces

4.1 Spectrum

Let A be an operator on $\mathcal{X}$. We define the *resolvent set* of A as

$$\mathrm{rs}A := \{z \in \mathbb{C} \ : \ z\mathbb{1} - A \ \text{ is invertible } \}.$$

We define the *spectrum* of A as $\mathrm{sp}A := \mathbb{C}\backslash\mathrm{rs}A$.

We say that $x \in \mathcal{X}$ is an *eigenvector* of A with *eigenvalue* $z \in \mathbb{C}$ iff $x \in \mathrm{Dom}\,A$, $x \neq 0$ and $Ax = zx$. The set of eigenvalues is called the point spectrum of A and denoted $\mathrm{sp}_{\mathrm{p}}A$. Clearly, $\mathrm{sp}_{\mathrm{p}}A \subset \mathrm{sp}A$.

Let $\mathbb{C} \cup \{\infty\}$ denote the *Riemann sphere* (the one-point compactification of $\mathbb{C}$). The *extended resolvent set* is defined as $\mathrm{rs}^{\mathrm{ext}}A := \mathrm{rs}A \cup \{\infty\}$ if $A \in B(\mathcal{X})$ and $\mathrm{rs}^{\mathrm{ext}}A := \mathrm{rs}A$, if A is unbounded. The *extended spectrum* is defined as

$$\mathrm{sp}^{\mathrm{ext}}A = \mathbb{C} \cup \{\infty\}\backslash\mathrm{rs}^{\mathrm{ext}}A.$$

If $A \in B(\mathcal{X})$, we set $(\infty - A)^{-1} = 0$.

Theorem 4.1 (1) *If* $\mathrm{rs}A$ *is nonempty, then* A *is closed.*

(2) *If* $z_0 \in \mathrm{rs}A$*, then* $\Big\{z : \ |z - z_0| < \|(z_0 - A)^{-1}\|^{-1}\Big\} \subset \mathrm{rs}A$.

(3) $\|(z - A)^{-1}\| \geq (\mathrm{dist}(z, \mathrm{sp}A))^{-1}$.

(4) *If* A *is bounded, then* $\{|z| > \|A\|\}$ *is contained in* $\mathrm{rs}A$.

(5) $\mathrm{sp}^{\mathrm{ext}}A$ *is a compact subset of* $\mathbb{C} \cup \{\infty\}$.

(6) *If* $z_1, z_2 \in \mathrm{rs}A$*, then*

$$(z_1 - A)^{-1} - (z_2 - A)^{-1} = (z_2 - z_1)(z_1 - A)^{-1}(z_2 - A)^{-1}.$$

(7) *If* $z \in \mathrm{rs}A$*, then*

$$\frac{\mathrm{d}}{\mathrm{d}z}(z-A)^{-1} = -(z-A)^{-2}.$$

(8) $(z-A)^{-1}$ *is analytic on* $\mathrm{rs}^{\mathrm{ext}}A$.

(9) $(z-A)^{-1}$ *cannot be analytically extended to a larger subset of* $\mathbb{C} \cup \{\infty\}$ *than* $\mathrm{rs}^{\mathrm{ext}}(A)$.

(10) $\mathrm{sp}^{\mathrm{ext}}(A) \neq \emptyset$

(11) $\mathrm{Ran}\,(z-A)^{-1}$ *does not depend on* $z \in \mathrm{rs}A$ *and equals* $\mathrm{Dom}\,A$.

(12) $\mathrm{Ker}(z-A)^{-1} = \{0\}$.

Proof. (1): If $\lambda \in \mathrm{rs}(A)$, then $\lambda - A$ is invertible, hence closed. $\lambda - A$ is closed iff A is closed.

(2): For $|z-z_0| < \|(z_0-A)^{-1}\|^{-1}$, we have $\|(z-z_0)(z_0-A)^{-1}\| < 1$ Hence, by Theorem 3.30, $z - A = z_0 - A + z - z_0$ is invertible.

By (2), $\mathrm{dist}(z_0, \mathrm{sp}A) \geq \|(z_0-A)^{-1}\|^{-1}$. This implies (3).

(4): We check that $\sum\limits_{n=0}^{\infty} z^{-n-1}A^n$ is convergent for $|z| > \|A\|$ and equals $(z-A)^{-1}$.

(5): By (2), $\mathrm{sp}^{\mathrm{ext}}A \cap \mathbb{C} = \mathrm{sp}A$ is closed in $\mathbb{C}$. For bounded A, $\mathrm{sp}^{\mathrm{ext}}A$ is bounded by (4). For unbounded A, $\infty \in \mathrm{sp}^{\mathrm{ext}}A$. So in both cases, $\mathrm{sp}^{\mathrm{ext}}A$ is closed in $\mathbb{C} \cap \{\infty\}$.

(6) follows from Thm 3.31. Note that it implies the continuity of the resolvent.

(7) follows from (6).

(8) follows from (7).

(9) follows from (3).

(10): For bounded A, $(z-A)^{-1}$ is an analytic function tending to zero at infinity. Hence it cannot be analytic everywhere, unless it is zero, which is impossible. For unbounded A, $\infty \in \mathrm{sp}^{\mathrm{ext}}A$.

(11) follow from (6).

(12) is an obvious property of the inverse of an invertible operator. □

Proposition 4.2 *Suppose that* $\mathrm{rs}A$ *is non-empty and* $\mathrm{Dom}\,A$ *is dense. Then* $\mathrm{Dom}\,A^2$ *is dense.*

Proof. Let $z \in \mathrm{rs}A$. $(z-A)^{-1}$ is a bounded operator with a dense range and $\mathrm{Dom}\,A$ is dense. Hence $(z-A)^{-1}\,\mathrm{Dom}\,A$ is dense. We will show that

$$(z-A)^{-1}\,\mathrm{Dom}\,A \subset \mathrm{Dom}\,A^2. \tag{4.1}$$

Indeed, obviously $(z-A)^{-1}\,\mathrm{Dom}\,A \subset \mathrm{Dom}\,A$. But $A(z-A)^{-1}\,\mathrm{Dom}\,A = (z-A)^{-1}A\,\mathrm{Dom}\,A \subset \mathrm{Dom}\,A$. Hence (4.1) is true. □

Proposition 4.3 *Let* A *and* B *be operators on* $\mathcal{X}$ *with* $A \subset B$, $A \neq B$*. Then* $\mathrm{rs}A \subset \mathrm{sp}B$*, and hence* $\mathrm{rs}B \subset \mathrm{sp}A$.

Proof. Let $\lambda \in \mathrm{rs}A$. Let $x \in \mathrm{Dom}\, B\backslash \mathrm{Dom}\, A$. We have $\mathrm{Ran}\,(\lambda - A) = \mathcal{X}$, hence there exists $y \in \mathrm{Dom}\, A$ such that $(\lambda - A)y = (\lambda - B)x$. Hence $(\lambda - B)y = (\lambda - B)x$. But $x \neq y$. Hence $\lambda \notin \mathrm{rs}B$. □

4.2 Spectral radius

Spectral radius of $A \in B(\mathcal{X})$ is defined as

$$\mathrm{sr}A := \sup_{\lambda\in\mathrm{sp}A} |\lambda|.$$

Lemma 4.4 *Let a sequence of reals* (c_n) *satisfy*

$$c_n + c_m \geq c_{n+m}.$$

Then

$$\lim_{n\to\infty} \frac{c_n}{n} = \inf \frac{c_n}{n}.$$

Proof. Fix $m \in \mathbb{N}$. Let $n = mq + r$, $r < m$. We have

$$c_n \leq qc_m + c_r.$$

So

$$\frac{c_n}{n} \leq \frac{qc_m}{n} + \frac{c_r}{n}.$$

Hence

$$\limsup_{n\to\infty} \frac{c_n}{n} \leq \frac{c_m}{m}.$$

Thus,

$$\limsup_{n\to\infty} \frac{c_n}{n} \leq \inf \frac{c_m}{m}.$$

□

Theorem 4.5 *Let* $A \in B(\mathcal{X})$. *Then*

$$\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} \tag{4.2}$$

exists and equals $\mathrm{sr}A$. *Besides,* $\mathrm{sr}A \leq \|A\|$.

Proof. Let

$$c_n := \log \|A^n\|.$$

Then

$$c_n + c_m \geq c_{n+m}$$

Hence there exists

$$\lim_{n\to\infty} \frac{c_n}{n}.$$

Consequently, there exists

$$r := \lim_{n\to\infty} \|A^n\|^{1/n}.$$

By the Cauchy criterion, the series

$$\sum_{n=0}^{\infty} A^n z^{-1-n}. \tag{4.3}$$

is absolutely convergent for $|z| > r$, and divergent for $|z| < r$. We easily check that (4.3) equals $(z-A)^{-1}$. □

4.3 Examples

Example 4.6 *Consider $l^2(\mathbb{Z})$ with the canonical basis e_j, $j \in \mathbb{Z}$, and the operator U defined by*

$$Ue_j = e_{j+1}.$$

Then $\mathrm{sp}U = \{|z| = 1\}$ *and* $\mathrm{sp}_\mathrm{p}U = \emptyset$.

Proof. Indeed, $\|U\| = \|U^{-1}\| = 1$,

$$(z-U)^{-1} = \begin{cases} \sum_{j=0}^{\infty} z^{-j-1}U^j, & |z| > 1, \\ \sum_{j=0}^{\infty} z^j U^{-1-j}, & |z| < 1. \end{cases}$$

Therefore, $\{|z| = 1\} \supset \mathrm{sp}U$.

Suppose that a sequence v satisfies $Uv = zv$. Then $v_j = cz^j$. However such v is not square integrable. Hence $\mathrm{sp}_\mathrm{p}U = \emptyset$.

For $\frac{1}{2} < t < 1$ and $|z| = 1$ set

$$v_z^t = \sqrt{\frac{t^{-1}-t}{t^{-1}+t}} \sum_{j=-\infty}^{\infty} z^j t^{|j|} e_j.$$

Then $\|v_z^t\| = 1$

$$\|(z-U)v_z^t\| \leq \max\left(t^{-1}-1, (1-t)\right)\|v_z^t\| \leq 2(1-t)\|v_z^t\|.$$

Hence $(z-U)$ is not invertible. Therefore, $\{|z| = 1\} \subset \mathrm{sp}U$. □

In what follows we consider $l^2(1,2,\dots)$ with the canonical basis $e_1, e_2, \dots$.

Example 4.7 *Let the operator T defined by*

$$Te_j := \begin{cases} e_{j-1}, & j \geq 2, \\ 0, & j = 1. \end{cases}.$$

Then $\mathrm{sp}T = \{|z| \leq 1\}$ *and* $\mathrm{sp}_\mathrm{p}T = \{|z| < 1\}$.

Proof. $\|T\| = 1$,
$$(z - T)^{-1} = \sum_{j=0}^{\infty} z^{-j-1} T^j, \ |z| > 1.$$
Therefore, $\{|z| < 1\} \supset \mathrm{sp}T$.

For $|z| < 1$ set
$$w_z := \sqrt{|z|^{-2} - 1} \sum_{j=1}^{\infty} z^j e_j.$$
Then $\|w_z\| = 1$ and $(z - T)w_z = 0$. Therefore, $\{|z| < 1\} \subset \mathrm{sp}_{\mathrm{p}}T \subset \mathrm{sp}T$. Using the fact that the spectrum is closed we obtain $\{|z| \leq 1\} \subset \mathrm{sp}T$.

We easily check that every eigenvector of T is proportional to w_z for some $|z| < 1$. Therefore, $\mathrm{sp}_{\mathrm{p}}T = \{|z| < 1\}$. $\Box$

Example 4.8 *Let the operator S defined by*
$$Se_j = e_{j+1}.$$
Then $\mathrm{sp}S = \{|z| < 1\}$.

Proof. $\|S\| = 1$, and we prove that $\{|z| \leq 1\} \supset \mathrm{sp}S$ the same way as for T.

Let $w \in l^2(1, 2, \dots)$ and let v_z be as above. We check that
$$(v_z|(z - S)w) = ((\overline{z} - T)v_z|w) = 0.$$
Hence $(z - S)$ is not invertible. Using the fact that the spectrum is closed we obtain $\{|z| \leq 1\} \subset \mathrm{sp}T$. $\Box$

Example 4.9 *Let (d_n) be a sequence convergent to 0. Let the operator D be defined by*
$$De_n = d_n e_n.$$
Set $N := SD$. Then $\mathrm{sp}N = \{0\}$. If all d_n are nonzero, then $\mathrm{sp}_{\mathrm{p}}N = \emptyset$.

Proof. We have
$$\|N^n\| = \sup_j |d_{j+n-1} \cdots d_j|.$$
Let $c := \sup |d_j|$. Let $\epsilon > 0$. We can find n_0 such that for $j > n_0$ $|d_j| \leq \epsilon$. Then
$$\|N^n\|^{1/n} \leq \epsilon^{(n-n_0)/n} c^{n_0/n}.$$
Therefore,
$$\limsup_{n\to\infty} \|N^n\|^{1/n} \leq \epsilon.$$
By the arbitrariness of $\epsilon > 0$, this implies $\lim\limits_{n\to\infty} \|N^n\|^{1/n} = 0$. $\Box$

We say that an operator N is nilpotent if for some n we have $N^n = 0$. Its degree of nilpotence is the smallest number $n \in \{0, 1, \dots\}$ such that $N^n = 0$.

We say that an operator N is quasinilpotent if $\mathrm{sp}N = \{0\}$, or equivalently

$$\lim_{n\to\infty} \|N^n\|^{1/n} = 0.$$

Clearly, every nilpotent operator is quasinilpotent. Moreover, if N is nilpotent, then $\mathrm{sp}_\mathrm{p} N = \{0\}$, because $\mathrm{Ran}\, N^{n-1} \subset \mathrm{Ker} N$, where n is the degree of the nilpotence of N.

4.4 Functional calculus

Let $K \subset \mathbb{C}$ be compact. By $\mathrm{Hol}(K)$ let us denote the set of analytic functions on a neighborhood of K. It is a commutative algebra.

More precisely, let $\widetilde{\mathrm{Hol}}(K)$ be the set of pairs $(f, \mathcal{D})$, where $\mathcal{D}$ is an open subset of $\mathbb{C}$ containing K and f is an analytic function on $\mathcal{D}$. We introduce the relation $(f_1, \mathcal{D}_1) \sim (f_2, \mathcal{D}_2)$ iff $f_1 = f_2$ on a neighborhood of K contained $\mathcal{D}_1 \cap \mathcal{D}_2$. We set $\mathrm{Hol}(K) := \widetilde{\mathrm{Hol}}(K)/\sim$.

Definition 4.10 *Let $A \in B(\mathcal{X})$. Let $f \in \mathrm{Hol}(\mathrm{sp}A)$. Let γ be a contour in a domain of f that encircles $\mathrm{sp}A$ counterclockwise. We define*

$$f(A) := \frac{1}{2\pi \mathrm{i}} \int_\gamma (z - A)^{-1} f(z) \mathrm{d}z \tag{4.4}$$

Clearly, the definition is independent of the choice of the contour.

Theorem 4.11

$$\mathrm{Hol}(\mathrm{sp}A) \ni f \mapsto f(A) \in B(\mathcal{X}) \tag{4.5}$$

is a linear map satisfying

(1) $fg(A) = f(A)g(A)$;

(2) $1(A) = \mathbb{1}$;

(3) $\mathrm{id} \in \mathrm{Hol}(\mathrm{sp}A)$ *for* $\mathrm{id}(z) = z$ *and* $\mathrm{id}(A) = A$.

(4) *If* $\lambda \in \mathrm{rs}A$ *and* $f_\lambda(z) = (\lambda - z)^{-1}$, *then* $f_\lambda \in \mathrm{Hol}(\mathrm{sp}A)$ *and* $f_\lambda(A) = (\lambda - A)^{-1}$;

(5) *If* $f(z) := \sum_{n=0}^\infty f_n z^n$ *is an analytic function defined by a series absolutely convergent in a disk of radius greater than* $\mathrm{sr}A$, *then*

$$f(A) = \sum_{n=0}^\infty f_n A^n;$$

(6) *(Spectral mapping theorem).* $\mathrm{sp}f(A) = f(\mathrm{sp}A)$

(7) $g \in \mathrm{Hol}(f(\mathrm{sp}A)) \Rightarrow g \circ f(A) = g(f(A))$,

(8) $\|f(A)\| \le c_{\gamma,A} \sup_{z\in\gamma} |f(z)|$.

Proof. It is clear that $f \to f(A)$ is linear. Let us show that it is multiplicative. Let $f_1, f_2 \in \mathrm{Hol}(\mathrm{sp}A)$. Choose a contour γ_2 around the contour γ_1, both in the domains of f_1 and f_2.

$$\begin{aligned}
&(2\pi\mathrm{i})^{-2} \int_{\gamma_1} f_1(z_1)(z_1 - A)^{-1}\mathrm{d}z_1 \int_{\gamma_2} f_2(z_2)(z_2 - A)^{-1}\mathrm{d}z_2 \\
&= (2\pi\mathrm{i})^{-2} \int_{\gamma_1} \int_{\gamma_2} f_1(z_1)f_2(z_2) \left((z_1 - A)^{-1} - (z_2 - A)^{-1}\right)(z_2 - z_1)^{-1}\mathrm{d}z_1\mathrm{d}z_2 \\
&= (2\pi\mathrm{i})^{-2} \int_{\gamma_1} f_1(z_1)(z_1 - A)^{-1}\mathrm{d}z_1 \int_{\gamma_2} (z_2 - z_1)^{-1} f_2(z_2)\mathrm{d}z_2 \\
&+(2\pi\mathrm{i})^{-2} \int_{\gamma_2} f_2(z_2)(z_2 - A)^{-1}\mathrm{d}z_2 \int_{\gamma_1} (z_1 - z_2)^{-1} f_1(z_1)\mathrm{d}z_1.
\end{aligned}$$

But

$$\begin{aligned}
&\int_{\gamma_1} (z_1 - z_2)^{-1} f_1(z_1)\mathrm{d}z_1 = 0, \\
&\int_{\gamma_2} (z_2 - z_1)^{-1} f_2(z_2)\mathrm{d}z_2 = 2\pi\mathrm{i} f_2(z_1).
\end{aligned}$$

Thus

$$f_1(A)f_2(A) = f_1f_2(A). \tag{4.6}$$

From the formula

$$(z - A)^{-1} = \sum_{n=0}^{\infty} z^{-n-1}A^n, \quad |z| > \mathrm{sr}(A),$$

we obtain $1(A) = \mathbb{1}$ and $\mathrm{id}(A) = A$.

Let $\lambda \in \mathrm{rs}A$. From the formula

$$(z - A)^{-1} = \sum_{n=0}^{\infty} \frac{(\lambda - z)^n}{(\lambda - A)^{n+1}}$$

we obtain $f_\lambda(A) = (\lambda - A)^{-1}$.

Let us prove the spectral mapping theorem. First we will show

$$\mathrm{sp}f(A) \subset f(\mathrm{sp}A). \tag{4.7}$$

If $\mu \notin f(\mathrm{sp}A)$, then the function $z \mapsto f(z) - \mu \neq 0$ on $\mathrm{sp}A$. Therefore, $z \mapsto (f(z) - \mu)^{-1}$ belongs to $\mathrm{Hol}(\mathrm{sp}A)$. Thus $f(A) - \mu$ is invertible and therefore, $\mu \notin \mathrm{sp}f(A)$. This implies (4.7).

Let us now show

$$\mathrm{sp}f(A) \supset f(\mathrm{sp}A). \tag{4.8}$$

Let $\mu \notin \mathrm{sp}f(A)$. This clearly implies that $\mu - f(A)$ is invertible.

If μ does not belong to the image of f, then of course it does not belong to $f(\mathrm{sp}A)$. Let us assume that $\mu = f(\lambda)$. Then the function

$$z \mapsto g(z) := (f(z) - \mu)(\lambda - z)^{-1}$$

belongs to $\mathrm{Hol}(\mathrm{sp}A)$. Hence $g(A)$ is well defined as an element of $B(\mathcal{X})$. Likewise, $z \mapsto (\mu - f(z))^{-1}$ belongs to $\mathrm{Hol}(\mathrm{sp}(A))$, and so we can define $(\mu - f(A))^{-1}$. Clearly, $g(z)(f(z) - \mu)^{-1} = (\lambda - z)^{-1}$. Hence, $g(A)(f(\lambda) - f(A))^{-1} = (\lambda - A)^{-1}$. Hence $\lambda \notin \mathrm{sp}A$. Thus $\mu \notin f(\mathrm{sp}A)$. Consequently, (4.8) holds.

Let us show now (7). Let γ be a contour around $\mathrm{sp}(A)$ and $\tilde{\gamma}$ around $g(\mathrm{sp}(A))$. Notice that if $w \notin f(\mathrm{sp}A)$, then the function $z \mapsto (w - f(z))^{-1}$ is analytic on a neighborhood of $\mathrm{sp}(A)$ and

$$(w - f(A))^{-1} = \frac{1}{2\pi\mathrm{i}} \int_\gamma (w - f(z))^{-1}(z - A)^{-1}\mathrm{d}z.$$

We compute

$$\begin{aligned}
&g(f(A)) \\
&= \tfrac{1}{2\pi\mathrm{i}} \int_{\tilde{\gamma}} g(w)(w - f(A))^{-1}\mathrm{d}w \\
&= \tfrac{1}{(2\pi\mathrm{i})^2} \int_{\tilde{\gamma}} \int_\gamma g(w)(w - f(z))^{-1}(z - A)^{-1}\mathrm{d}w\mathrm{d}z \\
&= \tfrac{1}{(2\pi\mathrm{i})^2} \int_\gamma (z - A)^{-1}\mathrm{d}z \int_{\tilde{\gamma}} g(w)(w - f(z))^{-1}\mathrm{d}w \\
&= \tfrac{1}{2\pi\mathrm{i}} \int_\gamma g(f(z))(z - A)^{-1}\mathrm{d}z.
\end{aligned}$$

□

Note that one can also define functional calculus for an unbounded operator A having nonempty resolvent set. One needs to consider functions holomorphic on a neighborhood of $\mathrm{sp}^{\mathrm{ext}}A$ inside $\mathbb{C} \cup \{\infty\}$. Thm 4.11 is then valid except for (3), and (2) needs to be replaced by $1(A) = 0$.

4.5 Idempotents

$P \in L(\mathcal{X})$ is called an idempotent if $P^2 = P$. Then $\mathcal{X}$ is the direct sum of $\mathcal{X}_1 := \mathrm{Ran}\, P$ and $\mathcal{X}_2 := \mathrm{Ker}P$. We then say that P is the projection onto $\mathcal{X}_1$ along $\mathcal{X}_2$.

Theorem 4.12 *Let $P \in L(\mathcal{X})$ be an idempotent. Then $P \in B(\mathcal{X})$ iff* $\mathrm{Ran}\, P$ *and* $\mathrm{Ker}P$ *are closed subspaces of $\mathcal{X}$. If this is the case,* $\mathrm{sp}P = \{0, 1\}$ *and*

$$(z - P)^{-1} = (z - 1)^{-1}P + z^{-1}(1 - P).$$

Proof. Let P be bounded. The kernel of a bounded operator is obviously closed. Hence $\mathrm{Ker}P$ and $\mathrm{Ran}\, P = \mathrm{Ker}(\mathbb{1} - P)$ are closed.

Let $\mathcal{X}_1 := \mathrm{Ker}P$ and $\mathcal{X}_2 := \mathrm{Ran}\, P$ be closed. Consider $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ endowed with the norm $\|x\|_0 := \|x_1\| + \|x_2\|$. Clearly, $\|\cdot\|_0$ makes $\mathcal{X}$ into a Banach space. Let J denote the identity on $\mathcal{X}$, where in the domain we use the norm $\|\cdot\|_0$ and in the image the norm $\|\cdot\|$. Obviously $\|x\| \leq \|x\|_0$, and hence J is bounded. It is also bijective. Hence J^{-1} is bounded. Therefore, there exists c such that

$$\|x\|_0 \leq c\|x\|.$$

Therefore, $\|P\| \leq c$. □

Theorem 4.13 *Let $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and P_1, P_2 be the corresponding projections. Then $A \in B(\mathcal{X})$ preserves $\mathcal{X}_1, \mathcal{X}_2$ iff it commutes with P_1, P_2. Let $A_i \in B(\mathcal{X}_i)$ be A restricted to $\mathcal{X}_i$. Then*

$$\mathrm{sp}(A) = \mathrm{sp}(A_1) \cup \mathrm{sp}(A_2). \tag{4.9}$$

Theorem 4.14 *Let $P, Q \in B(\mathcal{X})$ be idempotents such that $\mathrm{sr}(P-Q)^2 < 1$. Then there exists an invertible $U \in B(\mathcal{X})$ such that $P = UQU^{-1}$.*

Proof. Set

$$\tilde{U} := QP + (1-Q)(1-P), \quad \tilde{V} := PQ + (1-P)(1-Q).$$

We have

$$Q\tilde{U} = \tilde{U}P, \quad P\tilde{V} = \tilde{V}Q.$$

We also have

$$\tilde{V}\tilde{U} = \tilde{U}\tilde{V} = 1 - R,$$
$$R = (P-Q)^2 = P + Q - PQ - QP.$$

We check that P and Q commute with R (note in particular that $PR = P - PQP$, etc.).

Set $c := \mathrm{sr}R < 1$. Then on $\mathrm{sp}(1-R) \subset B(1,c)$, the function $z \mapsto z^{\frac{1}{2}}$ is well defined. Hence we can introduce the function

$$(1-R)^{-1/2}$$

(which can be defined by a convergent power series). We set

$$U := \tilde{U}(1-R)^{-1/2} = (1-R)^{-1/2}\tilde{U}, \quad V := \tilde{V}(1-R)^{-1/2} = (1-R)^{-1/2}\tilde{V}.$$

So $UV = VU = 1$, or $V = U^{-1}$ and

$$Q = UPU^{-1}.$$

Proposition 4.15 *Let $t \mapsto P(t)$ be a differentiable function with values in idempotents. Then*

$$P\dot{P}P = 0.$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}P = \frac{\mathrm{d}}{\mathrm{d}t}P^2 = \dot{P}P + P\dot{P}.$$

Hence $P\dot{P}P = 2P\dot{P}P$. □

4.6 Spectral idempotents

Let Ω_1 be a subset of $B \subset \mathbb{C}$. Set $\Omega_1 := B \setminus \Omega_1$. Ω_1 will be called an *isolated subset* of B, if $\Omega_1 \cap \Omega_2^{\mathrm{cl}} = \emptyset$ and $\Omega^{\mathrm{cl}} \cap \Omega_2 = \emptyset$ (or Ω is closed and open in the relative topology of B).

If B is in addition closed, then Ω_1 is isolated iff both Ω_1 and Ω_2 are closed in $\mathbb{C} \cup \{\infty\}$.

Let Ω_1 be an isolated subset of sp$A$. It is easy to see that we can find open non-intersecting neighbohoods of Ω_1 and Ω_2. Hence

$$\mathbb{1}_{\Omega_1}(z) := \begin{cases} 1 & z \text{ belongs to a neighborhood of } \Omega_1, \\ 0 & z \text{ belongs to a neighborhood of } \Omega_2. \end{cases}$$

defines an element of Hol(sp$A$). If γ is a counterclockwise contour around Ω_1 outside of Ω_2 then

$$\mathbb{1}_{\Omega}(A) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} (z - A)^{-1} \mathrm{d}z$$

This operator will be called the *spectral idempotent* of the operator A onto Ω. The same is true concerning Ω_2.

Clearly, $\mathbb{1}_{\Omega_i}^2 = \mathbb{1}_{\Omega_i}$, hence $\mathbb{1}_{\Omega_i}(A)$ are idempotents for $i = 1, 2$. Besides, $\mathbb{1}_{\Omega_1} \mathbb{1}_{\Omega_2} = 0$.

Setting $\operatorname{Ran} \mathbb{1}_{\Omega_i} = \mathcal{X}_1$ we obtain a direct sum decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ preserved by A. Let A_i be A restricted to $\mathcal{X}_i$.

Theorem 4.16 $\mathrm{sp}(A_i) = \Omega_i$, $i = 1, 2$.

Proof. Let $\lambda \notin \mathrm{sp}(A_1)$. For an arbitrary c, set

$$f(z) = \begin{cases} (z - \lambda)^{-1}, & z \in \text{neighborhood of } \Omega_1, \\ c & z \in \text{neighborhood of } \Omega_2. \end{cases} \tag{4.10}$$

Then f is holomorphic on a neighborhood of Ω and $f(A)$ restricted to $\mathcal{X}_1$ coincides with $(z - A_1)^{-1}$. Hence $z \notin \mathrm{sp}(A_1)$. Therefore, $\mathrm{sp}(A_1) \subset \Omega_1$. Likewise, $\mathrm{sp}(A_2) \subset \Omega_2$. Now we know by Theorem 4.13 that

$$\Omega_1 \cup \Omega_2 = \Omega = \mathrm{sp}(A) = \mathrm{sp}(A_1) \cup \mathrm{sp}(A_2). \tag{4.11}$$

and $\Omega_1 \cap \Omega_2 = \emptyset$. Therefore, $\Omega_i = \mathrm{sp}(A_i)$. $\square$

If Ω and Θ are two isolated subsets of sp$A$, then

$$\mathbb{1}_{\Omega}(A) \mathbb{1}_{\Theta}(A) = \mathbb{1}_{\Omega \cap \Theta}(A)$$

4.7 Isolated eigenvalues

Assume now that λ is an isolated point of sp$A$. Set

$$P := \mathbb{1}_{\lambda}(A), \qquad N := (A - \lambda)P.$$

Definition 4.17 *We say that λ is a semisimple eigenvalue if $N = 0$. If $N^n = 0$ and $N^{n-1} \neq 0$, then we say that λ is nilpotent of degree n. It is easy to see that if $A \in L(\mathcal{X})$, then the degree of nilpotence of λ is less than or equal to* $\dim P$.

Proposition 4.18 *The operator N is quasinilpotent, satisfies $PN = NP = N$ and can be written as*

$$N = f(A), \quad f(z) := (z - \lambda)\mathbb{1}_\lambda(z). \tag{4.12}$$

Besides,

$$(z - A)^{-1}P = (z - \lambda)^{-1}P + \sum_{j=1}^{\infty} N^j (z - \lambda)^{-j+1}.$$

and $(z - A)^{-1}(1 - P)$ is analytic in the neighborhood of λ. If N is nilpotent of degree n, then there exist $\delta > 0$ and C such that

$$\|(z - A)^{-1}\| \leq C|z - \lambda|^{-n}, \quad z \in B(\lambda, \delta). \tag{4.13}$$

Proof. Clearly, $AP = A1_\lambda(A)$ and $\lambda P = \lambda 1_\lambda(A)$. This shows (4.12). Then note that $f(z) = 0$ for $z \in \mathrm{sp}A$. Hence $\mathrm{sp}N = \{0\}$.

Using the Laurent series expansion we get

$$(z - A)^{-1} = \sum_{n=-\infty}^{\infty} C_n (z - \lambda)^n,$$

where

$$C_n = \frac{1}{2\pi \mathrm{i}} \int_\gamma (z - A)^{-1}(z - \lambda)^{-n-1} \mathrm{d}z.$$

Clearly, $C_{-1} = P$ and $C_{-2} = N$. Besides, by Theorem 4.11 we obtain

$$C_{-1-n} C_{-1-m} = C_{-1-n-m}.$$

□

4.8 Spectral theory in finite dimension

Suppose that $\mathcal{X}$ is finite dimensonal of dimension d and $A \in L(\mathcal{X})$. Then $\mathrm{sp}A$ has at most d elements. Let $\mathrm{sp}A = \{\lambda_1, \dots, \lambda_n\}$.

We say that A is diagonalizable iff

$$A = \sum_{j=1}^{n} \lambda_j \mathbb{1}_{\lambda_j}(A).$$

It is well known that in a finite dimension for every $A \in L(\mathcal{X})$, there exist unique diagonalizable D and nilpotent N satisfying $DN = ND$ such that $A = D + N$. Let m be the degree of nilpotence of N.

In fact, define two functions on a neighborhood of sp$A$: $d(z)$ is equal to λ_i on a neighborhood of $\lambda_i \in \mathrm{sp}A$ and $n(z) = z - \lambda_i$ on a neighborhood of $\lambda_i \in \mathrm{sp}A$. Both d and n belong to Hol(sp$A$). Clearly, and $D := d(A)$ and $N := n(A)$ satisfy the above requirements.

Clearly then $N = \sum_{j=1}^{n} N_j$ with $N_j = P_j N P_j$ also nilpotent. Let m_j be the degree of nilpotence of N_j. We have

$$\begin{aligned} f(A) &= \sum_{k=0}^{m} f^{(k)}(D)\frac{N^k}{k!} \\ &= \sum_{j=1}^{n}\sum_{k=0}^{m_j} f^{(k)}(\lambda_j)\frac{N_j^k}{k!}. \end{aligned}$$

4.9 Functional calculus for several commuting operators

Let $K \subset \mathbb{C}^n$ be compact. By Hol(K) let us denote the set of analytic functions on a neighborhood of K. It is a commutative algebra.

Let $\mathcal{X}$ be a Banach space.

Definition 4.19 *Let $A_1, \dots, A_n \in B(\mathcal{X})$ commute with one another. Let $F \in$ Hol$(\mathrm{sp}A_1 \times \cdots \times \mathrm{sp}A_n)$. Let $\gamma_1, \dots, \gamma_n$ be contours such that $\gamma_1 \times \cdots \times \gamma_n$ lies in the domain of F and each γ_j encircles* spA_j *counterclockwise. We define*

$$F(A_1, \dots, A_n) := \frac{1}{(2\pi i)^n}\int_{\gamma_1} \mathrm{d}z_1 \cdots \int_{\gamma_n} \mathrm{d}z_n (z_1 - A_1)^{-1} \cdots (z_n - A_n)^{-1} F(z_1, \dots, z_n). \tag{4.14}$$

Clearly, the definition is independent of the choice of the contour.

Theorem 4.20

$$\mathrm{Hol}(\mathrm{sp}A_1 \times \cdots \times \mathrm{sp}A_n) \ni F \mapsto F(A_1, \dots, A_n) \in B(\mathcal{X}) \tag{4.15}$$

is a linear map satisfying

(1) $FG(A_1, \dots, A_n) = F(A_1, \dots, A_n)G(A_1, \dots, A_n)$;

(2) $1(A_1, \dots, A_n) = \mathbb{1}$;

(3) $\mathrm{id}_j(A_1, \dots, A_n) = A_j$, *for* $\mathrm{id}_j(z_1, \dots, z_n) := z_j$;

(4) *If* $F(z_1, \dots, z_n) := \sum\limits_{m_1, \dots, m_n = 0}^{\infty} F_{m_1, \dots, m_n} z_1^{m_1} \cdots z_n^{m_n}$ *is an analytic function defined by a series absolutely convergent in a neighborhood of* $\mathrm{B}(\mathrm{sr}A_1) \times \cdots \times \mathrm{B}(\mathrm{sr}A_n)$, *then*

$$F(A_1, \dots, A_n) = \sum_{m_1, \dots, m_n = 0}^{\infty} F_{m_1, \dots, m_n} A_1^{m_1} \cdots A_n^{m_n};$$

(5) *(Weak version of the spectral mapping theorem).* $\mathrm{sp}F(A_1, \dots, A_n) \subset F(\mathrm{sp}A_1, \dots, \mathrm{sp}A_n)$

(6) $g \in \mathrm{Hol}(F(\mathrm{sp}A_1 \times \cdots \times \mathrm{sp}A_n)) \Rightarrow g \circ F(A_1, \ldots, A_n) = g(F(A_1, \ldots, A_n))$,

(7) $\|F(A_1, \ldots, A_n)\| \le c_{\gamma, A_1, \ldots, A_n} \sup_{z \in \gamma} |f(z)|$.

Proof. The proof is essentially the same as that of Theorem 4.11. Let us show for instance the weak version of the spectral mapping theorem. Let $\mu \notin F(\mathrm{sp}A_1, \ldots, \mathrm{sp}A_n)$. Then the function $(z_1, \ldots, z_n) \mapsto F(z_1, \ldots, z_n) - \mu \neq 0$ on $\mathrm{sp}A_1 \times \cdots \times \mathrm{sp}A_n$. Therefore, $(z_1, \ldots, z_n) \mapsto (F(z_1, \ldots, z_n) - \mu)^{-1}$ belongs to $\mathrm{Hol}(\mathrm{sp}A_1 \times \cdots \times A_n)$. Thus $F(A_1, \ldots, A_n) - \mu$ is inverible and therefore, $\mu \notin \mathrm{sp}F(A_1, \ldots, , A_n)$. $\square$

4.10 Examples of unbounded operators

Example 4.21 *Let I be an infinite set and let $(a_i)_{i \in I}$ be a complex sequence. Let $C_\mathrm{c}(I)$ be the space of sequences with a finite number of non-zero elements. Define the operator*

$$C_\mathrm{c}(I) \ni x \mapsto Ax \in C_\mathrm{c}(I)$$

by the formula

$$(Ax)_i = a_i x_i.$$

For $1 \le p < \infty$ let us treat $C_\mathrm{c}(I)$ as a subspace of the Banach space $L^p(I)$, or $C_\infty(I)$, so that A is a densely defined (partial) operator. The closure of A has the domain

$$\mathrm{Dom}\, A^{\mathrm{cl}} := \{(x_i)_{i \in I} \in L^p(I) \ : \ \textstyle\sum_{i \in I} |a_i x_i|^p < \infty\} \tag{4.16}$$

We then have

$$\mathrm{sp}_\mathrm{p}(A^{\mathrm{cl}}) = \{a_i \ : \ i \in I\},$$
$$\mathrm{sp}A^{\mathrm{cl}} = \{a_i \ : \ i \in I\}^{\mathrm{cl}}.$$

A is bounded iff the sequence a_i is bounded.

Proof. To prove this let $\mathcal{D}$ be the rhs of (4.16) and $x \in \mathcal{D}$. Then there exists a countable set I_1 such that $i \notin I_1$ implies $x_i = 0$. We enumerate the elements of I_1: $i_1, i_2, \ldots$. Define $x^n \in C_\mathrm{c}(I)$ setting $x^n_{i_j} = x_{i_j}$ for $j \le n$ and $x^n_i = 0$ for the remaining indices. Then $\lim_{n \to \infty} x^n = x$ and $Ax^n \to Ax$. Hence, $\{(x, Ax) \ : \ x \in \mathcal{D}\} \subset (\mathrm{Gr}\, A)^{\mathrm{cl}}$.

If x^n belongs to (4.16) and $(x^n, Ax^n) \to (x, y)$, then $x^n_i \to x_i$ and $a_i x^n_i = (Ax^n)_i \to y_i$. Hence $y_i = a_i x_i$. Using that $y \in L^p(I)$ we see that x belongs to (4.16). $\square$

Example 4.22 *Let $p^{-1} + q^{-1} = 1$, $1 < p \le \infty$ and let $(w_i)_{i \in I}$ be a sequence that does not belong to $L^q(I)$. Let $C_\mathrm{c}(I)$ be as above. Define*

$$L^p(I) \supset C_\mathrm{c}(I) \ni x \mapsto \langle w | x \rangle := \sum_{i \in I} x_i w_i \in \mathbb{C}.$$

Then $\langle w |$ is non-closable.

Proof. It is sufficient to assume that $I = \mathbb{N}$ and define $v_i^n := \frac{|w_i|^q}{w_i(\sum_{i=1}^n |w_i|^q)}$, $i \leq n$, $v_i^n = 0$, $i > n$. Then $\langle w|v^n\rangle = 1$ and $\|v^n\|_p = (\sum_{i=1}^n |w_i|^q)^{-\frac{1}{q}} \to 0$. Hence $(0,1)$ belongs to the closure of the graph of the operator. $\square$

4.11 Pseudoresolvents

Definition 4.23 *Let $\Omega \subset \mathbb{C}$ be open. Then the continuous function*

$$\Omega \ni z \mapsto R(z) \in B(\mathcal{X})$$

is called a pseudoresolvent if

$$R(z_1) - R(z_2) = (z_2 - z_1)R(z_1)R(z_2). \tag{4.17}$$

Evidently, if A is a closed operator and $\Omega \subset \mathrm{rs}A$, then $\Omega \ni z \mapsto (z - A)^{-1}$ is a pseudoresolvent.

Proposition 4.24 *Let $\Omega \ni z \mapsto R_n(z) \in B(\mathcal{X})$ be a sequence of pseudoresolvents and $R(z) := \mathrm{s-}\lim\limits_{n\to\infty} R_n(z)$. Then $R(z)$ is a pseudoresolvent.*

Theorem 4.25 *Let $\Omega \ni z \mapsto R(z) \in B(\mathcal{X})$ be a pseudoresolvent. Then*

(1) *$\mathcal{R} := \mathrm{Ran}\, R(z)$ does not depend on $z \in \Omega$.*

(2) *$\mathcal{N} := \mathrm{Ker} R(z)$ does not depend on $z \in \Omega$.*

(3) *$R(z)$ is an analytic function and*

$$\frac{\mathrm{d}}{\mathrm{d}z} R(z) = -R(z)^2.$$

(4) *$R(z)$ is a resolvent of a certain operator A iff $\mathcal{N} = \{0\}$. The operator A is uniquely defined and closed. Its domain is $\mathcal{R}$. For any $z \in \Omega$ and $y \in \mathcal{R}$,*

$$Ay = -R(z)^{-1}y + zy. \tag{4.18}$$

Proof. (1) Let $v \in \mathrm{Ran}\, R(z_1)$. Then for some w

$$v = R(z_1)w = R(z_2)(\mathbb{1} + (z_1 - z_2)R(z_1))w. \tag{4.19}$$

Hence $\mathrm{Ran}\, R(z_1) \subset \mathrm{Ran}\, R(z_2)$.

(2) Let $w \in \mathrm{Ker} R(z_1)$. Then

$$0 = (\mathbb{1} + (z_1 - z_2)R(z_2))R(z_1)w = R(z_2)w. \tag{4.20}$$

Hence $\mathrm{Ker} R(z_1) \subset \mathrm{Ker} R(z_2)$.

Let us prove (4)$\Leftarrow$. Fix $z_1 \in \Omega$. If $\mathcal{N} = \{0\}$, then every element of $\mathcal{R}$ can be uniquely represented as $R(z_1)x$, $x \in \mathcal{X}$. Then for $z = z_1$ (4.18) can be rewritten as $AR(z_1)x := -x + z_1 R(z_1)x$. Now for any $z_2 \in \Omega$,

$$\begin{aligned} AR(z_2)x &= AR(z_1)(x + (z_1 - z_2)R(z_2)x) \\ &= -x - (z_1 - z_2)R(z_2)x + z_1 R(z_1)\big(x + (z_1 - z_2)R(z_2)x\big) \\ &= -x + z_2 R(z_2)x. \end{aligned} \tag{4.21}$$

□

Chapter 5

One-parameter semigroups on Banach spaces

5.1 (M, β)-type semigroups

Let $\mathcal{X}$ be a Banach space.

Definition 5.1 $[0, \infty[\ni t \mapsto W(t) \in B(\mathcal{X})$ *is called a* strongly continuous one-parameter semigroup *iff*

(1) $W(0) = \mathbb{1}$;

(2) $W(t_1)W(t_2) = W(t_1 + t_2)$, $t_1, t_2 \in [0, \infty[$;

(3) $\lim\limits_{t \searrow 0} W(t)x = x$, $x \in \mathcal{X}$;

(4) *for some* $t_0 > 0$, $\|W(t)\| < M$, $0 \leq t \leq t_0$.

As a side remark we note that (4) can be removed from the above definition.

Proposition 5.2 *(4) follows from Def. 5.1 (2) and (3).*

Proof. Suppose that $t_0 > 0$ and

$$\sup\{\|W(t)\| \ : \ 0 \leq t \leq t_0\} = \infty. \tag{5.1}$$

Below, we will show that this implies the exitence of a sequence (s_n) such that

$$s_n \to 0, \text{ and } \|W(s_n)\| \to \infty \tag{5.2}$$

But by (2) we have s$-\lim\limits_{n \to \infty} W(s_n) = \mathbb{1}$. This is impossible by the Banach-Steinhaus Theorem (the Uniform Boundedness Principle).

Indeed, by (5.1) we can find a sequence (t_n) in $[0, t_0]$ such that $\|W(t_n)\| \to \infty$. In addition, we can assume that either $t_n \searrow t_\infty$ or $t_n \nearrow t_\infty$.

In the first case

$$\|W(t_n - t_\infty)\| \geq \frac{\|W(t_n)\|}{\|W(t_\infty)\|}$$

converges to ∞. Hence $s_n := t_n - t_\infty$ satisfies (5.2).

In the second case, we can assume in addition that $\frac{\|W(t_{n+1})\|}{\|W(t_n)\|} \to \infty$. Consequently,

$$\|W(t_{n+1} - t_n)\| \geq \frac{\|W(t_{n+1})\|}{\|W(t_n)\|}$$

converges to ∞. Hence $s_n := t_{n+1} - t_n$ satisfies (5.2). □

Theorem 5.3 *Let $W(t)$ e a strongly continuous semigroup. Then*

(1) *There exist constants M, β such that*

$$\|W(t)\| \leq M\mathrm{e}^{\beta t}; \tag{5.3}$$

(2) $[0, \infty[\times \mathcal{X} \ni (t, x) \mapsto W(t)x \in \mathcal{X}$ *is a continuous function.*

Proof. By (4), for $t \leq nt_0$ we have $\|W(t)\| \leq M^n$. Hence, $\|W(t)\| \leq M \exp(\frac{t}{t_0} \log M)$. Therefore, (5.3) is satisfied.

Let $t_n \to t$ and $x_n \to x$. Then

$$\begin{aligned}
\|W(t_n)x_n - W(t)x\| &\leq \|W(t_n)x_n - W(t_n)x\| + \|W(t_n)x - W(t)x\| \\
&\leq M\mathrm{e}^{\beta t_n}\|x_n - x\| + M\mathrm{e}^{\beta \min(t_n, t)}\|W(|t - t_n|)x - x\|.
\end{aligned}$$

□

We say that the semigroup $W(t)$ is (M, β)-*type*, if the condition (5.3) is satisfied.

Clearly, if $W(t)$ is (M, β)-type, then $W(t)\mathrm{e}^{-\beta t}$ is $(M, 0)$-type. Since $W(0) = \mathbb{1}$, no semigroups (M, β) exist for $M < 1$.

5.2 Generator of a semigroup

Let $W(t)$ be a strongly continuous one-parameter semigroup.

Definition 5.4 *We define*

$$\begin{aligned}
\mathrm{Dom}\, A &:= \{x \in \mathcal{X} \ : \ \text{there exists } \lim_{t \searrow 0} t^{-1}(W(t)x - x)\}, \\
Ax &:= \lim_{t \searrow 0} t^{-1}(W(t)x - x), \quad x \in \mathrm{Dom}\, A.
\end{aligned}$$

Theorem 5.5 (1) *A is a closed densely defined operator;*

(2) *$W(t)\,\mathrm{Dom}\, A \subset \mathrm{Dom}\, A$ and $W(t)A = AW(t)$;*

(3) *If $W_1(t)$, $W_2(t)$ are two different semigroups and A_1, A_2 are defined as above, then $A_1 \neq A_2$.*

A will be called the *generator* of $W(t)$. If $W(t)$ is the semigroup generated by A, then we will write $W(t) =: \mathrm{e}^{tA}$.

Proof of Theorem 5.5 (2). Let $x \in \mathrm{Dom}\, A$. Then

$$s^{-1}(W(s) - \mathbb{1})W(t)x = W(t)s^{-1}(W(s) - \mathbb{1})x. \tag{5.4}$$

But

$$W(t) \lim_{s \searrow 0} s^{-1}(W(s) - \mathbb{1})x = W(t)Ax$$

Hence $\lim\limits_{s \searrow 0}$ of the left hand side of (5.4) exists. Hence $W(t)x \in \mathrm{Dom}\, A$ and $AW(t)x = W(t)Ax$. □

Lemma 5.6 *For $x \in \mathcal{X}$ put*

$$B_t x := t^{-1} \int_0^t W(s)x\mathrm{d}s.$$

Then

(1) $\mathrm{s-}\lim\limits_{t \searrow 0} B_t = \mathbb{1}$.

(2) $B_t W(s) = W(s) B_t$.

(3) *For $x \in \mathrm{Dom}\, A$, $AB_t x = B_t Ax$.*

(4) *If $x \in \mathcal{X}$, then $B_t x \in \mathrm{Dom}\, A$,*

$$AB_t x = t^{-1}(W(t)x - x). \tag{5.5}$$

(5) *If $\lim\limits_{t \searrow 0} AB_t x$ exists, then $x \in \mathrm{Dom}\, A$ and $\lim\limits_{t \searrow 0} AB_t x = Ax$.*

Proof. (1) follows by

$$B_t x - x = t^{-1} \int_0^t (W(s)x - x)\mathrm{d}s \underset{t \searrow 0}{\rightarrow} 0.$$

(2) is obvious. (3) is proven as Theorem 5.5 (2). To prove (4) we note that

$$u^{-1}(W(u) - \mathbb{1})B_t x = t^{-1}(W(t) - \mathbb{1})B_u x \underset{u \searrow 0}{\rightarrow} t^{-1}(W(t)x - x),$$

where first we use a simple identity, and then we apply (1). (5) follows from (4). □

Proof of Theorem 5.5 (1) The density of $\mathrm{Dom}\, A$ follows by Lemma 5.6 (1) and (3).

Let us show that A is closed. Let $x_n \underset{n\to\infty}{\to} x$ and $Ax_n \underset{n\to\infty}{\to} y$. By (5.5), $B_t A = AB_t$ is bounded. Hence,

$$B_t y = \lim_{n\to\infty} B_t A x_n = \lim_{n\to\infty} AB_t x_n = AB_t x.$$

Thus,

$$y = \lim_{t\downarrow 0} B_t y = \lim_{t\downarrow 0} AB_t x. \tag{5.6}$$

By Lemma 5.6 (5), $x \in \mathrm{Dom}\, A$ and (5.6) equals Ax. □

Proposition 5.7 *Let $W(t)$ be a semigroup and A its generator. Then, for any $x \in \mathrm{Dom}\, A$ there exists a unique solution of*

$$[0, \infty[\ni t \mapsto x(t) \in \mathrm{Dom}\, A, \quad \frac{\mathrm{d}}{\mathrm{d}t} x(t) = Ax(t), \quad x(0) = x. \tag{5.7}$$

(for $t = 0$ the derivative is right-sided). The solution is given by $x(t) = W(t)x$.

Proof. Let us show that $x(t) := W(t)x$ solves (5.7), both for the left and right derivative. Let $u > 0$, in the latter case, assume also $u \leq t$. We have

$$\begin{aligned} u^{-1}(W(t+u)x - W(t)x) &= W(t)u^{-1}(W(u) - 1)x \underset{u\downarrow 0}{\to} W(t)Ax = AW(t)x, \\ u^{-1}(W(t-u)x - W(t)x) &= W(t-u)u^{-1}(W(u) - 1)x \underset{u\downarrow 0}{\to} W(t)Ax = AW(t)x. \end{aligned}$$

Let us show now the uniqueness. Let $t \mapsto x(t) \in \mathrm{Dom}\, A$ solve (5.7). Let $y(s) := W(t-s)x(s)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s} y(s) = W(t-s)Ax(s) - AW(t-s)x(s) = 0$$

Hence $y(s)$ does not depend on s. At $s = t$ it equals $x(t)$, and at $s = 0$ it equals $W(t)x$. □

Proof of Theorem 5.5 (3) By Prop. 5.7 (2), $W(t)$ is uniquely determined by A on $\mathrm{Dom}\, A$. But $W(t)$ is bounded and $\mathrm{Dom}\, A$ is dense, hence $W(t)$ is uniquely determined. □

5.3 One-parameter groups

Definition 5.8 $\mathbb{R} \ni t \mapsto W(t) \in B(\mathcal{X})$ *is called a* strongly continuous one-parameter group *iff*

(1) $W(0) = \mathbb{1}$;

(2) $W(t_1)W(t_2) = W(t_1 + t_2)$, $t_1, t_2 \in \mathbb{R}$;

(3) $\lim_{t\to 0} W(t)x = x$, $x \in \mathcal{X}$;

(4) *for some* $t_0 > 0$, $\|W(t)\| < M$, $|t| \le t_0$.

Proposition 5.9 (1) *Let* $\mathbb{R} \ni t \mapsto W(t)$ *be a* strongly continuous one-parameter group. *If* A *is the generator of the semigroup* $[0, \infty[\ni t \mapsto W(t)$, *then* $-A$ *is the generator of the semigroup* $[0, \infty[\ni t \mapsto W(-t)$..

(2) *Conversely, let* A *and* $-A$ *be generators of s.c. semigroups. Then*

$$W(t) := \begin{cases} e^{tA} & t \ge 0, \\ e^{t(-A)}, & t \le 0. \end{cases}$$

is a s.c. group.

Proof. (1) is immediate. To prove (2) it suffices to show that

$$e^{-tA}e^{tA} = \mathbb{1}. \tag{5.8}$$

But if $v \in \operatorname{Dom} A = \operatorname{Dom}(-A)$, then

$$\partial_t e^{-tA}e^{tA}v = e^{-tA}(-A + A)e^{tA}v = 0,$$

which proves (5.8). □

A will be called the *generator of the group* $\mathbb{R} \ni t \mapsto W(t)$. Note that it can be defined as in Def.5.4, where the derivative is both-sided.

5.4 Norm continuous semigroups

Theorem 5.10 (1) *If* $A \in B(\mathcal{X})$, *then* $\mathbb{R} \ni z \mapsto e^{tA} = \sum\limits_{n=0}^{\infty} \frac{t^n}{n!}A^n$ *is a norm continuous group and* A *is its generator.*

(2) *If a one-parameter semigroup* $W(t)$ *is norm continuous, then its generator is bounded.*

Proof. (1) follows by the functional calculus.

Let us show (2). $W(t)$ is norm continuous, hence $\lim\limits_{t \to 0} B_t = \mathbb{1}$. Therefore, for $0 < t < t_0$

$$\|B_t - \mathbb{1}\| < 1.$$

Hence B_t is then invertible.

We know that for $x \in \operatorname{Dom} A$

$$t^{-1}(W(t) - \mathbb{1})x = B_t Ax.$$

For $0 \le t < t_0$ we can write this as

$$Ax = t^{-1}B_t^{-1}(W(t) - 1)x.$$

Hence $\|Ax\| \le c\|x\|$. □

5.5 Essential domains of generators

Theorem 5.11 *Let $W(t)$ be a strongly continuous one-parameter semigroup and let A be its generator. Let $\mathcal{D} \subset \operatorname{Dom} A$ be dense in $\mathcal{X}$ and $W(t)\mathcal{D} \subset \mathcal{D}$, $t > 0$. Then $\mathcal{D}$ is dense in $\operatorname{Dom} A$ in the graph topology—in other words, $\mathcal{D}$ is an essential domain of A.*

We will write $\|x\|_A := \|Ax\| + \|x\|$ for the graph norm.

Lemma 5.12 (1) *For $x \in \mathcal{X}$, $\|B_t x\|_A \leq (Ct^{-1} + 1)\|x\|$;*

(2) *For $x \in \operatorname{Dom} A$, $\lim\limits_{t \searrow 0} \|B_t x - x\|_A = 0$;*

(3) *$W(t)$ is a strongly continuous semi-group on $\operatorname{Dom} A$ equipped with the graph norm.*

(4) *If $\tilde{\mathcal{D}}$ is a closed subspace in $\operatorname{Dom} A$ invariant wrt $W(t)$, then it is invariant also wrt B_t.*

Proof. (1) follows by Lemma 5.6 (3).

(2) follows by Lemma 5.6 (1) and because $B(t)$ commutes with A.

(3) follows from the fact that $W(t)$ is a strongly continuous semigroup on $\mathcal{X}$, preserves $\operatorname{Dom} A$ and commutes with A.

To show (4), note that $B_t x$ is defined using an integral involving $W(s)x$. $W(s)x$ depends continuously on s in the topology of $\operatorname{Dom} A$, as follows by (3). Hence this integral (as Riemann's integral) is well defined. Besides, $B_t x$ belongs to the closure of the space spanned by $W(s)x$, $0 \leq s \leq t$. □

Proof of Theorem 5.11. Let $x \in \operatorname{Dom} A$, $x_n \in \mathcal{D}$ and $x_n \underset{n\to\infty}{\to} x$ in $\mathcal{X}$. Let $\tilde{\mathcal{D}}$ be he closure of $\mathcal{D}$ in $\operatorname{Dom} A$. Then $B_t x_n \in \tilde{\mathcal{D}}$, by Lemma 5.12 (4). By Lemma 5.12 (1) we have

$$\|B_t x_n - B_t x\|_A \leq C_t \|x_n - x\|.$$

Hence $B_t x \in \tilde{\mathcal{D}}$. By Lemma 5.12 (2)

$$\|B_t x - x\|_A \underset{t\downarrow 0}{\to} 0.$$

Hence, $x \in \tilde{\mathcal{D}}$. □

5.6 Operators of (M, β)-type

Theorem 5.13 *Let A be a densely defned operator. Then the following conditions are equivalent:*

(1) $[\beta, \infty[\subset \operatorname{rs}(A)$ *and*

$$\|(x - A)^{-m}\| \leq M|x - \beta|^{-m}, \quad m = 1, 2, \ldots, \quad x \in \mathbb{R},\ x > \beta$$

(2) $\{z \in \mathbb{C} \ : \ \mathrm{Re} z > \beta\} \subset \mathrm{rs}(A)$ *and*

$$\|(z-A)^{-m}\| \leq M|\mathrm{Re} z - \beta|^{-m}, \quad m = 1, 2, \ldots, \quad z \in \mathbb{C}, \ \mathrm{Re} z > \beta.$$

Proof. It suffices to prove (1)⇒(2). Let (1) be satisfied. It suffices to assume that $\beta = 0$. Let $z = x + \mathrm{i}y$. Then for $t > 0$

$$\begin{aligned}(z-A)^{-m} &= (x+t-A)^m(\mathbb{1} + (\mathrm{i}y - t)(x+t-A)^{-1})^{-m} \\ &= \sum_{j=0}^{\infty}(x+t-A)^{-m-j}(\mathrm{i}y-t)^j \begin{pmatrix} -m \\ j \end{pmatrix}.\end{aligned}$$

Using the fact that $\left|\begin{pmatrix} -m \\ j \end{pmatrix}\right| = (-1)^j \begin{pmatrix} -m \\ j \end{pmatrix}$ we get

$$\begin{aligned}\|(z-A)^{-m}\| &\leq M \sum_{j=0}^{\infty} |x+t|^{-m-j}(-1)^j|\mathrm{i}y-t|^j \begin{pmatrix} -m \\ j \end{pmatrix} \\ &= M|x+t|^m\left(1 - \tfrac{|\mathrm{i}y-t|}{x+t}\right)^{-m} \\ &= M(x+t-|\mathrm{i}y-t|)^{-m} \underset{t\to\infty}{\to} Mx^{-m}.\end{aligned}$$

□

Definition 5.14 *We say that an operator A is* (M,β)-type*, iff the conditions of Theorem 5.13 are satisfied.*

Obviously, if A is of (M,β)-type, then $A - \beta$ is of $(M,0)$-type.

5.7 The Hille-Philips-Yosida theorem

Theorem 5.15 *If $W(t)$ is a semigroup of (M,β)-type, then its generator A is also of (M,β)-type. Besides,*

$$(z-A)^{-1} = \int_0^\infty \mathrm{e}^{-tz}W(t)\mathrm{d}t, \quad \mathrm{Re} z > \beta.$$

Proof. Set

$$R(z)x := \int_0^\infty \mathrm{e}^{-zt}W(t)x\mathrm{d}t.$$

Let $y = R(z)x$. Then

$$\begin{aligned}&u^{-1}(W(u)-\mathbb{1})y \\ = \quad &-u^{-1}\mathrm{e}^{zu}\int_0^u \mathrm{e}^{-zt}W(t)x\mathrm{d}t + u^{-1}(\mathrm{e}^{zu}-1)\int_0^\infty \mathrm{e}^{-zt}W(t)x\mathrm{d}t \underset{u\searrow 0}{\to} -x + zy.\end{aligned}$$

Hence $y \in \mathrm{Dom}\, A$ and $(z - A)R(z)x = x$.

Suppose now that $x \in \mathrm{Ker}(z - A)$. Then $x_t := \mathrm{e}^{zt}x \in \mathrm{Dom}\, A$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}x_t = Ax_t$. Hence $x_t = W(t)x$. But $\|x_t\| = \mathrm{e}^{\mathrm{Re}zt}\|x\|$, which is impossible.

By the formula

$$(z - A)^{-m} = \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-z(t_1+\cdots+t_m)} W(t_1 + \cdots + t_m)\mathrm{d}t_1 \cdots \mathrm{d}t_m$$

we get the estimate

$$\|(z - A)^{-m}\| \leq \int_0^\infty \cdots \int_0^\infty M\mathrm{e}^{-(z-\beta)(t_1+\cdots+t_m)}\mathrm{d}t_1 \cdots \mathrm{d}t_m = M|z - \beta|^{-m}.$$

□

Theorem 5.16 *If A is an operator of (M, β)-type, then it is the generator of a semigroup of (M, β)-type.*

To simplify, let us assume that $\beta = 0$ (which does not restrict the generality). Then we have the formula

$$\mathrm{e}^{tA} = \mathrm{s-}\lim_{n\to\infty} \left(1\!\!1 - \frac{t}{n}A\right)^{-n},$$

$$\left\| \mathrm{e}^{tA}x - \left(1\!\!1 - \frac{t}{n}A\right)^{-n} x \right\| \leq M\frac{t^2}{2}\|A^2x\|, \quad x \in \mathrm{Dom}\, A^2.$$

Proof. Set

$$V_n(t) := \left(1\!\!1 - \frac{t}{n}A\right)^{-n}.$$

Let us first show that

$$\mathrm{s-}\lim_{t\downarrow 0} V_n(t) = 1\!\!1. \tag{5.9}$$

To prove (5.9) it suffices to prove that

$$\mathrm{s-}\lim_{s\downarrow 0}(1\!\!1 - sA)^{-1} = 1\!\!1. \tag{5.10}$$

We have $(1\!\!1 - sA)^{-1} - 1\!\!1 = (s^{-1} - A)^{-1}A$. Hence for $x \in \mathrm{Dom}\, A$

$$\|(1\!\!1 - sA)^{-1}x - x\| \leq Ms^{-1}\|Ax\|,$$

which proves (5.10).

Let us list some other properties of $V_n(t)$: for $\mathrm{Re}t > 0$, $V_n(t)$ is holomorphic, $\|V_n(t)\| \leq M$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}V_n(t) = A\left(1\!\!1 - \frac{t}{n}A\right)^{-n-1}.$$

To show that $V_n(t)x$ is a Cauchy sequence for $x \in \mathrm{Dom}(A^2)$, we compute

$$\begin{aligned} V_n(t)x - V_m(t)x &= \lim_{s\downarrow 0} V_n(t-s)V_m(s)x - \lim_{s\uparrow t} V_n(t-s)V_m(s)x \\ &= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \tfrac{\mathrm{d}}{\mathrm{d}s} V_n(t-s)V_m(s)x \\ &= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \Big(- V_n'(t-s)V_m(s) + V_n(t-s)V_m'(s)\Big)x \\ &= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \Big(\tfrac{s}{n} - \tfrac{t-s}{m}\Big) \big(\mathbb{1} - \tfrac{t-s}{n}A\big)^{-n-1} \big(\mathbb{1} - \tfrac{s}{n}A\big)^{-m-1} A^2x. \end{aligned}$$

Hence for $x \in \mathrm{Dom}(A^2)$

$$\begin{aligned} \|V_n(t)x - V_m(t)x\| &\leq \|A^2x\| \int_0^t |\tfrac{s}{m} - \tfrac{t-s}{n}| M^2 \mathrm{d}s \\ &= M^2(\tfrac{1}{n} + \tfrac{1}{m})\tfrac{t^2}{2}. \end{aligned}$$

By the Proposition 4.2, $\mathrm{Dom}(A^2)$ is dense in $\mathcal{X}$. Therefore, there exists a limit uniform on $[0, t_0]$

$$\mathrm{s-}\lim_{n\to\infty} V_n(t) =: W(t),$$

which depends strongly continuously on t.

Finally, let us show that $W(t)$ is a semigroup with the generator A. To this end it suffices to show that for $x \in \mathrm{Dom}\, A$

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t)x = AW(t)x. \tag{5.11}$$

But $x \in \mathrm{Dom}\, A$

$$V_n(t+u)x = V_n(t)x + \int_t^{t+u} A\Big(\mathbb{1} - \frac{s}{n}A\Big)^{-1} V_n(s)x\mathrm{d}s$$

Hence passing to the limit we get

$$W(t+u)x = W(t)x + \int_t^{t+u} AW(s)x\mathrm{d}s.$$

This implies (5.11). □

5.8 Semigroups of contractions and their generators

Theorem 5.17 *Let A be a closed operator on $\mathcal{X}$. Then the following conditions are eqivalent:*

(1) *A is a generator of a semigroup of contractions, i.e. $\|\mathrm{e}^{tA}\| \leq 1$, $t \geq 0$.*

(2) *The operator A is of $(1,0)$-type.*

(3) $]0,\infty[\subset \mathrm{rs}(A)$ *and*

$$\|(\mu - A)^{-1}\| \leq \mu^{-1}, \quad \mu \in \mathbb{R},\ \mu > 0,$$

(4) $\{z \in \mathbb{C} \ :\ \mathrm{Re} z > 0\} \subset \mathrm{rs}(A)$ *and*

$$\|(z - A)^{-1}\| \leq |\mathrm{Re} z|^{-1}, \quad z \in \mathbb{C},\ \mathrm{Re} z > 0.$$

Proof. The equivalence of (1) and (2) is a special case of Theorems 5.15 and 5.16. The implications (2)⇒(3) and (2)⇒(4) are obvious, the converse implications are easy. □

Chapter 6

Hilbert spaces

6.1 Scalar product spaces

Let $\mathcal{V}$ be a vector space.

$$\mathcal{V} \times \mathcal{V} \ni (v, y) \mapsto (v|y) \in \mathbb{C}$$

is called a scalar product if

$$\begin{aligned}
&(v|y+z) = (v|y) + (v|z), \quad && (v|\lambda y) = \lambda(v|y), \\
&(v+y|z) = (v|z) + (y|z), \quad && (\lambda v|y) = \overline{\lambda}(v|y), \\
&(v|v) \geq 0, && \\
&(v|v) = 0 \Rightarrow v = 0. &&
\end{aligned}$$

Theorem 6.1 (The hermitian property.)

$$(v|y) = \overline{(y|v)}.$$

Proof. We use the polarization identity:

$$\begin{aligned}
(v|y) &= \tfrac{1}{4} \textstyle\sum_{n=0}^{3} (-\mathrm{i})^n (v + \mathrm{i}^n y | v + \mathrm{i}^n y), \\
(y|v) &= \tfrac{1}{4} \textstyle\sum_{n=0}^{3} \mathrm{i}^n (v + \mathrm{i}^n y | v + \mathrm{i}^n y).
\end{aligned}$$

□

We define

$$\|v\| := \sqrt{(v|v)}$$

Theorem 6.2 (The parallelogram identity.)

$$2(\|v\|^2 + \|y\|^2) = \|v+y\|^2 + \|v-y\|^2.$$

Theorem 6.3 (The Schwarz inequality.)

$$|(v|y)| \leq \|v\| \|y\|$$

Proof.

$$0 \quad \leq (v+ty|v+ty) = \|v\|^2 + t(v|y) + \overline{t(v|y)} + \|y\|^2|t|^2.$$

We set $t = -\frac{\overline{(v|y)}}{\|y\|^2}$ and we get

$$0 \leq \|v\|^2 - \frac{|(v|y)|^2}{\|y\|^2}.$$

□

Theorem 6.4 (The triangle inequality.)

$$\|v+y\| \leq \|v\| + \|y\|$$

Proof.

$$\|v+y\|^2 = \|v\|^2 + (v|y) + (y|v) + \|y\|^2 \leq (\|v\| + \|y\|)^2.$$

□

Hence $\| \cdot \|$ is a norm.

6.2 The definition and examples of Hilbert spaces

Definition 6.5 *A space with a scalar product is called a Hilbert space if it is complete.*

Example 6.6 *Let I be an arbitrary set of indices. Then $L^2(I)$ denotes the space of families $(v^i)_{i\in I}$ with values in $\mathbb{C}$ indexed by I such that*

$$\sum_{i\in I} |v^i|^2 < \infty$$

equipped with the scalar product

$$(v|w) = \sum_{i\in I} \overline{v^i} w^i.$$

The Schwarz inequality guarantees that the scalar product is well defined.

Example 6.7 *Let (X, μ) be a space with a measure. Then $L^2(X, \mu)$ equipped with the scalar product*

$$(v|w) := \int \overline{v}(x) w(x) \mathrm{d}\mu(x)$$

is a Hilbert space.

Theorem 6.8 *Let $\mathcal{V}_0$ be a space equipped with a scalar product (but not necessarily complete). Let $\mathcal{V}_0^{\mathrm{cpl}}$ be its completion (see Theorem 2.6). Then there exists a unique scalar product on $\mathcal{V}_0^{\mathrm{cpl}}$, which is compatible with the norm on $\mathcal{V}_0^{\mathrm{cpl}}$. $\mathcal{V}_0^{\mathrm{cpl}}$ with this scalar product is is a Hilbert space.*

6.3 Complementary subspaces

Suppose that (for the time being) $\mathcal{V}$ is a space with a scalar product (not necessarily complete).

If $A \subset \mathcal{V}$, then $A^\perp$ denotes

$$A^\perp := \{v \in \mathcal{V} \ : \ (v|z) = 0, \quad z \in A\}.$$

Proposition 6.9 (1) *$A^\perp$ is a closed subspace.*

(2) $A \subset B \ \Rightarrow A^\perp \supset B^\perp$

(3) $(A^\perp)^\perp \supset \mathrm{Span}(A)^{\mathrm{cl}}$

Proof. 1. and 2. are obvious. To prove 3. we note that $(A^\perp)^\perp \supset A$. But $(A^\perp)^\perp$ is a closed subspace by 1. Hence it contains the least closed subspace containing A, or $\mathrm{Span}(A)^{\mathrm{cl}}$. □

Suppose that $\mathcal{V}$ is Hilbert space.

Theorem 6.10 *Let $\mathcal{W}$ be a closed subspace of $\mathcal{V}$. Then $\mathcal{W}^\perp$ is a closed subspace and*

$$\mathcal{W} \oplus \mathcal{W}^\perp = \mathcal{V}, \quad (\mathcal{W}^\perp)^\perp = \mathcal{W}.$$

Proof. Let

$$\inf_{w \in \mathcal{W}} \|v - w\| =: d.$$

Then there exists a sequence $y_n \in \mathcal{W}$ such that

$$\lim_{n\to\infty} \|v - y_n\| = d.$$

Then using first the parallelogram identity and then $\frac{1}{2}(y_n + y_m) \in \mathcal{W}$ we get

$$\begin{aligned} \|y_n - y_m\|^2 \quad &= 2\|y_n - v\|^2 + 2\|y_m - v\|^2 - 4\|v - \tfrac{1}{2}(y_n + y_m)\|^2 \\ &\le 2\|y_n - v\|^2 + 2\|y_m - v\|^2 - 4d^2 \to 0. \end{aligned}$$

Therefore, (y_n) is a Cauchy sequence and hence

$$\lim_{n\to\infty} y_n =: y.$$

Clearly, $y \in \mathcal{W}$ and it is an element closest to v. We set $z := v - y$. We will show that $z \in \mathcal{W}^\perp$. Let $w \in \mathcal{W}$. Then

$$\begin{aligned} \|z\|^2 \quad &= \|v - y\|^2 \le \|v - (y + tw)\|^2 \\ &= \|z - tw\|^2 = \|z\|^2 - \bar{t}(w|z) - t\overline{(w|z)} + |t|^2\|w\|^2. \end{aligned}$$

We set $t = \frac{(w|z)}{\|w\|^2}$. We get

$$0 \le -\frac{|(w|z)|^2}{\|w\|^2}.$$

Thus $(w|z) = 0$. This shows that $\mathrm{Span}(\mathcal{W} \cup \mathcal{W}^\perp) = \mathcal{V}$.

$\mathcal{W} \cap \mathcal{W}^\perp = \{0\}$ is obvious. This implies the uniqueness of the pair $y \in \mathcal{W}$, $z \in \mathcal{W}^\perp$. This ends the proof of $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^\perp$.

Let us show now that $(\mathcal{W}^\perp)^\perp \subset \mathcal{W}$. Let $v \in (\mathcal{W}^\perp)^\perp$. Then $v = y + z$, where $y \in \mathcal{W}$, $z \in \mathcal{W}^\perp$. But $(z|v) = 0$ and $(z|y) = 0$. We have

$$(v|z) = (y|z) + (z|z).$$

Hence $(z|z) = 0$, or $z = 0$, therefore $v \in \mathcal{W}$ □

Corollary 6.11

$$A^{\perp\perp} = \mathrm{Span}(A)^{\mathrm{cl}}$$

Proof.

$$\mathrm{Span}(A)^{\mathrm{cl}\perp\perp} \supset A^{\perp\perp} \supset \mathrm{Span}(A)^{\mathrm{cl}}$$

follows by Proposition 6.9.

$$\mathrm{Span}(A)^{\mathrm{cl}\perp\perp} = \mathrm{Span}(A)^{\mathrm{cl}}$$

follows by Proposition 6.10. □

6.4 Orthonormal basis

Assume for the time being that $\mathcal{V}$ is a space with a scalar product.

Definition 6.12 *$A \subset \mathcal{V} \backslash \{0\}$ is an orthogonal system iff $e_1, e_2 \in A$, $e_1 \neq e_2$ implies $(e_1|e_2) = 0$. $A \subset \mathcal{V}$ is na orthonormal system if it is orthogonal and if $e \in A$, then $\|e\| = 1$.*

Theorem 6.13 *Let $(e_1, \dots, e_N)$ be an orthonormal system. We then have* **the Pythagoras Theorem**

$$\|v\|^2 = \sum_{n=1}^{N} |(v|e_n)|^2 + \|v - \sum_{n=1}^{N} (e_n|v) e_n\|^2$$

and **the Bessel inequality***:*

$$\|v\|^2 \geq \sum_{n=1}^{N} |(v|e_n)|^2.$$

Assume now that $\mathcal{V}$ is a Hilbert space.

Definition 6.14 *A maximal orthonormal system is called an orthonormal basis.*

Theorem 6.15 *Let $\{e_i\}_{i\in I}$ be an orthonormal system. It is an orthonormal basis iff one of the following conditions holds:*

(1) $\{e_i \; : \; i \in I\}^\perp = \{0\}$.

(2) $(\mathrm{Span}\{e_i \; : \; i \in I\})^{\mathrm{cl}} = \mathcal{V}$

Theorem 6.16 *Every orthonormal system can be completed to an orthonormal basis.*

Proof. Let $\mathcal{B}$ denote the family of all orthonormal systems ordered by inclusion. Let $\{A_i \; : \; i \in I\} \subset \mathcal{B}$ be a subset linearly ordered. Then

$$\cup_{i\in I} A_i$$

is also an orthonormal system. It is also an upper bound of the set $\{A_i \; : \; i \in I\}$. Hence we can apply the Kuratowski-Zorn lemma. $\square$

The definition of an orthogonal basis is similar. From an orthogonal basis $(w_i)_{i\in I}$ we can construct an orthonormal basis $\{\|w_i\|^{-\frac{1}{2}} w_i\}_{i\in I}$.

Theorem 6.17 *Let $(e_i)_{i\in I}$ be an orthonormal basis. Then*

(1)
$$v = \sum_{i\in I} (e_i|v) e_i, \tag{6.1}$$

and

$$\|v\|^2 = \sum_{i\in I} |(v|e_i)|^2.$$

(2) *If*

$$v = \sum_{i\in I} \lambda_i e_i,$$

then $\lambda_i = (e_i|v)$.

Proof. By the Bessel inequality, a finite number of coefficients is greater than $\epsilon > 0$. Hence a countable number of coefficients is non-zero. Let us enumerate the non-zero coefficients $(e_{i_n}|v), \; n = 1, 2, \ldots$ By the Bessel inequality, we get

$$\sum_{i=1}^{\infty} |(e_i|v)|^2 \leq \|v\|^2.$$

Set

$$v_N := \sum_{n=1}^{N} (e_i|v) e_i,$$

Then for $N < M$

$$\|v_M - v_M\|^2 = \sum_{i=N+1}^{M} |(e_i|v)|^2.$$

Hence by the completeness of $\mathcal{V}$ we get the convergence of v_N and thus the convergence of the series. Besides, the vector

$$v - \sum_{i \in I} e_i (e_i | v)$$

is orthogonal to the basis. Hence it is zero. This proves 1. □

Theorem 6.18 *Let B_1 and B_2 be orthonormal bases in $\mathcal{V}$. Then they have the same cardinality.*

Proof. First we prove this for finite B_1 or B_2.

For any $y \in B_1$ there exists a countable number of $x \in B_2$ such that $(x|y) \neq 0$. For every $x \in B_2$ we will find $y \in B_1$ such that $(x|y) \neq 0$. Hence there exists a function $f : B_2 \to B_1$ such that the preimage of every set is countable. Hence

$$|B_2| \leq |B_1 \times \mathbb{N}| = \max(|B_1|, \aleph_0).$$

Similarly we check that

$$|B_1| \leq \max(|B_2|, \aleph_0).$$

□

Definition 6.19 *The cardinality of this basis is called the dimension of the space.*

Definition 6.20 *We say that a linear operator $U : \mathcal{V}_1 \to \mathcal{V}_2$ is unitary iff it is a bijection and*

$$(Uw|Uv) = (w|v), \quad v, w \in \mathcal{V}_1.$$

We say that the Hilbert spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ are isomorphic iff there exists a unitary operator from $\mathcal{V}_1$ to H_2.

Theorem 6.21 *Two Hilbert spaces are isomorphic iff they have the same dimension.*

Proof. Let $\{x_i \,:\, i \in I\}$ be an orthonormal basis in $\mathcal{V}$. It suffices to show that $\mathcal{V}$ is isomorphic to $L^2(I)$. We define the unitary operator

$$(Uv)_i := (x_i|v).$$

□

6.5 The Riesz Lemma

Let $\mathcal{V}^*$ denote the space of antilinear bounded functionals on $\mathcal{V}$.

Theorem 6.22 (The Riesz Lemma) *The formula*

$$\langle Cv|x\rangle := (x|v)$$

defines a linear isometry from $\mathcal{V}$ onto $\mathcal{V}^*$.

Proof. Isometricity:

$$\|Cv\| = \sup_{\|x\|\leq 1} |(x|v)| \leq \|v\|.$$

It suffices to take $x = \frac{v}{\|v\|}$ to get the equality.

Surjectivity: Let $w \in \mathcal{V}^*$ and $\mathcal{W} := \mathrm{Ker} w$. If $\mathcal{W} = \mathcal{V}$, then $w = C0$. If not, then let $x_0 \in \mathcal{W}^\perp$, $\|x_0\| = 1$. Set

$$v := x_0 \langle w|x_0\rangle.$$

We will prove that $w = Cv$.

An arbitrary y can be represented as

$$y = \left(y - \frac{\overline{\langle w|y\rangle}}{\overline{\langle w|x_0\rangle}} x_0 \right) + \frac{\overline{\langle w|y\rangle}}{\overline{\langle w|x_0\rangle}} x_0$$

The first term belongs to $\mathcal{W}$. Hence

$$\langle Cv|y\rangle = (y|v) \quad = \quad \left(\frac{\overline{\langle w|y\rangle}}{\overline{\langle w|x_0\rangle}} x_0 | x_0 \langle w|x_0\rangle \right) = \langle w|y\rangle.$$

□

The space $\mathcal{V}^*$ has a natural structure of a Hilbert space:

$$(Cv|Cx) := (v|x), \quad v, x \in \mathcal{V},$$

so that C is a unitary map from $\mathcal{V}$ to $\mathcal{V}^*$.

6.6 Quadratic forms

Let $\mathcal{V}, \mathcal{W}$ be complex vector spaces.

Definition 6.23 $\mathfrak{a}$ *is called a sesquilinear form on $\mathcal{W} \times \mathcal{V}$ iff it is a map*

$$\mathcal{W} \times \mathcal{V} \ni (w, v) \mapsto \mathfrak{a}(w, v) \in \mathbb{C}$$

antilinear wrt the first argument and linear wrt the second argument.

If $\mathfrak{a}$ is a form, then we define $\lambda\mathfrak{a}$ by $(\lambda\mathfrak{a})(w,v) := \lambda\mathfrak{a}(w,v)$. and $\mathfrak{a}^*$ by $\mathfrak{a}^*(v,w) := \overline{\mathfrak{a}(w,v)}$. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are forms, then we define $\mathfrak{a}_1+\mathfrak{a}_2$ by $(\mathfrak{a}_1+\mathfrak{a}_2)(w,v) := \mathfrak{a}_1(w,v)+\mathfrak{a}_2(w,v)$.

Suppose that $\mathcal{V}=\mathcal{W}$. We will write $\mathfrak{a}(v) := \mathfrak{a}(v,v)$. We will call it a quadratic form. The knowledge of $\mathfrak{a}(v)$ determines $\mathfrak{a}(w,v)$:

$$\mathfrak{a}(w,v) = \frac{1}{4}\left(\mathfrak{a}(w+v)+\mathrm{i}\mathfrak{a}(w-\mathrm{i}v)-\mathfrak{a}(w-v)-\mathrm{i}\mathfrak{a}(w+\mathrm{i}v)\right). \tag{6.2}$$

Suppose now that $\mathcal{V},\mathcal{W}$ are Hilbert spaces. A form is bounded iff

$$|\mathfrak{a}(w,v)| \leq C\|w\|\|v\|.$$

Proposition 6.24 (1) *If $A\in B(\mathcal{V},\mathcal{W})$, then $(w|Av)$ is a bounded sesquilinear form on $\mathcal{W}\times\mathcal{V}$.*

(2) *Let $\mathfrak{a}$ be a bounded sesquilinear form on $\mathcal{W}\times\mathcal{V}$. Then there exists a unique operator $A\in B(\mathcal{V},\mathcal{W})$ such that*

$$\mathfrak{a}(w,v) = (w|Av).$$

Proof. (1) is obvious. To show (2) note that $w\mapsto\mathfrak{a}(w,v)$ is an antilinear functional on $\mathcal{W}$. Hence there exists $\eta\in\mathcal{W}$ such that $\mathfrak{a}(w,v)=(w|\eta)$. We put $Av := \eta$.

We will often identify bounded sesquilinear forms, bounded quadratic forms and bounded operators.

Theorem 6.25 *Suppose that $\mathcal{D},\mathcal{Q}$ are dense linear subspaces of $\mathcal{V},\mathcal{W}$ and $\mathfrak{a}$ is a bounded sesquilinear form on $\mathcal{D}\times\mathcal{Q}$. Then there exists a unique extension of $\mathfrak{a}$ to a bounded form on $\mathcal{V}\times\mathcal{W}$.*

6.7 Adjoint operators

Definition 6.26 *Let $A\in B(\mathcal{V},\mathcal{W})$. Then the operator A^* given (uniquely) by the formula*

$$(A^*w|v) := (w|Av)$$

is called the (hermitian) conjugate of A.

Note that the definition is correct, because $\mathfrak{a}(w,v) := (w|Av)$ is a bounded sesquilinear form, and hence so is $\mathfrak{a}^*$; and A^* is the operator associated with $\mathfrak{a}^*$.

Theorem 6.27 *The hermitian conjugation has the following properties*

1) $\|A^*\| = \|A\|$
2) $(\lambda A)^* = \overline{\lambda} A^*$
3) $(A+B)^* = A^* + B^*$,
4) $(AB)^* = B^* A^*$,
5) $A^{**} = A$,
6) $(\mathrm{Ran}\, A)^\perp = \mathrm{Ker} A^*$, *hence* $(\mathrm{Ker} A^*)^\perp = (\mathrm{Ran}\, A)^{\mathrm{cl}}$;
7) $(\mathrm{Ran}\, A^*)^\perp = \mathrm{Ker} A$, *hence* $(\mathrm{Ker} A)^\perp = (\mathrm{Ran}\, A^*)^{\mathrm{cl}}$;
8) A *is invertible* $\Leftrightarrow A^*$ *is invertible* $\Leftrightarrow \|Av\| \geq C\|v\|$ *and* $\|A^*v\| \geq C\|v\|$, *moreover,* $(A^{-1})^* = (A^*)^{-1}$.
9) $\mathrm{sp} A^* = \overline{\mathrm{sp} A}$.

6.8 Numerical range

Definition 6.28 *Let* $\mathfrak{t}$ *be a quadratic form on* $\mathcal{X}$. *The numerical range of* $\mathfrak{t}$ *is defined as*

$$\mathrm{Num}\, \mathfrak{t} := \{\mathfrak{t}(x) \in \mathbb{C} \ : \ x \in \mathcal{X}, \ \|x\| = 1\}.$$

Theorem 6.29 (1) *In a two-dimensional space the numerical range is always an elipse together with its interior.*

(2) $\mathrm{Num}\, \mathfrak{t}$ *is a convex set.*

(3) $\mathrm{Num}(\alpha\mathfrak{t} + \beta \mathbb{1}) = \alpha \mathrm{Num}(\mathfrak{t}) + \beta$.

(4) $\mathrm{Num}\, \mathfrak{t}^* = \overline{\mathrm{Num}\, \mathfrak{t}}$.

(5) $\mathrm{Num}(\mathfrak{t} + \mathfrak{s}) \subset \mathrm{Num}\, \mathfrak{t} + \mathrm{Num}\, \mathfrak{s}$.

Proof. (1) We write $\mathfrak{t} = \mathfrak{t}_{\mathrm{R}} + \mathrm{i}\mathfrak{t}_{\mathrm{I}}$, where $\mathfrak{t}_{\mathrm{R}}$, $\mathfrak{t}_{\mathrm{I}}$ are self-adjoint. We diagonalize $\mathfrak{t}_{\mathrm{I}}$. Thus if $\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ is the matrix of $\mathfrak{t}$, then $t_{12} = \overline{t}_{21}$. By multiplying one of the basis vectors with a phase factor we can guarantee that $t_{12} = t_{21}$ is real.

Now $\mathfrak{t}$ is given by a matrix of the form

$$c \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda & \mu \\ \mu & -\lambda \end{bmatrix} + \mathrm{i} \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}$$

Any normalized vector up to a phase factor equals $v = (\cos\alpha, \mathrm{e}^{\mathrm{i}\phi}\sin\alpha)$ and

$$\mathfrak{t}(v) - c = \lambda\cos 2\alpha + \mu\cos\phi\sin 2\alpha + \mathrm{i}\gamma\cos 2\alpha =: x + \mathrm{i}y. \tag{6.3}$$

Now it is an elementary exercise to check that $x + \mathrm{i}y$ are given by (6.3), iff they satisfy

$$(\gamma x - \lambda y)^2 + \mu^2 y^2 \leq \gamma^2 \mu^2.$$

(2) follows immediately from (1). $\square$

Let $\mathcal{V}$ be a Hilbert space. If A is an operator on $\mathcal{V}$, then the numerical range of A is defined as the numerical range of the form $v \mapsto (v|Av)$, that is

$$\mathrm{Num}A := \{(v|Av) \in \mathbb{C} \ : \ v \in \mathcal{V}, \ \|v\| = 1\}.$$

Theorem 6.30 *Let $A \in B(\mathcal{V})$. Then*

(1) $\mathrm{sp}A \subset (\mathrm{Num}A)^{\mathrm{cl}}$.

(2) *For $z \notin (\mathrm{Num}A)^{\mathrm{cl}}$,*

$$\|(z - A)^{-1}\| \leq \mathrm{dist}(z, \mathrm{Num}A)^{-1}.$$

Proof. Let $(z_0 \notin \mathrm{Num}A)^{\mathrm{cl}}$. Recall that $\mathrm{Num}(A)$ is convex. Hence, replacing A wih $\alpha A + \beta$ we can assume that $z_0 = \mathrm{i}\nu$ with $\nu = \mathrm{dist}(z, \mathrm{Num}(A))$ and $\mathrm{Num}A \subset \{\mathrm{Im}z \leq 0\}$. Now

$$\begin{aligned}
\|(z_0 - A)v\|^2 &= (Av|Av) + i\nu(v|Av) - \mathrm{i}\nu(Av|v) + |\nu|^2\|v\|^2 \\
&= (Av|Av) - 2\nu\mathrm{Im}(v|Av) + |\nu|^2\|v\|^2 \\
&\geq |\nu|^2\|v\|^2.
\end{aligned}$$

Next, $\mathrm{Num}A^* \subset \{\mathrm{Im}z \geq 0\}$.

$$\begin{aligned}
\|(\overline{z}_0 - A^*)v\|^2 &= (A^*v|A^*v) - i\nu(v|A^*v) + \mathrm{i}\nu(A^*v|v) + |\nu|^2\|v\|^2 \\
&= (A^*v|A^*v) - 2\nu\mathrm{Im}(v|Av) + |\nu|^2\|v\|^2 \\
&\geq |\nu|^2\|v\|^2.
\end{aligned}$$

Hence $z_0 - A$ is invertible and $z \in \mathrm{rs}A$. $\square$

6.9 Self-adjoint operators

Theorem 6.31 *Let $A \in B(\mathcal{V})$. The following conditions are equivalent:*

(1) $A = A^*$.

(2) $(Aw|v) = (w|Av), \ w, v \in \mathcal{V}$.

(3) $(w|Av) = \overline{(v|Aw)}, \ w, v \in \mathcal{V}$.

(4) $(v|Av) \in \mathbb{R}$.

Proof. (1)⇔(2)⇔(3)⇒(4) is obvious. To show (4)⇒(3) we use the polarization identity:

$$\begin{aligned}(w|Av) &= \tfrac{1}{4}\sum_{j=0}^{3}(-\mathrm{i})^j(w+\mathrm{i}^j v|A(w+\mathrm{i}^j v)),\\ \overline{(v|Aw)} &= \overline{\tfrac{1}{4}\sum_{j=0}^{3}(-\mathrm{i})^j(v+\mathrm{i}^j w|A(v+\mathrm{i}^j w))}\\ &= \tfrac{1}{4}\sum_{j=0}^{3}(-\mathrm{i})^j(w+\mathrm{i}^j v|A(w+\mathrm{i}^j v)).\end{aligned}$$

□

Definition 6.32 *An operator $A \in B(\mathcal{V})$ satisfying the conditions of Theorem 6.31 is called self-adjoint.*

An operator $A \in B(\mathcal{V})$ such that

$$(v|Av) \geq 0, \quad v \in \mathcal{V},$$

is called a positive operator.

By Theorem 6.31, positive operators are self-adjoint.

Clearly, if $A \in B(\mathcal{V})$, then A is self-adjoint iff $\mathrm{Num}A \subset \mathbb{R}$ and positive iff $\mathrm{Num}A \subset [0, \infty[$.

Theorem 6.33 *Let A be self-adjoint. Then $\mathrm{sp}A \subset \mathbb{R}$.*

Proof. This fact is a special case of Thm 15.1 (2). For those who omitted that theorem, we give the argument.

Let $\mu \neq 0$, $\mu, \lambda \in \mathbb{R}$. We have

$$\|(A - (\lambda + \mathrm{i}\mu))v\|^2 = \|(A - \lambda)v\|^2 + \mu^2\|v\|^2 \geq \mu^2\|v\|^2.$$

Besides, $(A - (\lambda + \mathrm{i}\mu))^* = A - (\lambda - \mathrm{i}\mu)$. Hence

$$\|(A - (\lambda + \mathrm{i}\mu))^* v\|^2 = \|(A - \lambda)v\|^2 + \mu^2\|v\|^2 \geq \mu^2\|v\|^2.$$

So $A - (\lambda + \mathrm{i}\mu)$ is invertible. □.

Theorem 6.34 *The operator A^*A is positive and*

$$\|A^*A\| = \|A\|^2. \tag{6.4}$$

Proof. A^*A is positive because

$$(v|A^*Av) = \|Av\|^2 \geq 0.$$

To show (6.4) we note that

$$\begin{aligned}\|A\|^2 = \|A^*\|\|A\| \geq \|A^*A\| &\geq \sup_{\|v\|=1}(v|A^*Av)\\ &= \sup_{\|v\|=1}\|Av\|^2 = \|A\|^2.\end{aligned}$$

□

Lemma 6.35 *Let A be self-adjoint. Then*

$$\|A\| = \sup_{\|v\|\leq 1} |(v|Av)|.$$

Proof. Let $w, v \in \mathcal{V}$. We will show first that

$$|(w|Av)| \leq \frac{1}{2}(\|w\|^2 + \|v\|^2) \sup_{\|y\|\leq 1} (y|Ay). \tag{6.5}$$

Replacing w with $\mathrm{e}^{i\alpha}w$ we can suppose that $(w|Av)$ is positive. Then

$$\begin{aligned}
(w|Av) &= \frac{1}{2}((w|Av) + (v|Aw)) \\
&= \frac{1}{4}\left((w+v|A(w+v) - (w-v|A(w-v))\right) \\
&\leq \frac{1}{4}\left(\|v+w\|^2 + \|v-w\|^2\right) \sup_{\|y\|=1} |(y|Ay)| \\
&= \frac{1}{2}\left(\|v\|^2 + \|w\|^2\right) \sup_{\|y\|=1} |(y|Ay)|
\end{aligned}$$

Hence (6.5) is true. Therefore,

$$\|A\| = \sup_{\|v\|=\|w\|=1} |(w|Av)| \leq \sup_{\|y\|=1} |(y|Ay)|.$$

□

Theorem 6.36 *If A is self-adjoint, then*

$$(\mathrm{Num}A)^{\mathrm{cl}} = \mathrm{ch}(\mathrm{sp}A), \tag{6.6}$$

where ch *denotes the convex hull.*

Proof. Step 1. Let A be self-adjoint and

$$-\inf(\mathrm{sp}A) = \sup(\mathrm{sp}A) =: a. \tag{6.7}$$

Clearly, $\mathrm{ch}(\mathrm{sp}A) = [-a, a]$ and $a = \|A\|$. By Lemma 6.35, $(\mathrm{Num}A)^{\mathrm{cl}} \subset [-a, a]$. Hence, $(\mathrm{Num}A)^{\mathrm{cl}} \subset \mathrm{ch}(\mathrm{sp}A)$. The converse inclusion follows from Theorem 15.1.

Step 2. Let A be self-adjoint. Let $a_- := \inf(\mathrm{sp}A)$, $a_+ := \sup(\mathrm{sp}A)$. Then $\tilde{A} := A - \frac{1}{2}(a_- + a_+)$ is self-adjoint and satisfies (6.7). Hence (6.6) holds for $\tilde{A}$. Hence (6.6) holds for A as well. □

6.10 Orthoprojections

Theorem 6.37 *Let $P \in B(\mathcal{V})$ be an idempotent. The following conditions are equivalent:*

(1) *P is self-adjoint.*

(2) $\mathrm{Ker}P = (\mathrm{Ran}\, P)^{\perp}$.

An idempotent P satisfing these conditions with $\mathrm{Ran}\, P = \mathcal{W}$ will be called the orthoprojection onto $\mathcal{W}$.

If $(w_i)_{i \in I}$ is an orthogonal basis in $\mathcal{W}$, then

$$Pv = \sum_{i \in I} \frac{(w_i|v)}{\|w_i\|^2} w_i.$$

Proposition 6.38 (Gramm-Schmidt ortogonalization) *Let $y_1, y_2, \ldots$ be a linearly independent system. Let P_n be a projection onto the (n-dimensional) space $\mathrm{Span}\{y_1, \ldots, y_n\}$. Then*

$$w_n := (1 - P_{n-1}) y_n$$

is an orthogonal system. An equivalent definition:

$$w_1 = y_1, \qquad w_n := y_n - \sum_{j=1}^{n-1} \frac{(w_j|y_n)}{\|w_j\|^2} w_j.$$

Theorem 6.39 *Let $P^* = P$ and $P^2 = P^3$. Then P is an orthoprojection.*

Proof. $(P^2 - P)^*(P^2 - P) = 0$, hence $P = P^2$. $\square$

6.11 Isometries and partial isometries

Definition 6.40 *An operator U is called a partial isometry if U^*U and are UU^* orthoprojections.*

Theorem 6.41 *U is a partial isometry iff U^*U is an orthoprojection.*

Proof. We check that $(UU^*)^3 = (UU^*)^2$. $\square$

Proposition 6.42 *If U is a partial isometry, then UU^* is an orthoprojection onto $\mathrm{Ran}\, U$ and U^*U is the orthoprojection onto $(\mathrm{Ker}U)^{\perp}$.*

Proof. It is easy to see that for any operator A we have $\mathrm{Ker}A = \mathrm{Ker}A^*A$. Therefore,

$$\mathrm{Ker}U = \mathrm{Ker}U^*U, \tag{6.8}$$

$$\mathrm{Ker}U^* = \mathrm{Ker}UU^*, \tag{6.9}$$

(6.8) means that U^*U is the orthoprojection onto $(\mathrm{Ker} U)^\perp$. (6.9) means that UU^* is the orthoprojection with the kernel $(\mathrm{Ker} U^*)^\perp$.

Let us prove that $\mathrm{Ran}\, U = \mathrm{Ran}\, UU^*$. Indeed, $\subset$ is obvious. $\supset$ follows from the fact that UU^* is a projection: $v \in \mathrm{Ran}\, UU^*$ iff $v = UU^*v$. Now the range of an orthoprojection is always closed. Hence $(\mathrm{Ker} U^*)^\perp = \mathrm{Ran}\, U$. □

Proposition 6.43 *Let $U \in B(\mathcal{V})$ be a partial isometry. Set $\mathcal{V}_1 := (\mathrm{Ker} U)^\perp$, $\mathcal{V}_2 := \mathrm{Ran}\, U$. Let $I_i : \mathcal{V}_i \to \mathcal{V}$ be the embeddings. Define $W \in B(\mathcal{V}_1, \mathcal{V}_2)$ by*

$$Wv = Uv, \quad v \in \mathcal{V}_1.$$

Then W is unitary and $U = I_2 W I_1^*$.

Theorem 6.44 *Let $U \in B(\mathcal{V}, \mathcal{W})$. The following properties are equivalent:*
1) $U^*U = \mathbb{1}$,
2) $(Uv|Uw) = (v|w), \ \ v, w \in \mathcal{V}$,
3) U is an isometry, that means $\|Uv\| = \|v\|$.

Definition 6.45 *An operator U satisfying the properties of Theorem 6.44 is called a linear isometry.*

Proof. 1)⇔2) is obvious, and so is2)⇒3). 3)⇒2) follows by the polarization identity:

$$\begin{aligned} (Uw|Uv) &= \tfrac{1}{4}\textstyle\sum_{j=0}^{3}(-\mathrm{i})^j (Uw + \mathrm{i}^j Uv | Uw + \mathrm{i}^j Uv), \\ (w|v) &= \tfrac{1}{4}\textstyle\sum_{j=0}^{3}(-\mathrm{i})^j (v + \mathrm{i}^j w | v + \mathrm{i}^j w). \end{aligned}$$

□

6.12 Unitary operators

Theorem 6.46 *Let $U \in B(\mathcal{V}, \mathcal{W})$. The following properties are equivalent:*
1) $U^*U = UU^* = \mathbb{1}$;
2) U is a surjective isometry;
3) U is bijective and $U^* = U^{-1}$.

Definition 6.47 *An operator satisfing the properties of Theorem 6.46 is called unitary.*

Proposition 6.48 *Let $\mathcal{V}$ be finite dimensional and $V \in B(\mathcal{V})$ isometric. Then V is unitary.*

Proof. We have $\dim \mathrm{Ker} V + \dim \mathrm{Ran}\, V = \dim \mathcal{V}$. $\mathrm{Ker} V = \{0\}$, since V is isometric. Hence $\dim \mathrm{Ran}\, V = \dim \mathcal{V}$. But $\mathcal{V}$ is finite dimensional, hence $\mathrm{Ran}\, V = \mathcal{V}$. □

Example 6.49 *Let (e_i), $i = 1, 2, \ldots$ be the canonical basis in $L^2(\mathbb{N})$. Put*

$$Te_i := e_{i+1}.$$

Then T is isometric but not unitary. It is called the unitalateral shift.

6.13 Normal operators

Let $A \in B(\mathcal{V}, \mathcal{W})$. We say that A is normal if $AA^* = A^*A$.

Theorem 6.50 *Let $A \in B(\mathcal{V})$ be normal. Then*

$$\mathrm{sr}(A) = \|A\|.$$

Proof. We compute using twice (6.4):

$$\|A^2\|^2 = \|A^{2*}A^2\| = \|(A^*A)^2\| = \|A^*A\|^2 = \|A\|^4.$$

Thus $\|A^{2^n}\| = \|A\|^{2^n}$. Hence, using the formula (4.2) for the spectral radius of A we get $\|A^{2^n}\|^{2^{-n}} = \|A\|$. □

Note that selfadjoint and unitary operators are normal. However, the set of normal operators is much more complicated than the set of self-adjoint operators, which is a real vector space, and the set of unitary operators, which is a group.

Theorem 6.51 (1) *U is unitary iff U is normal and* $\mathrm{sp}U \subset \{z \;:\; |z| = 1\}$*.*
(2) *A is self-adjoint iff A is normal and* $\mathrm{sp}A \subset \mathbb{R}$*.*

Proof. (1)⇒: Clearly, U is normal.
U is an isometry, hence $\mathrm{sp}U \subset \{|z| \leq 1\}$.
U^{-1} is also an isometry, hence $\mathrm{sp}U^{-1} \subset \{|z| \leq 1\}$. This implies $\mathrm{sp}U \subset \{|z| \geq 1\}$.
(1)⇐: Clearly, $\mathrm{sr}U = 1$. Likewise, using the spectral mapping theorem (Thm 4.11 (6)) we see that $\mathrm{sr}U^{-1} = 1$. Hence, by Thm 6.50 and the normality of U and U^{-1} we have $\|U\| = \|U^{-1}\| = 1$. Thus

$$\|v\| = \|U^{-1}Uv\| \leq \|Uv\| \leq \|v\|.$$

So, $\|v\| = \|Uv\|$. This means that U is an invertible isometry.
(2)⇒ was proven in Theorem 6.33.
(2)⇐: Let A be normal and $\mathrm{sp}A \subset \mathbb{R}$. We can find $\lambda > 0$ such that $\lambda\|A\| < 1$. Hence $1 + \mathrm{i}\lambda A$ is invertible. It is easy to check that $U := (1 - \mathrm{i}\lambda A)(1 + \mathrm{i}\lambda A)^{-1}$ is normal. By the spectral mapping theorem, $\mathrm{sp}U \subset \{|z| = 1\}$. Hence, by (1), it is unitary. Now

$$\begin{aligned} A = -\mathrm{i}\lambda^{-1}(1-U)(1+U)^{-1} \quad &= -\mathrm{i}\lambda^{-1}(U^*U - U)(UU^* + U)^{-1} \\ &= \mathrm{i}\lambda^{-1}(1 - U^*)(1 + U^*)^{-1} = A^*. \end{aligned}$$

□

Theorem 6.52 (Fuglede) *Let $A, B \in B(\mathcal{V})$ and let B be normal. Then $AB = BA$ implies $AB^* = B^*A$.*

Proof. For $\lambda \in \mathbb{C}$, the operator $U(\lambda) := \mathrm{e}^{\lambda B^* - \overline{\lambda} B} = \mathrm{e}^{-\overline{\lambda} B}\mathrm{e}^{\lambda B^*}$ is unitary. Moreover, $A = \mathrm{e}^{\overline{\lambda} B} A \mathrm{e}^{-\overline{\lambda} B}$. Hence

$$\mathrm{e}^{-\lambda B^*} A \mathrm{e}^{\lambda B^*} = U(-\lambda) A U(\lambda) \tag{6.10}$$

is a uniformly bounded analytic function. Hence is constant. Differentiating it wrt λ we get $[A, B^*] = 0$. □

6.14 Normal operators as multiplication operators

In finite dimensions we have the following elementary characterization of self-adjoint/unitary/normal operators.

Theorem 6.53 (1) *Suppose that $\mathcal{V}$ is a finite dimensional Hilbert space and $B \in B(\mathcal{V})$. Let $\mathrm{sp}B = \{b_1, \dots, b_k\}$. Then B is normal iff $1\!\!1_{b_j}(B)$ are orthogonal projections and*

$$B = \sum_{j=1}^{k} b_j 1\!\!1_{b_j}(B).$$

(2) *B is self-adjoint iff $b_j \in \mathbb{R}$.*

(3) *B is unitary iff $|b_j| = 1$.*

Example 6.54 *Let I be a set and let $(b_i)_{i \in I}$ be a bounded complex sequence. Define the operator B on $l^2(I)$ by*

$$(Bx)_i := b_i x_i, \quad i \in I.$$

We then have

$$\begin{aligned} \mathrm{sp_p}(B) &= \{b_i \ : \ i \in I\}, \\ \mathrm{sp}B &= \{b_i \ : \ i \in I\}^{\mathrm{cl}}, \\ \|B\| &= \sup\{|b_i| \ : \ i \in I\}. \end{aligned}$$

B is normal. B is self-adjoint iff b_i are real for all $i \in I$. B is unitary iff $|b_i| = 1$ for all $i \in I$.

Note that Thm 6.53 can be reformulated as follows: If the dimension of a Hilbert space is $n < \infty$, then a normal/self-adjoint/unitary is always unitarily eqivalent to an operator of the form described in Example (6.54) with $I = \{1, \dots, n\}$. If the dimension of a Hilbert space is infinite, normal/self-adjoint/unitary operators can be nonequivalent to an operator from Example (6.54). In the following example we show a more general form of such operators. In Chapter 7 we will show that in an arbtrary dimension every normal/self-adjoint/unitary operator is unitary equivalent to that described in example 6.55.

Example 6.55 *Let $(X, \mathcal{F}, \mu)$ be a space with a σ-finite measure and $f \in L^\infty(X)$. Define the operator T_f on $L^2(X, \mu)$ by*

$$(T_f x)(t) := f(t)x(t).$$

We then have

$$\begin{aligned}
\mathrm{sp_p}(T_f) &= \{z : \ \mu(f^{-1}\{z\}) > 0\}, \\
\mathrm{sp}T_f &= \left\{z : \ \mu\big(f^{-1}\{w \in \mathbb{C} \ : \ |w - z| < \epsilon\}\big) > 0, \ \ \textit{for all} \ \ \epsilon > 0\right\}, \\
\|T_f\| &= \|f\|_\infty.
\end{aligned}$$

T_f is normal. T_f is self-adjoint iff $f(x)$ are real for almost all $x \in X$. T_f is unitary iff $|f(x)| = 1$ for almost all $x \in X$.

Remark 6.56 *The σ-finiteness of the measure is needed only for the characterization of the point spectrum. More generally, it is enough to assume that the measure is* sum-finite*, with the same conclusions.*

The following two facts are obvious for operators of the form of Ex. 6.55. For general normal operators, the only way I know to prove them is to pass through the spectral theorem, which will be proven in the next chapter.

Proposition 6.57 *Let $A \in B(\mathcal{V})$ be normal and $\alpha, \beta \in \mathbb{C}$. Then*

$$\mathrm{sp}(\alpha A + \beta A^*) = \{\alpha z + \beta \overline{z} : \ z \in \mathrm{sp}A\}.$$

Theorem 6.58 *If $A \in B(\mathcal{V})$ is normal, then*

$$(\mathrm{Num}A)^{\mathrm{cl}} = \mathrm{ch}(\mathrm{sp}A). \tag{6.11}$$

Note that Thm 6.58 is a generalization of Thm 11.2.

6.15 Convergence

Let (A_j) be a sequence of operators in $B(\mathcal{V}, \mathcal{W})$.

(1) We say that (A_j) is norm convergent to A iff $\lim\limits_{j\to\infty} \|A_j - A\| = 0$. In this case we write

$$\lim_{j\to\infty} A_j = A.$$

(2) We say that (A_j) is strongly convergent to A iff $\lim\limits_{j\to\infty} \|A_j v - Av\| = 0$, $v \in \mathcal{V}$. In this case we write

$$\mathrm{s-}\lim_{j\to\infty} A_j = A.$$

(3) We say that (A_j) is weakly convergent to A iff $\lim\limits_{j\to\infty} |(w|A_j v)-(w|Av)| = 0$, $v \in \mathcal{V}$, $w \in \mathcal{W}$. In this case we write

$$\mathrm{w-}\lim_{j\to\infty} A_j = A.$$

Theorem 6.59 *Let (U_j) be a sequence of unitary operators*

(1) *If (U_j) is norm convergent, then its limit is unitary.*

(2) *If (U_j) is strongly convergent, then its limit is isometric.*

(3) *If (U_j) is weakly convergent, then its limit is a contraction.*

Theorem 6.60 (1) *Norm convergence implies strong convergence.*

(2) *Strong convergence implies weak convergence.*

(3) *Let (A_n) be a weakly convergent sequence of operators in $B(\mathcal{V})$. Then it is uniformly bounded.*

(4) *If (A_n) is a norm convergent sequence, then so is $(A_n)^*$ and*

$$\big(\lim_{n\to\infty} A_n\big)^* = \lim_{n\to\infty} A_n.$$

(5) *If (A_n) is a weakly convergent sequence, then so is $(A_n)^*$ and*

$$\big(\mathrm{w-}\lim_{n\to\infty} A_n\big)^* = \mathrm{w-}\lim_{n\to\infty} A_n.$$

(6) *If (A_n) and (B_n) are norm convergent sequences, then so is $A_n B_n$ and*

$$\lim_{n\to\infty} A_n \lim_{n\to\infty} B_n = \lim_{n\to\infty} A_n B_n.$$

(7) *If (A_n) and (B_n) are strong convergent sequences, then so is $(A_n B_n)$ and*

$$\Big(\mathrm{s-}\lim_{n\to\infty} A_n\Big)\Big(\mathrm{s-}\lim_{n\to\infty} B_n\Big) = \mathrm{s-}\lim_{n\to\infty} A_n B_n.$$

Proof. (3) follows from the uniform boundedness principle. □

Theorem 6.61 *Let (A_j) be a sequence of operators in $B(\mathcal{V})$ weakly convergent to A. Then*

$$\mathrm{Num} A \subset \bigcap_k \Big(\bigcup_{j>k} \mathrm{Num} A_j\Big)^{\mathrm{cl}}.$$

In particular, if A_j are self-adjoint, then so is A; if A_j are positive, then so is A.

Remark 6.62 *So far in this subsection we could almost everywhere replace the term "sequence " by "net". The exceptions are Thm 6.60 (3), which is in general not true for nets, and Thm 6.60 (7), where we need to assume that* (A_n) *is uniformly bounded.*

Example 6.63 *In* $L^2(\mathbb{N})$*, let* $(e_1, e_2, \dots)$ *be the canonical basis. Set*

$$\begin{aligned} U_n e_j &= e_{j+1}, \quad j = 1, \dots, n-1; \\ U_n e_n &= e_1; \\ U_n e_j &= e_j, \quad j = n+1, \dots; \\ U e_j &= e_{j+1}, \quad j = 1, \dots. \end{aligned}$$

Then U_n *are unitary,* $\mathrm{s}-\lim\limits_{n\to\infty} U_n = U$ *is not. Moreover.* $\mathrm{sp}U_n = \{\exp(\mathrm{i}2\pi/n) \ : \ j = 1, \dots, n\}$ *and* $\mathrm{sp}U = \{|z| \leq 1\}$*.*

Example 6.64 *In* $L^2(\mathbb{Z})$*, let* e_i*,* $i \in \mathbb{Z}$ *be the canonical basis. Set* $U_n e_j = e_{j+n}$*,* $j \in \mathbb{Z}$*. Then* U_n *are unitary,* $\mathrm{w}-\lim\limits_{n\to\infty} U_n = 0$*. Moreover,* $\mathrm{sp}U_n = \{|z| = 1\}$*,* $\mathrm{sp}U = \{0\}$*.*

6.16 Monotone convergence of selfadjoint operators

Theorem 6.65 (1) *Let* $\{A_\lambda \ : \ \lambda \in \Lambda\}$ *be a family of self-adjoint operators, which is uniformly bounded. Then there exists the smallest self-adjoint operator* A *such that* $A_\lambda \leq A$*. We will denote it* $\mathrm{lub}\{A_\lambda \ : \ \lambda \in \Lambda\}$ *(*lub *stands for the least upper bound).*

(2) *If* (A_n) *is an increasing bounded sequence of self-adjoint operators, then*

$$\mathrm{lub}\{A_n \ : \ n = 1, 2, \dots\} = \mathrm{s}-\lim_n A_n.$$

Proof. Let $\|A_\lambda\| \leq c$. For each $v \in \mathcal{V}$, $(v|A_\lambda v)$ is bounded by $c\|v\|^2$. Hence it has a supremum. Thus we can define $\mathfrak{a}(v) := \sup_\lambda (v|A_\lambda v)$.

Let $(v, w) \mapsto \mathfrak{a}(v, w)$ be defined by the polarization identity. Let $v, w \in \mathcal{V}$. We can find a sequence (A_n) in the family $\{A_\lambda \ : \ \lambda \in \Lambda\}$ such that

$$(v + \mathrm{i}^j w|A_n(v + \mathrm{i}^j w)) \quad \to \quad \mathfrak{a}(v + \mathrm{i}^j w), \quad j = 0, 1, 2, 3.$$

Then we see that

$$\mathfrak{a}(v, w) = \lim_{n\to\infty} (v|A_n w). \tag{6.12}$$

Thus $(v, w) \mapsto \mathfrak{a}(v, w)$ is a sesquilinear form. It is clearly bounded by c. Hence it defines a unique bounded operator A. It is evident that A is the smallest self-adjoint operator greater than A_n. This ends the proof of (1).

Let us prove (2). Since $A - A_n \geq 0$, we have

$$(A - A_n)^2 = (A - A_n)^{\frac{1}{2}}(A - A_n)(A - A_n)^{\frac{1}{2}} \leq \|A - A_n\|(A - A_n).$$

Besides, $\|A - A_n\| \leq 2c$. Now

$$\|(A - A_n)v\|^2 = (v|(A - A_n)^2 v) \leq \|A - A_n\|(v|(A - A_n)v) \to 0.$$

□

Chapter 7

Spectral theorems

7.1 Continuous functional calculus for self-adjoint and unitary operators

Let X be a compact Hausdorff space. The space of continuous functions on X with the norm $\|\cdot\|_\infty$ is denoted by $C(X)$. It is a complete normed commutative $*$-algebra.

Remark 7.1 *$C(X)$ is a commutative C^*-algebra. Note, however, that we will not use the theory of C^*-algebras. Compact Hausdorff spaces that we will use will be typically subsets of $\mathbb{R}^n$.*

In Sect. 4.4 we introduced a calculus for holomorphic functions of an arbitrary bounded operator on a Banach space. We will see that the holomorphic calculus extends to continuous functions for normal operators.

Let B be normal. Obviously

$$(z_1 - B)(\overline{z}_2 - B^*) = (\overline{z}_2 - B^*)(z_1 - B).$$

We multiply this with $(\overline{z}_2 - B^*)^{-1}(z_1 - B)^{-1}$ from the left and with $(z_1 - B)^{-1}(\overline{z}_2 - B^*)^{-1}$ from the right obtaining

$$(z_1 - B)^{-1}(\overline{z}_2 - B^*)^{-1} = (\overline{z}_2 - B^*)^{-1}(z_1 - B)^{-1}.$$

So $f(B)$ is normal for $f \in \mathrm{Hol}\big(\mathrm{sp}(B)\big)$. By the spectral mapping theorem, $\mathrm{sp}f(B) = f(\mathrm{sp}(B))$. Therefore, by Thm 6.50,

$$\|f(B)\| = \mathrm{sr}f(B) = \sup\{|f(z)| \ : \ z \in \mathrm{sp}B\} = \|f\|_\infty.$$

We first restrict ourselves to self-adjoint and unitary operators. We postpone the treatment of general normal operators to later sections.

Theorem 7.2 *Let $A \in B(\mathcal{V})$ be self-adjoint. Then there exists a unique continuous unital homomorphism*

$$C(\mathrm{sp}(A)) \ni f \mapsto f(A) \in B(\mathcal{V}) \tag{7.1}$$

such that

(1) $\mathrm{id}(A) = A$ *if* $\mathrm{id}(x) = x$, $x \in \mathrm{sp}(A)$.
Moreover, we have

(2) $f(A)^* = f^*(A)$, *where* $f^*(x) := \overline{f(x)}$, $x \in \mathrm{sp}A$.

(3) *If* $f \in \mathrm{Hol}(\mathrm{sp}(A))$, *then* $f(A)$ *coincides with* $f(A)$ *defined in (4.4).*

(4) $\mathrm{sp}(f(A)) = f(\mathrm{sp}(A))$.

(5) $g \in C(f(\mathrm{sp}(A))) \Rightarrow g \circ f(A) = g(f(A))$.

(6) $\|f(A)\| = \|f\|_\infty$.

(7) $f(A)$ *are normal.*

Proof. Clearly, (7.18) is uniquely defined on polynomials. Let $f \in C(\mathrm{sp}A)$. There exists a sequence f_n of polynomials such that $\|f_n - f\|_\infty \to 0$. Noting that $\|f_n(A) - f_m(A)\| = \|f_n - f_m\|_\infty$, we check that $f_n(A)$ is a Cauchy sequence. We set

$$f(A) := \lim_{n\to\infty} f_n(A).$$

We easily check that the definition of $f(A)$ does not depend on the choice of the sequence and verify all the properties described in the theorem. □

An almost identical theorem is true for unitary operators, with almost the same proof (where instead of usual polynomials we need to use polynomials in z and z^{-1}).

Theorem 7.3 *Let* $U \in B(\mathcal{V})$ *be unitary. Then there exists a unique continuous homomorphism*

$$C(\mathrm{sp}(U)) \ni f \mapsto f(U) \in B(\mathcal{V}) \tag{7.2}$$

such that

(1) $\mathrm{id}(U) = U$ *if* $\mathrm{id}(z) = z$, $z \in \mathrm{sp}(A)$.
Moreover, we have

(2) $f(U)^* = f^*(U)$, *where* $f^*(z) := \overline{f(z)}$, $z \in \mathrm{sp}U$.

and properties analogous to (3)-(7) of the previous theorem.

Remark 7.4 *Theorem 7.3 has a generalization (Thm 7.26) to an arbitrary normal operator* B. *However, this requires a more complicated proof, which will be given in later sections. What is easy and follows by an essentially the same proof is a weaker statement obtained from Theorem 7.3 by replacing* $C(\mathrm{sp}B)$ *with* $C_{\mathrm{hol}}(B)$.

Here we use the following notation: If K *is a compact subset of* $\mathbb{C}$, *then* $C_{\mathrm{hol}}(K)$ *denotes the completion of* $\mathrm{Hol}(K)$ *in* $C(K)$. *Note that if* K *is a subset of a line or a circle, then* $C_{\mathrm{hol}}(K) = C(K)$. *This simplifies functional calculus for self-adjoint and unitary operators.*

7.2 Projector valued measures

Let $(X, \mathcal{F})$ be a set with a σ-field. Let $\mathcal{V}$ be a Hilbert space. We say that

$$\mathcal{F} \ni D \mapsto P(D) \in \mathrm{Proj}(\mathcal{V}) \tag{7.3}$$

is an orthoprojection valued measure (PVM) on $\mathcal{V}$ iff

(1) $P(\emptyset) = 0$;

(2) If $D_1, D_2, \dots \in \mathcal{F}$ are disjoint, and $D = \overset{\infty}{\underset{i=1}{\cup}} D_i$, then $P(D) = \mathrm{s-}\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} P(D_j)$.

We call $P(X)$ the support of the orthoprojection valued measure (7.3).

Theorem 7.5 *For any $D, C \in \mathcal{F}$ we have*

$$P(D)P(C) = P(D \cap C).$$

Proof. First consider the case $D \cap C = \emptyset$. By (2)

$$P(D \cup C) = P(D) + P(C).$$

Hence $P(D) + P(C)$ is an orthoprojection. Hence $\big(P(D) + P(C)\big)^2 = P(D) + P(C)$. This implies

$$P(D)P(C) + P(C)P(D) = 0. \tag{7.4}$$

Multiplying from both sides by $P(C)$ we get $2P(C)P(D)P(C) = 0$ Multiplying (7.4) from the left by $P(C)$ we get $P(C)P(D) = -P(C)P(D)P(C)$. Thus $P(C)P(D) = 0$.

Next consider the case $D \subset C$. Then

$$P(C) = P(D) + P(C \backslash D).$$

Using $P(D)P(C\backslash D) = 0$ we see that $P(C)P(D) = P(D)$.

Finally, consider arbitrary D, C. Then

$$P(D)P(C) = \big(P(D\backslash C) + P(D \cap C)\big)\big(P(C\backslash D) + P(D \cap C)\big) = P(D \cap C).$$

□

Theorem 7.6 *Let $\mathcal{F} \ni D \mapsto P(D)$ be a PVM and let $\mathcal{L}^\infty(X)$ denote the space of bounded measurable functions on X. Then there exists a unique contractive $*$-homomorphism*

$$\mathcal{L}^\infty(X) \ni f \mapsto \int f(x) \mathrm{d}P(x) \in B(\mathcal{V}) \tag{7.5}$$

such that $\int 1_D(x) \mathrm{d}P(x) = P(D)$, $D \in \mathcal{F}$.

Proof. If f is an elementary function, that is a finite linear combination of characteristic functions of measurable sets

$$f = \sum_{j=1}^{n} \lambda_j 1_{D_j},$$

then clearly

$$\int f(x)\mathrm{d}P(x) = \sum_{j=1}^{n} \lambda_j P(D_j).$$

For such functions the multiplicativity of (7.5) is obvious.

Then we use the fact that elementary functions are dense in $\mathcal{L}^\infty(X)$ in the supremum norm. □

For any $w \in \mathcal{V}$ we define its spectral measure as

$$\mathcal{F} \ni D \mapsto \mu_w(D) := (w|P(D)w)$$

is a finite measure. Clearly, we have

Theorem 7.7 *For any $f \in \mathcal{L}^\infty(X)$,*

$$\int f(x)\mathrm{d}\mu_w(x) = \Big(w|\int f(x)\mathrm{d}P(x)w\Big).$$

Here is a version of the Lebesgue dominated convergence theorem for spectral integrals:

Theorem 7.8 *If $f_n \to f$ pointwise, $|f_n| \leq c$, then*

$$\mathrm{s}-\lim_{n\to\infty} \int f_n(x)\mathrm{d}P(x) = \int f(x)\mathrm{d}P(x).$$

Proof.

$$\begin{aligned} &\Big\|\Big(\int f(x)\mathrm{d}P(x) - f(x)\mathrm{d}P(x)\Big)v\Big\|^2 \\ = &\int \mathrm{d}\mu_v(x)|f(x) - f_n(x)|^2. \end{aligned} \tag{7.6}$$

Now $|f(x) - f_n(x)|^2 \leq 4c^2$, $\lim\limits_{n\to\infty} |f(x) - f_n(x)|^2 = 0$ and the measure $\mathrm{d}\mu_v$ is finite. Hence (7.6) converges to zero by the Lebesgue dominated convergence theorem. □

Theorem 7.9 *Let $(X_1, \mathcal{F}_1, P_1)$ and $(X_2, \mathcal{F}_2, P_2)$ be two spectral measures. Then there exsts a unique measure $(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1P_2)$ such that*

$$(P_1P_2)(D_1 \times D_2) = P_1(D_1)P_2(D_2).$$

7.3 Continuous and singular PVM's

Let $\mathcal{F} \ni D \mapsto P(D)$ be a PVM on $\mathcal{V}$.

Assume that all one-element sets (and hence all countable sets) belong to $\mathcal{F}$. We say that $\mathcal{F} \ni D \mapsto P(D)$ is continuous iff $P(\{x\}) = 0$ for all $x \in X$. It is pure point if $P(D) = \sum\limits_{x \in D} P_{\{x\}}$.

For any PVM $D \mapsto P(D)$ we set

$$P_{\rm p} := \sum_{x \in X} P_{\{x\}}, \quad P_{\rm c} := 1\!\!1 - P_{\rm p}.$$

Then

$$\begin{aligned} \mathcal{F} \ni D &\mapsto P_{\rm c}(D) := P_{\rm c} P(D), \\ \mathcal{F} \ni D &\mapsto P_{\rm p}(D) := P_{\rm p} P(D) \end{aligned}$$

are respectively continuous and pure point. They are called respectively the continuous and pure point part of the measure $D \mapsto P(D)$.

Theorem 7.10 *Suppose that $\mathcal{V}$ is separable. Then there exists a countable set $I \subset X$, such that $P_{\rm p} = P(I)$.*

Fix a measure μ on $(X, \mathcal{F})$. We say that $D \mapsto P(D)$ is μ-singular if

$$P(D) = \sup\{P(C) \ : \ C \subset D, \ \mu(C) = 0\}, \ D \in \mathcal{F}.$$

We say that P is μ-continuous if

$$\mu(D) = 0 \ \Rightarrow \ P(D) = 0. \tag{7.7}$$

For any PVM $D \mapsto P(D)$ we set

$$P_{\mu{\rm s}} := \sup\{P(N) \ : \ \mu(N) = 0\}, \quad P_{\mu{\rm c}} := 1\!\!1 - P_{\mu{\rm s}}.$$

Then

$$\begin{aligned} \mathcal{F} \ni D &\mapsto P_{\mu{\rm c}}(D) := P_{\mu{\rm c}} P(D), \\ \mathcal{F} \ni D &\mapsto P_{\mu{\rm s}}(D) := P_{\mu{\rm s}} P(D) \end{aligned}$$

are respectively absolutely continuous and the singular part of $D \mapsto P(D)$.

Theorem 7.11 *Suppose that $\mathcal{V}$ is separable. Then there exists a set $N \in \mathcal{I}$, such that $P_{\rm s} = P(N)$.*

Remark 7.12 *If μ is the counting measure then $P_{\mu{\rm s}} = P_{\rm p}$, $P_{\mu{\rm c}} = P_{\rm c}$.*

Remark 7.13 *We say that $\mathcal{I} \subset \mathcal{F}$ is an ideal if $\mathcal{I}$ is countably additive and $C \in \mathcal{F}$, $D \in \mathcal{I}$ and $C \subset D$ implies $C \in \mathcal{I}$. If μ is a measure then the family of measure zero sets is an ideal of $\mathcal{F}$. Obviously, the decomposition of $D \mapsto P(D)$ into its μ-continuous and μ-singular part does not need a measure, but only an ideal.*

The most important application of the above concepts is when μ is the Lebesgue measure. Then one-element sets are contained in the σ-field and have measure zero. In this case one says simply singular instead of μ-singular and absolutely continuous instead of μ-continuous. Instead of $P_{\mu\mathrm{c}}$ one writes P_{ac} and instead of $P_{\mu\mathrm{s}}$ one writes P_{s} Clearly,

$$P_{\mathrm{p}} \leq P_{\mathrm{s}}, \quad P_{\mathrm{c}} \leq P_{\mathrm{ac}}.$$

We set

$$P_{\mathrm{sc}} := P_{\mathrm{s}} P_{\mathrm{c}},$$

Thus

$$\mathbb{1} = P_{\mathrm{p}} + P_{\mathrm{sc}} + P_{\mathrm{ac}}$$

gives a decomposition of our PVM in its pure point part, singular continuous part and absolutely continuous part.

7.4 Projector valued Riesz-Markov theorem

Let X be a compact Hausdorff space, $\mathcal{V}$ a Hilbert space and $\gamma : C(X) \to B(\mathcal{V})$ a unital $*$-homomorphism.

We define the upper orthoprojection valued measure associated with γ as follows. For any open $U \subset X$ we define

$$P_U^{\mathrm{up}} := \sup\{\gamma(f) \ : \ 0 \leq f \leq 1_U, \ f \in C(X)\}.$$

For any $D \subset X$ we set

$$P_D^{\mathrm{up}} := \inf\{P_U^{\mathrm{up}} \ : \ U \text{ is open }, \ D \subset U\}.$$

We define the lower orthoprojection valued measure associated with γ as follows. For any closed $C \subset X$ we define

$$P_C^{\mathrm{low}} := \inf\{\gamma(f) \ : \ 1_C \leq f, \ f \in C(X)\}.$$

For any $D \subset X$ we set

$$P_D^{\mathrm{low}} := \sup\{P_C^{\mathrm{low}} \ : \ C \text{ is closed }, \ C \subset D\}.$$

We say that $D \subset X$ is γ-measurable if $P_D^{\mathrm{up}} = P_D^{\mathrm{low}}$. The family of γ-measurable sets is denoted $\mathcal{F}_\gamma$. For such sets D we set $P_D = P_D^{\mathrm{up}} = P_D^{\mathrm{low}}$.

Theorem 7.14 (1) *P_D^{up} and P_D^{low} are orthoprojections for any $D \subset X$.*

(2) *$\mathcal{F}_\gamma$ is a σ-field containing Borel sets.*

(3) *$\mathcal{F}_\gamma \ni D \mapsto P_D \in \mathrm{Proj}(\mathcal{V})$ is an orthoprojection valued measure with support $\mathbb{1}$.*

(4) *$C(X) \subset \mathcal{L}^\infty(X)$ and if $f \in C(X)$, then $\gamma(f) = \int f(x) \mathrm{d}P(x)$.*

7.5 Alternative approaches to the orthoprojection valued Riesz-Markov theorem

One can construct the spectral integral directly from γ as follows.

We define the upper integral as follows. If f is a lower semicontinuous function on X, we set

$$\int^{\mathrm{up}} f(x)\mathrm{d}P(x) := \sup\{\gamma(g) \ : \ g \in C(X), \quad g \leq f\}.$$

If f is an arbitrary function, we set

$$\int^{\mathrm{up}} f(x)\mathrm{d}P(x) := \inf\left\{\int^{\mathrm{up}} g(x)\mathrm{d}P(x) \ : \ g \text{ is lower semicontinuous and } f \leq g\right\}.$$

We define the lower integral as follows. If f is a upper semicontinuous function on X, we set

$$\int^{\mathrm{low}} f(x)\mathrm{d}P(x) := \inf\{\gamma(g) \ : \ g \in C(X), \quad f \leq g\}.$$

If f is an arbitrary function, we set

$$\int^{\mathrm{low}} f(x)\mathrm{d}P(x) := \sup\left\{\int^{\mathrm{low}} g(x)\mathrm{d}P(x) \ : \ g \text{ is upper semicontinuous and } g \leq f\right\}.$$

Theorem 7.15 *A function f on X is $\mathcal{F}_\gamma$-measurable iff*

$$\int^{\mathrm{up}} f(x)\mathrm{d}P(x) = \int^{\mathrm{low}} f(x)\mathrm{d}P(x) \tag{7.8}$$

and then (7.8) equals

$$\int f(x)\mathrm{d}P(x).$$

One can also construct the spectral integral using the Riesz-Markov for usual measures. For any $w \in \mathcal{V}$,

$$C(X) \ni f \mapsto (w|\gamma(f)w)$$

is a positive functional on X. By the Riesz-Markov theorem it defines a unique Radon measure on X, which we will call μ_w.

Theorem 7.16 *If f is γ-measurable, then it is measurable for measure μ_w for any $w \in \mathcal{V}$, and then,*

$$\left(w\middle|\int f(x)\mathrm{d}P(x)\ w\right) = \int f(x)\mathrm{d}\mu_w(x).$$

7.6 Spectral theorem for bounded Borel functions

If $A \in B(\mathcal{V})$ is a self-adjoint operator, then we have the unital $*$-homomorphism

$$C(\mathrm{sp}(A)) \ni f \mapsto f(A) \in B(\mathcal{V}).$$

Applying the projection valued Riesz-Markov Theorem we obtain a PVM $D \mapsto P(D)$. The σ-field of measurable sets contains all Borel subsets of $\mathrm{sp}(A)$. By Theorem 7.6 we can define for $f \in \mathcal{L}^\infty(\mathrm{sp}(A)$

$$f(A) := \int f(x)\mathrm{d}P(x).$$

In particular,

$$P(D) = 1_D(A),$$

for the characteristic function of a Borel set D. Thus, we do not need the notation P_D, instead we will write $1_D(A)$.

Thus we can extend the spectral theorem from continuous to bounded Borel functions:

Theorem 7.17 *Let $A \in B(\mathcal{V})$ be self-adjoint. Then there exists a unique continuous unital homomorphism*

$$C(\mathrm{sp}(A)) \ni f \mapsto f(A) \in B(\mathcal{V}) \tag{7.9}$$

such that

(1) $\mathrm{id}(A) = A$ *if* $\mathrm{id}(x) = x$, $x \in \mathrm{sp}(A)$.

(2) *If* $f_n \to f$ *pointwise,* $|f_n| \leq c$, *then*

$$\mathrm{s-}\lim_{n\to\infty} \int f_n(A) = \int f(A).$$

Moreover, we have

(3) $f(A)^* = f^*(A)$.

(4) *If* $f \in C(\mathrm{sp}(A))$, *then* $f(A)$ *coincides with* $f(A)$ *defined in (7.2).*

(5) $\mathrm{sp}(f(A)) \subset f(\mathrm{sp}(A))$.

(6) $g \in \mathcal{L}^\infty(f(\mathrm{sp}(A))) \Rightarrow g \circ f(A) = g(f(A))$.

(7) $\|f(A)\| \leq \|f\|_\infty$.

(8) $f(A)$ *are normal.*

We can define the projections $1_{\mathrm{ac}}(A)$, $1_{\mathrm{sc}}(A)$, $1_{\mathrm{p}}(A)$. Note that $1_{\mathrm{p}}(A)$ is the projection onto the closed span of eigenvectors of A.

7.7 Spectral theorem in terms of L^2 spaces

Theorem 7.18 *Let $A \in B(\mathcal{V})$ be a self-adjoint operator. Then there exists a family of Radon measures μ_i, $i \in I$, on* sp$A$ *and a unitary operator $U : \bigoplus_{i\in I} L^2(\mathrm{sp}A, \mu_i) \to \mathcal{V}$ such that*

$$\big(U^* A U \psi\big)_i(x) = x\psi_i(x).$$

Proof. Step 1. If $v \in \mathcal{V}$, the cyclic subspace for v is defined as $\mathcal{V}_v := \{f(A)v \ : \ f \in C(\mathrm{sp}A)\}^{\mathrm{cl}}$. Note that $\mathcal{V}_v$ is a closed linear subspace invariant wrt $f(A)$ and $\mathcal{V}_v^\perp$ is also invariant wrt $f(A)$.

We easily see that there exists a family of nonzero vectors $\{v_i \ : \ i \in I\}$ such that $\mathcal{V} = \bigoplus_{i\in I} \mathcal{V}_{v_i}$.

Step 2. Let μ_i be the spectral measure for the vector v_i. The unitary operator U is defined by $Uf := \sum_{i\in I} f(A)v_i$. □

Remark 7.19 *An essentially identical theorem is true if we replace the self-adjoint operator A by a unital $*$-homomorphism $\gamma : C(X) \to B(\mathcal{V})$ for a compact set X.*

7.8 Ideals in commutative C^*-algebras

Let Y be a closed subset of X. Let $C_Y(X)$ denote the set of functions vanishing on Y.

We view $C(X)$ as a commutative C^*-algebra.

Theorem 7.20 (1) *$C_Y(X)$ is a closed ideal of $C(X)$.*

(2)

$$C(X)/C_Y(X) \ni F + C_Y(X) \mapsto F\Big|_Y \in C(Y) \tag{7.10}$$

is an isometric $*$-homomorphism.

The following theorem describes a kind of a converse to above theorem:

Theorem 7.21 *Let $\mathfrak{N}$ be a closed ideal of $C(X)$. Set*

$$Y := \bigcap_{F\in\mathfrak{N}} F^{-1}(0)$$

or, equivalently,

$$x \notin Y \ \Leftrightarrow \ \text{there exists } H \in \mathfrak{N} \text{ such that } H(x) \neq 0.$$

Then Y is closed and $\mathfrak{N} = C_Y(X)$.

7.9 Spectrum of a $*$-homomorphisms of $C(X)$

Let X be a compact Hausdorff space. Let $\mathcal{V}$ be a Hilbert space and $\gamma : C(X) \to B(\mathcal{V})$ be a unital $*$-homomorphism. That means, $\gamma(FG) = \gamma(F)\gamma(G)$, $\gamma(1) = \mathbb{1}$ and $\gamma(\overline{F}) = \gamma(F)^*$.

Proposition 7.22 *γ is a contraction.*

Proof. Let $z \notin F(X)$. Then $(z - F)^{-1} \in C(X)$. Thus $\gamma((z - F)^{-1})$ is the inverse of $z - \gamma(F)$. Thus $\mathrm{sp}\gamma(F) \subset F(X)$, and hence $\mathrm{sr}\gamma(F) \leq \|F\|_\infty$.

Clearly, $\gamma(F)$ is normal, and hence $\|\gamma(F)\| = \mathrm{sr}\gamma(F)$. □

Clearly, $\mathrm{Ker}\gamma$ is a closed ideal of $C(X)$. We define the spectrum of the homomorphism γ as the closed subset of X associated with $\mathrm{Ker}\gamma$, that is

$$\mathrm{sp}\gamma = \bigcap_{F \in \mathrm{Ker}\gamma} F^{-1}(0). \tag{7.11}$$

Equivalently,

$$x \notin \mathrm{sp}\gamma \ \Leftrightarrow \ \text{there exists } H \text{ such that } \gamma(H) = 0 \text{ and } H(x) \neq 0.$$

Clearly, $\mathrm{sp}\gamma$ is a closed subset of X and $\mathrm{Ker}\gamma = C_{\mathrm{sp}\gamma}(X)$. Using the identification (7.10), we see that there exists a a unique unital $*$-isomorphism $\gamma_{\mathrm{red}} : C(\mathrm{sp}\gamma) \to B(\mathcal{V})$ such that

$$\gamma(F) = \gamma_{\mathrm{red}}\left(F\Big|_{\mathrm{sp}\gamma}\right), \quad F \in C(X)$$

Obviously, γ_{red} is injective. γ is injective iff $\mathrm{sp}\gamma = X$ iff $\gamma = \gamma_{\mathrm{red}}$.

Theorem 7.23 (1) *γ is injective iff it is isometric.*

(2) *γ_{red} is isometric.*

(3) *Let $F \in C(X)$. Then $F(\mathrm{sp}\gamma) = \mathrm{sp}\gamma(F)$.*

Proof. We first show (3).

$F(\mathrm{sp}\gamma) \subset \mathrm{sp}\gamma(F)$: Suppose that $z \notin \mathrm{sp}(\gamma(F))$. If $z \notin F(X)$, there is nothing to prove. Let $x_0 \in X$ such that $F(x_0) = z$. Let

$$U_c := \left\{ x \in X \ : \ |F(x) - z| < c\big\|\big(z - \gamma(F)\big)^{-1}\big\|^{-1} \right\}.$$

Let $c < 1$. There exists $H \in C(X)$ such that $\mathrm{supp}H \subset U_c$ and $H(x_0) = 1$. Choose c_1 such that $c < c_1 < 1$. We can find $G \in C(X)$, $0 \leq G \leq 1$, $G = 1$ on U_c and $\mathrm{supp}G \subset U_{c_1}$. Then

$$\begin{aligned} \|\gamma(G)\| &\leq \|(z - \gamma(F))^{-1}\|\|\gamma((z - F)G)\| \\ &\leq \|(z - \gamma(F))^{-1}\|\|(z - F)G\|_\infty \leq c_1 < 1. \end{aligned} \tag{7.12}$$

But $\gamma(H) = \Gamma(HG^n) = \gamma(H)\gamma(G^n)$ and $\gamma(G^n) \to 0$ by (7.12). Hence, $\gamma(H) = 0$. Hence $x \notin \mathrm{sp}\gamma$.

Let $z \notin F(\mathrm{sp}\gamma)$: $Z := \{x \in X \ : \ F(x) = z\}$ is a closed subset of X disjoint from $\mathrm{sp}\gamma$. Hence, there exists a function $G \in C(X)$ such that $G = 1$ on $\mathrm{sp}\gamma$ and $G = 0$ on a neighborhood of Z. Clearly, $G - 1 \in C_{\mathrm{sp}\gamma}(X)$, hence $\gamma(G) = 1$. Now $G(z - F)^{-1} \in C(X)$. We have

$$\gamma(z - F)\gamma((z - F)^{-1}G) = \gamma(G) = 1.$$

Hence $\gamma((z - F)^{-1}G)$ is the inverse of $z - \gamma(F)$. This means that $z \notin \mathrm{sp}\gamma(F)$. Thus (3) is proven.

By (3), $\mathrm{sr}\gamma(F) = \|F\big|_{\mathrm{sp}\gamma}\|_\infty$. By the normality of $\gamma(F)$, $\|\gamma(F)\| = \mathrm{sr}\gamma(F)$. This proves (2).

(1) follows from (2). □

7.10 Commuting self-adjoint operators

Suppose that $\{A_1, \dots, A_n\}$ is a family of commuting self-adjoint operators in $B(\mathcal{V})$. Clearly, if $f_i \in C(\mathrm{sp}(A_i))$, $i = 1, \dots, n$, then $f_i(A_i)$ commute with one another. The joint spectrum of this family, denoted by $\mathrm{sp}(A_1, \dots, A_n)$ is the subset of $\mathrm{sp}(A_1) \times \cdots \times \mathrm{sp}(A_n)$ defined as follows: $(x_1, \dots, x_n)$ does not belong to $\mathrm{sp}(A_1, \dots, A_n)$ iff there exist functions $f_i \in C(\mathrm{sp}A_i)$, with $f_i(x_i) \neq 0$, $j = 1, \dots, n$ such that $f_1(A_1) \cdots f_n(A_n) = 0$.

Theorem 7.24 (1) *There exists a unique continuous unital $*$-homomorphism*

$$C(\mathrm{sp}(A_1, \dots, A_n) \ni g \mapsto g(A_1, \dots, A_n) \in B(\mathcal{V}) \tag{7.13}$$

such that if $\mathrm{id}_j(z_i \ : \ i \in I) = z_j$, *then*

$$\mathrm{id}_j(A_1, \dots, A_n) = A_j.$$

(2) *(7.13) is injective and satisfies*

$$\|g(A_1, \dots, A_n)\| = \|g\|_\infty.$$

(3) $g(A_1, \dots, A_n)^* = g^*(A_1, \dots, A_n)$, *where* $g^*(x_1, \dots, x_n) := \overline{g(x_1, \dots, x_n)}$.

Proof. First we show that there exists a unique unital $*$-homomorphism

$$C\Big(\mathrm{sp}(A_1 \times \cdots \times \mathrm{sp}(A_n)\Big) \ni F \mapsto F(A_1, \dots, A_n) \in B(\mathcal{V}) \tag{7.14}$$

that satisfies (1), and (3), and instead of (2) satisfies

$$\|F(A_1, \dots, A_n)\| \leq \|F\|. \tag{7.15}$$

Indeed, on holomorphic functions we define (7.14) in the obvious way. By the weak spectral mapping theorem of Theorem 4.20,

$$\mathrm{sp}F(A_1,\dots,A_n)\subset F(\mathrm{sp}A_1\times\cdots\times A_n).$$

Hence, $\mathrm{sr}F(A_1,\dots,A_n)\ \le\ \|F\|_\infty$. But $F(A_1,\dots,A_n)$ is normal and hence $\|F(A_1,\dots,A_n)\|=\mathrm{sr}F(A_1,\dots,A_n)$. This proves (7.15) for holomorphic functions. By the Stone-Weierstrass Theorem, polynomials are dense in continuous functions, therefore we can extend the definition of (7.14) to
$C\Big(\mathrm{sp}(A_1\times\cdots\times\mathrm{sp}(A_n)\Big)$.

Thus we have a unital $*$-homomorphism from $C\Big(\mathrm{sp}(A_1\times\cdots\times\mathrm{sp}(A_n)\Big)$ to $B(\mathcal{V})$. We easily see that $\mathrm{sp}(A_1,\dots,A_n)$ is precisely the spectrum of this homomorphism, as defined in (7.11). Therefore, we can reduce (7.14), obtaining the isometric $*$-homomorphism (7.13). □

7.11 Functional calculus for a single normal operator

Let B be a normal operator. Then $B^{\mathrm{R}}:=\frac{1}{2}(B+B^*)$ and $B^{\mathrm{I}}:=\frac{1}{2\mathrm{i}}(B-B^*)$ are commuting self-adjoint operators. Therefore, we can define the joint spectrum $\mathrm{sp}(B^{\mathrm{R}},B^{\mathrm{I}})$ and the homomorphism

$$C\big(\mathrm{sp}(B^{\mathrm{R}},B^{\mathrm{I}})\big)\ni f\mapsto f(B^{R},B^{\mathrm{I}})\in B(\mathcal{V}).\tag{7.16}$$

Clearly, $B=B^{\mathrm{R}}+\mathrm{i}B^{\mathrm{I}}$. Define

$$\mathbb{R}^2\ni(x,y)\mapsto j(x,y):=x+\mathrm{i}y\in\mathbb{C}.\tag{7.17}$$

Proposition 7.25 *We have*

$$j(\mathrm{sp}(B^{\mathrm{R}},B^{\mathrm{I}}))=\mathrm{sp}B.$$

Proof. Let $(x_0,y_0)\not\in\mathrm{sp}(B^{\mathrm{R}},B^{\mathrm{I}})$. The function

$$(x,y)\mapsto(x_0+\mathrm{i}y_0-x-\mathrm{i}y)^{-1}$$

is continuous outside of (x_0,y_0). In particular, it belongs to $C\big(\mathrm{sp}(B^{\mathrm{R}},B^{\mathrm{I}})\big)$. Hence

$$(x_0+\mathrm{i}y_0-B^{\mathrm{R}}-\mathrm{i}B^{\mathrm{I}})^{-1}=(x_0+\mathrm{i}y_0-B)^{-1}$$

is well defined by Theorem 7.24. Therefore, $x_0+\mathrm{i}y_0\not\in\mathrm{sp}(B)$.

Let $x_0+\mathrm{i}y_0\not\in\mathrm{sp}B$. Suppose that $(x_0,y_0)\in\mathrm{sp}(B^{\mathrm{R}},B^{\mathrm{I}})$. Let $0<c<1$ $f\in C(\mathrm{sp}(B^{\mathrm{R}},B^{\mathrm{I}}))$ with $f(x_0,y_0)=1$, $\|f\|_\infty=1$ and

$$\{f\neq0\}\subset\big\{(x,y)\in\mathbb{R}^2\ :\ (x-x_0)^2+(y-y_0)^2<c^2\|(x_0+\mathrm{i}y_0-B)^{-1}\|^{-2}\big\}$$

Clearly,

$$\|f(B^{\mathrm{R}}, B^{\mathrm{I}})(x_0 + \mathrm{i}y_0 - B)\| \leq c\|(x_0 + \mathrm{i}y_0 - B)^{-1}\|^{-1}.$$

Hence,

$$\begin{aligned} \|f(B^{\mathrm{R}}, B^{\mathrm{I}})\| &\leq \|f(B^{\mathrm{R}}, B^{\mathrm{I}})(x_0 + \mathrm{i}y_0 - B)\|\|(x_0 + \mathrm{i}y_0 - B)^{-1}\| \leq c \\ &< 1 \leq \|f\|_\infty. \end{aligned}$$

But the functional calculus on the joint spectrum is isometric, hence this is a contradiction. Thus, $(x_0, y_0) \notin \mathrm{sp}(B^{\mathrm{R}}, B^{\mathrm{I}})$. $\square$

Theorem 7.26 *Let $B \in B(\mathcal{V})$ be normal. Then there exists a unique continuous homomorphism*

$$C(\mathrm{sp}(B)) \ni f \mapsto f(B) \in B(\mathcal{V}) \tag{7.18}$$

such that

(1) $\mathrm{id}(B) = B$ *if* $\mathrm{id}(z) = z$, $z \in \mathrm{sp}(B)$.

(2) $f(B)^* = f^*(B)$, *where* $f^*(z) := \overline{f(z)}$, $z \in \mathrm{sp}B$.
Moreover, we have

(3) *If $f \in \mathrm{Hol}(\mathrm{sp}(B))$, then $f(B)$ coincides with $f(B)$ defined in (4.4).*

(4) $\mathrm{sp}(f(B)) = f(\mathrm{sp}(B))$.

(5) $g \in C(f(\mathrm{sp}(B))) \Rightarrow g \circ f(B) = g(f(B))$.

(6) $\|f(B)\| = \|f\|_\infty$.

(7) $f(B)$ *are normal.*

Proof. For $g \in C(\mathrm{sp}B)$, using the functional calculus (7.16) and the map (7.17), we set

$$g(B) := g \circ j(\mathrm{Re}B, \mathrm{Im}B).$$

$\square$

7.12 Functional calculus for a family of commuting normal operators

Suppose that $B_1, \dots, B_n$ is a family of commuting normal operators in $B(\mathcal{V})$. Set $B_i^{\mathrm{R}} := \frac{1}{2}(B_i + B_i^*)$ and $B_i^{\mathrm{I}} := \frac{1}{2\mathrm{i}}(B_i - B_i^*)$. Then by the Fuglede theorem, $B_1^{\mathrm{R}}, B_1^{\mathrm{I}}, \dots, B_n^{\mathrm{R}}, B_n^{\mathrm{I}}$ is a family of commuting self-adjoint operators. Thus we have the $*$-homomorphism

$$C\big(\mathrm{sp}(B_1^{\mathrm{R}}, B_i^{\mathrm{I}}, \dots B_n^{\mathrm{R}}, B_n^{\mathrm{I}}) \ni G \mapsto G(B_1^{\mathrm{R}}, B_1^{\mathrm{I}}, \dots, B_n^{\mathrm{R}}, B_n^{\mathrm{I}}) \in B(\mathcal{V}) \tag{7.19}$$

We define

$$\begin{aligned} &\mathrm{sp}(B_1, \dots, B_n) \\ := \ &\{x_1 + \mathrm{i}y_1, \dots, x_n + \mathrm{i}y_n \ : \ (x_1, y_1, \dots, x_n, y_n) \in sp(B_1^{\mathrm{R}}, B_i^{\mathrm{I}}, \dots B_n^{\mathrm{R}}, B_n^{\mathrm{I}}). \end{aligned}$$

We obtain:

Theorem 7.27 *Let* $\{B_i \ : \ i \in I\}$ *be a family of commuting normal operators in a* $B(\mathcal{V})$*. Then*

(1) $\{(z_1, \dots, z_n) \in \mathrm{sp}B_1 \times \cdots \times \mathrm{sp}B_n$ *does not belong to* $\mathrm{sp}(B_1, \dots, B_n)$ *iff there exist functions* $f_1 \in C(\mathrm{sp}B_1), \dots, f_n \in C(\mathrm{sp}(B_n)$ *with* $f_i(z_i) \neq 0$, $j = 1, \dots, n$ *such that* $f_1(B_1) \cdots f_n(B_n) = 0$.

(2) *There exists a unique continuous unital* $*$*-homomorphism*

$$C(\mathrm{sp}(B_1, \dots, B_n) \ni g \mapsto g(B_1, \dots, B_n) \in B(\mathcal{V}) \tag{7.20}$$

such that if $\mathrm{id}_j(z_i \ : \ i \in I) = z_j$*, then*

$$\mathrm{id}_j(B_i \ : \ i \in I) = B_j.$$

(3) *(7.20) is injective and satisfies*

$$\|g(B_1, \dots, B_n)\| = \|g\|_\infty.$$

Example 7.28 *Let* $(X, \mathcal{F}, \mu)$ *be a space with a measure. Let* $f : X \to \mathbb{C}^n$ *a Borel function. We say that* $(z_1, \dots, z_n) \in \mathbb{C}^n$ *belongs to the essential range of* f*, denoted* $(z_1, \dots, z_n) \in \mathrm{essRan}\, f$*, iff for any neighborhood* U *of* $(z_1, \dots, z_n)$ *we have* $\mu(f^{-1}(U)) \neq 0$*. Note that if* $f : X \to \mathbb{C}$ *is Borel, then* $\|f\|_\infty = \sup\{|f(x)| \ : \ x \in \mathrm{essRan}\, f\}$.

Let $f \in L^\infty(X)$*. Then*

$$L^2(X) \ni h \mapsto T_f h := fh \in L^2(X)$$

is a bounded normal operator with $\mathrm{sp}T_f = \mathrm{essRan}\, f$ *and* $\|T_f\| = \|f\|_\infty$*. The operator* T_f *is self-adjoint iff* $\mathrm{essRan}\, f \subset \mathbb{R}$*. It is unitary iff* $\mathrm{essRan}\, f \subset \{|z| = 1\}$.

Suppose that $(f_1, \dots, f_n)$ *is a family of functions in* $L^\infty(X)$*. Clearly, the operators* T_{f_i} *are normal operators commuting with one another. We have*

$$\mathrm{sp}(T_{f_1}, \dots, T_{f_n}\} = \mathrm{essRan}\,(f_1, \dots, f_n).$$

Chapter 8

Compact operators

8.1 Finite rank operators

This subsection can be viewed as an elementary introduction to compact operators.

Definition 8.1 *An operator $K \in B(\mathcal{X}, \mathcal{Y})$ is called a finite rank operator iff* $\dim \mathrm{Ran}\, K < \infty$.

Theorem 8.2 *Let $K \in B(\mathcal{X}, \mathcal{Y})$ be a finite rank operator. Then*

$$\dim \mathrm{Ran}\, K = \dim X/\mathrm{Ker} K.$$

Proof. Let $y_1, \dots, y_n$ be a basis in $\mathrm{Ran}\, K$. We can find $x_1, \dots, x_n \in X$ such that $Kx_i = y_i$. Then $\mathrm{Span}\{x_1, \dots, x_n\} \cap \mathrm{Ker} K = \{0\}$. Assume that $z \in X$. Then $Kz = \sum c_i y_i$. Thus $z - \sum c_i x_i \in \mathrm{Ker} K$. Hence $z \in \mathrm{Span}\{x_1, \dots, x_n\} + \mathrm{Ker} K$. □

Theorem 8.3 *Let $K \in B(\mathcal{X})$ be a finite rank operator. Then* $\mathrm{sp} K = \mathrm{sp}_{\mathrm{p}} K$. *Moreover,* $\mathrm{sp}_{\mathrm{ess}} K = \emptyset$ *if* $\dim \mathcal{X} < \infty$, *otherwise* $\mathrm{sp}_{\mathrm{ess}} K = \{0\}$.

Proof. Using the fact that $\dim \mathcal{X}/\mathrm{Ker} K$ is finite, we can find a finite dimensional subspace $\mathcal{Z}$ such that $\mathcal{X} = \mathrm{Ker} K \oplus \mathcal{Z}$. $\mathcal{Z}_1 := \mathcal{Z} + \mathrm{Ran}\, K$ is also finite dimensional. We have $K\mathcal{Z}_1 \subset \mathcal{Z}_1$. We can find a subspace $\mathcal{Z}_2$ such that $\mathcal{Z}_1 \oplus \mathcal{Z}_2 = \mathcal{X}$. Obviously, $\mathcal{Z}_2 \subset \mathrm{Ker} K$. □

8.2 Compact operators on Banach spaces

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces.

Definition 8.4 *$K \in B(\mathcal{X}, \mathcal{Y})$ is called a compact operator iff for any bounded sequence $x_1, x_2, \dots \in \mathcal{X}$ we can find a convergent subsequence from the sequence $Kx_1, Kx_2, \dots \in \mathcal{Y}$.*

Equivalent definition: if $(\mathcal{X})_1$ denotes the unit ball in $\mathcal{X}$, then $(K(\mathcal{X})_1)^{\mathrm{cl}}$ is a compact set. The set of compact operators from $\mathcal{X}$ to $\mathcal{Y}$ will be denoted $B_\infty(\mathcal{X}, \mathcal{Y})$.

Theorem 8.5 (1) *Let K be a compact operator. Let $(x_i)_{i \in I}$ be a bounded net weakly convergent to x. Then $\lim\limits_{i \in I} Kx_i = Kx$. ($K$ is weak-norm continuous on the unit ball).*

(2) *Let K be a compact operator. Let (x_n) be a sequence weakly convergent to x. Then $\lim\limits_{n \to \infty} Kx_n = Kx$.*

(3) *If A is bounded, K is compact, then AK and KA are compact.*

(4) *If K_n are compact and $\lim_{n \to \infty} K_n = K$, then K is compact.*

(5) *If K is finite rank, then K is compact.*

Proof. (1) Let $(x_i)_{i \in I}$ be a bounded net weakly convergent to x. Then $\mathrm{w-}\lim_{i \in I} Kx_i = Kx$ (because K is bounded). Hence, if Kx_i is convergent in norm, its only limit can be Kx.

Suppose that Kx_i is not convergent. Then there exists a subnet x_{i_j} such that $\|Kx_{i_j} - Kx\| > \epsilon > 0$. By compactness, we can choose a subsubnet $x_{i_{j_m}}$ such that $Kx_{i_{j_m}}$ is convergent. But it can be convergent only to Kx, which is impossible.

(3) is obvious, if we note that A maps a ball into a ball and a convergent sequence onto a convergent sequence.

(4) Let $x_1, x_2, \dots$ be a bounded sequence so that $\|x_n\| \leq C$. Below we will construct a double sequence $x_{n,k}$ such that, for any n, $x_{n+1,1}, x_{n+1,2}, \dots$ is a subsequence of $x_{n,1}, x_{n,2}, \dots$ and

$$\|Kx_{n,m} - Kx_{n,k}\| < (\min(m,k,n))^{-1}.$$

Eventually, the sequence $x_{n,n}$ is a subsequence of x_n such that $Kx_{n,n}$ satisfies the Cauchy condition.

Suppose that we have constructed $x_{n,m}$ up to the index n. We can find N such that $\|K - K_N\| < \frac{1}{3C(n+1)}$. We put $x_{n+1,m} = x_{n,m}$ for $m = 1, \dots n$. For $m > n$, we choose $x_{n+1,m}$ as the subsequence of $x_{n,m}$ such that $\|K_N x_{n+1,m} - K_N x_{n+1,k}\| < \frac{1}{3(n+1)}$ for $k, m > n$. Then for $m > n$

$$\begin{aligned} \|Kx_{n+1,m} - Kx_{n+1,k}\| &\leq \|Kx_{n+1,m} - K_N x_{n+1,m}\| + \|K_N x_{n+1,m} - K_N x_{n+1,k}\| \\ &\quad + \|K_N x_{n+1,k} - Kx_{n+1,k}\| \leq \tfrac{2C}{C3(n+1)} + \tfrac{1}{3(n+1)} = (n+1)^{-1}. \end{aligned}$$

(5) follows by the compactness of the ball in a finite dimensional space $\operatorname{Ran} K$. □

Note that $B_\infty(X)$ is a closed ideal of $B(\mathcal{X})$.

8.3 Compact operators on a Hilbert space

Theorem 8.6 *Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert spaces and $K \in B(\mathcal{X}, \mathcal{Y})$. TFAE:*

(1) *K is compact (i.e. $(K(\mathcal{X})_1)^{\mathrm{cl}}$ is compact).*

(2) *K maps bounded weakly convergent nets onto norm convergent sequences (K is weak-norm continuous on the unit ball).*

(3) *$K(\mathcal{X})_1$ is compact.*

(4) *Let (x_n) be a sequence weakly convergent to x. Then $\lim_{n\to\infty} Kx_n = Kx$.*

(5) *If $|K| := (K^*K)^{1/2}$, then $\mathrm{sp}_{\mathrm{ess}}|K| \subset \{0\}$.*

(6) *There exist orthonormal systems $x_1, x_2, \dots \in \mathcal{X}$ and $y_1, y_2, \dots \in \mathcal{Y}$ and a sequence of positive numbers $k_1, k_2, \dots$ convergent to zero such that*

$$K = \sum_{n=1}^{\infty} k_n |y_n)(x_n|.$$

(7) *There exists a sequence of finite rank operators K_n such that $K_n \to K$.*

Proof. (1)⇒(2), by Theorem 8.5, is true even in Banach spaces.

(2)⇒(3). In a Hilbert space $(\mathcal{X})_1$ is weakly compact. The image of a compact set under a continuous map is compact.

(3)⇒(1) is obvious.

(2)⇒(4) is obvious.

(4)⇒(5). Suppose (5) is not true. This means that for some $\epsilon > 0$, $\mathrm{Ran}\, \mathbb{1}_{[\epsilon,\infty[}(|K|)$ is infinite dimensional. Let $x_1, x_2, \dots$ be an infinite orthonormal system in $\mathrm{Ran}\, 1_{[\epsilon,\infty[}(A)$. Then x_n goes weakly to zero, but $\|Kx_n\| \geq \epsilon$.

(5)⇒(6). Let $x_1, x_2, \dots$ be an orthonormal system of eigenvectors of $|K|$ with eigenvalues k_n. Then set $y_n := k_n^{-1} K x_n$.

(6)⇒(7). It suffices to set $K_\epsilon := K 1_{[\epsilon,\infty[}(|K|)$. Then

$$\|K - K_\epsilon\| = \||K| 1_{[0,\epsilon]}(|K|)\| \leq \epsilon.$$

(7)⇒(1), by Theorem 8.5, is true for Banach spaces. □

(1)⇒(6) is sometimes called the *Hilbert-Schmidt Theorem.*

Corollary 8.7 (Schauder) *Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert spaces and $K \in B_\infty(\mathcal{X}, \mathcal{Y})$. Then $K^* \in B_\infty(\mathcal{Y}, \mathcal{X})$.*

Proof. It follows immediately from Theorem 8.6 (7).

8.4 The Fredholm alternative

Theorem 8.8 (Analytic Fredholm Theorem) *Let $\mathcal{V}$ be a Hilbert space, $\Omega \subset \mathbb{C}$ is open and connected. Let $\Omega \ni z \mapsto A(z) \in B_\infty(\mathcal{V})$ be an analytic function. Let $S := \{z \in \Omega \ : 1 - A(z) \ \text{ is not invertible } \}$ Then either*

(1) *$S=\Omega$, or*

(2) *S is discrete in Ω. Moreover, for $z\in S$, $\mathrm{Ker}(1-A(z))\neq\{0\}$ and the coefficients at the negative powers of the Laurent expansion of $(1-A(z))^{-1}$ are of finite rank. In particular, the residuum is of finite rank.*

Proof. Let $z_0\in\Omega$. We can find a finite rank operator F with $\|A(z_0)-F\|<1/2$. Let $\epsilon>0$ with $\|A(z)-A(z_0)\|<1/2$ for $|z-z_0|<\epsilon$. Thus $\|A(z_0)-F\|<1$ for $|z-z_0|<1$.

Set $G(z):=F(1+F-A(z))^{-1}$. We have

$$(1-G(z))(1+F-A(z))=1-A(z).$$

Thus $1-A(z)$ is invertible iff $1-G(z)$ is invertible and $\mathrm{Ker}(1-A(z))=\{0\}$ iff $\mathrm{Ker}(1-G(z))=\{0\}$.

Let P be the orthoprojection onto $\mathrm{Ran}\,F$. Set

$$\begin{aligned} G_0(z) &:= G(z)P &&= PG(z)P,\\ G_1(z) &:= G(z)(1-P) &&= PG(z)(1-P). \end{aligned}$$

Then

$$1-G(z)=1-G_0(z)-G_1(z)=(1-G_1(z))(1-G_0(z)),$$

and $(1-G_1(z))^{-1}=1+G_1(z)$. Hence, $1-G(z)$ is invertible iff $1-G_0(z)$ is and $\mathrm{Ker}(1-G(z))=\{0\}$ iff $\mathrm{Ker}(1-G_0(z))=\{0\}$. Since $G_0(z)$ is an analytic function in a fixed finite dimensional space, $1-G_0(z)$ is invertible iff $\det(1-G_0(z))\neq 0$ iff $\mathrm{Ker}(1-G_0(z))=\{0\}$. Thus $S=\{z\in\Omega\ :\ \det(1-G_0(z))\neq 0\}$.

Now we have

$$(1-A(z))^{-1}=(1+F-A(z))^{-1}(1-G_0(z))^{-1}(1+G_0(z)).$$

The first and third factor on the rhs are analytic in the neighborhood of z_0. Suppose that the middle term has a singularity at z_0. Then it is a pole of the order at most $\dim\mathrm{Ran}\,F$ and all the coefficients at the negative powers of its Laurent expansion are finite rank. □

Corollary 8.9 (Riesz-Schauder) *Let K be a compact operator on a Hilbert space. Then $\mathrm{sp}_{\mathrm{ess}}K=\{0\}$ if the space is infinite dimensional and $\mathrm{sp}_{\mathrm{ess}}K=\emptyset$ otherwise.*

Proof. We apply the Analytic Fredholm Theorem to $1-z^{-1}K$. □

8.5 Positive trace class operators

Let $\{v_i\}_{i\in I}$ be an orthonormal basis of a Hilbert space $\mathcal{V}$. Let $A\in B(\mathcal{V})$ and $A\geq 0$. Define

$$\mathrm{Tr}A:=\sum_{i\in I}(v_i|Av_i). \tag{8.1}$$

Theorem 8.10 *(8.1) does not depend on the basis.*

Proof. First note that if $A_\alpha \in B(\mathcal{V})$ is an increasing net, then

$$\sum_{i\in I}(v_i|Av_i) = \sup_\alpha \sum_{i\in I}(v_i|A_\alpha v_i). \tag{8.2}$$

Let $\{v_i \ : \ i \in I\}$ and $\{w_j \ : \ j \in J\}$ are orthonormal bases. Assume that $c < \sum\limits_{i\in I}(v_i|Av_i)$. By (8.2), we can find a finite subset $J_0 \subset J$ such that if P_0 is the projection onto $\mathrm{Span}\{w_j \ : \ j \in J_0\}$, then

$$c \leq \sum_{i\in I}(v_i|P_0AP_0v_i).$$

Now

$$\begin{aligned}\sum_{i\in I}(v_i|P_0AP_0v_i) &= \sum_{i\in I}\sum_{j,k\in J_0}(v_i|w_j)(w_j|Aw_k)(w_k|v_i)\\ &= \sum_{j\in J_0}(w_j|Aw_j) \leq \sum_{j\in J}(w_j|Aw_j).\end{aligned} \tag{8.3}$$

Above we used the fact that for any j,k

$$\sum_{i\in I}|(v_i|w_j)(w_j|Aw_k)(w_k|v_i)| \leq \|A\|,$$

which together with the finiteness of J_0 imples that the second sum in (8.3) is absolutely convergent, and also

$$\sum_{i\in I}(v_i|w_j)(w_k|v_i) = \delta_{j,k}.$$

This shows

$$\sum_{i\in I}(v_i|Av_i) \leq \sum_{j\in J}(w_j|Aw_j).$$

Of course, we can reverse the argument. $\square$

We will write $B_+^1(\mathcal{V})$ for the set of $A \in B_+(\mathcal{V})$ such that $\mathrm{Tr}A < \infty$.

Theorem 8.11 (1) *If $A, B \in B_+(\mathcal{V})$, then $\mathrm{Tr}(A + B) = \mathrm{Tr}A + \mathrm{Tr}B$. If $\lambda \in [0, \infty[$, then $\mathrm{Tr}\lambda A = \lambda\mathrm{Tr}A$, where $0\infty = 0$.*

(2) *Let $B \in B(\mathcal{V}, \mathcal{W})$. Then $\mathrm{Tr}B^*B = \mathrm{Tr}BB^*$.*

(3) *If $A \in B_+^1(\mathcal{V})$, and $B \in B(\mathcal{W}, \mathcal{V})$. Then $B^*AB \in B_+^1(\mathcal{W})$ and $\mathrm{Tr}B^*AB \leq \|B\|^2\mathrm{Tr}A$.*

(4) *If $A \in B_+^1(\mathcal{V})$, then A is compact.*

(5) *Let $(A_i \ i \in I)$ be an increasing net in $B_+(\mathcal{V})$ and $A = \mathrm{lub}A_i$. Then*

$$\mathrm{Tr}A = \sup\{\mathrm{Tr}A_i \ : \ i \in I\}.$$

(6) $\mathrm{Tr}A = \sum_{n=1}^\infty s_n(A)$.

Proof. (2) Let (v_i) and (w_j) be bases of $\mathcal{V}$ and $\mathcal{W}$. Then

$$\begin{aligned} \mathrm{Tr} B^*B &= \sum_i \sum_j (v|B^*w_j)(w_j|Bv_i) \\ = & \quad \sum_j \sum_i (w_j|Bv_i)(v_i|B^*w_j) \quad = \mathrm{Tr} BB^*, \end{aligned}$$

where all the terms in the sum are positive, which justifies the exchange of the order of summation.

(3) By (2), we have $\mathrm{Tr} B^*AB = \mathrm{Tr} A^{1/2}BB^*A^{1/2}$. Besides $A^{1/2}BB^*A^{1/2} \leq \|B\|^2 A$.

(4) If A has continuous spectrum, then there exists an infinite dimensional orthoprojection P and $\epsilon > 0$ such that $A \geq \epsilon P$. Then $\mathrm{Tr} A \geq \epsilon \mathrm{Tr} P = \infty$.

Hence A has just point spectrum. We have $\mathrm{Tr} A = \sum\limits_{i \in I} a_i$, where a_i are eigenvalues of A (counting their multiplicities). □

8.6 Hilbert-Schmidt operators

For $A \in B(\mathcal{V}, \mathcal{W})$ set

$$\|A\|_2 := (\mathrm{Tr} A^*A)^{\frac{1}{2}} = (\mathrm{Tr} AA^*)^{\frac{1}{2}}.$$

$B^2(\mathcal{V}, \mathcal{W})$ denotes the set of operators with a finite norm $\|A\|_2$. Clearly,

$$\|A\|_2 = \left(\sum_{n=1}^{\infty} s_n(A)^2 \right)^{1/2}.$$

If $(v_i)_{i \in I}$ and $(w_j)_{j \in J}$ are bases in $\mathcal{V}$ and $\mathcal{W}$, then

$$\|A\|_2 = \sum_{i \in I} \sum_{j \in J} |(w_j|Av_i)|^2. \tag{8.4}$$

$B^2(\mathcal{V}, \mathcal{W})$ is equipped with the scalar product

$$(A|B)_2 = \sum_{i \in I} \sum_{j \in J} \overline{(w_j|Av_i)} (w_j|Bv_i), \tag{8.5}$$

where we used $(v_i)_{i \in I}$ and $(w_j)_{j \in J}$ orthonormal bases in $\mathcal{V}$ and $\mathcal{W}$.

Proposition 8.12 *Let $A, B \in B^2(\mathcal{V}, \mathcal{W})$. Then (8.5) is finite and does not depend on a choice of bases.*

Proof. Clearly, (8.4) is the norm for (8.5). Hence the finiteness of (8.5) follows by the Schwarz inequality: $|(A|B)_2| \leq \|A\|_2 \|B\|_2$.

Next note that, for any $v \in \mathcal{V}$,

$$\|(A + \mathrm{i}^k B)v\|^2 \leq 2\|Av\|^2 + 2\|Bv\|^2.$$

Therefore,

$$\|(A + \mathrm{i}^k B)\|_2^2 \leq \|A\|_2^2 + \|B\|_2^2.$$

Hence if A, B are Hilbert-Schmidt, then so are $A + \mathrm{i}^k B$. Then we note that (8.5) equals

$$(A|B)_2 := \sum_{k=0}^{3} \frac{\mathrm{i}^k}{4} \mathrm{Tr}(A + \mathrm{i}^k B)^*(A + \mathrm{i}^k B), \tag{8.6}$$

which is basis independent. □

Remark 8.13 *In the next subsection we extend the notion of trace and (8.6) will be written simply as* $\mathrm{Tr}A^*B$.

Theorem 8.14 (1) *If $A \in B^2(\mathcal{V}, \mathcal{W})$, then A is compact.*

(2) *$B^2(\mathcal{V}, \mathcal{W})$ is a Hilbert space.*

(3) *If $\{v_i\}_{i\in I}$ is a basis in $\mathcal{V}$ and $\{w_j\}_{j\in J}$ is a basis in $\mathcal{W}$, then $|w_j)(v_i|$ is a basis in $B^2(\mathcal{V}, \mathcal{W})$.*

(4) *$B^2(\mathcal{V}, \mathcal{W}) \ni A \mapsto A^* \in B^2(\mathcal{W}, \mathcal{V})$ is a unitary map.*

(5) *If $A \in B^2(\mathcal{V}, \mathcal{W})$ and $B \in B(\mathcal{W}, \mathcal{X})$, then $BA \in B^2(\mathcal{V}, \mathcal{X})$.*

(6) *If (X, μ) and (Y, ν) are spaces with measurs and $\mathcal{V} = L^2(X, \mu)$, $\mathcal{W} = L^2(Y, \nu)$, then every operator $A \in B^2(\mathcal{V}, \mathcal{W})$ has the integral kernel $A(\cdot, \cdot) \in L^2(Y \times X, \nu \otimes \mu)$, ie.*

$$(w|Av) = \int\int \overline{w(y)} A(y, x) v(x) \mathrm{d}\mu(y) \mathrm{d}\mu(x)$$

The transformation $B^2(\mathcal{V}, \mathcal{W}) \ni A \mapsto A(\cdot, \cdot) \in L^2(Y \times X, \nu \otimes \mu)$ that to an operator associates its integral kernel is unitary.

Proof. (1) The operator A^*A is trace class, hence is compact. We can represent A^*A as

$$A^*A = \sum_{j=1}^{\infty} b_j |v_j)(v_j|,$$

with $b_j \to 0$.

If we set $w_j := Av_j$, then

$$A = \sum_{j=1}^{\infty} a_j |w_j)(v_j|,$$

with $|a_j|^2 = b_j$. Hence, $a_j \to 0$.

Let us show (2) and (3). Set $E_{ji} := |w_j)(v_i|$. We first check that it is an orthonormal system. If $A \in B^2(\mathcal{V}, \mathcal{W})$ is orthogonal to all E_{ji}, then all its matrix elements are zero. Hence $A = 0$.

Then we check that if a_{ji} belongs to $L^2(J \times I)$, then $\sum_{j\in J,\ i\in I} a_{ji} E_{ji}$ is the integral kernel of an operator in $B^2(\mathcal{V}, \mathcal{W})$. Hence, $B^2(\mathcal{V}, \mathcal{W})$ is isomorphic to

$L^2(J\times I)$. Hence it is a Hilbert space and $\{E_{ij}\ :\ i\in I, j\in J\}$ is its orthonormal basis. This proves (2) and (3), $\Box$

Theorem 8.15 *Suppose that $f,g\in L^\infty(\mathbb{R}^d)$ converge to zero at infinity. Then the operator $g(D)f(x)$ is compact.*

Proof. Let

$$f_n(x) := \begin{cases} f(x), & |x|<n \\ 0 & |x|\geq n, \end{cases}$$

$$g_n(\xi) := \begin{cases} g(\xi), & |\xi|<n \\ 0 & |\xi|\geq n, \end{cases}$$

$$g(D)f(x) = \mathcal{F}^* g(x)\mathcal{F} f(x).$$

$$\begin{aligned} \|g(x)\mathcal{F}f(x) - g_n(x)\mathcal{F}f_n(x)\| &\leq \|(g(x)-g_n(x))\mathcal{F}f(x)\| \\ &+\|g_n(x)\mathcal{F}(f(x)-f_n(x)\| \to 0. \end{aligned}$$

It suffices to show the compactness of $g_n(x)\mathcal{F}f_n(x)$. But its integral kernel equals

$$(2\pi)^{-\frac{1}{2}d} g_n(x)\mathrm{e}^{-\mathrm{i}xy} f_n(y),$$

which is square integrable . $\Box$

8.7 Trace class operators

Lemma 8.16 *Let $A_+, A_+' \in B_+^1(\mathcal{V})$, $A_-, A_-' \in B_+(\mathcal{V})$ satisfy $A_+ - A_- = A_+' - A_-'$. Then*

$$\mathrm{Tr}A_+ - \mathrm{Tr}A_- = \mathrm{Tr}A_+' - \mathrm{Tr}A_-'.$$

Proof. Clearly, $A_+ + A_-' = A_- + A_+' \in B_+(\mathcal{V})$. Thus

$$\mathrm{Tr}A_+ + \mathrm{Tr}A_-' = \mathrm{Tr}(A_+ + A_-') = \mathrm{Tr}(A_- + A_+') = \mathrm{Tr}A_- + \mathrm{Tr}A_+'.$$

$\Box$

By Lemma 8.16, we can uniquely extend the definition of trace as a function with values in $[-\infty,\infty]$ to operators in $B_{\mathrm{sa}}(\mathcal{V})$ that admit a decomposition $A = A_+ - A_-$, where $A_+, A_- \in B_+(\mathcal{V})$ and either B_+ or B_- belongs to $B_+^1(\mathcal{V})$, by setting

$$\mathrm{Tr}(A_+ - A_-) := \mathrm{Tr}A_+ - \mathrm{Tr}A_-.$$

We define $B^1(\mathcal{V}) := \mathrm{Span}B_+^1(\mathcal{V})$. Clearly, $B_+(\mathcal{V})\cap B^1(\mathcal{V}) = B_+^1(\mathcal{V})$.
Obviously, Tr is well defined and finite on $B^1(\mathcal{V})$.

Theorem 8.17 *Let $A \in B^1(\mathcal{V})$. Then for any orthonormal basis (v_i) in $\mathcal{V}$,*

$$\mathrm{Tr}A = \sum_{i\in I}(v_i|Av_i), \tag{8.7}$$

where the above series is absolutely convergent.

Proof. Let $A = A_+ - A_-$, where $A_+, A_- \in B^1_+(\mathcal{V})$. Then for any orthonormal basis $\sum_{i\in I}(v_i|A_\pm v_i)$ is finite, hence absolutely convergent. Thus (8.7) is the sum of two absolutely convergent series, and hence, absolutely convergent. □

Theorem 8.18 *$B, C \in B^2(\mathcal{V},\mathcal{W})$ implies $B^*C \in B^1(\mathcal{V})$ and $(B|C)_2 = \mathrm{Tr}B^*C = \mathrm{Tr}CB^*$.*

Proof. We know that $B + \mathrm{i}^{-k}C \in B^2(\mathcal{V},\mathcal{W})$. Hence $B^*C \in B^1(\mathcal{V})$ follows immediately from (8.6). $(B|C)_2 = \mathrm{Tr}B^*C = \mathrm{Tr}CB^*$ also follows from (8.6).

Theorem 8.19 *If $A \in B^1(\mathcal{V})$ and $B \in B(\mathcal{V})$, then $AB, BA \in B^1(\mathcal{V})$ and*

$$\mathrm{Tr}AB = \mathrm{Tr}BA.$$

Proof. It suffices to assume that $A \in B^1_+(\mathcal{V})$. $A^{1/2}$ and $BA^{1/2}$ belong to $B^2(\mathcal{V})$. Hence, using Theorem 8.18, we obtain

$$\begin{aligned}\mathrm{Tr}BA = \mathrm{Tr}(BA^{1/2})A^{1/2} \quad &= \mathrm{Tr}A^{1/2}(BA^{1/2}) \\ &= \mathrm{Tr}(A^{1/2}B)A^{1/2} \quad = \mathrm{Tr}A^{1/2}(A^{1/2}B) = \mathrm{Tr}AB.\end{aligned}$$

□

Theorem 8.20 *TFAE*

(1) $A \in B^1(\mathcal{V})$.

(2) $|A| \in B^1_+(\mathcal{V})$.

(3) *There exist $B, C \in B^2(\mathcal{V},\mathcal{W})$ such that $A = B^*C$.*

(4) $\sum\limits_{n=1}^{\infty} s_n(A) < \infty$.

(5) *For any orthonormal basis (v_i) in $\mathcal{V}$,*

$$\sum_{i\in I} |(v_i|Av_i)| < \infty.$$

Proof. Let $A = U|A|$ be the polar decomposition of A.

(1)⇒(2). Let $A \in B^1(\mathcal{V})$. Then $U^*A = |A| \in B^1(\mathcal{V})$. Since $|A| \in B_+(\mathcal{V})$, this also means that $A \in B^1_+(\mathcal{V})$.

(1)⇐(2). Let $A \in B(\mathcal{V})$ with $A \in B^1(\mathcal{V})$. Then $A = U|A|$ shows that $A \in B^1(\mathcal{V})$.

(2)⇒(3). $A = U|A|^{1/2}|A|^{1/2}$ with $U|A|^{1/2}, |A|^{1/2} \in B^2(\mathcal{V})$.

(2)⇐(3) is Theorem 8.18.

(1)⇒(5). Write $A = A_1 + \mathrm{i}A_1 - A_3 - \mathrm{i}A_4$, with $A_i \in B^1(\mathcal{V})$. We have $\sum(v_i|A_k v_i) < \infty$. Thus $(v_i|Av_i)$ is a linear combination of 4 absolutely convergent series.

(1)⇐(5). First assume that A is self-adjoint. Then $A = A_+ - A_-$ with $A_+A_- = A_-A_+ = 0$ and $A_+, A_- \in B_+(\mathcal{V})$. We have the decomposition $\mathcal{V} = \mathrm{Ran}\, 1_{]-\infty,0[}A) \oplus \mathrm{Ker}A \oplus \mathrm{Ran}\, 1_{]0,\infty[}A)$. Let $(v_1^-, v_2^-, \dots, v_1^0, v_2^0, \dots, v_1^+, v_2^+, \dots)$ be a basis that respects this decomposition. Then we compute that

$$\infty > \sum_{\epsilon=-,0,+} \sum_i |(v_i^\epsilon|Av_i^\epsilon)| = \mathrm{Tr}A_+ + \mathrm{Tr}A_-.$$

Thus $A_+, A_- \in B^1_+(\mathcal{V})$. Hence $A \in B^1(\mathcal{V})$.

If A is not necessarily self-adjoint, then consider $\mathrm{Re}A := \frac{1}{2}(A + A^*)$, $A := \frac{1}{\mathrm{i}2}(A - A^*)$. Then

$$\sum |(v_i|\mathrm{Re}Av_i)| + \sum |(v_i|\mathrm{Im}Av_i)| \leq 2\sum |(v_i|Av_i)| < \infty$$

Thus (5) is satisfied for $\mathrm{Re}A$, $\mathrm{Im}A$, and hence $\mathrm{Re}A, \mathrm{Im}A \in B^1(\mathcal{V})$. But $A = \mathrm{Re}A + \mathrm{i}\mathrm{Im}A$. □

For $A \in B^1(\mathcal{V})$ we set

$$\|A\|_1 := \mathrm{Tr}|A| = \sum_{n=1}^{\infty} \mathrm{s}_n(A).$$

Theorem 8.21 (1) *If $A \in B^1(\mathcal{V})$, $B \in B(\mathcal{V})$, then*

$$\|AB\|_1 \leq \|A\|_1\|B\|, \quad \|BA\|_1 \leq \|A\|_1\|B\|.$$

(2) *$B^1(\mathcal{V})$ is a Banach algebra.*

Proof. (1) Let $BA = W|BA|$ be the polar decomposition of BA and $A = U|A|$ be the polar decomposition of A. Note that $BU|A|^{1/2} \in B^2(\mathcal{V})$. Thus

$$\mathrm{Tr}|BA| = \mathrm{Tr}W^*BU|A|^{1/2}|A|^{1/2}| \leq \|W^*BU|A|^{1/2}\|_2\||A|^{1/2}\|_2.$$

Now

$$\||A|^{1/2}\|_2 = (\mathrm{Tr}|A|)^{1/2},$$

$$\|W^*BU|A|^{1/2}\|_2 \leq \|W^*BU\|\||A|^{1/2}\|_2 \leq \|B\|(\mathrm{Tr}|A|)^{1/2}.$$

(2) Let us prove the subadditivity. Let $A, B \in B^1(\mathcal{V})$ and $A+B = W|A+B|$ be the polar decomposition of $A+B$. Then, using $|A+B| = W^*(A+B)$,

$$\begin{aligned}\|A+B\|_1 = &\quad \mathrm{Tr}W^*(A+B) \\ \leq &\quad |\mathrm{Tr}W^*A| + \mathrm{Tr}W^*B| \quad \leq \|W^*\|\mathrm{Tr}|A| + \|W^*\|\mathrm{Tr}|B| = \mathrm{Tr}|A| + \mathrm{Tr}|B|.\end{aligned}$$

Thus $B^1(\mathcal{V})$ is a normed space.

Using $\|A\| \leq \|A\|_1$ we see, that (1) implies

$$\|AB\|_1 \leq \|A\|_1 \|B\|_1.$$

Thus $B^1(\mathcal{V})$ is a normed algebra.

Let A_n be a Cauchy sequence in the $\|\cdot\|_1$ norm. Then it is also Cauchy in the $\|\cdot\|$ norm. Thus there exists $\lim_{n\to\infty} A_n =: A \in B(\mathcal{V})$. Let $A - A_n = U_n|A - A_n|$ be the polar decomposition of $A - A_n$. Let P be a finite projection. Clearly, for fixed n, $\|A_m - A_n\|_1$ is a Cauchy sequence and thus $\lim_{m\to\infty} \|A_m - A_n\|_1$ exists.

$$\begin{aligned} \|P|A - A_n|P\|_1 &= \mathrm{Tr} PU^*(A - A_n)P \\ &= \lim_{m\to\infty} \mathrm{Tr} PU^*(A_m - A_n)P \quad \leq \lim_{m\to\infty} \|A_m - A_n\|_1. \end{aligned}$$

Since P was arbitrary,

$$\|A - A_n\|_1 \leq \lim_{m\to\infty} \|A_m - A_n\|_1 \to 0.$$

Hence $B^1(\mathcal{V})$ is complete. □

Theorem 8.22 *Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be sequences of vectors with $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$, $\sum_{n=1}^{\infty} \|y_n\|^2 < \infty$. Then $\sum_{n=1}^{\infty} |y_n)(x_n|$ is trace class.*

Proof. Let $e_1, e_2, \ldots$ be an orthonormal system. Define $A := \sum_{n=1}^{\infty} |x_n)(e_n|$, $B := \sum_{n=1}^{\infty} |y_n)(e_n|$. Then $\mathrm{Tr} A^*A = \sum \|x_n\|^2$ and $\mathrm{Tr} B^*B = \sum \|y_n\|^2$. Hence A, B are Hilbert-Schmidt. But $C = BA^*$. □

Chapter 9

Unbounded operators on Hilbert spaces

9.1 Graph scalar product

Let $\mathcal{V}$, $\mathcal{W}$ be Hilbert spaces. Let $A : \mathcal{V} \to \mathcal{W}$ be an operator with domain $\operatorname{Dom} A$. It is natural to treat $\operatorname{Dom} A$ as a space with the *graph scalar product*

$$(v_1|v_2)_A := (v_1|v_2) + (Av_1|Av_2).$$

Clearly, $\operatorname{Dom} A$ is a Hilbert space with the graph scalar product iff A is closed.

9.2 The adjoint of an operator

Definition 9.1 *Let $A : \mathcal{V} \to \mathcal{W}$ have a dense domain. Then $w \in \operatorname{Dom} A^*$, iff the functional*

$$\operatorname{Dom} A \ni v \mapsto (w|Av)$$

is bounded (in the topology of $\mathcal{V}$). Hence there exists a unique $y \in \mathcal{V}$ such that

$$(w|Av) = (y|v), \quad v \in \mathcal{V}.$$

The adjoint *of A is then defined by setting*

$$A^*w = y.$$

Theorem 9.2 *Let $A : \mathcal{V} \to \mathcal{W}$ have a dense domain. Then*

(1) *A^* is closed.*

(2) $\operatorname{Dom} A^*$ *is dense in $\mathcal{W}$ iff A is closable.*

(3) $(\operatorname{Ran} A)^\perp = \operatorname{Ker} A^*$.

(4) $\operatorname{Dom} A \cap (\operatorname{Ran} A^*)^\perp \supset \operatorname{Ker} A$.

Proof. Let $j : \mathcal{V} \oplus \mathcal{W} \to \mathcal{W} \oplus \mathcal{V}$, $j(v, w) := (-w, v)$. Note that j is unitary. We have

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^\perp.$$

Hence $\mathrm{Gr}\, A^*$ is closed. This proves (1).

Let us prove (2).

$$\begin{aligned} w \in (\mathrm{Dom}\, A^*)^\perp \quad & \Leftrightarrow (0, w) \in (\mathrm{Gr}\, A^*)^\perp = j(\mathrm{Gr}\, A)^{\perp\perp} \\ & \Leftrightarrow (w, 0) \in (\mathrm{Gr}\, A)^{\perp\perp} = (\mathrm{Gr}\, A)^{\mathrm{cl}}. \end{aligned}$$

Proof of (3):

$$\begin{aligned} w \in \mathrm{Ker} A^* \quad & \Leftrightarrow (A^* w|v) = 0, \quad v \in \mathcal{V} \\ & \Leftrightarrow (A^* w|v) = 0, \quad v \in \mathrm{Dom}\, A \\ & \Leftrightarrow (w|Av) = 0, \quad v \in \mathrm{Dom}\, A \\ & \Leftrightarrow w \in (\mathrm{Ran}\, A)^\perp. \end{aligned}$$

Proof of (4)

$$\begin{aligned} v \in \mathrm{Ker} A \quad & \Leftrightarrow (w|Av) = 0, \quad w \in \mathcal{W} \\ & \Rightarrow (w|Av) = 0, \quad w \in \mathrm{Dom}\, A^* \\ & \Leftrightarrow (A^* w|v) = 0, \quad w \in \mathrm{Dom}\, A^* \\ & \Leftrightarrow v \in (\mathrm{Ran}\, A^*)^\perp. \end{aligned}$$

Theorem 9.3 *Let $A : \mathcal{V} \to \mathcal{W}$ be closable with a dense domain. Then*

(1) *A^* is closed with a dense domain.*

(2) $A^* = (A^{\mathrm{cl}})^*$.

(3) $(A^*)^* = A^{\mathrm{cl}}$

(4) $(\mathrm{Ran}\, A)^\perp = \mathrm{Ker} A^*$. *Hence A^* is injective iff $\mathrm{Ran}\, A$ is dense.*

(5) $(\mathrm{Ran}\, A^*)^\perp = \mathrm{Ker} A$. *Hence A is injective iff $\mathrm{Ran}\, A^*$ is dense.*

Proof. (1) was proven in Theorem 9.2.

To see (2) note that

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^\perp = j((\mathrm{Gr}\, A)^{\mathrm{cl}})^\perp = \mathrm{Gr}\, A^{\mathrm{cl}*}.$$

To see (3) we use

$$\mathrm{Gr}\, (A^{**}) = j^{-1}\left(j(\mathrm{Gr}\, A)^\perp\right)^\perp = (\mathrm{Gr}\, A)^{\perp\perp} = (\mathrm{Gr}\, A)^{\mathrm{cl}}.$$

(4) is proven in Theorem 9.2.

To prove (5) note that in the second line of the proof of Theorem 9.2 (4) we can use the fact that $\mathrm{Dom}\, A^*$ is dense in $\mathcal{W}$ to replace $\Rightarrow$ with $\Leftrightarrow$. $\square$

9.3 Inverse of the adjoint operator

Theorem 9.4 *Let A be densely defined, closed, injective and with a dense range. Then*

(1) *A^{-1} is densely defined, closed, injective and with a dense range.*

(2) *A^* is densely defined, closed, injective and with a dense range.*

(3) *$(A^*)^{-1} = (A^{-1})^*$.*

Proof. (1) and (2) sum up previously proven facts.

To prove (3), recall the maps $\tau, j : \mathcal{V} \oplus \mathcal{W} \to \mathcal{W} \oplus \mathcal{V}$. We have

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^{\perp}, \quad \mathrm{Gr}\, A^{-1} = \tau(\mathrm{Gr}\, A).$$

Hence

$$\mathrm{Gr}\, A^{-1*} = j(\tau(\mathrm{Gr}\, A))^{\perp} = \tau^{-1}(j(\mathrm{Gr}\, A)^{\perp}) = \mathrm{Gr}\, A^{*-1}.$$

□

Theorem 9.5 *Let $A : \mathcal{V} \to \mathcal{W}$ be densely defined and closed. Then the following conditions are equivalent:*

(1) *A is invertible.*

(2) *A^* is invertible.*

(3) *For some $c > 0$, $\|Av\| \geq c\|v\|$, $v \in \mathcal{V}$ and $\|A^*w\| \geq c\|v\|$, $w \in \mathcal{W}$.*

Proof. (1)⇒(2). Let A be invertible. Then $A^{-1} \in B(\mathcal{W}, \mathcal{V})$. Hence, $A^{-1*} \in B(\mathcal{V}, \mathcal{W})$.

Clearly, the assumptions of Theorem 9.4 are satisfied, and hence $A^{*-1} = A^{-1*}$. Therefore, $A^{*-1} \in B(\mathcal{V}, \mathcal{W})$.

(1)⇐(2). A^* is also densely defined and closed. Hence the same arguments as above apply.

It is obvious that (1) and (2) imply (3). Let us prove that (3)⇒(1). $\|A^*v\| \geq c\|v\|$ implies that $\mathrm{Ker} A^* = \{0\}$. Hence $(\mathrm{Ran}\, A)^{\perp}$ is dense. This together with $\|Av\| \geq c\|v\|$ implies that $\mathrm{Ran}\, A = \mathcal{W}$, and consequently, A is invertible. □

Theorem 9.6 *Let $A : \mathcal{V} \to \mathcal{W}$ be a densely defined and*

$$\|Av\| \geq c\|v\|, \ v \in \mathrm{Dom}\, A$$

Then the following are equivalent:

(1) *A is invertible.*

(2) *A is closable and $\mathrm{Ran}\, A = \mathcal{W}$.*

(3) *A is closed and $\mathrm{Ran}\, A$ is dense in $\mathcal{W}$.*

(4) *A is closed and $\mathrm{Ker} A^* = \{0\}$.*

Theorem 9.7 *Let $A : \mathcal{V} \to \mathcal{W}$ be densely defined and closed. Then $\mathrm{sp}^{\mathrm{ext}}(A) = \overline{\mathrm{sp}^{\mathrm{ext}}(A^*)}$.*

9.4 The adjoint of a product of operators

Proposition 9.8 (1) *Let A, B be densely defined operators, so that BA is also densely defined. Then*

$$(BA)^* \supset A^*B^*. \tag{9.1}$$

(2) *Suppose that A is densely defined and B is bounded everywhere defined. Then BA is densely defined and*

$$(BA)^* = A^*B^*. \tag{9.2}$$

Proof. (1): Let $u \in \mathrm{Dom}\, A^*B^*$. Then

$$\begin{aligned} u \in \mathrm{Dom}\, B^*, \quad & (B^*u|v) = (u|Bv),\ v \in \mathrm{Dom}\, B, \\ B^*u \in \mathrm{Dom}\, A^*, \quad & (A^*B^*u|w) = (B^*u|Aw),\ w \in \mathrm{Dom}\, A. \end{aligned}$$

Hence, if

$$w \in \mathrm{Dom}\, A, \quad Aw \in \mathrm{Dom}\, B, \tag{9.3}$$

then

$$(A^*B^*u|w) = (u|BAw). \tag{9.4}$$

But (9.3) means that $w \in \mathrm{Dom}\, BA$. Hence, by (9.4) $u \in \mathrm{Dom}(BA)^*$ and

$$(A^*B^*u|w) = \big((BA)^*u|w\big), \tag{9.5}$$

which means $A^*B^*u = (BA)^*u$.

(2): (See eg. Dereziński-Gérard). It is obvious that BA is densely defined. Let $u \in \mathrm{Dom}(BA)^*$. This means that

$$|(u|BAw)| \le c\|w\|, \quad w \in \mathrm{Dom}\, BA = \mathrm{Dom}\, A. \tag{9.6}$$

Using the boundedness of B, we can write

$$(u|BAw) = (B^*u|Aw), \tag{9.7}$$

and hence (9.6) implies

$$|(B^*u|Aw)| \le c'\|w\|, \quad w \in \mathrm{Dom}\, A. \tag{9.8}$$

Hence, $B^*u \in \mathrm{Dom}\, A^*$ and

$$(A^*B^*u|w) = (B^*u|Aw). \tag{9.9}$$

Therefore, $u \in \mathrm{Dom}\, A^*B^*$ and

$$(A^*B^*u|w) = (u|BAw). \tag{9.10}$$

This means that $A^*B^*u = (BA)^*u$. $\Box$

Proposition 9.9 (1) *Let A be closed and densely defined. Let C be bounded and everywhere defined. Assume also that AC is densely defined. Then*

$$(AC)^* = \big(C^*A^*\big)^{\mathrm{cl}}. \tag{9.11}$$

(2) *If A be closed and densely defined. Let C be bounded and invertible. Then AC is densely defined and*

$$(AC)^* = C^*A^*. \tag{9.12}$$

Proof. (1): A^* and C^* satisfy the assumptions of Prop. 9.8 (2). Hence

$$\big(C^*A^*\big)^* = A^{**}C^{**} = AC. \tag{9.13}$$

Now AC is densely defined. Hence

$$\big(C^*A^*\big)^{\mathrm{cl}} = \big(C^*A^*\big)^{**} = (AC)^*. \tag{9.14}$$

(2): C is invertible. Hence by Prop. 3.33 (1), C^*A^* is already closed. □

9.5 Numerical range and maximal operators

Let T be an operator on $\mathcal{V}$. Then we will write $\mathrm{Num}(T) := \mathrm{Num}\mathfrak{t}$, where $\mathfrak{t}$ is the quadratic form defined by T:

$$\mathfrak{t}(v) := (v|Tv), \quad v \in \mathrm{Dom}\, v.$$

In other words, $\mathrm{Num}T = \{(v|Tv) \ : \ v \in \mathrm{Dom}\, T,\ \|v\| = 1\}$.

Theorem 9.10 (1) $\|(z-T)v\| \geq \mathrm{dist}(z, \mathrm{Num}T)\|v\|, \quad v \in \mathrm{Dom}\, T$.

(2) *If T is a closed operator and $z \in \mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$, then $z - T$ has a closed range.*

(3) *If $z \in \mathrm{rs}T\backslash(\mathrm{Num}T)^{\mathrm{cl}}$, then $\|(z-T)^{-1}\| \leq |\mathrm{dist}(z, \mathrm{Num}T)|^{-1}$.*

(4) *Let Δ be a connected component of $\mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$. Then either $\Delta \subset \mathrm{sp}T$ or $\Delta \subset \mathrm{rs}T$.*

Proof. To prove (1), take $z \notin (\mathrm{Num}T)^{\mathrm{cl}}$. Recall that $\mathrm{Num}T$ is convex. Hence, replacing T wih $\alpha T + \beta$ for appropriate $\alpha, \beta \in \mathbb{C}$, we can assume that $z = \mathrm{i}\nu$ and $\nu = \mathrm{dist}(\mathrm{i}\nu, \mathrm{Num}T)$. Then,

$$0 \in (\mathrm{Num}T)^{\mathrm{cl}} \subset \{\mathrm{Im} z \leq 0\}.$$

Thus

$$\begin{aligned} \|(\mathrm{i}\nu - T)v\|^2 \quad &= (Tv|Tv) - i\nu(v|Tv) + \mathrm{i}\nu(Tv|v) + |\nu|^2\|v\|^2 \\ &= (Tv|Tv) - 2\nu\mathrm{Im}(v|Tv) + |\nu|^2\|v\|^2 \\ &\geq |\nu|^2\|v\|^2. \end{aligned}$$

(1) implies (2) and (3).

Let $z_0 \in \mathrm{rs}T \backslash \mathrm{Num}T$. By (3), if $r = \mathrm{dist}(z_0, \mathrm{Num}T)$, then $\{|z - z_0| < r\} \subset \mathrm{rs}T$. This proves (4). □

Definition 9.11 *An operator T is called* maximal, *if* $\mathrm{sp}T \subset (\mathrm{Num}T)^{\mathrm{cl}}$.

Clearly, if T is a maximal operator, and $z \notin (\mathrm{Num}T)^{\mathrm{cl}}$, then

$$\|(z - T)^{-1}\| \leq (\mathrm{dist}(z, \mathrm{Num}T))^{-1}.$$

If T is bounded, then T is maximal.

Theorem 9.12 *Suppose that T is an operator and Δ_i are the connected components of $\mathbb{C} \backslash (\mathrm{Num}T)^{\mathrm{cl}}$. Then the following conditions are necessary and sufficient for T to be maximal:*

(1) *For all i, there exists $z_i \in \Delta_i$ such that $z_i \notin \mathrm{sp}T$;*

(2) *T is closable and, for all i, there exists $z_i \in \Delta_i$ such that* $\mathrm{Ran}\,(z_i - T) = \mathcal{V}$.

(3) *T is closed and, for all i, there exists $z_i \in \Delta_i$ such that* $\mathrm{Ran}\,(z_i - T)$ *is dense in $\mathcal{V}$.*

(4) *T is closed and, for all i, there exists $z_i \in \Delta_i$ such that* $\mathrm{Ker}(\overline{z}_i - T^*) = \{0\}$.

If K is a closed convex subset of $\mathbb{C}$, then $\mathbb{C} \backslash K$ is either connected or has two connected components.

9.6 Dissipative operators

Definition 9.13 *We say that an operator A is* dissipative *iff*

$$\mathrm{Im}(v|Av) \leq 0, \quad v \in \mathrm{Dom}\, A.$$

Equivalently, A is dissipative iff $\mathrm{Num}A \subset \{\mathrm{Im}z \leq 0\}$.

Definition 9.14 *A is* maximally dissipative *or* m-dissipative *iff A is dissipative and*

$$\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}.$$

Theorem 9.15 *Let A be a densely defined operator. Then the following conditions are equivalent:*

(1) *$-\mathrm{i}A$ is the generator of a strongly continuous semigroup of contractions.*

(2) *A is maximally dissipative.*

Proof. (1) ⇒(2): We have

$$\mathrm{Re}(v|\mathrm{e}^{-\mathrm{i}tA}v) \leq |(v|\mathrm{e}^{-\mathrm{i}tA}v)| \leq \|v\|^2.$$

Hence, for $v \in \mathrm{Dom}\, A$,

$$\begin{aligned} \mathrm{Im}(v|Av) &= \mathrm{Re}(v| - \mathrm{i}Av) \\ &= \mathrm{Re} \lim_{t \nearrow 0} t^{-1} \left((v|\mathrm{e}^{-\mathrm{i}tA}v) - \|v\|^2 \right) \leq 0. \end{aligned}$$

Hence A is dissipative.

We know that the generators of contraction semigroups satisfy $\{\mathrm{Re} z > 0\} \subset \mathrm{rs}(-\mathrm{i}A)$.

(2)⇒(1): Let $\mathrm{Re} z > 0$. We have

$$\begin{aligned} \|v\| \|(z + \mathrm{i}A)v\| &\geq |(v|(z + \mathrm{i}A)v)| \\ &\geq \mathrm{Re}(v|(z + \mathrm{i}A)v) \geq \mathrm{Re} z \|v\|^2. \end{aligned}$$

Hence, noting that $z \in \mathrm{rs}(-\mathrm{i}A)$, we obtain $\|(z + \mathrm{i}A)^{-1}\| \leq \mathrm{Re} z^{-1}$. Therefore, $-\mathrm{i}A$ is an operator of the type $(1, 0)$, and hence generator of a contraction semigroup. □

Theorem 9.16 *Let A be densely defined and dissipative. Then the following conditions are sufficient and necessary for A to be maximally dissipative:*

(1) *A is closable and there exists z_+ with* $\mathrm{Im} z_+ > 0$ *and* $\mathrm{Ran}\,(z_+ - A) = \mathcal{V}$.

(2) *A is closed and there exists z_+ with* $\mathrm{Im} z_+ > 0$ *and* $\mathrm{Ran}\,(z_+ - A)$ *dense in* $\mathcal{V}$.

(3) *A is closed and there exists z_+ with* $\mathrm{Im} z_+ > 0$ *and* $\mathrm{Ker}(\overline{z}_+ - A^*) = \{0\}$.

9.7 Hermitian operators

Definition 9.17 *An operator $A : \mathcal{V} \to \mathcal{V}$ is* hermitian *iff*

$$(Aw|v) = (w|Av), \ \ w, v \in \mathrm{Dom}\, A.$$

Equivalently, A is hermitian iff

$$(Av|v) = (v|Av), \ \ v \in \mathrm{Dom}\, A,$$

iff $\mathrm{Num} A \subset \mathbb{R}$ iff A and $-A$ are dissipative.

If in addition A is densely defined, then it is hermitian iff $A \subset A^*$.

Remark 9.18 *In a part of literature the term "symmetric" is used instead of "hermitian".*

Theorem 9.19 *Let A be densely defined and hermitian. Then A is closable. Besides, one of the following possibilities is true:*

(1) $\mathrm{sp} A \subset \mathbb{R}$.

(2) $\mathrm{sp} A = \{\mathrm{Im} z \geq 0\}$.

(3) $\mathrm{sp}A = \{\mathrm{Im}z \leq 0\}$.

(4) $\mathrm{sp}A = \mathbb{C}$.

Proof. A is closable because $A \subset A^*$ and A^* is closed.

We know that $\mathrm{Num}A \subset \mathbb{R}$. If $\mathrm{Num}A \neq \mathbb{R}$, then $\mathbb{C}\backslash(\mathrm{Num}A)^{\mathrm{cl}}$ is connected. Hence then we have the possibilities (1) or (4).

If $\mathrm{Num}A = \mathbb{R}$, then $\mathbb{C}\backslash(\mathrm{Num}A)^{\mathrm{cl}}$ consists of two connected components, $\{\mathrm{Im}z > 0\}$ and $\{\mathrm{Im}z < 0\}$. Hence then we have the possibilities (1), (2), (3) and (4). □

Theorem 9.20 *Let A be a densely defined operator. Then the following conditions are equivalent:*

(1) $-\mathrm{i}A$ *is the generator of a strongly continuous semigroup of isometries.*

(2) *A is hermitian and* $\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}$.

Proof. (1)⇒(2): For $v \in \mathrm{Dom}\, A$,

$$0 = \partial_t(\mathrm{e}^{-\mathrm{i}tA}v|\mathrm{e}^{-\mathrm{i}tA}v)\Big|_{t=0} = \mathrm{i}(Av|v) - \mathrm{i}(v|Av).$$

Hence A is hermitian.

Isometries are contractions. Hence, by Thm 5.17, $\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}$.

(2)⇒(1): By Thm 9.10, $\|(z + \mathrm{i}A)^{-1}\| \leq |\mathrm{Re}z|^{-1}$, $\mathrm{Re}z > 0$. Hence, by Thm 5.17, $\mathrm{e}^{-\mathrm{i}tA}$ is the generator of a strongly continuous contractive semigroup.

For $v \in \mathrm{Dom}\, A$,

$$\partial_t(\mathrm{e}^{-\mathrm{i}tA}v|\mathrm{e}^{-\mathrm{i}tA}v) = \mathrm{i}(A\mathrm{e}^{-\mathrm{i}tA}v|\mathrm{e}^{-\mathrm{i}tA}v) - \mathrm{i}(\mathrm{e}^{-\mathrm{i}tA}v|A\mathrm{e}^{-\mathrm{i}tA}v) = 0.$$

Hence, for $v \in \mathrm{Dom}\, A$, $\|\mathrm{e}^{-\mathrm{i}tA}v\|^2 = \|v\|^2$. By density of $\mathrm{Dom}\, A$, $\mathrm{e}^{-\mathrm{i}tA}$ is a group of isometries. □

Theorem 9.21 *Let A be densely defined and hermitian. Then the following conditions are equivalent to* $\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}$*:*

(1) *There exists z_+ with* $\mathrm{Im}z_+ > 0$ *and* $z_+ \not\in \mathrm{sp}A$.

(2) *There exists z_+ with* $\mathrm{Im}z_+ > 0$ *and* $\mathrm{Ran}\,(z_+ - A) = \mathcal{V}$.

(3) *A is closed and there exists z_+ with* $\mathrm{Im}z_+ > 0$ *and* $\mathrm{Ran}\,(z_+ - A)$ *dense in* $\mathcal{V}$.

(4) *A is closed and there exists z_+ with* $\mathrm{Im}z_+ > 0$ *and* $\mathrm{Ker}(\overline{z}_+ - A^*) = \{0\}$.

9.8 Self-adjoint operators

Definition 9.22 *Let A be a densely defined operator on $\mathcal{V}$. A is* self-adjoint *iff* $A^* = A$.

In other words, A is self-adjoint if for $w \in \mathcal{W}$ there exists $y \in \mathcal{V}$ such that

$$(y|v) = (w|Av), \ \ v \in \mathrm{Dom}\, A,$$

then $w \in \mathrm{Dom}\, A$ and $Aw = y$.

Theorem 9.23 *Every self-adjoint operator is hermitian and closed. If $A \in B(\mathcal{V})$, then it is self-adjoint iff it is hermitian.*

Theorem 9.24 *Let A be densely defined hermitian. Then the following conditions are necessary and sufficient for A to be self-adjoint:*

(1) $\mathrm{sp}A \subset \mathbb{R}$.

(2) *There exist $z_\pm$ with $\pm\mathrm{Im}z_\pm > 0$ such that $z_\pm \not\in \mathrm{sp}A$.*

(3) *There exist $z_\pm$ with $\pm\mathrm{Im}z_\pm > 0$ such that* $\mathrm{Ran}\,(z_\pm - A) = \mathcal{V}$.

(4) *A is closed and there exist $z_\pm$ with $\pm\mathrm{Im}z_\pm > 0$ such that* $\mathrm{Ran}\,(z_\pm - A)$ *is dense in $\mathcal{V}$.*

(5) *A is closed and there exist $z_\pm$ with $\pm\mathrm{Im}z_\pm > 0$ such that* $\mathrm{Ker}(\overline{z}_\pm - A^*) = \{0\}$.

Theorem 9.25 *Let A be densely defined and hermitian. Then the following conditions are sufficient for A to be self-adjoint:*

(1) *There exists $z_0 \in \mathbb{R}$ such that $z_0 \not\in \mathrm{sp}A$.*

(2) *There exists $z_0 \in \mathbb{R}$ such that* $\mathrm{Ran}\,(z_0 - A) = \mathcal{V}$.

(3) *A is closed and there exists $z_0 \in \mathbb{R}$ such that* $\mathrm{Ran}\,(z_0 - A)$ *is dense in $\mathcal{V}$.*

(4) *A is closed and there exists $z_0 \in \mathbb{R}$ such that* $\mathrm{Ker}(z_0 - A^*) = \{0\}$.

Theorem 9.26 (Stone Theorem) *Let A be an operator. Then the following conditions are equivalent:*

(1) $-\mathrm{i}A$ *is the generator of a strongly continuous group of unitary operators.*

(2) *A is self-adjoint.*

Proof. To prove (1)⇒(2), suppose that $\mathbb{R} \mapsto U(t)$ is a strongly continuous unitary group. Let $-\mathrm{i}A$ be its generator. Then $[0, \infty[\ni t \mapsto U(t), U(-t)$ are semigroups of isometries with the generators $-\mathrm{i}A$ and $\mathrm{i}A$. By Theorem 9.20, A is hermitian and $\mathrm{sp}A \subset \{\mathrm{Im}z \geq 0\} \cap \{\mathrm{Im}z \leq 0\} = \mathbb{R}$. Hence A is self-adjoint.

(2)⇒(1): By Theorem 9.20, $\mp\mathrm{i}A$ generate semigroups of isometries $\mathrm{e}^{\mp\mathrm{i}tA}$. By (5.8), $\mathrm{e}^{\pm\mathrm{i}tA}$ is the inverse of $\mathrm{e}^{\mp\mathrm{i}tA}$. Hence these isometries are unitary. □

9.9 Spectral theorem

Definition 9.27 *Recall that $B \in B(\mathcal{V})$ is called* normal *if $B^*B = BB^*$.*

Let us recall one of the versions of the spectral theorem for bounded normal operators.

Let X be a Borel subset of $\mathbb{C}$. Let $\mathcal{M}(X)$ denote the space of measurable functions on X with values in $\mathbb{C}$. For $f \in \mathcal{M}(X)$ we set $f^*(x) := \overline{f(x)}$, $x \in X$. In particular, the function $X \ni z \mapsto \mathrm{id}(z) := z$ belongs to $\mathcal{M}(X)$.

$\mathcal{L}^\infty(X)$ will denote the space of bounded measurable functions on X.

Theorem 9.28 *Let B be a bounded normal operator on $\mathcal{V}$. Then there exists a unique linear map*

$$\mathcal{L}^\infty(\mathrm{sp}B) \ni f \mapsto f(B) \in B(\mathcal{V})$$

such that $1(B) = \mathbb{1}$, $\mathrm{id}(B) = B$, $fg(B) = f(B)g(B)$,
$f(B)^* = f^*(B)$, $\|f(B)\| \leq \sup |f|$,
if $f_n \to f$ *pointwise and* $|f_n| \leq c$ *then* $\mathrm{s}-\lim\limits_{n\to\infty} f_n(B) \to f(B)$.
Above, all functions $f, f_n, g \in \mathcal{L}^\infty(\mathrm{sp}B)$.

Theorem 9.29 *Let B be a bounded normal operator B. Let $f \in \mathcal{M}(\mathrm{sp}B)$. Set*

$$f_n(x) := \begin{cases} f(x) & |f(x)| \leq n, \\ 0, & |f(x)| > n. \end{cases}$$

$$\mathrm{Dom}(f(B)) \quad = \quad \{v \in \mathcal{V} \ : \ \sup \|f_n(B)v\| < \infty\}.$$

Then for $v \in \mathrm{Dom}\, B$ there exists the limit

$$f(B)v \quad := \quad \lim_{n\to\infty} f_n(B)v,$$

which defines a closed normal operator.

Let now A be a (possibly unbounded) self-adjoint operator on $\mathcal{V}$.

Theorem 9.30 *Then $U := (A + \mathrm{i})(A - \mathrm{i})^{-1}$ is a unitary operator with*

$$\mathrm{sp}U = (\mathrm{sp}^{\mathrm{ext}} A + \mathrm{i})(\mathrm{sp}^{\mathrm{ext}} A - \mathrm{i})^{-1}.$$

Proof. Using the fact that A is hermitian, for $v \in \mathrm{Dom}\, A$ we check that

$$\|(A \pm \mathrm{i})v\|^2 = \|Av\|^2 + \|v\|^2.$$

Therefore, $(A \pm \mathrm{i}) : \mathrm{Dom}\, A \to \mathcal{V}$ are isometric. Using $\mathrm{Ran}\,(A \pm \mathrm{i}) = \mathcal{V}$ we see that they are unitary. Hence so is $(A + \mathrm{i})(A - \mathrm{i})^{-1}$.

The location of the spectrum of U follows from

$$(z - U)^{-1} = (A - \mathrm{i})^{-1}(z - 1)^{-1}\Big(A - \mathrm{i}(z + 1)(z - 1)^{-1}\Big)^{-1}.$$

□

U is unitary, hence normal. If f is a measurable function on $\mathrm{sp}A$, we define

$$f(A) := g(U),$$

where $g(z) = f(\mathrm{i}(z + \mathrm{i})(z - 1)^{-1})$.

Theorem 9.31 *The map*

$$\mathcal{M}(\mathrm{sp}A) \ni f \mapsto f(A) \in B(\mathcal{V})$$

is linear and satisfies $1(A) = \mathbb{1}$, $\mathrm{id}(A) = A$, $fg(A) = f(A)g(A)$,
$f(A)^* = \overline{f}(A)$, $\|f(A)\| \leq \sup|f|$,
where $f, g \in \mathcal{M}(\mathrm{sp}A)$,

Definition 9.32 *A possibly unbounded densely defined operator A is called* normal *if* $\mathrm{Dom}\, A = \mathrm{Dom}\, A^*$ *and*

$$\|Av\|^2 = \|A^*v\|, \quad v \in \mathrm{Dom}\, A.$$

One can extend Thm 9.31 to normal unbounded operators in an obvious way.

Proposition 9.33 *Let A be normal. Then the closure of the numerical range is the convex hull of its spectrum.*

Proof. We can write $A = \int \lambda \mathrm{d}E(\lambda)$, where $E(\lambda)$ is a spectral measure. Then for $\|v\| = 1$, $(v|Av)$ is the center of mass of the measure $(v|\mathrm{d}E(\lambda)v)$. □

9.10 Essentially self-adjoint operators

Definition 9.34 *An operator* $A : \mathcal{V} \to \mathcal{V}$ *is essentially self-adjoint iff* A^{cl} *is self-adjoint.*

Theorem 9.35 (1) *Every essentially self-adjoint operator is hermitian and closable.*

(2) *A is essentially self-adjoint iff* A^* *is self-adjoint.*

Theorem 9.36 *Let A be hermitian. Then the following conditions are necessary and sufficient for A to be essentially self-adjoint:*

(1) *There exists* z_+ *with* $\mathrm{Im} z_+ > 0$ *and* z_- *with* $\mathrm{Im} z_- < 0$ *such that* $\mathrm{Ran}\,(z_+ - A)$ *and* $\mathrm{Ran}\,(z_- - A)$ *are dense in* $\mathcal{V}$.

(2) *There exists* z_+ *with* $\mathrm{Im} z_+ > 0$ *and* z_- *with* $\mathrm{Im} z_- < 0$ *such that* $\mathrm{Ker}(\overline{z}_+ - A^*) = \{0\}$ *and* $\mathrm{Ker}(\overline{z}_- - A^*) = \{0\}$.

Theorem 9.37 *Let A be hermitian. Let* $z_0 \in \mathbb{R}\backslash \mathrm{Num} A$. *Then the following conditions are sufficient for A to be essentially self-adjoint:*

(1) $\mathrm{Ran}\,(z_0 - A)$ *is dense in* $\mathcal{V}$.

(2) $\mathrm{Ker}(z_0 - A^*) = \{0\}$.

9.11 Rigged Hilbert space

Let $\mathcal{V}$ be a Hilbert space with the scalar product $(\cdot|\cdot)$. Suppose that T is a self-adjoint operator on $\mathcal{V}$ with $T \geq c_0 > 0$. Then $\operatorname{Dom} T$ can equipped with the scalar product

$$(Tv|Tw), \quad v, w \in \operatorname{Dom} T$$

is a Hilbert space embedded in $\mathcal{V}$. We will prove a converse construction, that leads from an embedded Hilbert space to a positive self-adjoint operator.

Let $\mathcal{V}^*$ denote the space of bounded antilinear functionals on $\mathcal{V}$. The Riesz lemma says that $\mathcal{V}^*$ is a Hilbert space naturally isomorphic to $\mathcal{V}$.

Suppose that $\mathcal{W}$ is a Hilbert space contained and dense in $\mathcal{V}$. We assume that for $c_0 > 0$

$$(w|w)_{\mathcal{W}} \geq c_0(w|w), \ w \in \mathcal{W}. \tag{9.15}$$

Of course, $\mathcal{W}^*$ is also a Hilbert naturally isomorphic to $\mathcal{W}$. However, we do not want to use this isomorphism.

Let $J : \mathcal{W} \to \mathcal{V}$ denote the embedding. By (9.15), it is bounded. Clearly $J^* : \mathcal{V} \to \mathcal{W}^*$ (where we use the identification $\mathcal{V} \simeq \mathcal{V}^*$). We have $\operatorname{Ker} J^* = (\operatorname{Ran} J)^\perp = \{0\}$ and $(\operatorname{Ran} J^*)^\perp = \operatorname{Ker} J = \{0\}$. Hence J^* is a dense embedding of $\mathcal{V}$ in $\mathcal{W}^*$. Thus we obtain a triplet of Hilbert spaces, sometimes called a *rigged Hilbert space*

$$\mathcal{W} \subset \mathcal{V} \subset \mathcal{W}^*.$$

Theorem 9.38 *There exists a unique positive injective self-adjoint operator T on $\mathcal{V}$ such that* $\operatorname{Dom} T = \mathcal{W}$ *and*

$$(w_1|w_2)_{\mathcal{W}} = (Tw_1|Tw_2), \quad w_1, w_2 \in \mathcal{W}. \tag{9.16}$$

Proof. Without loss of generality we will assume that $c_0 = 1$.

For $v \in \mathcal{V}$, $w \in \mathcal{W}$, we have

$$|(w|v)| \leq \|w\|\|v\| \leq \|w\|_{\mathcal{W}}\|v\|.$$

By the Riesz lemma, there exists $A : \mathcal{V} \to \mathcal{W}$ such that

$$(w|v) = (w|Av)_{\mathcal{W}}, \tag{9.17}$$

We treat A as an operator from $\mathcal{V}$ to $\mathcal{V}$. A is bounded, because

$$\|Av\|^2 \leq \|Av\|^2_{\mathcal{W}} = (Av|Av)_{\mathcal{W}} = (Av|v) \leq \|Av\|\|v\|.$$

A is positive, (and hence in particular self-adjoint) because

$$(Av|v) = (Av|Av)_{\mathcal{W}} \geq 0.$$

A has a zero kernel, because $Av = 0$ implies

$$0 = (w|Av)_{\mathcal{V}} = (w|v), \quad v \in \operatorname{Dom} \mathcal{W},$$

and $\mathcal{W}$ is dense.

Thus $T := A^{-1/2}$ defines a positive self-adjoint operator $\geq \mathbb{1}$. We have

$$(w|y)_{\mathcal{W}} = (w|T^2 y), \quad w \in \mathcal{W}, \quad y \in \operatorname{Dom} T^2 = \operatorname{Ran} A.$$

Using the lemma below, with two embedded Hilbert spaces $\mathcal{W}$ and $\operatorname{Dom} T$ having a common dense subspace $\operatorname{Dom} T^2$, we obtain $\mathcal{W} = \operatorname{Dom} T$ and the equality (9.16). □

Lemma 9.39 *Let $\mathcal{W}_+, \mathcal{W}_-$ be two Hilbert spaces embedded in a Hilbert space $\mathcal{V}$. Suppose that their norms satisfy*

$$\|w\| \leq \|w\|_+, \quad w \in \mathcal{W}_+, \quad \|w\| \leq \|w\|_-, \quad w \in \mathcal{W}_-.$$

Let $\mathcal{D} \subset \mathcal{W}_+ \cap \mathcal{W}_-$ be dense both in $\mathcal{W}_+$ and in $\mathcal{W}_-$. Suppose $\|\cdot\|_+ = \|\cdot\|_-$ in $\mathcal{D}$. Then $\mathcal{W}_+ = \mathcal{W}_-$ and $\|\cdot\|_+ = \|\cdot\|_-$.

Proof. Let $w_+ \in \mathcal{W}_+$. There exists $(w_n) \subset \mathcal{D}$ such that $\|w_n - w_+\|_+ \to 0$. This implies $\|w_n - w_+\| \to 0$.

Besides w_n is Cauchy in $\mathcal{W}_-$ Hence there exists $w_- \in \mathcal{W}_-$ such that $\|w_n - w_-\|_- \to 0$. This implies $\|w_n - w_-\| \to 0$. Hence $w_+ = w_-$. Besides, $\|w_+\|_+ = \lim \|w_n\|_+ = \lim \|w_n\|_- = \|w_-\|_-$.

Thus $\mathcal{W}_+ \subset \mathcal{W}_-$ and in $\mathcal{W}_+$ the norm $\|\cdot\|_+$ coincides with the norm $\|\cdot\|_-$. □

By functional calculus for self-adjoint operators we can define $S := T^2$. Clearly, $T = \sqrt{S}$ and

$$(v|Sw) = (v|w)_{\mathcal{W}}, \quad v \in \operatorname{Dom} \sqrt{S}, \quad w \in \operatorname{Dom} S.$$

We will say that the operator S is associated with the sesquilinear form $(\cdot|\cdot)_{\mathcal{W}}$.

9.12 Polar decomposition

Let A be a densely defined closed operator. Let $S+1$ be the positive operator associated with the sesquilinear form

$$(Av|Aw) + (v|w), \quad v, w \in \operatorname{Dom} A.$$

Theorem 9.40 *$S = A^*A$.*

In order to prove this theorem, introduce $\mathcal{V}_1 = (\mathbb{1} + T)^{-1}\mathcal{V}$ and $\mathcal{V}_{-1} = (\mathbb{1} + T)\mathcal{V}$, so that $\mathcal{V}_1 = \operatorname{Dom} A$ and $\mathcal{V}_{-1} = \mathcal{V}_1^*$. Denote by $A_{(1)}$ the operator A treated as an operator $\mathcal{V}_1 \to \mathcal{V}$. Clearly, $A_{(1)}$ is bounded, and so is $A_{(1)}^* : \mathcal{V} \to \mathcal{V}_{-1}$.

Proposition 9.41 (1) $\operatorname{Dom} A^* = \{v \in \mathcal{V} \ : \ A_{(1)}^* v \in \mathcal{V}\}$.

(2) *On $\operatorname{Dom} A^*$ the operators A^* and $A_{(1)}^*$ coincide.*

(3) $\mathrm{Dom}\, T^2 = \{v \in \mathrm{Dom}\, A \ : \ Av \in \mathrm{Dom}\, A^*\}$

(4) *For* $v \in \mathrm{Dom}\, T^2$, $T^2 v = A^* A v$.

Proof. (1). Let $w \in \mathcal{V}$. We have

$$w \in \mathrm{Dom}\, A^* \ \Leftrightarrow \ |(w|Av)| \leq C\|v\|, \quad v \in \mathrm{Dom}\, A. \tag{9.18}$$

But $\mathrm{Dom}\, A = \mathcal{V}_1$ and $(w|Av) = (A_{(1)}^* w|v)$. Hence, (9.18) is equivalent to

$$|(A_{(1)}^* w|v)| \leq C\|v\|, \quad v \in \mathrm{Dom}\, A, \tag{9.19}$$

which means $A_{(1)}^* w \in \mathcal{V}$.

In the proof of (3) we will use the operators $T_{(1)}$ and $T_{(1)}^*$ defined analogously as $A_{(1)}$ and $A_{(1)}^*$. We have

$$T_{(1)}^* T_{(1)} = A_{(1)}^* A_{(1)}. \tag{9.20}$$

In fact, for $v, w \in \mathcal{V}_1$

$$(w|T_{(1)}^* T_{(1)} v) = (T_{(1)} w|T_{(1)} v) = (A_{(1)} w|A_{(1)} v) = (w|A_{(1)}^* A_{(1)} v).$$

Now

$$\begin{aligned} \mathrm{Dom}\, T^2 &= \{v \in \mathcal{V}_1 \ : \ T_{(1)}^* T_{(1)} v \in \mathcal{V}\} \ \text{ by spectral theorem} \\ &= \{v \in \mathcal{V}_1 \ : \ A_{(1)}^* A_{(1)} v \in \mathcal{V}\} \ \text{ by (9.20)} \\ &= \{v \in \mathcal{V}_1 \ : \ A_{(1)} v \in \mathrm{Dom}\, A^*\} \ \text{ by (1).} \end{aligned}$$

□

Theorem 9.42 *Let A be closed. Then there exist a unique positive operator $|A|$ and a unique partial isometry U such that* $\mathrm{Ker} U = \mathrm{Ker} A$ *and* $A = U|A|$. *We have then* $\mathrm{Ran}\, U = \mathrm{Ran}\, A^{\mathrm{cl}}$.

Proof. The operator A^*A is positive. By the spectral theorem, we can then define

$$|A| := \sqrt{A^* A}.$$

On $\mathrm{Ran}\, |A|$ the operator U is defined by

$$U \ |A| v := Av.$$

It is isometric, because

$$\||A|v\|^2 = (v||A|^2 v) = (v|A^* A v) = \|Av\|^2,$$

and correctly defined. We can extend it to $(\mathrm{Ran}\, |A|)^{\mathrm{cl}}$ by continuity. On $\mathrm{Ker}|A| = (\mathrm{Ran}\, |A|)^{\mathrm{cl}}$, we extend it by putting $Uv = 0$. □

9.13 Scale of Hilbert spaces I

Let A be a positive self-adjoint operator on $\mathcal{V}$ with $A \geq 1$. We define the family of Hilbert spaces $\mathcal{V}_\alpha$, $\alpha \in \mathbb{R}$ as follows.

For $\alpha \geq 0$, we set $\mathcal{V}_\alpha := \operatorname{Ran} A^{-\alpha} = \operatorname{Dom} A^\alpha$ with the scalar product

$$(v|w)_\alpha := (v|A^{2\alpha}w).$$

Clearly, for $0 \leq \alpha \leq \beta$ we have the embedding $\mathcal{V}_\alpha \supset \mathcal{V}_\beta$.

For $\alpha \leq 0$ we set $\mathcal{V}_\alpha := \mathcal{V}_{-\alpha}^*$, If $\alpha \leq \beta \leq 0$ we have a natural inclusion $\mathcal{V}_\alpha \supset \mathcal{V}_\beta$.

Note that we have the identification $\mathcal{V} = \mathcal{V}^*$, hence both definitions give $\mathcal{V}_0 = \mathcal{V}$.

Thus we obtain

$$\mathcal{V}_\alpha \supset \mathcal{V}_\beta, \text{ for any } \alpha \leq \beta. \tag{9.21}$$

Note that for $\alpha \leq 0$ $\mathcal{V}$ is embedded in $\mathcal{V}_\alpha$ and for $v, w \in \mathcal{V}$

$$(v|w)_\alpha = \left(v|A^{2\alpha}w\right).$$

Moreover, $\mathcal{V}$ is dense in $\mathcal{V}_\alpha$.

Sometimes we will use a different notation: $A^{-\alpha}\mathcal{V} = \mathcal{V}_\alpha$.

By restriction or extension, we can reinterpret the operator A^β as a unitary operator

$$A^\beta_{(-\alpha)} : A^\alpha\mathcal{V} \to A^{\alpha+\beta}\mathcal{V}.$$

If B is a self-adjoint operator, then we will use the notation $\langle B\rangle := (1 + B^2)^{1/2}$. Clearly, B gives rise to a bounded operator

$$B_{(\alpha)} : \langle B\rangle^{-\alpha}\mathcal{V} \to \langle B\rangle^{-\alpha+1}\mathcal{V}.$$

Thus every self-adjoint operator can be interpreted in many ways, depending on β we choose. The standard choice corresponding to $\beta = 1$

$$B_{(1)} : \operatorname{Dom} B = \langle B\rangle^{-1}\mathcal{V} \to \mathcal{V}$$

can be called the "operator interpretation".

Another interpretation is often useful:

$$B_{(1/2)} : \langle B\rangle^{-1/2}\mathcal{V} \to \langle B\rangle^{1/2}\mathcal{V},$$

the "form interpretation". One often introduces the *form domain* $\mathcal{Q}(B) := \langle B\rangle^{-1/2}\mathcal{V}$. We obtain a sesquilinear form

$$\mathcal{Q}(B) \times \mathcal{Q}(B) \ni (v, w) \mapsto (v|B_{(1/2)}w).$$

9.14 Scale of Hilbert spaces II

We will write $A > 0$ if A is positive, self-adjoint and $\mathrm{Ker}A = \{0\}$. One can generalize the definition of the scale of spaces $A^\alpha\mathcal{V}$ to the case $A > 0$.

Set $\mathcal{V}_+ := \mathrm{Ran}\, 1\!\!1_{[1,\infty[}(A)$, $\mathcal{V}_- := \mathrm{Ran}\, 1\!\!1_{[0,1[}(A)$. Let $A_\pm := A\Big|_{\mathcal{V}_\pm}$. Then $A_+ \geq 1$ and $A_-^{-1} \geq 1$. Hence we can define the scales of spaces $A_+^\alpha\mathcal{V}_+$, $A_-^\alpha\mathcal{V}_- := (A_-^{-1})^{-\alpha}\mathcal{V}_-$, $\alpha \in \mathbb{R}$. We set

$$A^\alpha\mathcal{V} := A_+^\alpha\mathcal{V}_+ \oplus A_-^\alpha\mathcal{V}_-. \tag{9.22}$$

If A is not bounded away from zero, then the scale (9.22) does not have the nested property (9.21). However, for any $\alpha, \beta \in \mathbb{R}$, $A^\alpha\mathcal{V} \cap A^\beta\mathcal{V}$ is dense in $A^\alpha\mathcal{V}$. Again, we have a family of unitary operators

$$A_{(\alpha)}^\beta : A^\alpha\mathcal{V} \to A^{\alpha+\beta}\mathcal{V}.$$

9.15 Complex interpolation

Let us recall a classic fact from complex analysis:

Theorem 9.43 (Three lines theorem) *Suppose that a function $\{0 \leq \mathrm{Re}z \leq 1\} \ni z \mapsto f(z) \in \mathbb{C}$ is continuous, bounded, analytic in the interor of its domain, and satisfies the bounds*

$$\begin{aligned} |f(\mathrm{i}s)| &\leq c_0, \\ |f(1+\mathrm{i}s)| &\leq c_1, \quad s \in \mathbb{R}. \end{aligned} \tag{9.23}$$

Then

$$|f(t+\mathrm{i}s)| \leq c_0^{1-t}c_1^t, \quad t \in [0,1], \ s \in \mathbb{R}. \tag{9.24}$$

Theorem 9.44 *Let $A > 0$ on $\mathcal{V}$, $B > 0$ on $\mathcal{W}$. Consider an operator $C : \mathcal{V} \cap A^{-1}\mathcal{V} \to \mathcal{W} \cap B^{-1}\mathcal{W}$ that satisfies*

$$\begin{aligned} \|Cv\| &\leq c_0\|v\|, \\ \|BCv\| &\leq c_1\|Av\|, \quad v \in \mathcal{V} \cap A^{-1}\mathcal{V}. \end{aligned}$$

(In other words, C is bounded as an operator $\mathcal{V} \to \mathcal{W}$ with the norm $\leq c_0$ and $A^{-1}\mathcal{V} \to B^{-1}\mathcal{W}$ with the norm $\leq c_1$.) Then, for $0 \leq t \leq 1$,

$$\|B^tCv\| \leq c_0^{1-t}c_1^t\|A^tv\|, \tag{9.25}$$

and so C extends to a bounded operator

$$C : A^{-t}\mathcal{V} \to B^{-t}\mathcal{W},$$

with the norm $\leq c_0^{1-t}c_1^t$.

Proof. Let $w \in \mathcal{W} \cap B^{-1}\mathcal{W}$ and $v \in \mathcal{V} \cap A^{-1}\mathcal{V}$. The vector valued functions $z \mapsto B^z w$ and $z \mapsto A^z v$ are bounded on $\{0 \leq \mathrm{Re} z \leq 1\}$, and hence so is

$$f(z) := (B^{\overline{z}} w | C A^z v)$$

We have

$$\begin{aligned} |f(\mathrm{i}s)| &\leq c_0 \|w\| \|v\|, \\ |f(1 + \mathrm{i}s)| &\leq c_1 \|w\| \|v\|, \quad s \in \mathbb{R}. \end{aligned}$$

Hence,

$$|f(t)| \leq c_0^{1-t} c_1^t \|w\| \|v\|, \quad t \in [0, 1].$$

This implies (9.25), by the density of $\mathcal{W} \cap B^{-1}\mathcal{W}$. □

9.16 Relative operator boundedness

Let A be a closed operator and B an operator with $\mathrm{Dom}\, B \supset \mathrm{Dom}\, A$. Recall that the *(operator) A-bound of B* is

$$a_1 := \inf_{\nu > 0} \sup_{v \neq 0,\ v \in \mathrm{Dom}\, A} \left(\frac{\|Bv\|^2}{\|Av\|^2 + \nu^2 \|v\|^2} \right)^{\frac{1}{2}}. \tag{9.26}$$

In a Hilbert space

$$\|Av\|^2 + \nu^2 \|v\|^2 = \|(A^* A + \nu^2)^{1/2} v\|^2.$$

Therefore, (9.26) can be rewritten as

$$a_1 = \inf_{\nu > 0} \|B(A^* A + \nu^2)^{-1/2}\|. \tag{9.27}$$

If, moreover, A is self-adjoint, then, using the unitarity of $(A^2 + \nu^2)^{-1/2}(\pm \mathrm{i}\nu - A)$, we can rewrite (9.27) as

$$a_1 = \inf_{\nu \neq 0} \|B(\mathrm{i}\nu - A)^{-1}\|. \tag{9.28}$$

Using Prop. 3.22 we obtain

$$a_1 = \inf_{z \in \mathrm{rs} A} \|B(z - A)^{-1}\|. \tag{9.29}$$

Theorem 9.45 (Kato-Rellich) *Let A be self-adjoint, B hermitian. Let B be A-bounded with the $A-$bound < 1. Then*

(1) *$A + B$ is self-adjoint on* $\mathrm{Dom}\, A$.

(2) *If A is essentally self-adjoint on $\mathcal{D}$, then $A + B$ is essentially self-adjoint on $\mathcal{D}$.*

Proof. Clearly, $A + B$ is hermitian on $\mathrm{Dom}\, A$. Moreover, for some ν, $\|B(\pm \mathrm{i}\nu - A)^{-1}\| < 1$. Hence, $\mathrm{i}\nu - A - B$ and $-\mathrm{i}\nu - A - B$ are invertible. □

9.17 Relative form boundedness

Assume first that A is a positive self-adjoint operator. Let B be a bounded operator from $\mathrm{Dom}\, A^{1/2} = (\mathbb{1} + A)^{-1/2}\mathcal{V}$ to $(\mathbb{1} + A)^{1/2}\mathcal{V}$. Note that B defines a bounded quadratic form on $\mathcal{Q}(B) := (\mathbb{1} + A)^{-1/2}\mathcal{V}$

$$\mathcal{Q}(B) \ni u, v \mapsto (u|Bv).$$

Let us assume that this form is hermitian, that is

$$(u|Bv) = \overline{(v|Bu)}.$$

Definition 9.46 *We say that B is form-bounded relatively to A iff there exist constants a, b such that*

$$|(v|Bv)| \leq a(v|Av) + b(v|v), \quad v \in \mathrm{Dom}\, A^{1/2}. \tag{9.30}$$

The infimum of a satisfying (9.30) is called the A-bound of B.

In other words: the A-form bound of B equals

$$a_2 := \inf_{c>0} \sup_{v \in \mathrm{Dom}\, A^{1/2} \setminus \{0\}} \frac{(v|Bv)}{(v|Av) + c(v|v)}.$$

This can be rewritten as

$$a_2 = \inf_{c>0} \|(A + c)^{-1/2} B (A + c)^{-1/2}\|.$$

Theorem 9.47 *A is a positive self-adjoint operator. Let B have the form A-bound less than* 1. *Then*

$$R(\mu) := \sum_{j=0}^{\infty} (\mu - A)^{-1/2} \big((\mu - A)^{-1/2} B (\mu - A)^{-1/2}\big)^j (\mu - A)^{-1/2}$$

is convergent for large negative μ. Moreover, $R(z)$ is a resolvent of a self-adjoint bounded from below operator, which will be called the form sum of A and B *and denoted, by the abuse of notation, $A + B$. We have* $\mathrm{Dom}\, |A + B|^{\frac{1}{2}} = \mathrm{Dom}\, |A|^{\frac{1}{2}}$.

We can generalize the concept of the form boundedness to the context of not necessarily positive operators as follows. Let A be a self-adjoint operator. Let B be a bounded operator from $\langle A\rangle^{-1/2}\mathcal{V}$ to $\langle A\rangle^{1/2}\mathcal{V}$. We assume that the form given by B is hermitian.

Definition 9.48 *The* improved form A-bound of B *is*

$$a_2' := \inf_{\nu>0, \mu} \|((A - \mu)^2 + \nu^2)^{-\frac{1}{4}} B ((A - \mu)^2 + \nu^2)^{-\frac{1}{4}}\|. \tag{9.31}$$

(9.31) can be rewritten as

$$a_2' = \inf_{\nu>0, \mu} \|(\mu + \mathrm{i}\nu - A)^{-\frac{1}{2}} B (\mu + \mathrm{i}\nu - A)^{-\frac{1}{2}}\|. \tag{9.32}$$

Theorem 9.49 *Let A be a self-adjoint operator. Let B have the improved A-form bound less than 1. Then there exists open subsets in the upper and lower complex half-plane such that the series*

$$R(z) := \sum_{j=0}^{\infty} (z-A)^{-1/2}\big((z-A)^{-1/2}B(z-A)^{-1/2}\big)^j (z-A)^{-1/2}$$

is convergent. Moreover, $R(z)$ is a resolvent of a self-adjoint operator, which will be called the form sum of A and B and denoted, by the abuse of notation, $A+B$.

The form boundedness is stronger than the operator boundedness. Indeed, suppose that B is a hermitian operator on $\mathcal{V}$ with $\mathrm{Dom}\, B \supset \mathrm{Dom}\, A$ and

$$\|B\big((A-\mu)^2+\nu^2\big)^{1/2}\| \leq a.$$

This means that B is bounded as an operator $\big((A-\mu)^2+\nu^2\big)^{-1/2}\mathcal{V} \to \mathcal{V}$ and as an operator $\mathcal{V} \to \big((A-\mu)^2+\nu^2\big)^{1/2}\mathcal{V}$, in both cases with norm $\leq a$. By the complex interpolation, it is bounded as an operator $\big((A-\mu)^2+\nu^2\big)^{-1/4}\mathcal{V} \to \big((A-\mu)^2+\nu^2\big)^{1/4}\mathcal{V}$ with norm $\leq a$. In particular, we have $a_2' \leq a_1$, where a_1 is the operator A-bound and a_2' is the improved form A-bound.

9.18 Discrete and essential spectrum

Let $\mathcal{X}$ be a Banach space and $A \in B(\mathcal{X})$. We say that $e \in \mathrm{sp}A$ belongs to the discrete spectrum of A if it is an isolated point of $\mathrm{sp}A$ and $\dim \mathbb{1}_{\{e\}}(A) < \infty$. The discrete spectrum is denoted by $\mathrm{sp}_{\mathrm{d}}(A)$. The essential spectrum is defined as

$$\mathrm{sp}_{\mathrm{ess}}A := \mathrm{sp}A \backslash \mathrm{sp}_{\mathrm{d}}A.$$

Assume now that $\mathcal{H}$ is a Hilbert space and A is an operator on $\mathcal{H}$. Then

Theorem 9.50 *Let A be self-adjoint and $\lambda \in \mathrm{sp}A$. Then*

(1) $\lambda \in \mathrm{sp}_{\mathrm{d}}A$ *iff there exists* $\epsilon > 0$ *such that* $\dim \mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(A) < \infty$.

(2) $\lambda \in \mathrm{sp}_{\mathrm{ess}}(A)$ *iff for every* $\epsilon > 0$ *we have* $\dim \mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(A) = \infty$.

Theorem 9.51 *Let A be normal and $\lambda \in \mathrm{sp}A$. Then*

(1) $\lambda \in \mathrm{sp}_{\mathrm{d}}A$ *iff there exists* $\epsilon > 0$ *such that* $\dim \mathbb{1}_{B(\lambda,\epsilon)}(A) < \infty$.

(2) $\lambda \in \mathrm{sp}_{\mathrm{ess}}(A)$ *iff for every* $\epsilon > 0$ *we have* $\dim \mathbb{1}_{B(\lambda,\epsilon)}(A) = \infty$.

Proposition 9.52 *Let A be a normal operator and $\lambda \in \mathbb{C}$. Then the following are equivalent:*

(1) $\lambda \in \mathrm{sp}_{\mathrm{ess}}(A)$.

(2) *There exists a sequence of vectors* (v_n) *such that* $\mathrm{w}-\lim_{n\to\infty} v_n = 0$, $\|v_n\| = 1$ *and* $\lim_{n\to\infty} \|(H-\lambda)v_n\| = 0$.

Proof. Fix $\epsilon > 0$ and set $P_\epsilon := 1\!\!1_{B(\lambda,\epsilon)}(A)$. Then

$$\|(1-P_\epsilon)v_n\| \leq \epsilon^{-1}\|(A-\lambda)v_n\| \to 0. \tag{9.33}$$

Thus, after dropping a finite number of elements of the sequence, we can assume that $\|(1-P_\epsilon)v_n\| < \frac{1}{2}$, and hence $\|P_\epsilon v_n\| > \frac{1}{2}$. Set $w_n := \frac{1}{\|P_\epsilon v_n\|}P_\epsilon v_n$. Then $\|w_n\| = 1$, $w_n \in \operatorname{Ran} P_\epsilon$, $\mathrm{w}-\lim_{n\to\infty} w_n = 0$.

Suppose that $\operatorname{Ran} P_\epsilon$ is finite dimensional. Then $\{w \in \operatorname{Ran} P_\epsilon \mid \|w\| = 1\}$ is compact. Hence, passing to a subsequence, we can assume that w_n is convergent (in norm). But it is weakly convergent to 0. So it is convergent in norm to 0. But this is in contradiction with $\|w_n\| = 1$. □

9.19 The mini-max and max-min principle

We will need the following lemma:

Lemma 9.53 *Let* $\mathcal{X}, \mathcal{Y}$ *be finite dimensional subspaces. Then*

$$\dim \mathcal{X} \cap \mathcal{Y}^\perp \geq \dim \mathcal{X} - \dim \mathcal{Y}. \tag{9.34}$$

Proof. It is well-known that

$$\dim \mathcal{X} + \dim \mathcal{W} = \dim(\mathcal{X} + \mathcal{W}) + \dim \mathcal{X} \cap \mathcal{W}. \tag{9.35}$$

Assume for a moment that $\mathcal{X}, \mathcal{W}$ are contained in a finite dimensional space $\mathcal{V}$. Then

$$\dim \mathcal{Y}^\perp = \dim \mathcal{V} - \dim \mathcal{Y}. \tag{9.36}$$

Hence, setting $\mathcal{W} = \mathcal{Y}^\perp$, we obtain

$$\dim \mathcal{X} \cap \mathcal{Y}^\perp = \dim \mathcal{X} + \dim \mathcal{Y}^\perp - \dim(\mathcal{X} + \mathcal{Y}^\perp) \tag{9.37}$$

$$\geq \dim \mathcal{X} + \dim \mathcal{Y}^\perp - \dim \mathcal{V} = \dim \mathcal{X} - \dim \mathcal{Y}. \tag{9.38}$$

But enlarging $\mathcal{V}$ only makes $\mathcal{Y}^\perp$ bigger. □

If H is self-adjoint, we will write

$$\inf H := \inf \operatorname{sp}(H), \qquad \sup H := \sup \operatorname{sp}(H). \tag{9.39}$$

Let H be a bounded from below self-adjoint operator on a Hilbert space $\mathcal{V}$. It is easy to see that

$$\inf H = \inf\{(v|Hv) \ : \ \|v\| = 1, \ v \in \mathcal{V}\}. \tag{9.40}$$

For an operator H on $\mathcal{V}$ and $\mathcal{W}$, a closed subspace $\mathcal{W}$ of $\mathcal{V}$, we will write $H_{\mathcal{W}} := I_{\mathcal{W}}^* H\big|_{\mathcal{W}}$, where $I_{\mathcal{W}}$ denotes the embedding of $\mathcal{W}$ into $\mathcal{V}$. Then if H is bounded and self-adjoint, then so is $H_{\mathcal{W}}$. If H is only bounded from below, then so is $H_{\mathcal{W}}$.

(9.40) allows us to compute the *ground state energy* of a Hamiltonian. Let us extend (9.40) to next eigenvalues. We define

$$\mu_n(H) := \inf \Big\{ \sup H_{\mathcal{L}} \qquad \mathcal{L} \text{ is an } n\text{-dim. subspace of } \mathcal{V} \Big\}, \quad n = 1, 2, \dots;$$
$$\Sigma(H) := \inf \mathrm{sp}_{\mathrm{ess}}(H),$$
$$N(H) := \dim 1\!\!1_{]-\infty,\Sigma[}(H)$$

Theorem 9.54 *$\mu_n(H)$ for $n \le N$ are the consecutive eigenvalues of H, counting the multiplicity. For $n > N(H)$ we have $\mu_n(H) = \Sigma(H)$.*

Proof. Let $a \in \mathrm{sp}(H)$. Let $\mathcal{W} := \mathrm{Ran}\, 1\!\!1_{]-\infty,a[}(H)$, $\mathcal{X} := \mathrm{Ran}\, 1\!\!1_{]-\infty,a]}(H)$, Let $n \in \mathbb{N}$ satisfy

$$\dim \mathcal{W} < n \le \dim \mathcal{X}. \tag{9.41}$$

and $\dim \mathcal{L} = n$. Then

$$\dim \mathcal{L} \cap \mathcal{W}^{\perp} \ge \dim \mathcal{L} - \dim \mathcal{W} > 1. \tag{9.42}$$

Hence there exists $w \in \mathcal{L} \cap \mathcal{W}^{\perp}$, $\|w\| = 1$. So

$$\sup H_{\mathcal{L}} \ge (w|Hw) \ge a. \tag{9.43}$$

On the other hand, if $\mathcal{L}$ is n-dimensional and $\mathcal{W} \subset \mathcal{L} \subset \mathcal{X}$, then $\sup H_{\mathcal{L}} = a$. Hence $\mu_n = a$. □

Theorem 9.55 (The Rayleigh-Ritz method) *We have*

$$\mu_n(H) \le \mu_n(H_{\mathcal{W}}).$$

Proof.

$$\begin{aligned}
\mu_n(H) &= \inf\{\sup H_{\mathcal{L}} \ : \ \dim \mathcal{L} = n\} && (9.44)\\
&\le \inf\{\sup H_{\mathcal{L}} \ : \ \dim \mathcal{L} = n, \quad \mathcal{L} \subset \mathcal{W}\} && (9.45)\\
&\le \inf\{\sup (H_{\mathcal{W}})_{\mathcal{L}} \ : \ \dim \mathcal{L} = n, \quad \mathcal{L} \subset \mathcal{W}\} = \mu_n(H_{\mathcal{W}}). && (9.46)
\end{aligned}$$

□

Theorem 9.56 (1) *Let $H \le G$. Then $\mu_n(H) \le \mu_n(G)$.*
(2) $|\mu_n(H) - \mu_n(G)| \le \|H - G\|$.

Remark 9.57 *The theorems of this subsection remain true if the operators are only bounded from below (but not necessarily bounded). In this case, if v does not belong to the form domain of A, then we set $(v|Av) = \infty$. Hence, if $\mathcal{L}$ is not contained in the form domain of A, then $\sup A_{\mathcal{L}} = \infty$, and the above theorems remain true.*

Notice also that if $\mathcal{D}$ is an essential domain for the quadratic form generated by A, then

$$\mu_n(A) := \inf\{\sup A_{\mathcal{L}} \ : \ \mathcal{L} \text{ is an } n\text{-dim. subspace of } \mathcal{D}\}.$$

9.19.1 Weyl Theorem on essential spectrum

Theorem 9.58 *Suppose H_0, H are self-adjoint and for all $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$(z-H)^{-1}-(z-H_0)^{-1}$$

is compact. Then $\mathrm{sp}_{\mathrm{ess}}(H)=\mathrm{sp}_{\mathrm{ess}}(H_0)$.

Proof. We have for $z_0 \in \mathbb{C}\backslash\mathbb{R}$ and $r < \mathrm{Im} z_0$,

$$(z_0-H)^{-n}=\frac{1}{2\pi \mathrm{i} n!}\int_{\partial K(z_0,r)}(z_0-z)^{-n}(z-H)^{-1}\mathrm{d}z. \tag{9.47}$$

Hence

$$(z_0-H)^{-n}-(z_0-H_0)^{-n}=\frac{1}{2\pi \mathrm{i} n!}\int_{\partial K(z_0,r)}(z_0-z)^{-n}\Big((z-H)^{-1}-(z-H_0)^{-1}\Big)\mathrm{d}z$$

is compact as well. But every $f \in C_{\mathrm{c}}(\mathbb{R})$ can be approximated in the supremum norm by linear combinations of $(z_0-H)^{-n}$, $(\overline{z}_0-H)^{-n}$, $n=1,2,\dots$. Hence $f(H)-f(H_0)$ is compact.

In particular, let $\lambda \notin \mathrm{sp}_{\mathrm{ess}}(H)$. Then there exists $f \in C_{\mathrm{c}}(\mathbb{R})$, $f(\lambda)\neq 0$ such that $f(H)$ is compact. But $f(H)-f(H_0)$ is compact. Hence $f(H_0)$ is compact. Hence $\lambda \notin \mathrm{sp}_{\mathrm{ess}}(H_0)$. Therefore, $\mathrm{sp}_{\mathrm{ess}}(H_0)\subset \mathrm{sp}_{\mathrm{ess}}(H)$. $\Box$

9.20 Singular values of an operator

Let A be a bounded operator on a Hilbert space $\mathcal{V}$. We define for $n=1,2,\dots$

$$\mathrm{s}_n(A) := \sup\{\inf\{(\|Av\| \ : \ \|v\|=1, \ v \in \mathcal{L}\} \ : \ \mathcal{L} \ n\text{-dim. subspace of } \mathcal{V}\}.$$

Clearly, for $|A| := (A^*A)^{1/2}$,

$$\mathrm{s}_n(A)=\mathrm{s}_n(|A|)=-\mu_n(-|A|),$$

and $\mathrm{s}_1(A)=\|A\|$.

9.21 Convergence of unbounded operators

Recall that lim denotes the norm convergence and s$-$ lim the strong convergence (of bounded operators). Recall also that $C_\infty(\mathbb{R})$ denotes the space of continuous functions on $\mathbb{R}$ vanishing at infinity and $C_{\mathrm{b}}(\mathbb{R})$ the space of bunded functions on $\mathbb{R}$.

Let (A_n) be a sequence of (possibly unbounded) operators. We say that

(1) $A_n \to A$ in the norm resolvent sense if for $z \in \mathbb{C}\backslash\mathbb{R}$

$$\lim_{n\to\infty}(z-A_n)^{-1}=(z-A)^{-1};$$

(2) $A_n \to A$ in the strong resolvent sense if for $z \in \mathbb{C}\backslash\mathbb{R}$

$$\text{s}-\lim_{n\to\infty}(z-A_n)^{-1} = (z-A)^{-1};$$

Theorem 9.59 (1) *$A_n \to A$ in the norm resolvent sense iff for any $f \in C_\infty(\mathbb{R})$ we have $\lim_{n\to\infty} f(A_n) = f(A)$.*

(2) *$A_n \to A$ in the strong resolvent sense iff for any $g \in C_\mathrm{b}(\mathbb{R})$ we have $\text{s}-\lim_{n\to\infty} g(A_n) = g(A)$.*

Proof. The $\Leftarrow$ implications are obvious. Let us prove the other implications.

(1): Let $z_0 \in \mathbb{C}\backslash\mathbb{R}$, $k = 1, 2, \dots$, and $r < \mathrm{Im} z_0$. We have

$$(z_0 - A)^{-k} = \frac{1}{2\pi \mathrm{i} k!}\int_{\partial K(z_0,r)} (z_0 - z)^{-k}(z-A)^{-1}\mathrm{d}z, \tag{9.48}$$

and similarly with A replaced by A_n. Hence $\lim_{n\to\infty}(z_0 - A_n)^{-k} = (z_0 - A)^{-k}$. Likewise, $\lim_{n\to\infty}(\overline{z}_0 - A_n)^{-k} = (\overline{z}_0 - A)^{-k}$. Now, by the Stone-Weierstrass Theorem, linear combinations of $x \mapsto (z_0 - x)^{-k}$ and $x \mapsto (\overline{z}_0 - x)^{-k}$ with $k = 1, 2, \dots$ are dense in $C_\infty(\mathbb{R})$ in the supremum norm. This easily implies (1).

(2): We first prove (2) for $g \in C_\infty(\mathbb{R})$, following the proof of (1).

Let $g \in C_\mathrm{b}(\mathbb{R})$, $v \in \mathcal{V}$ and $\epsilon > 0$. We can find $f \in C_\infty(\mathbb{R})$ such that

$$\|(f(A) - 1)v\| < \frac{\epsilon}{4\|g\|_\infty}.$$

Since $f, gf \in C_\infty(\mathbb{R})$, we can also find n_0 such that for $n > n_0$

$$\begin{aligned}\|(g(A)f(A) - g(A_n)f(A_n))v\| &< \frac{\epsilon}{4},\\ \|(f(A) - f(A_n))v\| &< \frac{\epsilon}{4\|g\|_\infty}.\end{aligned}$$

Now

$$\begin{aligned}\|g(A)v - g(A_n)v\| \le& \|g(A)(f(A)-1)v\| + \|g(A)f(A) - g(A_n)f(A_n)v\|\\ &+ \|g(A_n)(f(A_n) - f(A))v\| + \|g(A_n)(f(A)-1)v\| < \epsilon.\end{aligned}$$

This proves (2). $\square$

Chapter 10

Positive forms

10.1 Quadratic forms

Let $\mathcal{V}, \mathcal{W}$ be complex vector spaces.

Definition 10.1 $\mathfrak{a}$ *is called a* sesquilinear form *on* $\mathcal{W} \times \mathcal{V}$ *iff it is a map*

$$\mathcal{W} \times \mathcal{V} \ni (w, v) \mapsto \mathfrak{a}(w, v) \in \mathbb{C}$$

antilinear wrt the first argument and linear wrt the second argument.

If $\lambda \in \mathbb{C}$, then λ can be treated as a sesquilinear form $\lambda(w, v) := \lambda(w|v)$. If $\mathfrak{a}$ is a form, then we define $\lambda\mathfrak{a}$ by $(\lambda\mathfrak{a})(w, v) := \lambda\mathfrak{a}(w, v)$. and $\mathfrak{a}^*$ by $\mathfrak{a}^*(v, w) := \overline{\mathfrak{a}(w, v)}$. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are forms, then we define $\mathfrak{a}_1 + \mathfrak{a}_2$ by $(\mathfrak{a}_1 + \mathfrak{a}_2)(w, v) := \mathfrak{a}_1(w, v) + \mathfrak{a}_2(w, v)$.

Suppose that $\mathcal{V} = \mathcal{W}$. We will write $\mathfrak{a}(v) := \mathfrak{a}(v, v)$. We will call it a *quadratic form.* The knowledge of $\mathfrak{a}(v)$ determines $\mathfrak{a}(w, v)$:

$$\mathfrak{a}(w, v) = \frac{1}{4}\left(\mathfrak{a}(w + v) + \mathrm{i}\mathfrak{a}(w - \mathrm{i}v) - \mathfrak{a}(w - v) - \mathrm{i}\mathfrak{a}(w + \mathrm{i}v)\right). \tag{10.1}$$

Suppose now that $\mathcal{V}, \mathcal{W}$ are Hilbert spaces. A form is bounded iff

$$|\mathfrak{a}(w, v)| \leq C\|w\|\|v\|.$$

Proposition 10.2 (1) *Let $\mathfrak{a}$ be a bounded sesquilinear form on $\mathcal{W} \times \mathcal{V}$. Then there exists a unique operator $A \in B(\mathcal{V}, \mathcal{W})$ such that*

$$\mathfrak{a}(w, v) = (w|Av).$$

(2) *If $A \in B(\mathcal{V}, \mathcal{W})$, then $(w|Av)$ is a bounded sesquilinear form on $\mathcal{W} \times \mathcal{V}$.*

Proof. (2) is obvious. To show (1) note that $w \mapsto \mathfrak{a}(w|v)$ is an antilinear functional on $\mathcal{W}$. Hence there exists $\eta \in \mathcal{W}$ such that $\mathfrak{a}(w, v) = (w|\eta)$. We put $Av := \eta$.

Theorem 10.3 *Suppose that $\mathcal{D}, \mathcal{Q}$ are dense linear subspaces of $\mathcal{V}, \mathcal{W}$ and $\mathfrak{a}$ is a bounded sesquilinear form on $\mathcal{D} \times \mathcal{Q}$. Then there exists a unique extension of $\mathfrak{a}$ to a bounded form on $\mathcal{V} \times \mathcal{W}$.*

10.2 Sesquilinear quasiforms

Let $\mathcal{V}, \mathcal{W}$ be complex spaces. We say that $\mathfrak{t}$ is a *sesquilinear quasiform on* $\mathcal{W} \times \mathcal{V}$ iff there exist subspaces $\mathrm{Dom_l}\,\mathfrak{t} \subset \mathcal{W}$ and $\mathrm{Dom_r}\,\mathfrak{t} \subset \mathcal{V}$ such that

$$\mathrm{Dom_l}\mathfrak{t} \times \mathrm{Dom_r}\mathfrak{t} \ni (w, v) \mapsto \mathfrak{t}(w, v) \in \mathbb{C}$$

is a sesquilinear map. From now on by a sesquilinear form we will mean a sesquilinear quasiform.

We define a form $\mathfrak{t}^*$ with the domains $\mathrm{Dom_l}\,\mathfrak{t}^* := \mathrm{Dom_r}\,\mathfrak{t}$, $\mathrm{Dom_r}\,\mathfrak{t}^* := \mathrm{Dom_l}\,\mathfrak{t}$, by the formula $\mathfrak{t}^*(v, w) := \overline{\mathfrak{t}(w, v)}$. If $\mathfrak{t}_1$ are $\mathfrak{t}_2$ forms, then we define $\mathfrak{t}_1 + \mathfrak{t}_2$ with the domain $\mathrm{Dom_l}(\mathfrak{t}_1 + \mathfrak{t}_2) := \mathrm{Dom_l}\,\mathfrak{t}_1 \cap \mathrm{Dom_l}\,\mathfrak{t}_1$, $\mathrm{Dom_r}(\mathfrak{t}_1 + \mathfrak{t}_2) := \mathrm{Dom_r}\,\mathfrak{t}_1 \cap \mathrm{Dom_r}\,\mathfrak{t}_1$ by $(\mathfrak{t}_1 + \mathfrak{t}_2)(w, v) := \mathfrak{t}_1(w, v) + \mathfrak{t}_2(w, v)$. We write $\mathfrak{t}_1 \subset \mathfrak{t}_2$ if $\mathrm{Dom_l}\,\mathfrak{t}_1 \subset \mathrm{Dom_l}\,\mathfrak{t}_2$, $\mathrm{Dom_r}\,\mathfrak{t}_1 \subset \mathrm{Dom_r}\,\mathfrak{t}_2$, and $\mathfrak{t}_1(w, v) = \mathfrak{t}_2(w, v)$, $w \in \mathrm{Dom_l}\,\mathfrak{t}_1$, $v \in \mathrm{Dom_r}\,\mathfrak{t}_1$.

From now on, we will usually assume that $\mathcal{W} = \mathcal{V}$ and $\mathrm{Dom_l}\,\mathfrak{t} = \mathrm{Dom_r}\,\mathfrak{t}$ and the latter subspace will be simply denoted by $\mathrm{Dom}\,\mathfrak{t}$. We will then write $\mathfrak{t}(v) := \mathfrak{t}(v, v)$, $v \in \mathrm{Dom}\,\mathfrak{t}$.

The numerical range of the form $\mathfrak{t}$ is defined as

$$\mathrm{Num}\mathfrak{t} := \{\mathfrak{t}(v) \ : \ v \in \mathrm{Dom}\,\mathfrak{t},\ \|v\| = 1\}.$$

We proved that $\mathrm{Num}\mathfrak{t}$ is a convex set.

With every operator T on $\mathcal{V}$ we can associate the form

$$\mathfrak{t}_1(w, v) := (w|Tv), \quad w, v \in \mathrm{Dom}\,T.$$

Clearly, $\mathrm{Num}\mathfrak{t}_1 = \mathrm{Num}T$. If T is self-adjoint, we will however prefer to associate a different form to it, see Theorem 10.11.

The form $\mathfrak{t}$ is bounded iff $\mathrm{Num}\mathfrak{t}$ is bounded. Equivalently, $|\mathfrak{t}(v)| \leq c\|v\|^2$.

$\mathfrak{t}$ is hermitian iff $\mathrm{Num}\mathfrak{t} \subset \mathbb{R}$. An equivalent condition: $\mathfrak{t}(w, v) = \overline{\mathfrak{t}(v, w)}$.

A form $\mathfrak{t}$ is bounded from below, if there exists c such that

$$\mathrm{Num}\mathfrak{t} \subset \{z \ : \ \mathrm{Re}z > c\}.$$

A form $\mathfrak{t}$ is positive if $\mathrm{Num}\mathfrak{t} \subset [0, \infty[$. In this section we develop the basics of the theory of positive forms.

Note that many of the concepts and facts about positive forms generalize to hermitian bounded from below forms. In fact, if $\mathfrak{t}$ is bounded from below hermitian, then for some $c \in \mathbb{R}$ we have a positive form $\mathfrak{t} + c$. We leave these generalizations to the reader.

10.3 Closed positive forms

Let $\mathfrak{s}$ be a positive form.

Definition 10.4 *We say that* $\mathfrak{s}$ *is a closed form iff* $\mathrm{Dom}\,\mathfrak{s}$ *with the scalar product*

$$(w|v)_{\mathfrak{s}} := (\mathfrak{s} + 1)(w, v), \quad w, v \in \mathrm{Dom}\,\mathfrak{s}, \tag{10.2}$$

is a Hilbert space. We will then write $\|v\|_{\mathfrak{s}} := \sqrt{(v|v)_{\mathfrak{s}}}$.

Clearly, the scalar product (10.2) is equivalent with

$$(\mathfrak{s} + c)(w, v), \quad w, v \in \mathrm{Dom}\,\mathfrak{s},$$

for any $c > 0$.

Theorem 10.5 *The form $\mathfrak{s}$ is closed iff for any sequence (v_n) in $\mathrm{Dom}\,\mathfrak{s}$, if $v_n \to v$ and $\mathfrak{s}(v_n - v_m) \to 0$, then $v \in \mathrm{Dom}\,\mathfrak{s}$ and $\mathfrak{s}(v_n - v) \to 0$.*

Example 10.6 *Let A be an operator. Then*

$$(Aw|Av), \quad w, v \in \mathrm{Dom}\,A,$$

is a closed form iff A is closed.

10.4 Closable positive forms

Let $\mathfrak{s}$ be a positive form.

Definition 10.7 *We say that $\mathfrak{s}$ is a* closable form *iff there exists a closed form $\mathfrak{s}_1$ such that $\mathfrak{s} \subset \mathfrak{s}_1$.*

Theorem 10.8 (1) *The form $\mathfrak{s}$ is closable $\Leftrightarrow$ for any sequence $(v_n) \subset \mathrm{Dom}\,\mathfrak{s}$, if $v_n \to 0$ and $\mathfrak{s}(v_n - v_m) \to 0$, then $\mathfrak{s}(v_n) \to 0$.*

(2) *If $\mathfrak{s}$ is closable, then there exists the smallest closed form $\mathfrak{s}_1$ such that $\mathfrak{s} \subset \mathfrak{s}_1$. We will denote it by $\mathfrak{s}^{\mathrm{cl}}$.*

(3) $\mathrm{Num}\mathfrak{s}$ *is dense in* $\mathrm{Num}\mathfrak{s}^{\mathrm{cl}}$

Proof. (1) $\Rightarrow$ follows immediately from Theorem 10.5.

To prove (1) $\Leftarrow$, define $\mathfrak{s}_1$ as follows: $v \in \mathrm{Dom}\,\mathfrak{s}_1$, iff there exists a sequence $(v_n) \subset \mathrm{Dom}\,\mathfrak{s}$ such that $v_n \to v$ and $\mathfrak{s}(v_n - v_m) \to 0$. From $\mathfrak{s}(v_n) \leq (\sqrt{\mathfrak{s}(v_1)} + \sqrt{\mathfrak{s}(v_n - v_1)})^2$ it follows that $\big(\mathfrak{s}(v_n)\big)$ is bounded. From $|\mathfrak{s}(v_n) - \mathfrak{s}(v_m)| \leq \sqrt{\mathfrak{s}(v_n - v_m)}\big(\sqrt{\mathfrak{s}(v_n)} + \sqrt{\mathfrak{s}(v_n)}\big)$ it follows that $\big(\mathfrak{s}(v_n)\big)$ is a Cauchy sequence. Hence we can set $\mathfrak{s}_1(v) := \lim\limits_{n\to\infty} \mathfrak{s}(v_n)$

To show that the definition is correct, suppose that $(w_n) \in \mathrm{Dom}\,\mathfrak{s}$, $w_n \to v$ and $\mathfrak{s}(w_n - w_m) \to 0$. Then $\mathfrak{s}(v_n - w_n - (v_m - w_m)) \to 0$ and $v_n - w_n \to 0$. By the hypothesis we get $\mathfrak{s}(v_n - w_n) \to 0$. Hence, $\lim\limits_{n\to\infty} \mathfrak{s}(v_n) = \lim\limits_{n\to\infty} \mathfrak{s}(w_n)$. Thus the definition of $\mathfrak{s}_1$ does not depend on the choice of the sequence v_n. It is clear that $\mathfrak{s}_1$ is a closed form containing $\mathfrak{s}$. Hence $\mathfrak{s}$ is closable.

To prove (2) note that the form $\mathfrak{s}_1$ constructed above is the smallest closed form containg $\mathfrak{s}$. $\Box$

Example 10.9 *Let A be an operator. Then*

$$(Aw|Av), \quad w, v \in \mathrm{Dom}\,A,$$

is closable iff A is a closable operator. Then

$$(A^{\mathrm{cl}}w|A^{\mathrm{cl}}v), \quad w, v \in \mathrm{Dom}\,A^{\mathrm{cl}}$$

is its closure.

Definition 10.10 *We say that a linear subspace $\mathcal{Q}$ is an* essential domain *of the form $\mathfrak{s}$ if* $\left(\mathfrak{s}\Big|_{\mathcal{Q}\times\mathcal{Q}}\right)^{\mathrm{cl}} = \mathfrak{s}$.

10.5 Operators associated with positive forms

Let S be a self-adjoint operator. We define the form $\mathfrak{s}$ as follows:

$$\mathfrak{s}(v,w) := (|S|^{1/2}v|\mathrm{sgn}(S)|S|^{1/2}w), \quad v,w \in \mathrm{Dom}\,\mathfrak{s} := \mathrm{Dom}\,|S|^{1/2}.$$

We will say that $\mathfrak{s}$ is the *form associated with the operator S.*

Theorem 10.11 (1) $\mathrm{Num}S$ *is dense in* $\mathrm{Num}\mathfrak{s}$.

(2) *If S is positive, then $\mathfrak{s}$ is a closed positive form and* $\mathrm{Dom}\,S$ *is its essential domain.*

The next theorem describes the converse construction. It follows immediately from Thm 9.41.

Theorem 10.12 (Lax-Milgram Theorem) *Let $\mathfrak{s}$ be a densely defined closed positive form. Then there exists a unique positive self-adjoint operator S such that*

$$\mathfrak{s}(v,w) := (S^{1/2}v|S^{1/2}w), \quad v,w \in \mathrm{Dom}\,\mathfrak{s} := \mathrm{Dom}\,S^{1/2}.$$

Proof. By Thm 9.38 applied to $\mathrm{Dom}\,\mathfrak{s}$ there exists a positive self-adjoint operator T such that

$$\mathfrak{s}(v,w) := (Tv|Tw), \quad v,w \in \mathrm{Dom}\,\mathfrak{s} := \mathrm{Dom}\,T.$$

We set $S := T^2$. $\Box$

We will say that S is the *operator associated with the form $\mathfrak{s}$.*

10.6 Perturbations of positive forms

Theorem 10.13 *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be positive forms.*

(1) $\mathfrak{t}_1 + \mathfrak{t}_2$ *is also a positive form.*

(2) *If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closed, then $\mathfrak{t}_1 + \mathfrak{t}_2$ is closed as well.*

(3) *If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closable, then $\mathfrak{t}_1 + \mathfrak{t}_2$ is closable as well and* $(\mathfrak{t}_1 + \mathfrak{t}_2)^{\mathrm{cl}} \subset \mathfrak{t}_1^{\mathrm{cl}} + \mathfrak{t}_2^{\mathrm{cl}}$.

Definition 10.14 *Let $\mathfrak{p}$, $\mathfrak{t}$ be hermitian forms. Let $\mathfrak{t}$ be positive. We say that $\mathfrak{p}$ is $\mathfrak{t}$-bounded iff* $\mathrm{Dom}\,\mathfrak{t} \subset \mathrm{Dom}\,\mathfrak{p}$ *and*

$$b := \inf_{c>0} \sup_{v\in\mathrm{Dom}\,\mathfrak{t}} \frac{|\mathfrak{p}(v)|}{\mathfrak{t}(v) + c\|v\|^2} < \infty.$$

The number b is called the $\mathfrak{t}$-bound of $\mathfrak{p}$.

Theorem 10.15 *Let $\mathfrak{t}$ be positive and let $\mathfrak{p}$ be $\mathfrak{t}$-bounded with the $\mathfrak{t}$-bound < 1. Then*

(1) *The form $\mathfrak{t} + \mathfrak{p}$ (with the domain $\mathrm{Dom}\,\mathfrak{t}$) is bounded from below.*

(2) *$\mathfrak{t}$ is closed $\Leftrightarrow$ $\mathfrak{t} + \mathfrak{p}$ is closed.*

(3) *$\mathfrak{t}$ is closable $\Leftrightarrow$ $\mathfrak{t} + \mathfrak{p}$ is closable, and then $\mathrm{Dom}(\mathfrak{t} + \mathfrak{p})^{\mathrm{cl}} = \mathrm{Dom}\,\mathfrak{t}^{\mathrm{cl}}$.*

Proof. Let us prove (1). For some $b < 1$, we have

$$(\mathfrak{t} + \mathfrak{p})(v) \geq \mathfrak{t}(v) - |\mathfrak{p}(v)| \geq (1-b)\mathfrak{t}(v) - c\|v\|^2. \tag{10.3}$$

This proves that $\mathfrak{t} + \mathfrak{p}$ is bounded from below.

To see (2) and (3), note that (10.3) and

$$(1+b)\mathfrak{t}(v) + c\|v\|^2 \geq (\mathfrak{t} + \mathfrak{p})(v)$$

prove that the norms $\|\cdot\|_{\mathfrak{t}}$ and $\|\cdot\|_{\mathfrak{t}+\mathfrak{p}}$ are equivalent. $\Box$

10.7 Friedrichs extensions

Theorem 10.16 *Let T be a positive densely defined operator. Then the form*

$$\mathfrak{t}(w, v) := (w|Tv), \quad w, v \in \mathrm{Dom}\,\mathfrak{t} := \mathrm{Dom}\,T$$

is closable.

Proof. Suppose that $w_n \in \mathrm{Dom}\,T$, $w_n \to 0$, $\lim\limits_{n,m\to\infty} \mathfrak{t}(w_n - w_m) = 0$. Then

$$\begin{aligned} |\mathfrak{t}(w_n)| \quad &\leq |\mathfrak{t}(w_n - w_m, w_n)| + |\mathfrak{t}(w_m, w_n)| \\ &\leq \sqrt{\mathfrak{t}(w_n)}\sqrt{\mathfrak{t}(w_n - w_m)} + (w_m|Tw_n). \end{aligned}$$

For any $\epsilon > 0$ there exists N such that for $n, m > N$ we have $\mathfrak{t}(w_n - w_m) \leq \epsilon^2$. Besides, $\lim\limits_{m\to\infty}(w_m|Tw_n) = 0$. Therefore, for $n > N$,

$$|\mathfrak{t}(w_n)| \leq \epsilon|\mathfrak{t}(w_n)|^{1/2}.$$

Hence $\mathfrak{t}(w_n) \to 0$. $\Box$

Thus there exists a unique postive self-adjoint operator T^{Fr} associated with the form $\mathfrak{t}^{\mathrm{cl}}$. The operator T^{Fr} is called the Friedrichs extension of T.

Clearly, $\mathrm{Dom}\,T$ is then essential form domain of T^{Fr}. However in general it is not an essential operator domain of T^{Fr}. The theorem says nothing about essential operator domains.

For example, consider any open $\Omega \subset \mathbb{R}^d$. Note that $C_{\mathrm{c}}^{\infty}(\Omega)$ is dense in $L^2(\Omega)$. The equation

$$(f| - \Delta g) = \int \overline{\nabla f(x)} \nabla g(x) \mathrm{d}x, \quad f \in C_{\mathrm{c}}^{\infty}(\Omega)$$

shows that $-\Delta$ on $C_c^\infty(\Omega)$ is a positive operator. Its Friedrichs extension is called the laplacian on Ω with the Dirichlet boundary conditions.

If V is any positive bounded from below function we can consider $\Delta + V(x)$ and define its Friedrichs extension.

Chapter 11

Non-maximal operators

11.1 Defect indices

If $\mathcal{V}$ is a finite dimensional Hilbert space and $\mathcal{V}_1, \mathcal{V}_2$ its two subspaces such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$, then we have the following obvious inequalities:

$$\begin{array}{rcl} \dim \mathcal{V}_1 + \dim \mathcal{V}_2 & \leq & \dim \mathcal{V}, \\ \dim \mathcal{V}_1 & \leq & \dim \mathcal{V}_2^{\perp}, \\ \dim \mathcal{V}_2 & \leq & \dim \mathcal{V}_1^{\perp}. \end{array}$$

If $\dim \mathcal{V} = \infty$, then clearly the first inequality loses its interest. However the other two inequalities, which are still true, may be interesting.

Let A be an operator on a Hilbert space $\mathcal{V}$.

Theorem 11.1 $\dim \mathrm{Ran}\,(z - A)^{\perp} = \dim \mathrm{Ker}(\overline{z} - A^*)$ *is a constant function on connected components of* $\mathbb{C}\backslash(\mathrm{Num}A)^{\mathrm{cl}}$.

Proof. Let us show that if $|z - z_1| < \mathrm{dist}(z, \mathrm{Num}A)$, then

$$\mathrm{Ran}\,(z - A) \cap \mathrm{Ran}\,(z_1 - A)^{\perp} = \{0\}. \tag{11.1}$$

Let $w \in \mathrm{Ran}\,(z - A)$. Then there exists $v \in \mathrm{Dom}\, A$ such that

$$w = (z - A)v$$

and $\|v\| \leq c\|w\|$, where $c = \big(\mathrm{dist}(z, \mathrm{Num}A)\big)^{-1}$. If moreover, $w \in \mathrm{Ran}\,(z_1 - A)^{\perp} = \mathrm{Ker}(\overline{z}_1 - A^*)$, then

$$\begin{aligned} 0 &= \big((z_1 - A^*)w|v\big) \\ &= \big(w|(z - A)v\big) + (z_1 - z)(w|v) \\ &= \|w\|^2 + (z - z_1)(w|v). \end{aligned}$$

But

$$\big|\|w\|^2 + (z_1 - z)(w|v)\big| \geq (1 - |z_1 - z|c)\|w\|^2 > 0,$$

which is a contradiction and completes the proof of (11.1).

Now (11.1) implies that $\dim \mathrm{Ran}\,(z - A)^\perp \leq \dim \mathrm{Ran}\,(z_1 - A)^\perp$. □

11.2 Extensions of hermitian operators

Let A be closed hermitian.

Theorem 11.2 *The so-called* deficiency indices of A

$$n_\pm := \dim \mathrm{Ker}(z - A^*), \;\; z \in \mathbb{C}_\pm$$

do not depend on z. Then A possesses a self-adjoint extension iff $n_+ = n_-$. Moreover, one of the following possibilities is true:

(1) $\mathrm{Num}A \neq \mathbb{R}$.

- (i) $\mathrm{sp}A \subset \mathbb{R}$, $n_+ = n_- = 0$ *and A is self-adjoint.*
- (ii) $\mathrm{sp}A = \mathbb{C}$, $n_+ = n_- > 0$.

(2) $\mathrm{Num}A = \mathbb{R}$.

- (i) $\mathrm{sp}A \subset \mathbb{R}$, $n_+ = n_- = 0$, *A is self-adjoint.*
- (ii) $\mathrm{sp}A = \{\mathrm{Im}z \geq 0\}$, $n_+ > 0$, $n_- = 0$, *A is not self-adjoint.*
- (iii) $\mathrm{sp}A = \{\mathrm{Im}z \leq 0\}$, $n_+ = 0$, $n_- > 0$, *A is not self-adjoint.*
- (iv) $\mathrm{sp}A = \mathbb{C}$, $n_+ > 0$, $n_- > 0$, *A is not self-adjoint.*

Proof. The existence of self-adjoint extensions for $n_+ = n_-$ follows from Theorem 11.4.

The remaining statements are essentially a special case of Theorem 11.1. □

Definition 11.3 *Define on* $\mathrm{Dom}\,A^*$ *the following scalar product:*

$$(v|w)_{A^*} := (v|w) + (A^*v|A^*w)$$

and the following antihermitian form:

$$[v|w]_{A^*} := (A^*v|w) - (v|A^*w).$$

The A^*-closedness and the A^*-orthogonality is defined using the scalar product $(\cdot|\cdot)_{A^*}$.

Theorem 11.4 (1) *Every closed extension of A is a restriction of A^* to an A^*-closed subspace in* $\mathrm{Dom}\,A^*$ *containing* $\mathrm{Dom}\,A$.

(2)
$$\mathrm{Dom}\,A^* = \mathrm{Dom}\,A \oplus \mathrm{Ker}(A^* + \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i})$$

and the components in the above direct sum are A^*-closed, A^*-orthogonal and

$$\begin{aligned}
(w_0 \oplus w_+ \oplus w_- | v_0 \oplus v_+ \oplus v_-)_{A^*} &= (w_0|v_0) + (Aw_0|Av_0) + 2(w_+|v_+) + 2(w_-|v_-), \\
[w_0 \oplus w_+ \oplus w_- | v_0 \oplus v_+ \oplus v_-]_{A^*} &= 2\mathrm{i}(w_+|v_+) - 2\mathrm{i}(w_-|v_-).
\end{aligned}$$

Proof. (1) is obvious. In (2) the A^*-orthogonality and the A^*-closedness are easy.

Let $w \in \mathrm{Dom}\,A^*$ and
$$w \perp_{A^*} \mathrm{Dom}\,A \oplus \mathrm{Ker}(A^* + \mathrm{i}).$$
In particular, for $v \in \mathrm{Dom}\,A$ we have
$$0 = (A^*w|A^*v) + (w|v) = (A^*w|Av) + (w|v).$$
Hence $A^*w \in \mathrm{Dom}\,A^*$ and
$$A^*A^*w = -w.$$
Therefore,
$$(A^* + \mathrm{i})(A^* - \mathrm{i})w = 0.$$
Thus
$$(A^* - \mathrm{i})w \in \mathrm{Ker}(A^* + \mathrm{i}). \tag{11.2}$$
If $y \in \mathrm{Ker}(A^* + \mathrm{i})$, then
$$\mathrm{i}(y|(A^* - \mathrm{i})w) = (A^*y|A^*w) + (y|w) = (y|w)_{A^*} = 0$$
In particular, by (11.2) we can set $y = (A^* - \mathrm{i})w$. We get $w \in \mathrm{Ker}(A^* - \mathrm{i})$. □

$\mathrm{Dom}\,A$ belongs to the kernel of the antisymmetric form $[\cdot,\cdot]_{A^*}$. Therefore, in what follows we restrict this form to
$$\mathcal{V}_{\mathrm{def}} := \mathrm{Ker}(A^* + \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i}).$$
We will write
$$\mathcal{Z}^{\mathrm{per}} := \{v \in \mathcal{V}_{\mathrm{def}} \ : \ [z, v]_{A^*} = 0,\ z \in \mathcal{Z}\}.$$
We will say that a subspace $\mathcal{Z}$ of $\mathcal{V}_{\mathrm{def}}$ is A^*-isotropic iff $[\cdot|\cdot]_{A^*}$ vanishes on $\mathcal{Z}$ and A^*-Lagrangian if $\mathcal{Z}^{\mathrm{per}} = \mathcal{Z}$.

Every A^*-closed subspace of $\mathcal{V}$ containing $\mathrm{Dom}\,A$ is of the form $\mathrm{Dom}\,A \oplus \mathcal{Z}$, where $\mathcal{Z} \subset \mathcal{V}_{\mathrm{def}}$. If
$$A \subset B \subset A^*,$$
then the subspace $\mathcal{Z}$ corresponding to B will be denoted by $\mathcal{Z}_B$.

Theorem 11.5 (1) *We have*

$$\mathcal{Z}_{B^*} = (\mathcal{Z}_B)^{\mathrm{per}}.$$

(2) *B is hermitian iff $\mathcal{Z}_B$ is A^*-isotropic iff there exists a partial isometry $U : \mathrm{Ker}(A^* + \mathrm{i}) \to \mathrm{Ker}(A^* - \mathrm{i})$ such that*

$$\mathcal{Z} := \{w_+ \oplus Uw_+ \ : \ w_+ \in \mathrm{Ran}\, U^*U\}.$$

(3) *B is self-adjoint iff $\mathcal{Z}_B$ is A^*-Lagrangian iff there exists a unitary U : $\mathrm{Ker}(A^* + \mathrm{i}) \to \mathrm{Ker}(A^* - \mathrm{i})$ such that*

$$\mathcal{Z} := \{w_+ \oplus Uw_+ \ : \ w_+ \in \mathrm{Ker}(A^* + \mathrm{i})\}.$$

11.3 Extension of positive operators

(This subsection is based on unpublished lectures of S.L.Woronowicz).

Theorem 11.6 *Let $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ and*

$$B = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}$$

be an operator in $B(\mathcal{V})$ with B_{11} invertible. Then B is positive iff $B_{11} \geq 0$, $B_{01} = B_{10}^*$ and $B_{00} \geq B_{01}B_{11}^{-1}B_{10}$.

Proof. Let $v_0 \in \mathcal{V}_0$, $v_1 \in \mathcal{V}_1$. For $v_z = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$. Then

$$\begin{aligned} 0 \leq (v|Bv) &= (v_0 B_{00} v_0) + (v_0|B_{01}v_1) + (v_1|B_{10}v_0) + (v_1|B_{11}v_1) \\ &= \big(v_0|(B_{00} - B_{01}B_{11}^{-1}B_{10})v_0\big) + \|B_{11}^{-1/2}B_{10}v_0 + B_{11}^{1/2}v_1\|^2 \end{aligned}$$

This proves $\Rightarrow$.

Let us prove $\Leftarrow$. The necessity of $B_{11} \geq 0$ is obvious. Given v_0, we can choose $v_1 = -B_{11}^{-1}B_{10}v_0$. This shows that $B_{00} - B_{01}B_{11}^{-1}B_{10}$ has to be positive. $\square$

Suppose that G is hermitian, positive and closed. We would like to describe its positive self-adjoint extensions. Thus we are looking for positive self-adjoint H such that $G \subset H$.

The operator $G + \mathbb{1}$ is injective and has a closed range. Define $\mathcal{V}_1 := \mathrm{Ran}\, G$ and set $\mathcal{V}_0 := \mathcal{V}_1^\perp$, so that $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$. Let $A \in B(\mathcal{V}_1, \mathcal{V})$ be the left inverse of $G + \mathbb{1}$. We can write it as

$$A = \begin{bmatrix} A_{01} \\ A_{11} \end{bmatrix}$$

We are looking for a bounded operator

$$(\mathbb{1} + H)^{-1} = B = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \in B(\mathcal{V})$$

that extends A and $0 \leq B \leq \mathbb{1}$. Clearly, $B_{11} = A_{11}$, $B_{01} = A_{01}$, $B_{10} = A_{01}^*$. By Theorem 11.6,

$$\begin{aligned} B_{00} &\geq B_{01} B_{11}^{-1} B_{10}, \\ \mathbb{1}_{00} - B_{00} &\geq B_{01} (\mathbb{1}_{11} - B_{11})^{-1} B_{10}. \end{aligned}$$

Thus we can choose any $B_{00} \in B(\mathcal{V}_0)$ satisfying

$$\mathbb{1}_{00} - A_{01}(\mathbb{1}_{11} - A_{11})^{-1} A_{01}^* \geq B_{00} \geq A_{01} A_{11}^{-1} A_{01}^*.$$

This condition has two extreme solutions: The smallest $A_{01} A_{11}^{-1} A_{01}^*$ yields the largest extension, called the *Friedrichs extension* H^{Fr}. The largest $\mathbb{1}_{00} - A_{01}(\mathbb{1}_{11} - A_{11})^{-1} A_{01}^*$, gives the smallest positive extension, called the *Krein extension* H^{Kr}. We have the following formula for both extensions:

$$\begin{aligned} & (\mathbb{1} + H^{\mathrm{Fr}})^{-1} \\ := \ & \big(A_{11}^{1/2} + A_{01} A_{11}^{-1/2}\big)\big(A_{11}^{1/2} + A_{01} A_{11}^{-1/2}\big)^*, \\ & \mathbb{1} - (\mathbb{1} + H^{\mathrm{Kr}})^{-1} \\ := \ & \big((\mathbb{1}_{11} - A_{11})^{1/2} - A_{01}(\mathbb{1}_{11} - A_{11})^{-1/2}\big)\big((\mathbb{1}_{11} - A_{11})^{1/2} - A_{01}(\mathbb{1}_{11} - A_{11})^{-1/2}\big)^*. \end{aligned}$$

Chapter 12

Aronszajn-Donoghue Hamiltonians and their renormalization

12.1 Construction

Recall that the operators $(h|$ and $|h)$ are defined by

$$\begin{aligned} &\mathcal{H} \ni v \mapsto (h|v := (h|v) \in \mathbb{C}, \\ &\mathbb{C} \ni \alpha \mapsto |h)\alpha := \alpha h \in \mathcal{H}. \end{aligned} \tag{12.1}$$

In particular, $|h)(h|$ equals the orthogonal projection onto h times $\|h\|^2$.

Let H_0 be a self-adjoint operator on $\mathcal{H}$, $h \in \mathcal{H}$ and $\lambda \in \mathbb{R}$.

$$H_\lambda := H_0 + \lambda|h)(h|, \tag{12.2}$$

is a rank one perturbation of H_0. We will call (12.2) the *Aronszajn Donoghue Hamiltonian.*

We would like to describe how to define the Aronszajn-Donoghue Hamiltonian if h is not necessarily a bounded functional on $\mathcal{H}$. It will turn out that it is natural to consider 3 types of h:

$$\text{I. } h \in \mathcal{H}, \quad \text{II. } h \in \langle H_0\rangle^{1/2}\mathcal{H} \setminus \mathcal{H}, \quad \text{III. } h \in \langle H_0\rangle\mathcal{H} \setminus \langle H_0\rangle^{1/2}\mathcal{H}, \tag{12.3}$$

where $\langle H_0\rangle := (1 + H_0^2)^{1/2}$.

Clearly, in the case I H_λ is self-adjoint on $\operatorname{Dom} H_0$. We will see that in the case II one can easily define H_λ as a self-adjoint operator, but its domain is no longer equal to $\operatorname{Dom} H_0$. In the case III, strictly speaking, the formula (12.2) does not make sense. Nevertheless, it is possible to define a *renormalized Aronszajn-Donoghue Hamiltonian.* To do this one needs to *renormalize the* *parameter* λ. This procedure resembles the *renormalization of the charge* in quantum field theory. In this case usually the parameter λ looses its meaning, so we will abandon the notation H_λ. Instead, one can label the Hamiltonian by various parameters, which we will put in brackets.

Lemma 12.1 *In Case I with $\lambda \neq 0$, the resolvent of H_λ equals*

$$\begin{aligned} R_\lambda(z) &:= (z - H_\lambda)^{-1} \\ &= (z - H_0)^{-1} - g_\lambda(z)^{-1}(z - H_0)^{-1}|h)(h|(z - H_0)^{-1}, \quad (12.4) \end{aligned}$$

where

$$g_\lambda(z) := -\lambda^{-1} + (h|(z - H_0)^{-1}h). \quad (12.5)$$

defined for $z \notin \mathrm{sp}H_0$.

Proof. We have

$$\begin{aligned} R(z) - (z - H_0)^{-1} &= \lambda R(z)|h)(h|(z - H_0)^{-1} \\ &= \lambda(z - H_0)^{-1}|h)(h|R(z). \quad (12.6) \end{aligned}$$

Hence the range of (12.6) is $\mathbb{C}(z - H_0)^{-1}h$, and the kernel is $\{(z - H_0)^{-1}h\}^\perp$. Therefore, (12.6) has the form

$$-g(z)^{-1}(z - H_0)^{-1}|h)(h|(z - H_0)^{-1} \quad (12.7)$$

for some complex function $g(z)$. Thus it remains to determine $g(z)$ in (12.4). We insert (12.4) into

$$\lambda(z - H_0)^{-1}|h)(h|R(z) = -g(z)^{-1}(z - H_0)^{-1}|h)(h|(z - H_0)^{-1},$$

and we obtain the formula for g, sometimes called *Krein's formula.* $\Box$

For $\lambda = 0$, clearly

$$R_0(z) = (z - H_0)^{-1}. \quad (12.8)$$

The following theorem describes how to define the Aronszajn-Donoghue Hamiltonian also in cases II and III:

Theorem 12.2 *Assume that:*
(A) $h \in \langle H_0\rangle^{1/2}\mathcal{H}$, $\lambda \in \mathbb{R} \cup \{\infty\}$. Let $R_\lambda(z)$ be given by (12.8) or (12.4) with $g_\lambda(z)$ given by (12.5),
or
(B) $h \in \langle H_0\rangle\mathcal{H}$, $\gamma \in \mathbb{R}$. Let $R_{\diamond,\gamma}(z)$ be given by (12.4) where instead of g_λ we set $g_{\diamond,\gamma}(z)$, the solution of

$$\partial_z g_{\diamond,\gamma}(z) = -\left(h|(z - H_0)^{-2}h\right), \quad (12.9)$$

$$\frac{1}{2}\left(g_{\diamond,\gamma}(\mathrm{i}) + g_{\diamond,\gamma}(-\mathrm{i})\right) = \gamma. \quad (12.10)$$

Then, (dropping the subscript λ or $\diamond,\gamma$) for $z \in \mathbb{C}\backslash\mathrm{sp}H_0$ such that $g(z) \neq 0$

(1) $z \mapsto R(z)$ *is a pseudoresolvent (a function with values in bounded operators that fulfill the first resolvent formula);*

(2) $\mathrm{Ker} R(z) = \{0\}$, *unless* $h \in \mathcal{H}$ *and* $\lambda = \infty$*;*

(3) $\mathrm{Ran}\, R(z)$ *is dense in* $\mathcal{H}$*, unless* $h \in \mathcal{H}$ *and* $\lambda = \infty$*;*

(4) $R(z)^* = R(\overline{z})$.

Hence, except for the case $h \in \mathcal{H}$, $\lambda = \infty$*, there exists a unique densely defined self-adjoint operator* H *such that* $R(z)$ *is the resolvent of* H*.*

The symbol $\diamond$, is supposed to symbolize the initial condition in (12.10), which can be called the *renormalization condition*. It is easy to solve (12.10) obtaining

$$g_{\diamond,\gamma}(z) = \gamma + \big(h|\big((z - H_0)^{-1} + H_0(1 + H_0^2)^{-1}\big)h\big) .$$

If $g(\beta) = 0$ and $\beta \notin \mathrm{sp}H_0$, then H has an eigenvalue at β, and the corresponding eigenprojection is

$$1_{\{\beta\}}(H) = (h|(\beta - H_0)^{-2}h)^{-1}(\beta - H_0)^{-1}|h)(h|(\beta - H_0)^{-1}.$$

In Case I and II the function $\mathbb{R} \cup \{\infty\} \ni \lambda \mapsto H_\lambda$ is increasing.

In Case III we rename H_0 as $H_{(\infty)}$.

12.2 Cut-off method

Another way to define H for the case $h \in \langle H_0 \rangle \mathcal{H}$ is the cut-off method. For $\Lambda > 0$ we define

$$h_\Lambda := \mathbb{1}_{[-\Lambda,\Lambda]}(H_0)\, h, \tag{12.11}$$

where $\mathbb{1}_{[-\Lambda,\Lambda]}(H_0)$ is the spectral projection for H_0 onto $[-\Lambda, \Lambda] \subset \mathbb{R}$. Note that $h_\Lambda \in \mathcal{H}$.

We fix the *running coupling constant* by

$$-\lambda_\Lambda^{-1} := \gamma + (h_\Lambda|H_0(1 + H_0^2)^{-1}h_\Lambda)$$

and set the *cut-off Hamiltonian* to be

$$H_\Lambda := H_0 + \lambda_\Lambda |h_\Lambda)(h_\Lambda|. \tag{12.12}$$

Then the resolvent for H_Λ is given by

$$R_\Lambda(z) = (z - H_0)^{-1} - g_\Lambda(z)^{-1}(z - H_0)^{-1}|h_\Lambda)(h_\Lambda|(z - H_0)^{-1}, \tag{12.13}$$

where

$$g_\Lambda(z) := -\lambda_\Lambda^{-1} + \big(h_\Lambda|(z - H_0)^{-1}h_\Lambda\big) . \tag{12.14}$$

Note that λ_Λ is chosen in such a way that the renormalization condition

$$\frac{1}{2}\left(g_\Lambda(\mathrm{i}) + g_\Lambda(-\mathrm{i})\right) = \gamma. \tag{12.15}$$

holds. The cut-off Hamiltonian converges to the renormalized Hamiltonian:

Theorem 12.3 *Assume that* $h \in \langle H_0 \rangle \mathcal{H}$*. Then* $\lim\limits_{k \to \infty} R_\Lambda(z) = R_{\diamond,\gamma}(z)$*.*

12.3 Extensions of hermitian operators

Let H_0 be as above and $h \in \langle H_0\rangle\mathcal{H}\backslash\mathcal{H}$. (Thus we consider jointly Case II and III.) Define H_{min} to be the restriction of H_0 to

$$\mathrm{Dom}(H_{\mathrm{min}}) := \{v \in \mathrm{Dom}(H_0) = \langle H_0\rangle^{-1}\mathcal{H} \;:\; (h|v) = 0\}.$$

Then H_{min} is a closed densely defined Hermitian operator. Set $H_{\mathrm{max}} := H_{\mathrm{min}}^*$. Then for any $z_0 \in \mathrm{rs}H_0$

$$\mathrm{Dom}(H_{\mathrm{max}}) = \mathrm{Span}\big(\,\mathrm{Dom}\,H_0 \cup \{(z_0 - H_0)^{-1}h\}\big).$$

Note that $\mathrm{Ker}(H_{\mathrm{max}} \pm \mathrm{i})$ is spanned by

$$v_\pm := (\pm\mathrm{i} - H_0)^{-1}h.$$

Thus the deficiency indices of H_{min} are $(1,1)$.

The operators $H_{\diamond,\gamma}$ described in the previous subsection are self-adjoint extensions of H_{min}. To obtain $H_{\diamond,\gamma}$ it suffices to increase the domain of H_{min} by adding the vector

$$\frac{\gamma + (h|H_0(1+H_0^2)^{-1}h)}{\gamma - \mathrm{i}(h|(1+H_0^2)^{-1}h)}(\mathrm{i} - H_0)^{-1}h - \frac{\gamma + (h|H_0(1+H_0^2)^{-1}h)}{\gamma + \mathrm{i}(h|(1+H_0^2)^{-1}h)}(\mathrm{i} + H_0)^{-1}h,$$

If $H_{\diamond,\gamma}$ has an eigenvalue β outside of $\mathrm{sp}H_0$, then instead we can add the vector

$$(\beta - H_0)^{-1}h.$$

12.4 Positive H_0

Let us consider the special case $H_0 > 0$.

Clearly, g is analytic on $\mathbb{C}\backslash[0,\infty[$. g restricted to $]-\infty,0[$ is a decreasing function (in all cases I, II and III). Therefore, H can possess at most one negative eigenvalue.

If we use the cut-off method in Case III, then $\lambda_\Lambda \nearrow 0$. Thus we should think of λ as infinitesimally small negative.

We distinguish subcases of Cases I, II and III

We assume additionally that

$$\mathrm{Ia},\ \mathrm{IIa} : h \in H_0^{\frac12}\mathcal{H}, \qquad \mathrm{IIIa} : h \in H_0^{\frac12}\langle H_0\rangle^{\frac12}\mathcal{H}, \tag{12.16}$$

$$\mathrm{Ia},\ \mathrm{IIa} : h \notin H_0^{\frac12}\mathcal{H}, \qquad \mathrm{IIIa} : h \notin H_0^{\frac12}\langle H_0\rangle^{\frac12}\mathcal{H}. \tag{12.17}$$

In Case Ia and IIa we set

$$\lambda_{\mathrm{Kr}} := -(h|H_0^{-1}h)^{-1}. \tag{12.18}$$

Note that λ_{Kr} is negative. (In all other cases one could interpret $(h|H_0^{-1}h)$ as $+\infty$, and therefore one can then set $\lambda_{\mathrm{Kr}} := 0$). We have

$$\lim_{x\to-\infty} g_\lambda(x) = -\lambda^{-1}, \quad g_\lambda(0) = -\lambda^{-1} + \lambda_{\mathrm{Kr}}^{-1}.$$

Therefore, H_λ is positive for $\lambda_{\mathrm{Kr}} \leq \lambda \leq \infty$. For $\lambda < \lambda_{\mathrm{Kr}}$, H_λ has a single negative eigenvalue β, which is the solution of

$$\lambda(h|(H_0 - \beta)^{-1}h) = -1. \tag{12.19}$$

In Case IIa $H_{\lambda_{\mathrm{Kr}}}$ is the Krein extension of $H_{\min}$ and H_∞ is the Friedrichs extension.

In Case Ib and IIb we have

$$\lim_{x \to -\infty} g_\lambda(x) = -\lambda^{-1}, \quad g_\lambda(0) = -\infty.$$

H_λ is positive for $0 \leq \lambda \leq \infty$. For $\lambda < 0$, H_λ has a single negative negative eigenvalue β, which is the solution of (12.19). In Case IIb H_0 is the Krein extension of $H_{\min}$ and H_∞ is its Friedrichs extension.

In Case III H_0 is the Friedrichs extension of $H_{\min}$.

In Case IIIa we have it is natural to replace the renormalization condition (12.10) by

$$g_{0,\gamma}(0) =: \gamma. \tag{12.20}$$

Thus

$$g_{0,\gamma}(z) \quad = \quad \gamma - \big(h|(H_0 - z)^{-1}H_0^{-1}h\big)\, z.$$

One can rename H_0 as $H_{0,\infty}$, which is the Friedrichs extension of $H_{\min}$. The Krein extension is $H_{0,0}$.

$H_{0,\gamma}$ is an increasing function of $\gamma \in \mathbb{R} \cup \{\infty\}$. It is positive for $0 \leq \gamma$. It has a single negative eigenvalue at β solving

$$\gamma = (h|(H_0 - \beta)^{-1}H_0^{-1}h)\beta$$

for $\gamma < 0$. β is an increasing function of γ.

In Case IIIb the condition (12.20) does not work and should be replaced e.g. by

$$g_{-1,\gamma}(-1) =: \gamma, \tag{12.21}$$

This yields

$$g_{-1,\gamma}(z) \quad = \quad \gamma - \big(h|(H_0 - z)^{-1}(H_0 + 1)^{-1}h\big)\,(z + 1).$$

One can rename H_0 as $H_{-1,\infty}$, and it is both the Krein extension and Friedrichs extension of $H_{\min}$. $H_{-1,\gamma}$ is an increasing function of $\gamma \in \mathbb{R} \cup \{\infty\}$.

$H_{-1,\gamma}$ has a single negative eigenvalue β for all $\gamma \in \mathbb{R}$. β is an increasing function of γ.

Chapter 13

Friedrichs Hamiltonians and their renormalization

13.1 Construction

Let H_0 be again a self-adjoint operator on the Hilbert space $\mathcal{H}$. Let $\epsilon \in \mathbb{R}$ and $h \in \mathcal{H}$. The following operator on the Hilbert space $\mathbb{C} \oplus \mathcal{H}$ is often called the *Friedrichs Hamiltonian*:

$$G := \begin{bmatrix} \epsilon & (h| \\ |h) & H_0 \end{bmatrix}. \tag{13.1}$$

We would like to describe how to define the Friedrichs Hamiltonian if h is not necessarily a bounded functional on $\mathcal{H}$. It will turn out that it is natural to consider 3 types of h:

$$\text{I. } h \in \mathcal{H}, \quad \text{II. } h \in \langle H_0\rangle^{1/2}\mathcal{H} \setminus \mathcal{H}, \quad \text{III. } h \in \langle H_0\rangle\mathcal{H} \setminus \langle H_0\rangle^{1/2}\mathcal{H}, \tag{13.2}$$

Clearly, in case I G is self-adjoint on $\mathbb{C}\oplus\mathrm{Dom}\, H_0$. We will see that in case II one can easily define G as a self-adjoint operator, but its domain is no longer $\mathbb{C}\oplus \mathrm{Dom}\, H_0$. In case III, strictly speaking, the formula (13.1) does not make sense. Nevertheless, it is possible to define a *renormalized Friedrichs Hamiltonian*. To do this one needs to *renormalize the parameter* ϵ. This procedure resembles the *renormalization of mass* in quantum field theory.

Let us first consider the case $h \in \mathcal{H}$. As we said earlier, the operator G with $\mathrm{Dom}\, G = \mathbb{C} \oplus \mathrm{Dom}\, H_0$ is self-adjoint. It is well known that the resolvent of G can be computed exactly. In fact, for $z \notin \mathrm{sp}H_0$ define the analytic function

$$f(z) := \epsilon + (h|(z - H_0)^{-1}h). \tag{13.3}$$

Then for $z \in \mathbb{C}\backslash \mathrm{sp}H_0$, $f(z) \neq z$ the resolvent $Q(z) := (z-G)^{-1}$ is given by

$$\begin{aligned} Q(z) &= \begin{bmatrix} 0 & 0 \\ 0 & (z-H_0)^{-1} \end{bmatrix} \\ &+\Big(z-f(z)\Big)^{-1} \begin{bmatrix} 1\!\!1 & (h|(z-H_0)^{-1} \\ (z-H_0)^{-1}|h) & (z-H_0)^{-1}|h)(h|(z-H_0)^{-1} \end{bmatrix}. \end{aligned} \tag{13.4}$$

Theorem 13.1 *Assume that:*
(A) $h \in \langle H_0\rangle^{1/2}\mathcal{H}$, $\epsilon \in \mathbb{R}$. *Let* $Q(z)$ *be given by (13.4) with* $f(z)$ *defined by (13.3),*
or
(B) $h \in \langle H_0\rangle\mathcal{H}$, $\gamma \in \mathbb{R}$. *Let* $Q(z)$ *be given by (13.4) with* $f(z)$ *defined by*

$$\begin{cases} \partial_z f(z) = -\big(h|(z-H_0)^{-2}h\big)\,, \\ \frac{1}{2}\big(f(\mathrm{i}) + f(-\mathrm{i})\big) = \gamma. \end{cases} \tag{13.5}$$

Then for all $z \in \mathbb{C}\backslash \mathrm{sp}H_0$, $f(z) \neq z$ *:*

(1) $Q(z)$ *is a pseudoresolvent;*

(2) $\mathrm{Ker}Q(z) = \{0\}$*;*

(3) $\mathrm{Ran}\, Q(z)$ *is dense in* $\mathbb{C}\oplus\mathcal{H}$*;*

(4) $Q(z)^* = Q(\overline{z})$.

Therefore, there exists a unique densely defined self-adjoint operator G *such that* $Q(z) = (z-G)^{-1}$.

Proof. Let $z \in \mathbb{C}\backslash \mathrm{sp}H_0$, $f(z) \neq z$. It is obvious that $Q(z)$ is bounded and satisfies (4). We easily see that both in the case (A) and (B) the function $f(z)$ satisfies

$$f(z_1) - f(z_2) = -(z_1 - z_2)(h|(z_1-H_0)^{-1}(z_2-H_0)^{-1}|h). \tag{13.6}$$

Direct computations using (13.6) show the first resolvent formula.

Let $(\alpha, f) \in \mathbb{C}\oplus\mathcal{H}$ be such that $(\alpha, f) \in \mathrm{Ker}Q(z)$. Then

$$0 = (z-f(z))^{-1}\Big(\alpha + (h|(z-H_0)^{-1}f)\Big), \tag{13.7}$$

$$0 = (z-H_0)^{-1}f + (z-H_0)^{-1}h(z-f(z))^{-1}\Big(\alpha + (h|(z-H_0)^{-1}f)\Big). \tag{13.8}$$

Inserting (13.7) into (13.8) we get $0 = (z-H_0)^{-1}f$ and hence $f = 0$. Now (13.7) implies $\alpha = 0$, so $\mathrm{Ker}Q(z) = \{0\}$.

Using (2) and (4) we get $(\mathrm{Ran}\, Q(z))^{\perp} = \mathrm{Ker}Q(z)^* = \mathrm{Ker}Q(\overline{z}) = \{0\}$. Hence (3) holds. □

It is easy to solve (13.5):

$$\begin{aligned} f(z) :=&\ \ \gamma + \big(h|((z-H_0)^{-1} + H_0(1+H_0^2)^{-1})h\big) \\ =&\ \ \gamma + \Big(h|(\tfrac{\mathrm{i}-z}{2(z-H_0)(\mathrm{i}-H_0)} - \tfrac{\mathrm{i}+z}{2(z-H_0)(-\mathrm{i}-H_0)})h\Big) \end{aligned} \tag{13.9}$$

13.2 The cut-off method

Let $h \in \langle H_0\rangle\mathcal{H}$ and $\gamma \in \mathbb{R}$. We can also use the cut-off method. For all $\Lambda > 0$ we define h_Λ as in (12.11), that is $h_\Lambda := 1\!\!1_{[-\Lambda,\Lambda]}(H_0)\,h,$. We set

$$\epsilon_\Lambda := \gamma + (h_\Lambda|H_0(1+H_0^2)^{-1}h_\Lambda).$$

For all $\Lambda > 0$, the cut-off Friedrichs Hamiltonian

$$G_\Lambda := \begin{bmatrix} \epsilon_\Lambda & (h_\Lambda| \\ |h_\Lambda) & H_0 \end{bmatrix}$$

is well defined and we can compute its resolvent, $Q_\Lambda(z) := (z - G_\Lambda)^{-1}$:

$$\begin{aligned} Q_\Lambda(z) \quad &= \quad \begin{bmatrix} 0 & 0 \\ 0 & (z-H_0)^{-1} \end{bmatrix} \qquad (13.10) \\ &+ \Big(z - f_\Lambda(z)\Big)^{-1} \begin{bmatrix} 1 & (h_\Lambda|(z-H_0)^{-1} \\ (z-H_0)^{-1}|h_\Lambda) & (z-H_0)^{-1}|h_\Lambda)(h_\Lambda|(z-H_0)^{-1} \end{bmatrix}. \end{aligned}$$

where

$$f_\Lambda(z) := \epsilon_\Lambda + (h_\Lambda|(z-H_0)^{-1}h_\Lambda). \qquad (13.11)$$

Note that ϵ_Λ is chosen such a way that the following *renormalization condition* is satisfied: $\frac{1}{2}\left(f_\Lambda(\mathrm{i}) + f_\Lambda(-\mathrm{i})\right) = \gamma$.

Theorem 13.2 *Assume that $h \in \langle H_0\rangle\mathcal{H}$. Then $\lim\limits_{k\to\infty} Q_\Lambda(z) = Q(z)$, where $Q(z)$ is given by (13.4) and $f(z)$ is given by (13.9). If H_0 is bounded from below, then $\lim\limits_{k\to\infty} \epsilon_\Lambda = \infty$.*

Proof. The proof is obvious if we note that $\lim\limits_{k\to\infty} \|(z-H_0)^{-1}h - (z-H_0)^{-1}h_\Lambda\| = 0$ and $\lim\limits_{k\to\infty} f_\Lambda(z) = f(z)$. $\Box$

Thus the cut-off Friedrichs Hamiltonian is norm resolvent convergent to the renormalized Friedrichs Hamiltonian.

13.3 Eigenvectors and resonances

Let $\beta \notin \mathrm{sp}H_0$, If $\beta = f(\beta) = 0$ then G has an eigenvalue at β. The corresponding eigenprojection equals

$$1\!\!1_\beta(G) = (1+(h|(\beta-H_0)^{-2}|h))^{-1} \begin{bmatrix} 1 & (h|(\beta-H_0)^{-1} \\ (\beta-H_0)^{-1}|h) & (\beta-H_0)^{-1}|h)(h|(\beta-H_0)^{-1} \end{bmatrix}.$$

It may happen that $\mathbb{C}\backslash\mathrm{sp}H_0 \ni z \mapsto f(z)$ extends to an analytic multivalued function accross some parts of $\mathrm{sp}H_0$. Then so does the resolvent $(z-G)^{-1}$ sandwiched between a certain class of vectors, in particular, between

$$w := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (13.12)$$

$$\big(w\big|(z-G)^{-1}w\big) = (z-f(z))^{-1}.$$

It may happen that we obtain a solution of

$$f(\beta)=\beta$$

in this *non-physical sheet of the complex plane*. This gives a pole of the resolvent called a *resonance*.

Suppose that we replace h with λh and ϵ with $\epsilon_0+\lambda^2\alpha$ and assume that we have Case I or II with λ small.

Then if $\epsilon_0 \notin \mathrm{sp}H_0$, we have an approximate expression for the eigenvalue for small λ:

$$\epsilon_\lambda = \epsilon_0 + \lambda^2\alpha + \lambda^2(h|(\epsilon_0-H_0)^{-1}h) + O(\lambda^4).$$

If $\epsilon_0 \in \mathrm{sp}H_0$, then the eigenvalue typically disappears and we obtain an approximate formula for the resonance:

$$\begin{aligned}\epsilon_\lambda &= \epsilon_0 + \lambda^2\alpha + \lambda^2(h|(\epsilon_0+\mathrm{i}0-H_0)^{-1}h) + O(\lambda^4)\\ &= \epsilon_0 + \lambda^2\alpha + \lambda^2(h|\mathcal{P}(\epsilon_0-H_0)^{-1}h) - \lambda^2\mathrm{i}\pi(h|\delta(H_0)h) + O(\lambda^4).\end{aligned}$$

Suppose now that $\epsilon_0=0$. Then we have the *weak coupling limit*:

$$\lim_{\lambda\searrow 0}(w|\mathrm{e}^{-\mathrm{i}\frac{t}{\lambda^2}G_\lambda}w) = \exp\left(-\mathrm{i}t\alpha + \mathrm{i}t(h|\mathcal{P}(H_0^{-1})h) - t\pi(h|\delta(H_0)h)\right).$$

13.4 Dissipative semigroup from a Friedrichs Hamiltonian

Consider $L^2(\mathbb{R})$, $\epsilon\in\mathbb{R}$, $\lambda\in\mathbb{C}$ and

$$H_0 v(k) := k v(k), \quad v\in L^2(\mathbb{R}), \quad k\in\mathbb{R}.$$

Then $\mathbb{R}\ni k\mapsto 1(k)=1$ does not belong to $\langle H_0\rangle^{1/2}L^2(\mathbb{R})$, however it belongs to $\langle H_0\rangle L^2(\mathbb{R})$. We will see that

$$G = \begin{bmatrix} \epsilon & \lambda(1| \\ \overline{\lambda}|1) & H_0 \end{bmatrix} \tag{13.13}$$

is a well defined Friedrichs Hamiltonian without renormalizing λ, even though it is only type III.

Set $1_\Lambda(k) := \mathbb{1}_{[-\Lambda,\Lambda]}(k)$. We approximate (13.13) by

$$G_\Lambda = \begin{bmatrix} \epsilon & \lambda(1_\Lambda| \\ \overline{\lambda}|1_\Lambda) & H_0 \end{bmatrix} \tag{13.14}$$

Note that (13.14) has a norm resolvent limit, which can be denoted (13.13). In fact,

$$f(z) = \epsilon + \lim_{\Lambda\to\infty}\int_\Lambda^{-\Lambda}\frac{|\lambda|^2}{z-k}\mathrm{d}k = \begin{cases}\epsilon-\mathrm{i}\pi|\lambda|^2 & \mathrm{Im}z>0,\\ \epsilon+\mathrm{i}\pi|\lambda|^2 & \mathrm{Im}z<0.\end{cases}$$

If w is the distinguished vector (13.12), then

$$\begin{aligned}\left(w|(z-G)^{-1}w\right) &= (z-\epsilon\pm \mathrm{i}\pi|\lambda|^2)^{-1}, \quad \pm\mathrm{Im} z>0,\\ \left(w|\mathrm{e}^{-\mathrm{i}tG}w\right) &= \mathrm{e}^{-\mathrm{i}\epsilon t-\pi|\lambda|^2|t|}.\end{aligned}$$

Chapter 14

Convolutions and Fourier transformation

14.1 Introduction to convolutions

In this chapter notes X will denote the space $\mathbb{R}^d$ equipped with the Lebesgue measure.

Let us recall two estimates, which we will often use, whose validity is not restricted to $\mathbb{R}^d$:

The Hölder inequality Let $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$:

$$\int |f(x)g(x)|\mathrm{d}x \leq \|f\|_p \|g\|_q,$$

The generalized Minkowski inequality

$$\left(\int \mathrm{d}y \left| \int f(x,y)\mathrm{d}\, x \right|^p \right)^{\frac{1}{p}} \leq \int \mathrm{d}x \left(\int |f|^p(x,y)\mathrm{d}y \right)^{\frac{1}{p}}$$

If g, h are functions on $\mathbb{R}^d$, then their convolution is formally defined by

$$g * h(x) := \int g(x-y)h(y)\mathrm{d}y,$$

provided this makes sense. In what follows we will give a number of conditions when the convolution is well defined.

14.2 Modulus of continuity

Lemma 14.1 *For $1 \leq p < \infty$, $f \in L^p(X)$, set*

$$\omega_{p,f}(y) := \left(\int |f(x+y) - f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}};$$

and for $p=\infty$, $f \in C_\infty(\mathbb{R}^n)$

$$\omega_{\infty,f}(y) := \sup_x |f(x+y) - f(x)|.$$

Then $\omega_{p,f}(y)$ *is bounded and*

$$\lim_{y\to 0} \omega_{p,f}(y) = 0.$$

Proof. The boundedness follows from the Minkowski inequality. In fact, $\omega_{p,f}(y) \le 2\|f\|_p$.

The convergence to zero is obvious for $f \in C_{\mathrm{c}}(\mathbb{R}^n)$. But C_{c} is dense in L^p for $1 \le p < \infty$ and in C_∞. $\square$

14.3 The special case of the Young inequality with $\frac{1}{p} + \frac{1}{q} = 1$

Theorem 14.2 *Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$, $g \in L^q$. *Then*

$$f * g \in C_\infty.$$

If $f \in L^1$, $g \in L^\infty$, *then* $f * g$ *is uniformly continuous.*

Proof. By the Hölder inequality, $f * g(x)$ is defined for all x and depends continuously on $f \in L^p(X)$ and $g \in L^q(X)$. Moreover,

$$\begin{aligned}
&f * g(x_1) - f * g(x_2) \\
&= \int (f(x_1 - y) - f(x_2 - y)) g(y) \mathrm{d}y \\
&\le \left(\int |f(x_1 - y) - f(x_2 - y)|^p \mathrm{d}y\right)^{\frac{1}{p}} \|g\|_q \\
&= \omega_{p,f}(x_1 - x_2) \|g\|_q.
\end{aligned}$$

Hence $f * g$ is uniformly continuous.

For $f \in C_{\mathrm{c}}(X)$ obviously $f * g \in C_{\mathrm{c}}(X)$. If $p, q < \infty$, then $C_{\mathrm{c}}(X)$ is dense in $L^p(X)$, $L^q(X)$. Hence for such p, q, $f * g$ belongs to the closure of $C_{\mathrm{c}}(X)$ in $L^\infty(X)$, which is $C_\infty(X)$. $\square$

14.4 Convolution by an L^1 function

Theorem 14.3 *Let* $g \in L^p(X)$ *and* $h \in L^1(X)$. *Then* $g * h$ *is well defined almost everywhewre and*

$$\|g * h\|_p \le \|h\|_1 \|g\|_p.$$

Proof. In the generalized Minkowski inequality set $X = Y = \mathbb{R}^n$ and $f(x, y) = h(y)g(x - y)$. $\square$

Theorem 14.4 *Let $\phi \in L^1(\mathbb{R}^n)$ and $\int \phi(x)\mathrm{d}x = 1$. Set*

$$\phi_\epsilon(x) := \epsilon^{-n}\phi(\epsilon^{-1}x), \ \epsilon > 0.$$

Then

$$\lim_{\epsilon\to 0} \|f * \phi_\epsilon - f\|_p = 0, \quad f \in L^p(\mathbb{R}^n), \ 1 \le p < \infty,$$
$$\lim_{\epsilon\to 0} \|f * \phi_\epsilon - f\|_\infty = 0, \quad f \in C_\infty(\mathbb{R}^n).$$

Proof.

$$f * \phi_\epsilon(x) - f(x) = \int (f(x - y) - f(x))\phi_\epsilon(y)\mathrm{d}y.$$

$$\begin{aligned}
&\|f * \phi_\epsilon(x) - f(x)\|_p \\
&\le \int \mathrm{d}y \left(\int |f(x - y) - f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} |\phi_\epsilon(y)| \\
&= \int \omega_{p,f}(y)\phi_\epsilon(y)\mathrm{d}y = \int \omega_{p,f}(\epsilon y)\phi(y)\mathrm{d}y \to_{\epsilon\to 0} 0.
\end{aligned}$$

$\square$

14.5 The Young inequality

Theorem 14.5 *Let $1 \le p, q, r \le \infty$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, $f, g, h \in \mathcal{M}_+(X)$ (positive, measurable functions on X). Then*

$$\int\int f(x)g(x - y)h(y)\mathrm{d}x\mathrm{d}y \le \|f\|_p\|g\|_q\|h\|_r.$$

Proof. Let $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Set

$$\begin{aligned}
\alpha(x, y) &:= f(x)^{p/r'} g(x - y)^{q/r'}, \\
\beta(x, y) &:= g(x - y)^{q/p'} h(y)^{r/p'}, \\
\gamma(x, y) &:= f(x)^{p/q'} h(y)^{r/q'}.
\end{aligned}$$

Then

$$\begin{aligned}
\int\int f(x)g(x - y)h(y)\mathrm{d}x\mathrm{d}y \quad &= \int\int f(x)^{p(2-\frac{1}{q}-\frac{1}{r})} g(x - y)^{q(2-\frac{1}{p}-\frac{1}{r})} h(y)^{r(2-\frac{1}{p}-\frac{1}{q})} \\
&= \int\int f(x)^{p(\frac{1}{q'}+\frac{1}{r'})} g(x - y)^{q(\frac{1}{p'}+\frac{1}{r'})} h(y)^{r(\frac{1}{p'}+\frac{1}{q'})} \\
&= \int\int \alpha(x, y)\beta(x, y)\gamma(x, y)\mathrm{d}x\mathrm{d}y \le \|\alpha\|_{r'}\|\beta\|_{p'}\|\gamma\|_{q'},
\end{aligned}$$

where in the last step we used the Hölder inequality noting that $\frac{1}{r'} + \frac{1}{p'} + \frac{1}{q'} = 1$. Finally,

$$\|\alpha\|_{r'} = \left(\int\int f(x)^p g(x - y)^q \mathrm{d}x\mathrm{d}y\right)^{1/r'} = \|f\|_p^{p/r'} \|g\|_q^{q/r'}.$$

$\square$

Corollary 14.6 *If $\frac{1}{q}+\frac{1}{r}=1+\frac{1}{s}$, $h \in L^r(X)$, $g \in L^q(X)$, then for almost all x*

$$y \mapsto g(x-y)h(y)$$

belongs to $L^1(X)$ and

$$g * h(x) = \int g(x-y)h(y)\mathrm{d}y$$

belongs to $L^s(X)$ and

$$\|g * h\|_s \leq \|g\|_q \|h\|_r. \tag{14.1}$$

Proof. We know that for $f \in L^p(X)$, $\frac{1}{p}+\frac{1}{s}=1$ we have

$$\int |f(x)|\mathrm{d}x \int |g(x-y)h(y)|\mathrm{d}y \leq \|f\|_p \|g\|_q \|h\|_r < \infty.$$

Hence for a.a x

$$|f(x)| \int |g(x-y)h(y)|\mathrm{d}y < \infty.$$

Hence for a.a. x

$$\int |g(x-y)h(y)|\mathrm{d}y < \infty.$$

From

$$|\int f(x) g * h(x)\mathrm{d}x| \leq \|f\|_p \|g\|_q \|h\|_r.$$

we obtain (14.1). □

14.6 Fourier transformation on $L^1 \cup L^2(\mathbb{R}^d)$

For

$$f \in L^1(\mathbb{R}^d)$$

we define its Fourier transform as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int \mathrm{e}^{-\mathrm{i}x\xi} f(x)\mathrm{d}x.$$

We also introduce the following notation:

$$\check{f}(x) := f(-x), \quad \tau_y f(x) := f(x-y), \quad \rho_a f(x) := f(ax)$$

Theorem 14.7 (1) $\|\hat{f}\|_\infty \leq \|f\|_1$;

(2) $\hat{\check{f}}(\xi) = \check{\hat{f}}(\xi) = \int \mathrm{e}^{\mathrm{i}x\xi} f(x)\mathrm{d}x$.

(3) $\overline{\hat{f}} = \check{\hat{\overline{f}}}$;

(4) $\hat{\rho}_a f(x) = a^{-d}\hat{f}(a^{-1}x)$;

(5) $\widehat{\tau_y f}(\xi) = \mathrm{e}^{-\mathrm{i}y\xi}\hat{f}(\xi)$;

(6) $\left(f\mathrm{e}^{\mathrm{i}\eta\cdot}\right)\hat{}\,(\xi) = \hat{f}(\xi - \eta)$.

Example 14.8 (1) $f(x) = \mathrm{e}^{-\frac{x^2}{2}}$, $\hat{f}(\xi) = (2\pi)^{\frac{n}{2}}\mathrm{e}^{-\frac{x^2}{2}}$.

(2) $f(x) = \mathrm{e}^{-\epsilon x}x^\alpha\theta(x)$, $\hat{f}(\xi) = \frac{\Gamma(\alpha+1)}{(\epsilon+i\xi)^{\alpha+1}}$, $\mathrm{Re}\epsilon > 0$.

(3) $f(x) = \chi_{[-1,1]}(x)$, $\hat{f}(\xi) = \frac{2\sin\xi}{\xi}$.

(4) $f(x) = \mathrm{e}^{-|x|}$, $\hat{f} = \frac{1}{1+\xi^2}$.

Theorem 14.9 (The Riemann-Lebesgue Lemma) *If $f \in L^1$, then $\hat{f} \in C_\infty$.*

Proof. We know that the Fourier transformation is continuous from L^1 to L^∞. C_∞ is a closed subspace of L^∞.

Combinations of characteristic functions of intervals are dense in L^1. Their Fourier transforms, which we computed explicitly, belong to C_∞. □

Theorem 14.10 *Let $f, g \in L^1$. Then*

(1) $\int \hat{f}(\xi)g(\xi)\mathrm{d}\xi = \int f(x)\hat{g}(x)\mathrm{d}x$.

(2) $(f\hat{g})\check{} = \check{\hat{f}} * g$.

(3) $(f * g)\hat{} = \hat{f}\hat{g}$.

Proof. (2) For $f_\eta(x) = f(x)\mathrm{e}^{\mathrm{i}x\eta}$, we have $\hat{f}_\eta(\xi) = \check{\hat{f}}(\eta - \xi)$. Hence

$$\int \hat{f}_\eta(\xi)g(\xi)\mathrm{d}\xi = \check{\hat{f}} * g(\eta).$$

Besides,

$$\int f_\eta(x)\hat{g}(x)\mathrm{d}x = (h\hat{g})\check{}(\eta)$$

Therefore, it suffices to apply (1). □

Theorem 14.11 (Parseval) *Let $g, \hat{g} \in L^1$. Then*

$$\check{\hat{g}} = (2\pi)^d g.$$

Proof. Let

$$\phi_\epsilon(x) := \mathrm{e}^{-\frac{\epsilon x^2}{2}}.$$

We have

$$0 \le \phi_\epsilon \le 1, \quad \lim_{\epsilon \to 0}\phi_\epsilon = 1.$$

Using that $\hat{g} \in L^1$, by the Lebesgue Theorem we obtain

$$\phi_\epsilon\hat{g} \to \hat{g}$$

in the sense of L^1. Therefore,

$$(\phi_\epsilon \hat{g})\check{}(x) \to \hat{\check{g}}(x), \quad (\phi_\epsilon \hat{g})\check{}(x) \to \check{\hat{\check{g}}}(x),$$

in the sense of L^∞

Moreover,

$$\int \phi(\xi) = (2\pi)^d, \qquad \hat{\phi}_\epsilon(\xi) = \left(\frac{2\pi}{\epsilon}\right)^{\frac{d}{2}} \mathrm{e}^{-\frac{\xi^2}{2\epsilon}}.$$

Using that $g \in L^1$ we obtain

$$\hat{\phi}_\epsilon * g \to (2\pi)^d g$$

in the sense of L^1.

Finally, we use

$$\hat{\phi}_\epsilon * g = \check{\hat{\hat{\phi}}}_\epsilon * g = (\phi_\epsilon \hat{g})\hat{}. \tag{14.2}$$

(14.2) converges to $\check{\hat{\hat{g}}}$ in the sense of L^∞ and to $(2\pi)^d g$ in the sense of L^1. It is easy to see that these two functions have to coincide. □

Theorem 14.12 *Let $f \in L^1$, $\hat{f} \geq 0$ and let f be continuous at 0. Then $\hat{f} \in L^1$ and we have*

$$\int \hat{f}(\xi)\mathrm{d}\xi = (2\pi)^d f(0)$$

Proof. If ϕ_ϵ is as in the proof of the Parseval Theorem, then

$$\int \phi_\epsilon(\xi)\hat{f}(\xi)\mathrm{d}\xi = \int \hat{\phi}_\epsilon(x) f(x)\mathrm{d}x.$$

The left hand side is increasing and converges to $\int \hat{f}(\xi)\mathrm{d}\xi$. The right hand side goes to $(2\pi)^d f(0)$. By the Fatou Lemma, $\hat{f}$ is integrable. □

Theorem 14.13 *Let $f \in L^1 \cap L^2$. Then*

$$\|\hat{f}\|_2 = (2\pi)^{\frac{d}{2}} \|f\|_2.$$

Proof. The function $h := \check{\overline{f}} * f$ belongs to L^1 as the convolution of functions from L^1 and is continuous as the convolution of functions from L^2. Besides,

$$\hat{h} = \left(\check{\overline{f}} * f\right)\hat{} = \hat{\check{\overline{f}}}\hat{f} = \overline{\hat{f}}\hat{f} \geq 0.$$

Hence, by Theorem 14.12, $\hat{h} \in L^1$ and

$$(2\pi)^d h(0) = \int \hat{h}(\xi)\mathrm{d}\xi.$$

Finally,

$$(2\pi)^d \int |f(x)|^d \mathrm{d}x = (2\pi)^d h(0) = \int \hat{h}(\xi)\mathrm{d}\xi = \int |\hat{f}(\xi)|^2 \mathrm{d}\xi.$$

□

Let $f \in L^2$. Then for any sequence $f_n \in L^1 \cap L^2$ such that

$$\lim_{n\to\infty} f_n = f$$

in L^2, there exists $\lim_{n\to\infty} \hat{f}_n = \hat{f}$. The operator

$$f \mapsto (2\pi)^{-\frac{d}{2}} \hat{f}$$

is unitary.

Theorem 14.14 *If $f \in L^1$ and $xf \in L^1$, then $\hat{f} \in C^1$ and*

$$\partial_\xi \hat{f}(\xi) = (xf)\hat{}(\xi).$$

Proof. We use the theorem about differentiation of an integral depending on a parameter. □

14.7 Tempered distributions on $\mathbb{R}^d$

Typical spaces of functions (measures) on $\mathbb{R}^d$ are

$$C_\infty(X), \quad L^p(X), \ \mathrm{Ch}(X).$$

where $\mathrm{Ch}(X)$ denotes Borel complex charges of finite variation. We have

$$C_\infty^\#(X) = \mathrm{Ch}(X), \ L^p(X)^\# = L^q(X), \quad p^1 + q^{-1} + 1, \quad 1 \leq p < \infty.$$

We have a bilinear and sesquilinear forms

$$\langle a, b\rangle = \int a(x)b(x)\mathrm{d}x, \quad (a, b) = \int \overline{a}(x)b(x)\mathrm{d}x.$$

Lemma 14.15

$$\|f\|_\infty \leq C\|(1+|x|)^{-p} f\|_1 + C\|\partial_{x_1} \dots \partial_{x_d} f\|_1, \ p > d$$

$$\|f\|_q \leq C\|(1+|x|)^{-k} f\|_p, \ \frac{1}{q} < \frac{k}{d} + \frac{1}{p}.$$

Theorem 14.16 *The following set does not depend on $1 \leq p \leq \infty$:*

$$\bigcap_{\alpha, m > 0} \{f \ : \ \|\partial^\alpha (1+|x|^2)^{m/2} f\|_p < \infty\}. \tag{14.3}$$

The space $\mathcal{S}(\mathbb{R}^d)$ is defined as (14.3). It is a Frechet space.

For the dual of $\mathcal{S}(\mathbb{R}^d)$ we will use the traditional notation $\mathcal{S}'(\mathbb{R}^d)$.

Example 14.17 *Elements of $\mathcal{S}'(X)$ satisfying*

$$|\langle v, \phi\rangle| \leq C\|x^m\phi\|_\infty$$

have the form

$$\langle v, \phi\rangle = \int \phi(x)\mathrm{d}\mu$$

for a certain Borel charge μ for which there exists m such that $\mu(1+|x|)^{-m} \in$ $\mathrm{Ch}(X)$.

The operator ∂ is continuous on $\mathcal{S}(X)$. For $v \in \mathcal{S})(X)$ we define $\partial v \in \mathcal{S}'(X)$ by

$$\langle v, \partial\phi\rangle = -\langle \partial v, \phi\rangle.$$

Theorem 14.18 *Any $v \in \mathcal{S}'(X)$ has the form*

$$\sum_{\alpha<N} \partial_x^\alpha \mu_\alpha$$

for some Borel charge μ such that for some m we have $\mu(1+|x|)^{-m} \in \mathrm{Ch}(X)$.

Proof. For some α, β,

$$\langle v, \phi\rangle \leq C \sum_{|\alpha|,|\beta|\leq N} \|x^\alpha \partial_x^\beta \phi\|_\infty.$$

Introduce the locally compact space

$$\tilde{X} = \prod_{|\alpha|,|\beta|\leq N} X$$

and the map

$$\mathcal{S}(X) \ni \phi \mapsto j(\phi) = \sum_{|\alpha|,|\beta|\leq N}^{\oplus} x^\alpha \partial^\beta \phi \in C_\infty(\tilde{X})$$

Any distribution v determines a bounded functional on $j(\mathcal{S}(X))$. By the Hahn-Banach Theorem, this functional can be extended to a bounded functional $\tilde{v}$ on $C_\infty(\tilde{X})$. By the Riesz-Markov Theorem, there exists a finite Borel charge on $\tilde{X}$ Such that

$$\tilde{v}(\phi_{\alpha,\beta}) = \sum_{|\alpha|,|\beta|\leq N} \int \phi(x)\mathrm{d}\eta_{\alpha,\beta}(x).$$

□

Clearly, $\mathcal{S}(X) \subset L^1(X)$. Hence the Fourier transform is defined on $\mathcal{S}(X)$.

Theorem 14.19 *If $\phi \in \mathcal{S}(X)$, then $\hat{\phi} \in \mathcal{S}(X)$.*

Recall that for $\psi \in \mathcal{S}(X)$, $\phi \in \mathcal{S}(X)$ we have

$$\langle \psi, \hat{\phi} \rangle = \langle \hat{\psi}, \phi \rangle.$$

For $v \in \mathcal{S}'(X)$ we define

$$\langle \hat{v}, \phi \rangle := \langle v, \hat{\phi} \rangle, \ \phi \in \mathcal{S}(X).$$

Clearly, $L^1(X) \cup L^2 \subset \mathcal{S}'(X)$ and the Fourier transformation previously defined coincides with the presently defined on $L^1(X) \cup L^2$.

Theorem 14.20

$$\check{\hat{v}} = (2\pi)^d v, \quad v \in \mathcal{S}'(X), \tag{14.4}$$

14.8 Spaces of sequences

Below we list a couple of typical spaces of sequences indexed by $\mathbb{Z}^d$:

$$L^1(\mathbb{Z}^d) \subset L^p(\mathbb{Z}^d) \subset L^q(\mathbb{Z}^d) \subset C_\infty(\mathbb{Z}^d) \subset L^\infty(\mathbb{Z}^d), \ p \leq q$$

We have

$$C_\infty(\mathbb{Z}^d)^\# = L^1(\mathbb{Z}^d), \quad L^p(\mathbb{Z}^d)^\# = L^q(\mathbb{Z}^d), \quad p^{-1} + q^{-1} = 1, \quad 1 \leq p < \infty.$$

We have natural bilinear and sesquilinear forms:

$$\langle a|b \rangle = \sum a_n b_n, \quad (a|b) = \sum \overline{a}_n b_n.$$

Lemma 14.21

$$\|a\|_p \leq \|a\|_q, \ p \geq q,$$

$$\|a\|_q \leq \|(1+n)^{-k} a\|_p, \quad \frac{1}{q} < \frac{k}{d} + \frac{1}{p}.$$

Theorem 14.22 *The following set does not depend on $1 \leq p \leq \infty$:*

$$\underset{m>0}{\cap} \{a \ : \ \|(1+|n|^2)^{m/2} a\|_p < \infty\}.$$

The above space is a Frechet space, which will be denoted $\mathcal{S}(\mathbb{Z}^d)$.

Theorem 14.23 *The space dual to $\mathcal{S}(\mathbb{Z}^d)$, denoted $\mathcal{S}'(\mathbb{Z}^d)$, equals*

$$\underset{m>0}{\cup} \{a \ : \ \|(1+|n|^2)^{-m/2} a\|_p < \infty\}.$$

Theorem 14.24 *$\mathcal{S}(\mathbb{Z}^d)$ is dense in $\mathcal{S}'(\mathbb{Z}^d)$.*

14.9 The oscillator representation of $\mathcal{S}(X)$ and $\mathcal{S}'(X)$

For simplicity, we discuss $X = \mathbb{R}$.

Lemma 14.25

$$\lim_{n\to\infty}\left\|\mathrm{e}^{ix\xi}\mathrm{e}^{-\frac{x^2}{2}} - \sum_{j=0}^{n}\frac{(ix\xi)^j}{j!}\mathrm{e}^{-\frac{x^2}{2}}\right\| = 0$$

Proof.

$$\left|\mathrm{e}^{ix\xi}\mathrm{e}^{-\frac{x^2}{2}} - \sum_{j=0}^{n}\frac{(ix\xi)^j}{j!}\mathrm{e}^{-\frac{x^2}{2}}\right| \leq \frac{\xi^{n+1}x^{n+1}}{(n+1)!}\mathrm{e}^{-\frac{x^2}{2}}.$$

Hence the norm of the difference is estimated by

$$\int\frac{\xi^{2(n+1)}x^{2(n+1)}}{((n+1)!)^2}\mathrm{e}^{-x^2}\mathrm{d}x = \xi^{2(n+1)}\int_0^{\infty}\frac{s^{n+\frac{1}{2}}\mathrm{e}^{-s}\mathrm{d}s}{((n+1)!)^2} = \frac{\xi^{2(n+1)}\Gamma(n+\frac{1}{2})}{((n+1)!)^2}.$$

□

Theorem 14.26 *Linear combinations of*

$$x^n\mathrm{e}^{-\frac{x^2}{2}} \tag{14.5}$$

are dense in $L^2(\mathbb{R})$.

Proof. Let f be orthogonal to the space spanned by (14.5). Then for any ξ

$$\int f(x)\mathrm{e}^{ix\xi}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x = 0.$$

Hence, the Fourier transform of $f\mathrm{e}^{-\frac{x^2}{2}}$ is zero. Therefore, $f = 0$ almost everywhere. □

Let

$$A^* := \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right),\ A := \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)$$

$$\phi_n := \pi^{-\frac{1}{4}}(n!)^{-\frac{1}{2}}(A^+)^n\mathrm{e}^{-\frac{x^2}{2}} = (2^2n!)^{-\frac{1}{2}}(-1)^n\pi^{-\frac{1}{4}}\mathrm{e}^{\frac{x^2}{2}}\partial_x^n\mathrm{e}^{-x^2}$$

$$N := A^*A + AA^* = x^2 + D^2.$$

Theorem 14.27 ϕ_n *is an orthonormal basis obtained by the Gramm-Schmidt orthonormalization of* $x^n\mathrm{e}^{-\frac{x^2}{2}}$*. They are eigenvectors of* N *and* $\mathcal{F}$*:*

$$N\phi_n = \left(n + \frac{1}{2}\right)\phi_n,\quad \mathcal{F}\phi_n = i^n(2\pi)^d\phi_n.$$

Theorem 14.28 *Suppose that for $v \in \mathcal{S}'(\mathbb{R})$*

$$v_n := \langle v, \phi_n \rangle$$

Then there exists m such that

$$|v_n| \leq C(1+n)^m,$$

or, in other words, $(v_n) \in \mathcal{S}'(\mathbb{N})$. The map

$$\mathcal{S}'(\mathbb{R}) \ni v \to (v_n) \in \mathcal{S}'(\mathbb{N})$$

is an isomorphism. $v \in \mathcal{S}(\mathbb{R})$, iff

$$|v_n| \leq C(1+n)^{-m}, \quad m = 0, 1, \dots$$

The map

$$\mathcal{S}(\mathbb{R}) \ni v \to (v_n) \in \mathcal{S}(\mathbb{N})$$

is an isomorphism and

$$\mathcal{S}(\mathbb{R}) = \cap_{n=0}^{\infty} \mathrm{Dom}(N^n).$$

Proof. Clearly, the seminorms $\|N^m\phi\|$ can be estimated by linear combinations of seminorms $\|\phi\|_{\alpha,\beta,2}$. Hence,

$$\mathcal{S}(\mathbb{R}) \supset \cap_{n=0}^{\infty} \mathrm{Dom}(N^n).$$

To show the inverse estimate note first that $\|\phi\|_{\alpha,\beta,2}$ can be bounded by

$$(\phi, A_1^\natural \dots A_n^\natural \phi),$$

where $A_i^\natural = A$ or $A_i^\natural = A^*$. After commuting we can estimate them by linear combinations

$$\begin{aligned} &(\phi A^k, A^{+m}\phi) \\ &\leq \tfrac{1}{2}\|A^{+k}\phi\|^2 + \tfrac{1}{2}\|A^{+m}\phi\|^2 \\ &\leq C \textstyle\sum_{j=1}^{\max\{k,m\}} \|N^j\phi\|^2. \end{aligned}$$

Hence

$$\mathcal{S}(\mathbb{R}) \subset \cap_{n=0}^{\infty} \mathrm{Dom}(N^n).$$

□

Corollary 14.29 (The Schwartz Kernel Theorem) *Every continuous bilinear functional*

$$\mathcal{S}(X_1) \times \mathcal{S}(X_2) \ni (\phi, \psi) \mapsto T(\phi, \psi)$$

has the form

$$\langle T, \phi \otimes \psi \rangle$$

for some $T \in \mathcal{S}'(X_1 \times X_2)$

Proof. We have
$$\langle T, \phi \otimes \psi \rangle = \sum t_{k,m} \phi_k \otimes \psi_m,$$
where
$$|t_{k,m}| \leq (1+|k|)^n (1+|m|)^n.$$
Hence,
$$|t_{k,m}| \leq (1+|k|+|m|)^{2n}.$$
□

14.10 Convolution of distributions

Theorem 14.30 *The following space does not depend on $1 \leq p \leq \infty$:*
$$\bigcap_{\alpha} \bigcup_{m_\alpha} \{ f \in C^\infty(\mathbb{R}^d) \ : \ \|(1+|x|)^{-m_\alpha} D^\alpha f\|_p < \infty \}. \tag{14.6}$$

The space (14.6), which is an inductive limit of Frechet space, is denoted $\mathcal{O}(\mathbb{R}^d)$. Its dual space, for which we will use the traditional notation $\mathcal{O}'(\mathbb{R}^d)$, is called the space of rapidly decreasing distributions.

We have the inclusions
$$\mathcal{S} \subset \mathcal{O} \subset \mathcal{S}', \quad \mathcal{S} \subset \mathcal{O}' \subset \mathcal{S}'$$

Example 14.31 *If μ is a Borel charge and for any m*
$$\int (1+|x|)^m |\mathrm{d}\mu|(x) < \infty,$$
then $\mu \in \mathcal{O}'$.

Clearly, if $f \in \mathcal{O}$, then
$$\mathcal{S} \ni \phi \mapsto f\phi \in \mathcal{S} \tag{14.7}$$
is continuous. For $v \in \mathcal{S}'$ we define $fv \in \mathcal{S}'$ as the adjoint of (14.7), that is
$$\langle v, f\phi \rangle = \langle fv, \phi \rangle.$$

The operator ∂ is continuous also on $\mathcal{O}$ and $\mathcal{O}'$.

For $\phi \in \mathcal{S}$ we define
$$\check{\phi}(x) := \phi(-x).$$
Clearly,
$$\langle \psi, \check{\phi} \rangle = \langle \check{\psi}, \phi \rangle$$
For $v \in \mathcal{S}'$ we introduce
$$\langle v, \check{\phi} \rangle = \langle \check{v}, \phi \rangle$$

Note that for $\phi, \psi, \chi \in \mathcal{S}$ we have
$$\langle \chi, \psi * \phi \rangle = \langle \chi * \check{\psi}, \phi \rangle.$$
For $v \in \mathcal{S}'$, $\psi \in \mathcal{S}$ we define
$$\langle v * \psi, \phi \rangle := \langle v, \check{\psi} * \phi \rangle.$$

Theorem 14.32 *For $v \in \mathcal{S}'$, $\phi \in \mathcal{S}$ we define*

$$\phi_y(x) := \phi(x - y).$$

Then

$$v * \phi(x) := \langle v, (\check{\phi})_{-x} \rangle.$$

and

$$v * \psi \in \mathcal{O}. \tag{14.8}$$

Proof. Let us show (14.8):

$$\begin{aligned} &|\partial_x^\alpha v * \phi(x)| = |\langle v | \partial_y^\alpha \check{\phi}_{-x} \rangle| \\ &\leq C \| y^n \partial_y^{\alpha+\gamma} \phi_{-x} \|_\infty \\ &\leq C(1 + |x|)^n \| y^n \partial_y^{\alpha+\gamma} \phi \|_\infty. \end{aligned}$$

□

Hence we can extend the definition of the convolution as follows. Let $w \in \mathcal{S}'$, $v \in \mathcal{O}'$. Then

$$\langle v * w, \phi \rangle := \langle v, \check{w} * \phi \rangle, \quad \phi \in \mathcal{S}.$$

Using the convolution we can easily show that $\mathcal{S}$ is dense in $\mathcal{S}'$.

Theorem 14.33 *If $v \in \mathcal{O}'$, then $\hat{v} \in \mathcal{O}$.*

Proof. Note first that

$$\partial_\xi^\beta \hat{v}(\xi) = \langle v, x^\beta \mathrm{e}^{-\mathrm{i}\xi \cdot} \rangle.$$

We know that

$$|\langle v, \phi \rangle| \leq \sum_{|\alpha| \leq N} \|(1 + x^2)^{-\frac{|\beta|}{2}} \partial_x^\alpha \phi \|_\infty.$$

Hence,

$$|\partial_\xi^\beta \hat{v}(\xi)| \leq \sum_{|\alpha| \leq N} |\xi|^\alpha.$$

□

Theorem 14.34

$$(v * w)\hat{} = \hat{v} \hat{w}, \quad v \in \mathcal{S}', \; w \in \mathcal{O}' \tag{14.9}$$

Proof. First prove (14.9) for $w \in \mathcal{S}$. Let $\phi \in \mathcal{S}$. Then

$$\begin{aligned}
&\langle (v * w)\hat{}, \phi \rangle \\
&= \langle v * w, \hat{\phi} \rangle \\
&= \langle v, \check{w} * \hat{\phi} \rangle \\
&= (2\pi)^{-d} \langle v, (\check{w} * \hat{\phi})\hat{\check{}} \rangle \\
&= (2\pi)^{-d} \langle \hat{\check{v}}, \hat{\check{w}} \hat{\hat{\phi}} \rangle \\
&= \langle \hat{\check{v}}, \hat{\check{w}} \check{\phi} \rangle \\
&= \langle \hat{\check{v}} \check{\hat{w}}, \check{\phi} \rangle \\
&= \langle \hat{v} \hat{w}, \phi \rangle .
\end{aligned}$$

Then we assume that $v \in \mathcal{S}'$, $w \in \mathcal{O}'$ and we repeat the same reasoning. □

14.11 The Hardy-Littlewood-Sobolev inequality

Let θ denote the Heaviside function, that is

$$\theta(t) := \begin{cases} 0 & t < 0, \\ 1 & t > 0. \end{cases}$$

Let $0 \leq \lambda \leq n$. Then

$$|x|^{-\lambda} \theta(|x| - 1) \in L^p(X), \quad \infty \geq p > \frac{n}{\lambda},$$
$$|x|^{-\lambda} \theta(1 - |x|) \in L^p(X), \quad 1 \leq p < \frac{n}{\lambda}.$$

Theorem 14.35 $1 < p, r < \infty$, $0 < \lambda < n$, $\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{r} = 2$, $f, h \in \mathcal{M}_+(X)$. *Then*

$$\int \int f(x) |x - y|^{-\lambda} h(y) \mathrm{d}x \mathrm{d}y \leq C_{n,\lambda,r} \|f\|_p \|h\|_r .$$

Corollary 14.36 *If* $\frac{\lambda}{n} + \frac{1}{r} = 1 + \frac{1}{s}$, $h \in L^r(X)$, *then for almost all* x

$$y \mapsto |x - y|^{-\lambda} h(y)$$

belongs to $L^1(X)$ *and*

$$x \mapsto \int |x - y|^{-\lambda} h(y) \mathrm{d}y$$

belongs to $L^s(X)$ *and for* $g(x) = |x|^{-\lambda}$,

$$\|g * h\|_s \leq C_{n,\lambda,r} \|h\|_r . \tag{14.10}$$

Proof of Theorem 14.35 We will write $g(x) := |x|^{-\lambda}$. Set

$$v(a) := \int 1_{\{f>a\}}(x)\mathrm{d}x, \quad w(b) := \int 1_{\{h>b\}}(x)\mathrm{d}x, \quad u(c) := \int 1_{\{g>c\}}(x)\mathrm{d}x.$$

Note that

$$u(c) = C_n c^{-n/\lambda}, \quad u^{-1}(t) = \tilde{C}_n t^{-\lambda/n}.$$

We can assume that

$$1 = \|f\|_p^p = p\int_0^\infty a^{p-1}v(p)\mathrm{d}a, \quad 1 = \|h\|_r^r = r\int_0^\infty b^{r-1}w(b)\mathrm{d}b$$

Now

$$\begin{aligned} I \quad &:= \int\int f(x)g(x-y)h(y)\mathrm{d}x\mathrm{d}y = \int\int\int\int\int 1_{\{f>a\}}(x)1_{\{h>b\}}(y)1_{\{g>c\}}(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}a\mathrm{d}b\mathrm{d}c \\ &= \int\int\int_{w(b)\le v(a)} \mathrm{d}a\mathrm{d}b\mathrm{d}y 1_{\{h>b\}}(y)\int\int \mathrm{d}c\mathrm{d}x 1_{\{f>a\}}(x)1_{\{g>c\}}(x-y) \\ &+ \int\int\int_{w(b)\ge v(a)} \mathrm{d}a\mathrm{d}b\mathrm{d}x 1_{\{f>a\}}(x)\int\int \mathrm{d}c\mathrm{d}y 1_{\{h>b\}}(y)1_{\{g>c\}}(x-y). \end{aligned}$$

Now

$$\begin{aligned} \int\int \mathrm{d}c\mathrm{d}x 1_{\{f>a\}}(x)1_{\{g>c\}}(x-y) \quad &\le \int\int_{v(a)\ge u(c)} \mathrm{d}c\mathrm{d}x 1_{\{f>a\}}(x) + \int\int_{v(a)\le u(c)} \mathrm{d}c\mathrm{d}x 1_{\{g>c\}}(x-y) \\ &= v(a)\int_0^{u^{-1}(v(a))}\mathrm{d}c + \int_{u^{-1}(v(a))}^\infty u(c)\mathrm{d}c \\ &= v(a)u^{-1}(v(a)) + c_{n,\lambda}(u^{-1}(v(a)))^{1-n/\lambda} \\ &= c_{n,\lambda}v(a)^{1-\lambda/n}. \end{aligned}$$

Therefore,

$$\begin{aligned} I \quad &\le c_{n,\lambda}\int\int_{w(b)\le v(a)} \mathrm{d}a\mathrm{d}b w(b)v(a)^{1-\lambda/n} + c_{n,\lambda}\int\int_{w(b)\ge v(a)} \mathrm{d}a\mathrm{d}b v(a)w(b)^{1-\lambda/n} \\ &= c_{n,\lambda}\int\int \mathrm{d}a\mathrm{d}b \min\Big(w(b)v(a)^{1-\lambda/n}, v(a)w(b)^{1-\lambda/n}\Big) \\ &\le c_{n,\lambda}\int_0^\infty \mathrm{d}a v(a)\int_0^{a^{p/r}} \mathrm{d}b w(b)^{1-\lambda/n} + c_{n,\lambda}\int_0^\infty \mathrm{d}b w(b)\int_0^{b^{r/p}} \mathrm{d}a v(a)^{1-\lambda/n} \end{aligned}$$

Now setting $m := (r-1)(1-\lambda/n)$, we get

$$\begin{aligned} \int_0^{a^{p/r}} w(b)^{1-\lambda/n}\mathrm{d}b \quad &= \int_0^{a^{p/r}} w(b)^{1-\lambda/n}b^m b^{-m}\mathrm{d}b \\ &\le \Big(\int_0^{a^{p/r}} w(b)b^{r-1}\mathrm{d}b\Big)^{1-\lambda/n}\Big(\int_0^{a^{p/r}} b^{-mn/\lambda}\mathrm{d}b\Big)^{\lambda/n} \\ &\le C\Big(\int_0^\infty w(b)b^{r-1}\mathrm{d}b\Big)^{1-\lambda/n} a^{p-1}. \end{aligned}$$

Hence

$$\begin{aligned} I \quad &\le c_{n,\lambda,r}\int v(a)a^{p-1}\mathrm{d}a\Big(\int_0^\infty w(b)b^{r-1}\mathrm{d}b\Big)^{1-\lambda/n} \\ &+ c_{n,\lambda,r}\int_0^\infty w(b)b^{r-1}\mathrm{d}b\Big(\int v(a)a^{p-1}\mathrm{d}a\Big)^{1-\lambda/n} = 2c_{n,\lambda,r} \end{aligned}$$

14.12 Self-adjointness of Schrödinger operators

The following lemma is a consequence of the Hölder inequality:

Lemma 14.37 *Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then the operator of multiplication by $V \in L^p(\mathbb{R}^d)$ is bounded as a map $L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ with norm equal to $\|V\|_q$.*

The following two lemmas follow from the Hardy-Littlewood-Sobolev inequality:

Lemma 14.38 *The operator $(\mathbb{1} - \Delta)^{-1}$ is bounded from $L^2(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ in the following cases:*

(1) *For $d = 1, 2, 3$ if $\frac{1}{\infty} \leq \frac{1}{q} \leq \frac{1}{2}$.*

(2) *For $d = 4$ if $\frac{1}{\infty} < \frac{1}{q} \leq \frac{1}{2}$.*

(3) *For $d \geq 5$ if $\frac{1}{2} - \frac{2}{d} \leq \frac{1}{q} \leq \frac{1}{2}$.*

Lemma 14.39 *The operator $(\mathbb{1} - \Delta)^{-\frac{1}{2}}$ is bounded from $L^2(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ in the following cases:*

(1) *For $d = 1$ if $\frac{1}{\infty} \leq \frac{1}{q} \leq \frac{1}{2}$.*

(2) *For $d = 2$ if $\frac{1}{\infty} < \frac{1}{q} \leq \frac{1}{2}$.*

(3) *For $d \geq 3$ if $\frac{1}{2} - \frac{1}{d} \leq \frac{1}{q} \leq \frac{1}{2}$.*

Proposition 14.40 *Let $V \in L^p + L^\infty(\mathbb{R}^d)$,where*

(1) *for $d = 1, 2, 3$, $p = 2$,*

(2) *for $d = 4$, $p > 2$,*

(3) *for $d \geq 5$, $p = \frac{d}{2}$.*

Then the $-\Delta$-bound of V is zero. Hence $-\Delta + V(x)$ is self-adjoint on $\mathrm{Dom}(-\Delta)$.

Proof. We need to show that

$$\lim_{c \to \infty} V(x)(c - \Delta)^{-1} = 0, \tag{14.11}$$

where (14.11) is understood as an operator on $L^2(\mathbb{R}^d)$.

For any $\epsilon > 0$ we can write $V = V_\infty + V_p$, where $V_\infty \in L^\infty(\mathbb{R}^d)$, $V_p \in L^p(\mathbb{R}^d)$ and $\|V_p\|_p \leq \epsilon$. Now

$$V(x)(c - \Delta)^{-1} \quad = \quad V_\infty(x)(c - \Delta)^{-1} + V_p(x)(c - \Delta)^{-1}.$$

The first term has the norm $\leq \|V_\infty\|_\infty c^{-1}$. Consider the second term. Let

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$$

$\|V_p(x)_{L^q \to L^2} = \|V_p\|_p \leq \epsilon$, and $\|(c - \Delta)^{-1}_{L^2 \to L^q}\|$ is uniformly finite for $c > 1$ by Lemma 14.39. □

Proposition 14.41 *Let* $V \in L^p + L^\infty(\mathbb{R}^d)$*,where*

(1) *for* $d = 1$*,* $p = 1$*,*

(2) *for* $d = 2$*,* $p > 1$*,*

(3) *for* $d \geq 3$*,* $p = \frac{d}{2}$*.*

Then the form $-\Delta$*-bound of* V *is zero. Hence* $-\Delta + V(x)$ *can be defned in the sense of the form sum with the form domain* $\mathrm{Dom}(\sqrt{-\Delta})$*.*

Proof. We need to show that

$$\lim_{c\to\infty}(c-\Delta)^{-1/2}V(x)(c-\Delta)^{-1/2} = 0, \tag{14.12}$$

where (14.12) is understood as an operator on $L^2(\mathbb{R}^d)$. For any $\epsilon > 0$ we can write $V = V_\infty + V_p$, where $V_\infty \in L^\infty(\mathbb{R}^d)$, $V_p \in L^p(\mathbb{R}^d)$ and $\|V_p\|_p \leq \epsilon$. Now

$$\begin{aligned}(c-\Delta)^{-1/2}V(x)(c-\Delta)^{-1/2} &= (c-\Delta)^{-1/2}V_\infty(x)(c-\Delta)^{-1/2}\\ &\quad+\big(|V_p(x)|^{1/2}(c-\Delta)^{-1/2}\big)^*\mathrm{sgn}V_p(x)|V_p(x)|^{1/2}(c-\Delta)^{-1/2}.\end{aligned}$$

The first term has the norm $\leq \|V_\infty\|_\infty c^{-1}$. Consider the second term. Let

$$\frac{1}{q}+\frac{2}{p}=\frac{1}{2}.$$

$\||V_p(x)|^{1/2}_{L^q(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}\| = \sqrt{\|V_p\|_p} \leq \sqrt{\epsilon}$ and $\|(c-\Delta)^{-1/2}_{L^2\to L^q}\|$ is uniformly finite for $c > 1$ by Lemma 14.39. □

Chapter 15

Momentum in one dimension

15.1 Distributions on $\mathbb{R}$

The space of distributions on $\mathbb{R}$ is denoted $\mathcal{D}'(\mathbb{R})$. Note that $L^1_{\mathrm{loc}}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$. Obviously $C(\mathbb{R}) \subset L^1_{\mathrm{loc}}(\mathbb{R})$.

For every $T \in \mathcal{D}'(\mathbb{R})$, one can define its support, which is a closed subset of $\mathbb{R}$. Clearly, if $T \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $\mathrm{supp}T$ in the sense of L^1_{loc} and $\mathcal{D}'$ coincide.

Proposition 15.1 (1) *Let $g \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then*

$$\int_0^x g(y)\mathrm{d}y =: f(x) \tag{15.1}$$

is a continuous function and $f' = g$, where we use the derivative in the distributional sense.

(2) *If $g \in L^p(\mathbb{R})$ with $1 \leq p$, then $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ and so $f(x)$ defined in (15.1) is a continuous function.*

(3) *If $f' = g \in C(\mathbb{R})$, then $f \in C^1(\mathbb{R})$ and $f' = g$ is true in the classical sense.*

(4) *The differentiation does not increase the support of a distribution.*

We will consider sometimes L^1_{loc} functions defined on closed subsets of $\mathbb{R}$, eg. $[0,\infty[$. Clearly, $L^1_{\mathrm{loc}}[0,\infty[\subset L^1_{\mathrm{loc}}(\mathbb{R})$, hence we know what it means to take the distributional derivative of elements of $L^1_{\mathrm{loc}}[0,\infty[$.

θ will denote the Heavyside function.

15.2 Momentum on the line

Consider the Hilbert space $L^2(\mathbb{R})$.

The equation

$$U(t)f(x) := f(x-t), \quad f \in L^2(\mathbb{R}), \quad t \in \mathbb{R},$$

defines a unitary strongly continuous group.

The momentum operator p is defined on the domain

$$\mathrm{Dom}\, p := \{f \in L^2(\mathbb{R}) \;:\; f' \in L^2(\mathbb{R})\}$$

by

$$pf(x) := \frac{1}{\mathrm{i}}\partial_x f(x), \quad f \in \mathrm{Dom}\, p. \tag{15.2}$$

Its graph scalar product is

$$(f|g)_p = \int_{-\infty}^{\infty} \big(\overline{f(x)}g(x) + \overline{f'(x)}g'(x)\big)\mathrm{d}x.$$

Theorem 15.2 (1) $U(t) = \mathrm{e}^{-\mathrm{i}tp}$.

(2) *p is a self-adjoint operator.*

(3) *$C_{\mathrm{c}}^\infty(\mathbb{R})$ is an essential domain of p.*

(4) $\mathrm{sp}\, p = \mathbb{R}$, $\mathrm{sp}_{\mathrm{p}} p = \emptyset$.

(5) *The integral kernel of $(z-p)^{-1}$ equals*

$$R(z,x,y) = \begin{cases} -\mathrm{i}\theta(x-y)\mathrm{e}^{iz(x-y)}, & \mathrm{Im} z > 0, \\ +\mathrm{i}\theta(y-x)\mathrm{e}^{iz(x-y)}, & \mathrm{Im} z < 0. \end{cases}$$

Proof. (1): Let $U(t) = \mathrm{e}^{tA}$.

Let $f \in \mathrm{Dom}\, A$. Then for any $\phi \in C_{\mathrm{c}}^\infty(\mathbb{R})$

$$\begin{aligned} (\phi|Af) \leftarrow \frac{1}{t}(\phi|U(t)f - f) \quad &= \quad \frac{1}{t}\int \big(\overline{\phi(x+t)} - \overline{\phi(x)}\big)f(x)\mathrm{d}x \\ &\to \quad \int \overline{\phi'(x)}f(x)\mathrm{d}x = -\int \overline{\phi(x)}f'(x)\mathrm{d}x \end{aligned}$$

Therefore, $Af = -\partial_x f$ (in the distributional sense). Thus, $f \in \mathrm{Dom}\, p$ and $Af = -\mathrm{i}pf$.

Let $f \in \mathrm{Dom}\, p$, which means $f \in L^2$, $g := f' \in L^2$. Then $f \in C(\mathbb{R})$ and

$$\frac{1}{t}\big(f(x-t) - f(x)\big) = \frac{1}{t}\int_{x-t}^{x} g(y)\mathrm{d}y = j_t * g \to g, \tag{15.3}$$

where we $j_t := \begin{cases} 1/t, & y \in [-t,0], \\ 0 & y \notin [-t,0], \end{cases}$ and (15.3) is understood in the L^2 sense. Therefore, $f \in \mathrm{Dom}\, A$.

(2): p is self-adjoint because $-\mathrm{i}p$ generates a unitary group.

(3): $C_{\rm c}^\infty(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$ left invariant by $U(t)$. Therefore, it is an essential domain.

(5): For ${\rm Im} z > 0$

$$(z-p)^{-1} = -{\rm i}\int_0^\infty {\rm e}^{{\rm i}zt} U(t){\rm d}t.$$

Hence

$$(z-p)^{-1} f(x) = -{\rm i}\int_0^\infty {\rm e}^{{\rm i}zt} f(x-t){\rm d}t = -{\rm i}\int_{-\infty}^\infty {\rm e}^{{\rm i}(x-y)z}\theta(x-y)f(y){\rm d}y.$$

For ${\rm Im} z < 0$ we can use

$$(z-p)^{-1*} = (\overline{z}-p)^{-1}.$$

(4): Let $k \in \mathbb{R}$. Consider $f_{\epsilon,k} = \sqrt{\pi\epsilon}{\rm e}^{-\epsilon x^2 + {\rm i}kx}$. Then $\|f_{\epsilon,k}\| = 1$, $f_{\epsilon,k} \in {\rm Dom}\, p$ and $(k-p)f_{\epsilon,k} \to 0$ as $\epsilon \to 0$. Hence $k \in {\rm sp} p$.

Suppose that $f \in {\rm Dom}\, p$ and $pf = kf$. The only solution is $f = c{\rm e}^{{\rm i}kx}$, which does not belong to $L^2(\mathbb{R})$. Hence ${\rm sp}_{\rm p} p = \emptyset$. □

Proposition 15.3 ${\rm Dom}\, p \subset C_\infty(\mathbb{R})$ *and* ${\rm Dom}\, p \ni f \mapsto f(x) \in \mathbb{C}$ *is a continuous functional.*

Proof. ${\rm Dom}\, p = {\rm Ran}\,({\rm i}-p)^{-1}$. Now $({\rm i}-p)^{-1}$ is the convolution with $-{\rm i}\theta(x){\rm e}^{-|x|}$, which belongs to $L^2(\mathbb{R})$. The convolution of two $L^2(\mathbb{R})$ functions belongs to $C_\infty(\mathbb{R})$. □

Proposition 15.4 (1) *The spaces*

$$\{f \in {\rm Dom}\, p \ : \ f(x) = 0,\ x < 0\}, \tag{15.4}$$
$$\{f \in {\rm Dom}\, p \ : \ f(x) = 0,\ x > 0\}. \tag{15.5}$$

are mutually orthogonal in ${\rm Dom}\, p$.

(2) *The orthogonal complement of the direct sum of (15.4) and (15.5) is spanned by* ${\rm e}^{-|x|}$.

Proof. (2): We easily check the orthogonality of ${\rm e}^{-|x|}$ to (15.4) and (15.5).

Let $f \in L_1^2(\mathbb{R})$. Set $f_\pm(x) := \theta(\pm x)\big(f(x) - f(0){\rm e}^{-|x|}\big)$. Then

$$f(x) = f(0){\rm e}^{-|x|} + f_-(x) + f_+(x).$$

□

15.3 Momentum on the half-line

Consider the Hilbert space $L^2([0,\infty[)$.

Define the semigroups

$$U_\leftarrow(t)f(x) := f(x+t), \quad t \geq 0.$$

$$U_\rightarrow(t)f(x) := \begin{cases} f(x-t), & x \geq t \geq 0. \\ 0, & t > x, \end{cases}$$

If we embed $L^2[0,\infty[$ in $L^2(\mathbb{R})$, then, for $t \geq 0$,

$$\begin{aligned} U_\leftarrow(t) &= \mathbb{1}_{[0,\infty[}(x)U(-t)\mathbb{1}_{[0,\infty[}(x), \\ U_\rightarrow(t) &= \mathbb{1}_{[0,\infty[}(x)U(t)\mathbb{1}_{[0,\infty[}(x). \end{aligned}$$

Define $p_{\max}$ by

$$\begin{aligned} p_{\max}f(x) &:= \frac{1}{\mathrm{i}}\partial_x f(x), \\ f \in \operatorname{Dom} p_{\max} &:= \{f \in L^2[0,\infty[\ :\ f' \in L^2[0,\infty[\}. \end{aligned} \tag{15.6}$$

The graph scalar product of $p_{\max}$ is

$$(f|g)_{p_{\max}} = \int_0^\infty \big(\overline{f(x)}g(x) + \overline{f'(x)}g'(x)\big)\mathrm{d}x,$$

$\operatorname{Dom} p_{\max} \subset C[0,\infty[$, and for $x \in [0,\infty[$

$$\operatorname{Dom} p_{\max} \ni f \mapsto f(x)$$

is a continuous functional.

Define the operator $p_{\min}$ as the restriction of $p_{\max}$ to the domain

$$\operatorname{Dom} p_{\min} := \{f \in \operatorname{Dom} p_{\max}\ :\ f(0) = 0\}.$$

If we embed $L^2[0,\infty[$ in $L^2(\mathbb{R})$, then

$$\begin{aligned} \operatorname{Dom} p_{\max} &= \{\mathbb{1}_{[0,\infty[}f\ :\ f \in \operatorname{Dom} p\}, \\ \operatorname{Dom} p_{\min} &= \{f \in \operatorname{Dom} p\ :\ f(x) = 0,\ x < 0\}. \end{aligned}$$

Theorem 15.5 (1) *We have* $U_\leftarrow(t) = \mathrm{e}^{\mathrm{i}tp_{\max}}$ *and* $U_\rightarrow(t) = \mathrm{e}^{-\mathrm{i}tp_{\min}}$.

(2) $p_{\min} \subset p_{\max}$, $p_{\min}^* = p_{\max}$, $p_{\max}^* = p_{\min}$*; the operators* $p_{\min}$ *and* $-p_{\max}$ *are m-dissipative (in particular, they are closed); the operator* $p_{\min}$ *is hermitian.*

(3) $\operatorname{Dom} p_{\min}$ *is a subspace of* $\operatorname{Dom} p_{\max}$ *of codimension 1 and its orthogonal complement is spanned by* $\mathbb{1}_{[0,\infty[}(x)\mathrm{e}^{-x}$.

(4) $C_{\mathrm{c}}^\infty([0,\infty[)$ *is an essential domain of* $p_{\max}$ *and* $C_{\mathrm{c}}^\infty(]0,\infty[)$ *is an essential domain of* $p_{\min}$.

(5) $\mathrm{sp} p_{\max} = \mathrm{sp_p} p_{\max} = \{\mathrm{Im} z \geq 0\}$, $\mathrm{sp} p_{\min} = \{\mathrm{Im} z \leq 0\}$, $\mathrm{sp_p} p_{\min} = \emptyset$,

$$p_{\max} \mathrm{e}^{\mathrm{i}zx} = z \mathrm{e}^{\mathrm{i}zx}, \quad \mathrm{e}^{\mathrm{i}zx} \in \mathrm{Dom}\, p_{\max}, \quad \mathrm{Im} z > 0; \tag{15.7}$$

(6) *The integral kernels of* $(z - p_{\max})^{-1}$ *and* $(z - p_{\min})^{-1}$ *are equal*

$$R_{\max}(z, x, y) = \mathrm{i}\theta(y - x)\mathrm{e}^{\mathrm{i}z(x-y)}, \ \mathrm{Im} z < 0.$$

$$R_{\min}(z, x, y) = -\mathrm{i}\theta(x - y)\mathrm{e}^{\mathrm{i}z(x-y)}, \ \mathrm{Im} z > 0.$$

15.4 Momentum on an interval I

Consider the Hilbert space $L^2([-\pi, \pi])$.

Define $p_{\max}$ as an operator with domain

$$\mathrm{Dom}\, p_{\max} := \{f \in L^2[-\pi, \pi] \ : \ f' \in L^2[-\pi, \pi]\}$$

$$p_{\max} f(x) := \frac{1}{\mathrm{i}} \partial_x f(x), \quad f \in \mathrm{Dom}\, p_{\max}. \tag{15.8}$$

Note that the graph scalar product for $p_{\max}$ is

$$(f|g)_{p_{\max}} = \int_{-\pi}^{\pi} \big(\overline{f(x)} g(x) + \overline{f'(x)} g'(x)\big) \mathrm{d}x, \quad f, g \in \mathrm{Dom}\, p_{\max},$$

$C[-\pi, \pi] \subset \mathrm{Dom}\, p_{\max}$, and for $x \in [-\pi, \pi]$

$$\mathrm{Dom}\, p_{\max} \ni f \mapsto f(x)$$

is a continuous functional. Define the operator $p_{\min}$ as the restriction of $p_{\max}$ to the domain

$$\mathrm{Dom}\, p_{\min} := \{f \in \mathrm{Dom}\, p_{\max} \ : \ f(-\pi) = f(\pi) = 0\}.$$

Theorem 15.6 (1) *Neither* $p_{\max}$ *nor* $p_{\min}$ *generate a semigroup.*

(2) $p_{\min} \subset p_{\max}$, $p_{\min}^* = p_{\max}$, $p_{\max}^* = p_{\min}$*; the operators* $p_{\min}$ *and* $p_{\max}$ *are closed; the operator* $p_{\min}$ *is hermitian.*

(3) $C^\infty([-\pi, \pi])$ *is an essential domain of* $p_{\max}$ *and* $C_{\mathrm{c}}^\infty(]-\pi, \pi[)$ *is an essential domain of* $p_{\min}$.

(4) $\mathrm{sp} p_{\max} = \mathrm{sp_p} p_{\max} = \mathbb{C}$, $\mathrm{sp} p_{\min} = \mathbb{C}$, $\mathrm{sp_p} p_{\min} = \emptyset$,

$$p_{\max} \mathrm{e}^{\mathrm{i}zx} = z \mathrm{e}^{\mathrm{i}zx}, \qquad z \in \mathbb{C}, \tag{15.9}$$

15.5 Momentum on an interval II

Let $\kappa \in \mathbb{C}$. Define the family of groups on $L^2([-\pi,\pi])$ by

$$U_\kappa(t)\phi(x) = \mathrm{e}^{\mathrm{i}2\pi n\kappa}\phi(x-t), \quad -(2n-1)\pi < x-t < -(2n+1)\pi,\ n \in \mathbb{Z}.$$

Let the operator p_κ be defined as the restriction of $p_{\max}$ to

$$\mathrm{Dom}\, p_\kappa = \{f \in \mathrm{Dom}\, p_{\max}\ :\ \mathrm{e}^{\mathrm{i}2\pi\kappa} f(-\pi) = f(\pi)\}.$$

Theorem 15.7 (1) $U_\kappa(t) = \mathrm{e}^{-\mathrm{i}tp_\kappa}$.

(2) $\|U_\kappa(t)\| = \mathrm{e}^{2\pi n \mathrm{Im}\kappa}$, $2\pi(n-1) < t \leq 2\pi n$, $n \in \mathbb{Z}$.

(3) *The semigroup* $[0,\infty[\ni t \mapsto U_\kappa(t)$ *is of type* $(1,0)$ *for* $\mathrm{Im}\kappa \leq 0$ *and of type* $(\mathrm{e}^{2\pi\mathrm{Im}\kappa}, \mathrm{Im}\kappa)$ *for* $\mathrm{Im}\kappa \geq 0$.

(4) $p_\kappa^* = p_{\overline{\kappa}}$, $\quad p_\kappa = p_{\kappa+1}$; $\quad p_{\min} \subset p_\kappa \subset p_{\max}$. *Operators* p_κ *are closed. For* $\kappa \in \mathbb{R}$ *they are self-adjoint.*

(5) $\{f \in C^\infty([-\pi,\pi])\ :\ \mathrm{e}^{\mathrm{i}2\pi\kappa} f(-\pi) = f(\pi)\}$ *is an essential domain of* p_κ.

(6) $\mathrm{sp} p_\kappa = \mathrm{sp}_{\mathrm{p}} p_\kappa = \mathbb{Z} + \kappa$,

$$p_\kappa \mathrm{e}^{\mathrm{i}(n+\kappa)x} = (n+\kappa)\mathrm{e}^{\mathrm{i}(n+\kappa)x},\ n \in \mathbb{Z}.$$

(7) *The integral kernel of* $(z-p_\kappa)^{-1}$ *equals*

$$R_\kappa(z,x,y) = \frac{1}{2\sin\pi(z-\kappa)}\left(\mathrm{e}^{-\mathrm{i}(z-\kappa)\pi}\mathrm{e}^{\mathrm{i}z(x-y)}\theta(x-y) + \mathrm{e}^{\mathrm{i}(z-\kappa)\pi}\mathrm{e}^{\mathrm{i}z(x-y)}\theta(y-x)\right).$$

(8) *The operators* p_κ *are similar to one another up to an additive constant:*

$$\mathrm{Dom}\, p_\kappa = \mathrm{e}^{\mathrm{i}\kappa x}\,\mathrm{Dom}\, p_0, \qquad p_\kappa = \mathrm{e}^{\mathrm{i}\kappa x} p_0 \mathrm{e}^{-\mathrm{i}\kappa x} + \kappa. \tag{15.10}$$

15.6 Momentum on an interval III

Define the contractive semigroups on $L^2([-\pi,\pi])$:

$$U_\leftarrow(t)f(x) := \begin{cases} f(x+t), & |x+t| \leq \pi, \\ 0 & |x+t| > \pi. \end{cases}$$

$$U_\rightarrow(t)f(x) := \begin{cases} f(x-t), & |x-t| \leq \pi, \\ 0 & |x-t| > \pi. \end{cases}.$$

Let the operator $p_{\pm\mathrm{i}\infty}$ be defined as the restriction of $p_{\max}$ to

$$\mathrm{Dom}\, p_{\pm\mathrm{i}\infty} = \{f \in \mathrm{Dom}\, p_{\max}\ :\ f(\pm\pi) = 0\}.$$

Theorem 15.8 (1) $U_\leftarrow(t) = \mathrm{e}^{\mathrm{i}tp_{+\mathrm{i}\infty}}$ *and* $U_\rightarrow(t) = \mathrm{e}^{-\mathrm{i}tp_{-\mathrm{i}\infty}}$.

(2) $p_{\pm\mathrm{i}\infty}^* = p_{\mp\mathrm{i}\infty}$; $p_{\min} \subset p_{\pm\mathrm{i}\infty} \subset p_{\max}$. *Operators* $p_{\pm\mathrm{i}\infty}$ *are closed.*

(3) $\mathrm{sp} p_{\pm\mathrm{i}\infty} = \emptyset$.

(4) *The integral kernel of* $(z-p_{\pm\mathrm{i}\infty})^{-1}$ *equals*

$$R_{\pm\mathrm{i}\infty}(z,x,y) = \pm\mathrm{i}\mathrm{e}^{\mathrm{i}z(x-y\pm\pi)}\theta(\pm y \mp x), \quad z \in \mathbb{C}.$$

Chapter 16

Laplacian

16.1 Sobolev spaces in one dimension

For $\alpha \in \mathbb{R}$ let $\langle p\rangle^{-\alpha}L^2(\mathbb{R})$ be the scale of Hilbert spaces associated with the operator p. It is called the scale of *Sobolev spaces.* We will focus in the case $\alpha \in \mathbb{N}$.

Theorem 16.1 (1)

$$\langle p\rangle^{-n}L^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \ : \ f^{(n)} \in L^2(\mathbb{R})\}.$$

(2) $\langle p\rangle^{-n}L^2(\mathbb{R}) \subset C^{n-1}(\mathbb{R})$ *and* $\langle p\rangle^{-n}L^2(\mathbb{R}) \ni f \mapsto f^{(j)}(x)$ *for* $j = 0, \dots, n-1$ *are continuous functionals depending continuously on* $x \in \mathbb{R}$.

Proof. We use induction. The step $n = 1$ was proven before.

Suppose that we know that $\langle p\rangle^{-n}L^2(\mathbb{R}) \subset C^n(\mathbb{R})$. Let $f \in \langle p\rangle^{-(n+1)}L^2(\mathbb{R})$. Then $(\mathrm{i} - p)f = g \in \langle p\rangle^{-n}L^2(\mathbb{R})$. Clearly, $\langle p\rangle^{-n-1}L^2(\mathbb{R}) \subset \langle p\rangle^{-n}L^2(\mathbb{R})$, hence $f \in C^{n-1}(\mathbb{R})$. Likewise, $g \in C^{n-1}(\mathbb{R})$, by the induction assumption. Now $pf = -g + \mathrm{i}f \in C^{n-1}(\mathbb{R})$. Hence, by Prop. 15.3 (3) $f \in C^n(\mathbb{R})$. □

16.2 Laplacian on the line

Define the form $\mathfrak{d}$ by

$$\mathfrak{d}(f, g) := \int \overline{f'(x)} g'(x)\mathrm{d}x, \quad f, g \in \mathrm{Dom}\,\mathfrak{d} := \langle p\rangle^{-1}L^2(\mathbb{R}).$$

The operator p^2 on $L^2(\mathbb{R})$ will be denoted $-\Delta$. Thus

$$-\Delta f(x) = -\partial_x^2 f(x), \quad f \in \mathrm{Dom}(-\Delta) = \langle p\rangle^{-2}L^2(\mathbb{R}).$$

Theorem 16.2 (1) $-\Delta$ *is a positive self-adjoint operator.*

(2) $\mathrm{sp_p}(-\Delta) = \emptyset$.

(3) $\mathrm{sp}(-\Delta) = [0, \infty[$.

(4) *The integral kernel of* $(k^2 - \Delta)^{-1}$, *for* $\mathrm{Re}k > 0$, *is*

$$R(k, x, y) = \frac{1}{2k} \mathrm{e}^{-k|x-y|}.$$

(5) *The integral kernel of* $\mathrm{e}^{t\Delta}$ *is*

$$K(t, x, y) = (4\pi t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{(x-y)^2}{4t}}.$$

(6) *The form* $\mathfrak{d}$ *is closed and associated with the operator* $-\Delta$.

(7) $\{f \in C^2(\mathbb{R}) \cap L^2(\mathbb{R}) \ : \ f', f'' \in L^2(\mathbb{R})\}$ *is contained in* $\mathrm{Dom}(-\Delta)$ *and on this set*

$$-\Delta f(x) = -\partial_x^2 f(x).$$

(8) $C_\mathrm{c}^\infty(\mathbb{R})$ *is an essential domain of* $-\Delta$.

Proof. (4) Let $\mathrm{Re}k > 0$. Then

$$(\mathrm{i}k - p)^{-1}(x, y) = -\mathrm{i}\theta(x - y)\mathrm{e}^{-k|x-y|}, \quad (-\mathrm{i}k - p)^{-1}(x, y) = \mathrm{i}\theta(y - x)\mathrm{e}^{-k|x-y|}.$$

Now

$$\begin{aligned} (k^2 - \Delta)^{-1} &= (\mathrm{i}k - p)^{-1}(-\mathrm{i}k - p)^{-1} \\ &= (-2\mathrm{i}k)^{-1} \left((\mathrm{i}k - p)^{-1} - (-\mathrm{i}k - p)^{-1}\right). \end{aligned} \tag{16.1}$$

The integral kernel of (16.1) equals $(2k)^{-1}\mathrm{e}^{-k|x-y|}$.

(5) We have

$$\mathrm{e}^{t\Delta} = (2\pi\mathrm{i})^{-1} \int_\gamma (z - \Delta)^{-1} \mathrm{e}^{tz} \mathrm{d}z,$$

where γ is a contour of the form $\mathrm{e}^{-\mathrm{i}\alpha}]0, \infty[\cup \mathrm{e}^{\mathrm{i}\alpha}[0, \infty[$ bypassing 0, where $\pi/2 < \alpha < \pi$. Hence

$$\mathrm{e}^{t\Delta}(x, y) = (2\pi\mathrm{i})^{-1} \int_{\tilde{\gamma}} \mathrm{e}^{-k|x-y|+tk^2} \mathrm{d}k$$

where $\tilde{\gamma}$ is a contour of the form $\mathrm{e}^{-\mathrm{i}\alpha/2}[0, \infty[\cup \mathrm{e}^{\mathrm{i}\alpha/2}[0, \infty[$. We put $k = \mathrm{i}u$ and obtain

$$\mathrm{e}^{t\Delta}(x, y) = (2\pi\mathrm{i})^{-1} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u|x-y|-tu^2} \mathrm{i}\mathrm{d}u$$

$\square$

16.3 Laplacian on the halfline I

Consider the space $L^2([0,\infty[)$. Define $-\Delta_{\max}$ by

$$-\Delta_{\max}f = -\partial_x^2 f,\quad f\in \mathrm{Dom}\left(-\Delta_{\max}\right) := \{1\!\!1_{[0,\infty[}f \ :\ f\in\langle p\rangle^{-2}L^2(\mathbb{R})\}.$$

Likewise, define $-\Delta_{\min}$ as the restriction of $-\Delta_{\max}$ to

$$\mathrm{Dom}\left(-\Delta_{\min}\right) := \{f\in\langle p\rangle^{-2}L^2(\mathbb{R})\ :\ f(x)=0,\ x<0\}.$$

(Both $\mathrm{Dom}\left(-\Delta_{\max}\right)$ and $\mathrm{Dom}\left(-\Delta_{\min}\right)$ are defined using the space $L^2(\mathbb{R})$. It is easy to see that they are contained in $L^2([0,\infty[)$.)

Theorem 16.3 (1) $-\Delta_{\min}^* = -\Delta_{\max}, \qquad -\Delta_{\min}\subset-\Delta_{\max}$.

(2) *The operators $-\Delta_{\min}$ and $-\Delta_{\max}$ are closed and $-\Delta_{\min}$ is hermitian.*

(3) $\mathrm{sp_p}(-\Delta_{\max}) = \mathbb{C}\backslash[0,\infty[, \qquad \mathrm{sp_p}(-\Delta_{\min})=\emptyset$

$$-\Delta_{\max}\mathrm{e}^{\mathrm{i}kx} = k^2\mathrm{e}^{\mathrm{i}kx},\ \mathrm{Im}k>0,\qquad \mathrm{e}^{\mathrm{i}kx}\in\mathrm{Dom}(-\Delta_{\max}).$$

(4) $\mathrm{sp}(-\Delta_{\max})=\mathbb{C},\ \ \mathrm{sp}(-\Delta_{\min})=\mathbb{C}$.

(5) $-\Delta_{\min} = \left(p_{\min}\right)^2, \quad -\Delta_{\max} = \left(p_{\max}\right)^2$.

16.4 Laplacian on the halfline II

Let $\mu\in\mathbb{C}\cup\{\infty\}$. Let $-\Delta_\mu$ be the restriction of $-\Delta_{\max}$ to

$$\mathrm{Dom}(-\Delta_\mu) = \{f\in\mathrm{Dom}(-\Delta_{\max})\ \ :\ \ \mu f(0)=f'(0)\}. \tag{16.2}$$

(If $\mu=\infty$, these are the Dirichlet boundary conditions, that means $f(0)=0$, if $\mu=0$, these are the Neumann boundary conditions, that means $f'(0)=0$).

Define also the form $\mathfrak{d}_\mu$ as follows. If $\mu\in\mathbb{R}$, then

$$\mathfrak{d}_\mu(f,g) := \mu\overline{f(0)}g(0) + \int\overline{f'(x)}g'(x)\mathrm{d}x,\quad f,g\in\mathrm{Dom}\,\mathfrak{d}_\mu := \mathrm{Dom}\,p_{\max}.$$

For $\mu=\infty$,

$$\mathfrak{d}_\infty(f,g) := \int\overline{f'(x)}g'(x)\mathrm{d}x,\quad f,g\in\mathrm{Dom}\,\mathfrak{d}_\infty := \mathrm{Dom}\,p_{\min}.$$

Theorem 16.4 (1) $-\Delta_{\min}\subset-\Delta_\mu\subset-\Delta_{\max}$.

(2) $-\Delta_\mu^* = -\Delta_{\overline{\mu}}$.

(3) *The operator $-\Delta_\mu$ is a generator of a group. For $\mu\in\mathbb{R}\cup\{\infty\}$ it is self-adjoint.*

(4) $\mathrm{sp_p}(-\Delta_\mu) = \begin{cases}\{-\mu^2\}, & \mathrm{Re}\mu<0;\\ \emptyset, & \text{otherwise};\end{cases}$

$-\Delta_\mu\mathrm{e}^{\mu x} = -\mu^2\mathrm{e}^{\mu x},\ \mathrm{Re}\mu<0,\qquad \mathrm{e}^{\mu x}\in\mathrm{Dom}(-\Delta_\mu)$.

(5) $\mathrm{sp}(-\Delta_\mu) = \begin{cases} \{-\mu^2\} \cup [0, \infty[, & \mathrm{Re}\mu < 0, \\ [0, \infty[, & \text{otherwise.} \end{cases}$

(6) $-\Delta_0 = p_{\max}^* p_{\max}, \quad -\Delta_\infty = p_{\min}^* p_{\min}$.

(7) *The forms* $\mathfrak{d}_\mu$ *are closed and associated with the operator* $-\Delta_\mu$.

(8) *Let* $\mathrm{Re}k > 0$. *The integral kernel of* $(k^2 - \Delta_\mu)^{-1}$ *is equal*

$$R_\mu(k, x, y) = \frac{1}{2k} \mathrm{e}^{-k|x-y|} + \frac{1}{2k} \frac{(k - \mu)}{(k + \mu)} \mathrm{e}^{-k(x+y)},$$

in particular, for the Dirichlet boundary conditions,

$$R_\infty(k, x, y) = \frac{1}{2k} \mathrm{e}^{-k|x-y|} - \frac{1}{2k} \mathrm{e}^{-k(x+y)},$$

and for the Neumann boundary conditions

$$R_0(k, x, y) = \frac{1}{2k} \mathrm{e}^{-k|x-y|} + \frac{1}{2k} \mathrm{e}^{-k(x+y)}.$$

(9) *The semigroups* $\mathrm{e}^{t\Delta_\mu}$ *have the integral kernel*

$$K_\mu(t, x, y) = (4\pi t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{(x-y)^2}{4t}} + (2\pi)^{-1} \int_{-\infty}^{\infty} \frac{\mathrm{i}u - \mu}{\mathrm{i}u + \mu} \mathrm{e}^{-\mathrm{i}u(x+y) - tu^2} \mathrm{d}u,$$

In particular, in the Dirichlet case

$$K_\infty(t, x, y) = (4\pi t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{(x-y)^2}{4t}} - (4\pi t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{(x+y)^2}{4t}},$$

and in the Neumann case

$$K_0(t, x, y) = (4\pi t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{(x-y)^2}{4t}} + (4\pi t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{(x+y)^2}{4t}}.$$

The group $\mathrm{e}^{\mathrm{i}t\Delta_\mu}$ for $\mu \in \mathbb{R} \cup \{\infty\}$ describes a quantum particle with a potential well or bump at the end of the halfline.

The semigroup $\mathrm{e}^{t\Delta_\mu}$ for $\mu \in \mathbb{R}$ describes the diffusion with a sink or source at the end of the halfline. Note that $\mathrm{e}^{t\Delta_\mu}$ preserves the pointwise positivity. If $p_t = \mathrm{e}^{t\Delta_\mu} p_0$, $0 < a < b$, then

$$\partial_t \int_a^b p_t(x) \mathrm{d}x = p'(b) - p'(a).$$

$$\partial_t \int_0^a p_t(x) \mathrm{d}x = p'(a) - \mu p(0).$$

Thus at 0 there is a sink of p with the rate μ.

16.5 Neumann Laplacian on a halfline with the delta potential

On $L^2([0,\infty[)$ we define the cosine transform

$$U_{\rm N}f(k) := \sqrt{2/\pi}\int_0^\infty \cos kx f(x){\rm d}x, \quad k\geq 0.$$

Note that $U_{\rm N}$ is unitary and $U_{\rm N}^2 = 1$.

Let $\Delta_{\rm N}$ be the Laplacian on $L^2([0,\infty[)$ with the Neumann boundary condition. Clearly,

$$-U_{\rm N}\Delta_{\rm N}U_{\rm N}^* = k^2.$$

Let $|\delta)(\delta|$ be the quadratic form given by

$$(f_1|\delta)(\delta|f_2) = \overline{f_1(0)}f_2(0),$$

Note that it can be formally written as

$$\int_0^\infty \overline{f(x)}\delta(x)g(x){\rm d}x,$$

and thus is interpreted as a "potential".

Let $(1|$ denote the functional on $L^2([0,\infty[)$ given by

$$(1|g) = \int_0^\infty g(k){\rm d}k.$$

Using $\delta(x) = \pi^{-1}\int_0^\infty \cos kx{\rm d}x$ we deduce that

$$U_{\rm N}|\delta)(\delta|U_{\rm N}^* = \pi^{-1}|1)(1|.$$

Then

$$U_{\rm N}\left(-\Delta_{\rm N} + \lambda|\delta)(\delta|\right)U_{\rm N}^* = k^2 + \lambda\pi^{-1}|1)(1|$$

is an example of an Aronszajn-Donoghue Hamiltonian of type IIb, because

$$\int_0^\infty 1{\rm d}k = \infty, \quad \int_0^\infty \frac{1}{1+k^2}{\rm d}k < \infty, \quad \int_0^\infty \frac{1}{k^2}{\rm d}k = \infty.$$

16.6 Dirichlet Laplacian on a halfline with the δ' potential

On $L^2([0,\infty[)$ we define the sine transform

$$U_{\rm D}f(k) := \sqrt{2/\pi}\int_0^\infty \sin kx f(x){\rm d}x, \quad k\geq 0.$$

Note that $U_{\rm D}$ is unitary and $U_{\rm D}^2 = 1$

Let Δ_{D} be the Laplacian on $L^2([0,\infty[)$ with the Dirichlet boundary condition. Clearly,

$$-U_{\mathrm{D}}\Delta_{\mathrm{D}}U_{\mathrm{D}}^* = k^2.$$

Using $-\delta'(x) = \pi^{-1}\int_0^\infty \sin kx\mathrm{d}x$ we deduce that

$$U_{\mathrm{D}}|\delta')(\delta'|U_{\mathrm{D}}^* = \pi^{-1}|k)(k|.$$

Here $|\delta')(\delta'|$ is the quadratic form given by

$$(f_1|\delta')(\delta'|f_2) = \overline{f_1'(0)}f_2'(0),$$

and $(k|$ is the functional on $L^2([0,\infty[)$ given by

$$(k|g) = \int_0^\infty kg(k)\mathrm{d}k.$$

Thus

$$U_{\mathrm{D}}\left(-\Delta_{\mathrm{D}} + \lambda|\delta')(\delta'|\right)U^* = k^2 + \lambda\pi^{-1}|k)(k|$$

is an example of an Aronszajn-Donoghue Hamiltonian of type IIIa, because

$$\int_0^\infty \frac{k^2}{1+k^2}\mathrm{d}k = \infty, \quad \int_0^\infty \frac{k^2}{(1+k^2)^2}\mathrm{d}k < \infty, \quad \int_0^\infty \frac{k^2}{(1+k^2)k^2}\mathrm{d}k < \infty.$$

16.7 Laplacian on $L^2(\mathbb{R}^d)$ with the delta potential

On $L^2(\mathbb{R}^d)$ we consider the unitary operator $U = (2\pi)^{d/2}\mathcal{F}$, where $\mathcal{F}$ is the Fourier transformation. Note that U is unitary.

Let Δ be the usual Laplacian. Clearly,

$$-U\Delta U^* = k^2.$$

Let $|\delta)(\delta|$ be the quadratic form given by

$$(f_1|\delta)(\delta|f_2) = \overline{f_1(0)}f_2(0).$$

Note that again it can be also written as

$$\int \overline{f(x)}\delta(x)g(x)\mathrm{d}x,$$

and thus is interpreted as a "potential". Let $(1|$ denote the functional on $L^2(\mathbb{R}^d)$ given by

$$(1|g) = \int g(k)\mathrm{d}k.$$

Using $\delta(x) = (2\pi)^{-d}\int \mathrm{e}^{\mathrm{i}kx}\mathrm{d}x$ we deduce that

$$U|\delta)(\delta|U^* = (2\pi)^{-d}|1)(1|.$$

Consider

$$U\,(-\Delta+\lambda|\delta)(\delta|)\,U^* = k^2+\lambda(2\pi)^{-d}|1)(1|$$

as an example of an Aronszajn-Donoghue Hamiltonian. We compute:

$$\begin{aligned}
&\int\frac{\mathrm{d}^dk}{1+k^2}<\infty \;\Leftrightarrow\; d=1,\\
&\int\frac{\mathrm{d}^dk}{(1+k^2)^2}<\infty \;\Leftrightarrow\; d=1,2,3,\\
&\int\frac{\mathrm{d}^dk}{k^2(1+k^2)}<\infty \;\Leftrightarrow\; d=3.
\end{aligned}$$

Thus

(1) for $d=1$ it is of type IIb, so it can be defined in the form sense using the parameter λ (as we have already seen),

(2) for $d=2$ it is of type IIIb. It can be renormalized.

(3) for d=3 it is of type IIIa. It can be renormalized.

(4) for $d\geq 4$ there is no nontrivial renormalization procedure.

Consider dimension $d=2$. Let us compute the resolvent for $z=-p^2$. We have

$$\begin{aligned}
g_{-1,\gamma}(-p^2) &= \gamma+(p^2-1)\frac{(1|(H_0+p^2)^{-1}(H_0+1)^{-1}|1)}{(2\pi)^2}\\
&= \gamma+(p^2-1)\int\frac{\mathrm{d}^2k}{(2\pi)^2(k^2+p^2)(k^2+1)} \;=\; \gamma+\frac{\ln p^2}{4\pi}.
\end{aligned}$$

Using that the Fourier transform of $k\mapsto\frac{1}{k^2+p^2}$ equals $x\mapsto 2\pi K_0(p|x|)$, where K_0 is the 0th MacDonald function, we obtain the following expression for the integral kernel of $(p^2+H)^{-1}$:

$$\frac{1}{2\pi}K_0(p|x-y|)+\frac{K_0(p|x|)K_0(p|y|)}{(2\pi)^2(\gamma+\frac{\ln p^2}{4\pi})}. \tag{16.3}$$

In the physics literature one usually introduces the parameter $a=\mathrm{e}^{\gamma/2\pi}$ called the *scattering length*. There is a bound state $K_0(|x|/a)$ with eigenvalue $-a^{-2}$.

Note that

$$\{f\in(1-\Delta)^{-1}L^2(\mathbb{R}^2):\ f(0)=0\} \tag{16.4}$$

is a closed subspace of $(1-\Delta)^{-1}L^2(\mathbb{R}^2)$. The domain of H is spanned by (16.4) and

$$(-a^{-2}-\Delta)^{-1}|1), \tag{16.5}$$

which is in $L^2(\mathbb{R}^2)\backslash(1-\Delta)^{-1}L^2(\mathbb{R}^2)$. In the position representation (16.5) is $x\mapsto 2\pi K_0(|x|/a)$ Around $r\sim 0$ we have the asymptotics $K_0(r)\simeq-\log(r/2)-\gamma$. Therefore, the domain of H contains functions that behave at zero as $C\big(\log(|x|/2a)+\gamma\big)$.

Consider dimension $d = 3$. Let us compute the resolvent for $z = -p^2$. We have

$$\begin{aligned} g_{0,\gamma}(-p^2) &= \gamma + p^2 \frac{(1|(H_0+p^2)^{-1}H_0^{-1}|1)}{(2\pi)^3} \\ &= \gamma + p^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3(k^2+p^2)k^2} \;=\; \gamma + \frac{p}{4\pi} \end{aligned}$$

Using that the Fourier transform of $k \mapsto \frac{1}{k^2+p^2}$ equals $x \mapsto 2\pi^2 \frac{\mathrm{e}^{p|x|}}{|x|}$, we obtain the following expression for the integral kernel of $(p^2 + H)^{-1}$:

$$\frac{\mathrm{e}^{-p|x-y|}}{4\pi|x-y|} + \frac{\mathrm{e}^{-p|x|}\mathrm{e}^{-p|y|}}{(4\pi)^2(\gamma_0 + \frac{p}{4\pi})|x||y|}. \tag{16.6}$$

In the physics literature one usually introduces the parameter $a = -(4\pi\gamma)^{-1}$ called the *scattering length*.

$$\{f \in (1-\Delta)^{-1}L^2(\mathbb{R}^3) : \; f(0) = 0\} \tag{16.7}$$

is a closed subspace of $(1-\Delta)^{-1}L^2(\mathbb{R}^3)$. The domain of H is spanned by (16.7)

$$(a\mathrm{e}^{\mathrm{i}\pi/4} - \mathrm{i})(\mathrm{i}-\Delta)^{-1}|1) + (a\mathrm{e}^{-\mathrm{i}\pi/4} + \mathrm{i})(-\mathrm{i}-\Delta)^{-1}|1) \tag{16.8}$$

In the position representation $(\pm\mathrm{i}-\Delta)^{-1}|1)$ equals $x \mapsto 2\pi^2 \frac{\exp(\mathrm{e}^{\pm\mathrm{i}\pi/4}|x|)}{|x|}$. Therefore, the Hamiltonian with the scattering length a has the domain whose elements around zero behave as $C(1 - a/|x|)$.

For $a > 0$ there is a bound state $\frac{\mathrm{e}^{-|x|/a}}{|x|}$ with eigenvalue $-a^{-2}$. To get the domain, instead of (16.8), we can adjoin this bound state to (16.7).

Note that the Hamiltonian is increasing wrt $\gamma \in]-\infty, \infty]$. It is also increasing wrt a separately on $[-\infty, 0]$ and $]0, \infty]$. At 0 the monotonicity is lost. $a = 0$ corresponds to the usual Laplacian.

The following theorem summarizes a part of the above results.

Theorem 16.5 *Consider* $-\Delta$ *on* $C_\mathrm{c}^\infty(\mathbb{R}^d\backslash\{0\})$

(1) *It has the defficiency index* $(2,2)$ *for* $d = 1$.

(2) *It has the defficiency index* $(1,1)$ *for* $d = 2, 3$.

(3) *It is essentially self-adjoint for* $d \geq 4$.

(4) *For* $d = 1$ *its Friedrichs extension is* $-\Delta_\mathrm{D}$ *and its Krein extension is* $-\Delta$.

(5) *For* $d = 2$ *its Friedrichs and Krein extension is* $-\Delta$.

(6) *For* $d = 3$ *its Friedrichs extension is* $-\Delta$ *an its Krein extension corresponds to* $a = \infty$.

Let us sketch an alternative approach. The Laplacian in d dimensions written in spherical coordinates equals

$$\Delta = \partial_r^2 + \frac{d-1}{r}\partial_r + \frac{\Delta_{\mathrm{LB}}}{r^2},$$

where Δ_{LB} is the Laplace-Beltrami operator on the sphere. For $d \geq 2$, the eigenvalues of Δ_{LB} are $-l(l+d-2)$, for $l = 0, 1, \dots$. For $d = 1$ instead of the Laplace-Beltrami operator we consider the parity operator with the eigenvalues ± 1. We will write $l = 0$ for parity $+1$ and $l = 1$ for parity -1. Hence the radial part of the operator is

$$\partial_r^2 + \frac{d-1}{r}\partial_r - \frac{l(l+d-2)}{r^2}.$$

The indicial equation of this operator reads

$$\lambda(\lambda + d - 2) - l(l + d - 2) = 0.$$

It has the solutions $\lambda = l$ and $\lambda = 2 - l - d$.

For $l \geq 2$ only the solutions behaving as r^l around zero are locally square integrable, the solutions behaving as r^{2-1-d} have to be discarded. For $l = 0, 1$ we have the following possible square integrable behaviors around zero:

	$l = 0$	$l = 1$	$l \geq 2$
$d = 1$	r^0, r^1	r^0, r^1	$--$
$d = 2$	$r^0, r^0 \ln r$	r^1	r^l
$d = 3$	r^0, r^{-1}	r^1	r^l
$d \geq 4$	r^0	r^1	r^l

In dimension $d = 1$ in both parity sectors we have non-uniqueness of boundary conditions. In dimensions $d = 2, 3$ this non-uniqueness appears only in the spherically symmetric sector. There is no nonuniqueness in higher dimensions.

16.8 Approximating delta potentials by separable potentials

Set $1_\Lambda(k) := \mathbb{1}_{[0,\Lambda]}(|k|)$. The Laplacian with a delta potential can be conveniently approximated by a *separable potential*

$$-\Delta + \frac{\lambda}{(2\pi)^d}|1_\Lambda)(1_\Lambda|. \tag{16.9}$$

In dimension $d = 1$ and $d = 2$ (16.9) has a (single) negative bound state iff $\lambda < 0$.

Clearly, in dimension $d = 1$ (16.9) converges to $-\Delta + \lambda\delta$ in the norm resolvent sense for all $\lambda \in \mathbb{R}$.

In dimension $d = 2$ it is easy to check that

$$-\Delta - \big(\gamma_{-1} + \pi \log(1 + \Lambda^2)\big)^{-1}|1_\Lambda)(1_\Lambda| \tag{16.10}$$

converges to $-\Delta_{(\gamma_{-1})}$ for all $\gamma_{-1} \in \mathbb{R}$.

In dimension $d = 3$ (16.9) has a (single) negative bound state for all $\frac{\lambda}{(2\pi)^3} < -(\Lambda 4\pi)^{-1}$. It is easy to check that

$$-\Delta - \big(\gamma_0 + 4\pi\Lambda\big)^{-1} |1_\Lambda)(1_\Lambda| \tag{16.11}$$

converges to $-\Delta_{(\gamma_0)}$ for all $\gamma_0 \in \mathbb{R}$.

Chapter 17

Orthogonal polynomials

17.1 Orthogonal polynomials

Let $-\infty \le a < b \le \infty$. Let $\rho > 0$ be a fixed positive integrable function on $]a, b[$ called a *weight*. Let x denote the generic variable in $\mathbb{R}$.

We will denote by Pol the space of complex polynomials of the real variable. We assume that

$$\int_a^b |x|^n \rho(x) \mathrm{d}x < \infty, \quad n = 0, 1, \dots. \tag{17.1}$$

Then Pol is contained in $L^2([a, b], \rho)$.

The monomials $1, x, x^2, \dots$ form a linearly independent sequence in $L^2([a, b], \rho)$. Applying the *Gram-Schmidt orthogonalization* to this sequence we obtain the orthogonal polynomials $P_0, P_1, P_2, \dots$. Note that $\deg P_n = n$. There exist a simple criterion that allows us to check whether this is an *orthogonal basis*.

Theorem 17.1 *Suppose that there exists $\epsilon > 0$ such that*

$$\int_a^b \mathrm{e}^{\epsilon|x|} \rho(x) \mathrm{d}x < \infty.$$

Then Pol *is dense in $L^2([a, b], \rho)$. Therefore, $P_0, P_1, \dots$ form an orthogonal basis of $L^2([a, b], \rho)$.*

Proof. Let $h \in L^2([a, b], \rho)$. Then for $|\mathrm{Im} z| \le \frac{\epsilon}{2}$

$$\int_a^b |\rho(x) h(x) \mathrm{e}^{\mathrm{i}xz}| \mathrm{d}x \quad \le \quad \left(\int_a^b \rho(x) \mathrm{e}^{\epsilon|x|} \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_a^b \rho(x) |h(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}} < \infty.$$

Hence, for $|\mathrm{Im} z| \le \frac{\epsilon}{2}$ we can define

$$F(z) := \int_a^b \rho(x) \mathrm{e}^{-\mathrm{i}zx} h(x) \mathrm{d}x.$$

F is analytic in the strip $\{z \in \mathbb{C} \ : \ |\mathrm{Im} z| < \frac{\epsilon}{2}\}$. Let $(x^n|h) = 0$, $n = 0, 1, \dots$. Then

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} F(z)\Big|_{z=0} = (-\mathrm{i})^n \int_a^b x^n \rho(x) h(x) \mathrm{d}x = (-\mathrm{i})^n (x^n|h) = 0.$$

But an analytic function vanishing with all derivatves at one point vanishes in its whole (connected) domain. Hence $F = 0$ in the whole strip, and in particular on the real line. Hence $\hat{h} = 0$. Applying the inverse Fourier transformation we obtain $h = 0$.

Therefore, there are no nonzero vectors orthogonal to Pol. □

17.2 Classical orthogonal polynomials

We will classify the so called *classical orthogonal polynomials*, that is orthogonal polynomials that are eigefunctions of a certain second order differential operator. We will show that all classical orthogonal polynomials essentially fall into one of the following 3 classes:

(1) *Hermite polynomials* $H_n(x) = \frac{(-1)^n}{n!} \mathrm{e}^{x^2} \partial_x \mathrm{e}^{-x^2}$, which form an orthogonal basis in $L^2(\mathbb{R}, \mathrm{e}^{-x^2})$ and satisfy

$$(\partial_x^2 - 2x\partial_x + 2n)H_n(x) = 0.$$

(2) *Laguerre polynomials* $L_n^\alpha(x) = \frac{1}{n!} \mathrm{e}^x \partial_x^n \mathrm{e}^{-x} x^{n+\alpha}$, which form an orthogonal basis in $L^2(]0, \infty[, \mathrm{e}^{-x} x^\alpha)$ for $\alpha > -1$ and satisfy

$$(x\partial_x^2 + (\alpha + 1 - x)\partial_x + n)L_n^\alpha(x) = 0.$$

(3) *Jacobi polynomials* $P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1 - x)^{-\alpha}(1 + x)^{-\beta} \partial_x^n (1 - x)^{\alpha+n}(1 + x)^{\beta+n}$, which form an orthogonal basis in $L^2(] - 1, 1[, (1 - x)^\alpha (1 + x)^\beta)$ for $\alpha, \beta > -1$ and satisfy

$$(1 - x^2)\partial_x^2 + (\beta - \alpha - (\alpha + \beta + 2)x)\partial_x + n(n + \alpha + \beta + 1)P_n^{\alpha,\beta}(x) = 0.$$

An important role in the proof is played by unbounded operators. More precisely, we use the fact that eigenvectors of hermitian operators with distinct eigenvalues are orthogonal.

Note that the proof is quite elementary – it has been routinely used in courses for physics students of 2nd year of University of Warsaw. In particular, one does not need to introduce the concept of a self-adjoint or essentially self-adjoint operator: one can limit oneself to the concept of a hermitian operator, which is much less technical and acceptable for students without sophisticated mathematical training.

17.3 Reminder about hermitian operators

In this chapter we will need some minimal knowledge about *hermitian* operators. In order to make it essentially self-contained, we recall that an operator A is hermitian if

$$(w|Av) = (Aw|v), \quad v, w \in \mathrm{Dom}\, A.$$

Theorem 17.2 *Let A be a hermitian operator.*

(1) *If $v \in \mathrm{Dom}\, A$ is its eigenvector with eigenvalue λ, that is $Av = \lambda v$, then $\lambda \in \mathbb{R}$.*

(2) *If $\lambda_1 \neq \lambda_2$ are its eigenvalues with eigenvectors v_1 and v_2, then v_1 is orthogonal to v_2.*

Proof. To prove (1), we note that

$$\lambda(v|v) = (v|Av) = (Av|v) = \overline{\lambda}(v|v).$$

then we divide by $(v|v) \neq 0$.

Proof of (2):

$$(\lambda_1 - \lambda_2)(v_1|v_2) = (Av_1|v_2) - (v_1|Av_2) = (v_1|Av_2) - (v_1|Av_2) = 0.$$

□

Remark 17.3 *In finite dimension we can always find an orthonormal basis consisting of eigenvectors of a hermitian operators. In infinite dimension this is not always the case. If it happens then the operator is essentially self-adjoint.*

17.4 2nd order differential operators

A general 2nd order differential operator without a 0th order term can be written as

$$\mathcal{C} := \sigma(x)\partial_x^2 + \tau(x)\partial_x, \tag{17.2}$$

for some functions $\sigma(x)$ and $\tau(x)$.

It is often convenient to rewrite $\mathcal{C}$ in a different form. Let $\rho(x)$ satisfy

$$\sigma(x)\rho'(x) = (\tau(x) - \sigma'(x))\rho(x). \tag{17.3}$$

We have then

$$\mathcal{C} = \rho(x)^{-1}\partial_x\rho(x)\sigma(x)\partial_x. \tag{17.4}$$

The form (17.4) of the operator $\mathcal{C}$ is convenient for the study of its hermiticity.

To simplify the exposition, in the remaining part of this subsection we will assume that $a = 0$ and $b = \infty$, which will illustrate the two possible types of endpoints. The generalization to arbitrary $a < b$ will be obvious.

Theorem 17.4 *Assume (17.1). Suppose also that*

(1) *ρ and σ are real differentiable functions on $]0,\infty[$ and $\rho > 0$;*

(2) *at the boundaries of the interval we have*

$$\begin{aligned} \sigma(0)\rho(0) &= 0, \\ \lim_{x\to\infty}\sigma(x)\rho(x)|x|^n &= 0, \ \ n = 0, 1, 2, \dots. \end{aligned}$$

Then $\mathcal{C}$ as an operator on $L^2([0,\infty[,\rho)$ with domain Pol *is hermitian.*

Proof.

$$\begin{aligned} (g|\mathcal{C}f) &= \int_0^\infty \rho(x)\overline{g}(x)\rho(x)^{-1}\partial_x\sigma(x)\rho(x)\partial_x f(x)\mathrm{d}x \\ &= \lim_{R\to\infty}\int_0^R \overline{g(x)}\partial_x\sigma(x)\rho(x)\partial_x f(x)\mathrm{d}x \\ &= \lim_{R\to\infty}\overline{g(x)}\rho(x)\sigma(x)f'(x)\Big|_0^R - \lim_{R\to\infty}\int_0^R(\partial_x\overline{g(x)})\sigma(x)\rho(x)\partial_x f(x)\mathrm{d}x \\ &= -\lim_{R\to\infty}\overline{g'(x)}\rho(x)\sigma(x)f(x)\Big|_0^R + \lim_{R\to\infty}\int_0^R(\partial_x\rho(x)\sigma(x)\partial_x\overline{g(x)})f(x)\mathrm{d}x \\ &= \int_0^\infty \rho(x)\overline{(\rho(x)^{-1}\partial_x\sigma(x)\rho(x)\partial_x g(x))}f(x)\mathrm{d}x = (\mathcal{C}g|f). \end{aligned}$$

□

Self-adjoint operators of the form (17.4) are often called *Sturm-Liouville operators.*

17.5 Hypergeometric type operators

We are looking for 2nd order differential operators whose eigenfunctions are polynomials. This restricts severely the form of such operators.

Theorem 17.5 *Let*

$$\mathcal{C} := \sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta(z) \tag{17.5}$$

Suppose there exist polynomials P_0, P_1, P_2 *of degree* $0, 1, 2$ *respectively, satisfying*

$$\mathcal{C}P_n = \lambda_n P_n.$$

Then

(1) $\sigma(z)$ *is a polynomial of degree* ≤ 2,

(2) $\tau(z)$ *is a polynomial of degree* ≤ 1,

(3) $\eta(z)$ *is a polynomial of degree* ≤ 0 *(in other words, it is a number).*

Proof. $\mathcal{C}P_0 = \eta(z)P_0$, hence $\deg \eta = 0$.
$\mathcal{C}P_1 = \tau(z)P_1' + \eta P_1$, hence $\deg \tau \leq 1$.
$\mathcal{C}P_2 = \sigma(z)P_2'' + \tau(z)P_2'(z) + \eta P_2$, hence $\deg \sigma \leq 2$. □

Clearly, the number η can be included in the eigenvalue. Therefore, it is enough to consider operators of the form

$$\mathcal{C} := \sigma(z)\partial_z^2 + \tau(z)\partial_z, \tag{17.6}$$

where $\deg \sigma \leq 2$ and $\deg \tau \leq 1$. We will show that for a large class of (17.6) there exists for every $n \in \mathbb{N}$ a polynomial P_n of degree n that is an eigenfunction of (17.6).

The eigenvalue equation of (17.6), that is equations of the form

$$\big(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \lambda\big)f(z) = 0,$$

will be called *hypergeometric type equations*. Solutions of these equations will be called *hypergeometric type functions*. Polynomial solutions will be called *hypergeometric type polynomials*.

17.6 Generalized Rodrigues formula

Some of the properties of hypergeometric type polynomials can be introduced in a unified way. Let ρ satisfy

$$\sigma(z)\partial_z\rho(z) = \left(\tau(z) - \sigma'(z)\right)\rho(z). \tag{17.7}$$

Note that ρ can be expressed by elementary functions.

Let us fix σ. We will however make explicit the dependence on ρ. The operator $\mathcal{C}(\rho)$ can be written as

$$\begin{aligned} \mathcal{C}(\rho) &= \rho^{-1}(z)\partial_z\sigma(z)\rho(z)\partial_z && (17.8)\\ &= \partial_z\rho^{-1}(z)\sigma(z)\partial_z\rho(z) - \tau' + \sigma''. && (17.9) \end{aligned}$$

The following is a generalization of the *Rodrigues formula*, originally given in the case of Legendre polynomials:

$$\begin{aligned} P_n(\rho; z) &:= \frac{1}{n!}\rho^{-1}(z)\partial_z^n\sigma^n(z)\rho(z) && (17.10)\\ &= \frac{1}{2\pi \mathrm{i}}\rho^{-1}(z)\int_{[0^+]}\sigma^n(z+t)\rho(z+t)t^{-n-1}\mathrm{d}t. && (17.11) \end{aligned}$$

Theorem 17.6 *P_n is a polynomial, typically of degree n, more precisely its degree is given as follows:*

(1) *If $\sigma'' = \tau' = 0$, then $\deg P_n = 0$.*

(2) *If $\sigma'' \neq 0$ and $-\frac{2\tau'}{\sigma''} + 1 = m$ is a positive integer, then*

$$\deg P_n = \begin{cases} n, & n = 0, 1, \ldots, m; \\ n - m - 1, & n = m+1, m+2, \ldots. \end{cases}$$

(3) *Otherwise, $\deg P_n = n$.*

We have

$$\begin{aligned} \big(\sigma(z)\partial_z^2 + \tau(z)\partial_z\big) P_n(\rho; z) &= (n\tau' + n(n-1)\frac{\sigma''}{2})P_n(\rho; z), && (17.12)\\ (\sigma(z)\partial_z + \tau(z) - \sigma'(z)) P_n(\rho; z) &= (n+1)P_{n+1}(\rho\sigma^{-1}; z), && (17.13)\\ \partial_z P_n(\rho; z) &= \left(\tau' + (n-1)\frac{\sigma''}{2}\right) P_{n-1}(\rho\sigma; z), && (17.14)\\ \frac{\rho(z + t\sigma(z))}{\rho(z)} &= \sum_{n=0}^{\infty} t^n P_n(\rho\sigma^n; z). && (17.15) \end{aligned}$$

Proof. Introduce the following *creation and annihilation operators*:

$$\begin{aligned} \mathcal{A}^+(\rho) &:= \sigma(z)\partial_z + \tau(z) = \rho^{-1}(z)\partial_z\rho(z)\sigma(z),\\ \mathcal{A}^- &:= \partial_z. \end{aligned}$$

Note that

$$\begin{aligned} \mathcal{C}(\rho) &= \mathcal{A}^+(\rho)\mathcal{A}^-\\ &= \mathcal{A}^-\mathcal{A}^+(\rho\sigma^{-1}) - \tau' + \sigma''. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{C}(\rho)\mathcal{A}^+(\rho) &= A^+(\rho)A^-A^+(\rho)\\ &= A^+(\rho)\big(\mathcal{C}(\rho\sigma) + \tau'\big). \end{aligned}$$

Therefore, if $\mathcal{C}(\rho\sigma^n)F_0 = \lambda_0 F_0$, then

$$\begin{aligned}
&\mathcal{C}(\rho)\ A^+(\rho)\cdots A^+(\rho\sigma^{n-1})F_0 \\
= \quad &\Big(\lambda_0 + n\tau' + n(n-1)\frac{\sigma''}{2}\Big)A^+(\rho)\cdots A^+(\rho\sigma^{-1})F_0.
\end{aligned}$$

Using

$$\begin{aligned}
A^+(\rho) &= \rho^{-1}(z)\partial_z\rho(z)\sigma(z), \\
A^+(\rho\sigma) &= \rho^{-1}(z)\sigma^{-1}(z)\partial_z\rho(z)\sigma^2(z), \\
\cdots &= \cdots \\
A^+(\rho\sigma^{n-1}) &= \rho^{-1}(z)\sigma^{-(n-1)}\partial_z\rho(z)\sigma^n(z),
\end{aligned}$$

we obtain

$$A^+(\rho)\cdots A^+(\rho\sigma^{n-1})F_0 \quad = \quad \rho(z)^{-1}\partial_z^n\rho(z)\sigma^n(z)F_0(z).$$

Take $F_0 = 1$, for which $\lambda_0 = 0$. We then obtain (17.12). $\Box$

17.7 Classical orthogonal polynomials as eigenfunctions of a Sturm-Liouville operator

We are looking for $-\infty \leq a < b \leq \infty$ and weights $]a,b[\ni x \mapsto \rho(x)$ with the following properties: There exist polynomials $P_0, P_1, \dots$ satisfying $\deg P_n = n$ which form an orthogonal basis of $L^2(]a,b[,\rho)$ and are eigenfunctions of a certain 2nd order differential operator $\mathcal{C} := \sigma(x)\partial_x^2 + \tau(x)\partial_x$, that is, for some $\lambda_n \in \mathbb{R}$

$$\big(\sigma(x)\partial_x^2 + \tau(x)\partial_x + \lambda_n\big)\,P_n(x) = 0. \tag{17.16}$$

In particular, we want $\mathcal{C}$ to be hermitian on Pol.

We know that one has to satisfy the following conditions:

(1) For any $n \in \mathbb{N}$
$$\int_a^b \rho(x)|x|^n \mathrm{d}x < \infty, \tag{17.17}$$
which guarantees that $\mathrm{Pol} \subset L^2(]a,b[,\rho)$.

(2) σ has to be a polynomial of degree at most 2 and τ a polynomial of degree at most 1. (See Thm 17.5).

(3) The weight ρ has to solve
$$\sigma(x)\rho'(x) = (\tau(x) - \sigma'(x))\rho(x), \tag{17.18}$$
to be positive, σ has to be real. (See Thm 17.4 (1)).

(4) We have to check the boundary conditions

(i) If an endpoint, say, a is a finite number, we check whether $\rho(a)\sigma(a) = 0$.

(ii) If an endpoint is infinite, say $a = -\infty$, then

$$\lim_{x \to -\infty} |x|^n \sigma(x)\rho(x) = 0, \quad n = 0, 1, 2, \ldots.$$

(see Thm 17.4 (2).)

We will find all weighted spaces $L^2(]a, b[, \rho)$ satisfying the conditions (1)-(4). It will turn out that in all cases the condition

$$\int_a^b \mathrm{e}^{\epsilon|x|}\rho(x)\mathrm{d}x < \infty \tag{17.19}$$

for some $\epsilon > 0$ will hold, which will guarantee that we obtain an orthogonal basis (see Thm 17.1).

We will simplify our answers to standard forms

(1) by changing the variable $x \mapsto \alpha x + \beta$ for $\alpha \neq 0$;

(2) by dividing (both the differential equation and the weight) by a constant.

As a result, we will obtain all *classical orthogonal polynomials*.

17.8 Classical orthogonal polynomials for $\deg \sigma = 0$

We can assume that $\sigma(x) = 1$.

If $\deg \tau = 0$, then

$$\mathcal{C} = \partial_y^2 + c\partial_y.$$

It is easy to discard this case.

Hence $\deg \tau = 1$. Thus

$$\mathcal{C} = \partial_y^2 + (ay + b)\partial_y.$$

Let us set $x = \sqrt{\frac{|a|}{2}}\left(y + \frac{b}{a}\right)$. We obtain

$$\mathcal{C} = \partial_x^2 + 2x\partial_x, \qquad a > 0; \tag{17.20}$$

$$\mathcal{C} = \partial_x^2 - 2x\partial_x, \qquad a < 0. \tag{17.21}$$

Thus $\rho(x) = \mathrm{e}^{\pm x^2}$.

$\sigma(x)\rho(x) = \mathrm{e}^{\pm x^2}$ is never zero, hence the only possible interval is $]-\infty, \infty[$.

If $a > 0$, we have $\rho(x) = \mathrm{e}^{x^2}$, which is impossible because of (4ii).

If $a < 0$, we have $\rho(x) = \mathrm{e}^{-x^2}$ and the interval $]-\infty, \infty[$ is admissible, and even satisfes (17.19). We obtain *Hermite polynomials*

17.9 Classical orthogonal polynomials for $\deg \sigma = 1$

We can assume that $\sigma(y) = y$.

If $\deg \tau = 0$, then

$$\mathcal{C} = y\partial_y^2 + c\partial_y$$

Such a $\mathcal{C}$ always decreases the degree of a polynomial. Therefore, if P is a polynomial and $\mathcal{C}P = \lambda P$, then $\lambda = 0$. Hence $P(x) = x^{-c}$. Therefore, we do not obtain polynomials of all degrees as eigenfunctions.

Thus $\deg \tau = 1$. Hence, for $b \neq 0$,

$$y\partial_y^2 + (a + by)\partial_y. \tag{17.22}$$

After rescaling, we obtain the operator:

$$\mathcal{C} = -x\partial_x^2 + (-\alpha - 1 + x)\partial_x.$$

We compute: $\rho = x^\alpha \mathrm{e}^{-x}$. $\rho(x)\sigma(x) = x^{\alpha+1}\mathrm{e}^{-x}$ is zero only for $x = 0$ i $\alpha > -1$. The interval $[-\infty, 0]$ is eliminated by (4ii). The interval $[0, \infty]$ is admissible for $\alpha > -1$, and even it satisfies 17.19. We obtain *Laguerre polynomials*.

17.10 Classical orthogonal polynomials for $\deg \sigma = 2$, σ has a double root

We can assume that $\sigma(x) = x^2$.

If $\tau(0) = 0$, then

$$\mathcal{C} = x^2\partial_x^2 + cx\partial_x.$$

x^n are eigenfunctions of this operator, but the weight $\rho(x) = x^{c-2}$ is not good.

Let us assume now that $\tau(0) \neq 0$. After rescaling we can suppose that

$$\tau(x) = 1 + (\gamma + 2)x.$$

This gives $\rho(x) = \mathrm{e}^{-\frac{1}{x}}x^\gamma$. The only point where $\rho(x)\sigma(x) = \mathrm{e}^{-\frac{1}{x}}x^{\gamma+2}$ can be zero is $x = 0$. Hence the only possible intervals are $]-\infty, 0[$ and $]0, \infty[$. Both are eliminated by (4ii).

17.11 Classical orthogonal polynomials for $\deg \sigma = 2$, σ has two roots

If both roots are imaginary, it suffices to assume that $\sigma(x) = 1 + x^2$. We can suppose that $\tau(x) = a + (b + 2)x$. Then $\rho(x) = \mathrm{e}^{a \arctan x}(1 + x^2)^b$. $\sigma(x)\rho(x)$ is nowhere zero and therefore the only admissble interval is $[-\infty, \infty]$. This has to be rejected, because $\lim_{|x|\to\infty} \rho(x)|x|^n(1+x^2) = \infty$ for large enough n.

Thus we can assume that the roots are real. It suffices to assume that $\sigma(x) = 1 - x^2$. Let

$$\tau(x) = \beta - \alpha - (\alpha + \beta - 2)x,$$

which corresponds to the operator

$$(1 - x^2)\partial_x^2 + (\beta - \alpha - (\alpha + \beta - 2)x\partial_x,$$

We obtain $\rho(x) = |1-x|^\beta |1+x|^\alpha$. (4ii) eliminates the intervals $]-\infty, -1[$ and $]1, \infty[$. There remains only the interval $[-1, 1]$, which satisfies (4i) for $\alpha, \beta > -1$. We obtain *Jacobi polynomials*.

Bibliography

[Da] Davies, E. B.: One parameter semigroups, Academic Press 1980

[Ka] Kato, T.: Perturbation theory for linear operators, Springer 1966

[RS1] Reed, M., Simon, B.: Methods of Modern Mathematics, vol. 1, Academic Press 1972

[RS2] Reed, M., Simon, B.: Methods of Modern Mathematics, vol. 4, Academic Press 1978