Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians

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June 1997

Abstract

Spectral and scattering theory of massive Pauli-Fierz Hamiltonians is studied. Asymptotic completeness of these Hamiltonians is shown. The proof consists of three parts. The first is a construction of asymptotic fields and a proof of their Fock property. The second part is a geometric analysis of observables. Its main result is what we call geometric asymptotic completeness. Finally, the last part is a proof of asymptotic completeness itself.

1 Introduction

Our paper is devoted to a class of Hamiltonians used in physics to describe a quantum system ("matter" or "an atom") interacting with a bosonic field ("radiation"). $\mathcal{K}$ and K are respectively the Hilbert space and the Hamiltonian describing the matter. The bosonic field is described by a Fock space $\Gamma(\mathfrak{h})$ with the one-particle space $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$, where $\mathbb{R}^d$ is the momentum space, and a free Hamiltonian of the form

$$\mathrm{d}\Gamma(\omega(k)) = \int \omega(k) a^*(k) a(k) \mathrm{d}k.$$

The function $\omega(k)$ is called the *dispersion relation*. The interaction of the "matter" and the bosons is described by the operator

$$V = \int a^*(k) v(k) \mathrm{d}k + \mathrm{hc},$$

where $\mathbb{R}^d \ni k \to v(k)$ is a function with values in operators on $\mathcal{K}$. Thus, the system is described by the Hilbert space $\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h})$ and the Hamiltonian

$$H = K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k)) + V. \tag{1.1}$$

The class of such Hamiltonians is very common in the physics literature. It is also quite natural from the mathematical point of view, as in particular we will see in our paper. Nevertheless, it does not seem to have a generally accepted name. We will call the Hamiltonians of the form (1.1) *Pauli-Fierz Hamiltonians.* In the thirties, Pauli and Fierz wrote a paper on nonrelativistic quantum electrodynamics [PF], where a Hamiltonian of the form (1.1) was obtained, and since then the name Pauli-Fierz Hamiltonian has been occasionally used in this context (see for example [Bl]).

Let us describe some typical examples of Pauli-Fierz Hamiltonians.

If $\dim \mathcal{K} = 1$, then they are exactly solvable – by a Bogolyubov transformation they are equivalent to a quadratic bosonic Hamiltonian.

If $\dim \mathcal{K} = 2$, $K = \sigma_z$ and $v(k) = g(k)\sigma_x$, where σ_z, σ_x are Pauli matrices and $g(k)$ is a real function on $\mathbb{R}^d$, then the Hamiltonian H goes under the name of a *spin-boson Hamiltonian.* In a sense, it is the simplest non-trivial example of a Pauli-Fierz Hamiltonian.

After a certain approximation (dropping interaction terms quadratic in the fields) nonrelativistic quantum electrodynamics can also be put in the form (1.1). In this case $\omega(k) = |k|$ and K is a Schrödinger Hamiltonian (see [CT, BFS]).

If the bosonic field describes a relativistic particle of mass m, then the dispersion relation is of the form $\omega(k) = \sqrt{m^2 + k^2}$.

Various branches of physics, such as solid state theory and quantum optics, furnish more examples of Hamiltonians of the form (1.1). The bosonic field may describe effective quasiparticles, eg. phonons. $\omega(k)$ is then a phenomenological dispersion relation and can be, to a large extent, an arbitrary function. The matter Hamiltonian K and the interaction V may also vary depending on the model. Therefore, from the physical point of view, it seems natural to consider the class of Pauli-Fierz Hamiltonians under as broad conditions as possible.

Let us now describe the assumptions that we will impose on the Hamiltonian H in our paper.

First of all, we will assume that the function $\hat{v}(x)$ decays sufficiently fast in the space variables. We call this the *short-range condition.* Physically, it means that the interaction is well localized. This assumption is needed to prove the existence of asymptotic fields. Note, however, that the results about the location of spectrum (our analog of the HVZ theorem) and the Mourre estimate hold under weaker decay condition on $\hat{v}(x)$.

Secondly, we will assume that the dispersion relation is positive and bounded away from zero, that is

$$\inf \omega(k) := m > 0. \tag{1.2}$$

The number m is sometimes called the *mass* of the field and (1.2) is the positive mass assumption. Besides, we will make some other technical assumptions on $\omega(k)$ (which in general can be relaxed): we will assume that zero is the only critical point of $\omega(k)$, all the derivatives of $\omega(k)$ are bounded and $\lim_{|k|\to\infty} \omega(k) = +\infty$. Thus, a typical dispersion relation satisfying our assumptions is $\omega(k) = \sqrt{m^2 + k^2}$. Unfortunately, due to the assumption (1.2), the dispersion relation $\omega(k) = |k|$ is not covered by our paper. We hope that our results, appropriately modified, can be extended to this case – under suitable conditions on the decay of $v(k)$ as $k \to 0$. The assumption (1.2) means that there is no "infra-red problem". This assumption plays an important role in our considerations and relaxing it will entail additional technical difficulties.

Finally, we assume that the matter Hamiltonian K has a compact resolvent. Physically, this means that the Hilbert space $\mathcal{K}$ is supposed to describe a confined system, eg. $\mathcal{K}$ is finite dimensional or $K = -\frac{1}{2}\Delta + W(x)$ with $\lim_{|x|\to\infty} W(x) = \infty$. Note also that this assumption plays a role only in the so-called HVZ theorem, the Mourre estimate and its consequences, and in the last stage of the proof of the asymptotic completness. The existence of asymptotic fields, the Fock property of wave operators and the geometric asymptotic completeness are true without this assumption.

In Section 3 we describe some general properties of the Pauli-Fierz Hamiltonians. We prove the self-adjointness of these Hamiltonians and some other technical properties.

In Section 4 we impose the condition that the resolvent of K is compact. Under this condition, we show an analog of the HVZ theorem. This theorem says that the essential spectrum of H equals $[E_0 + m, \infty[$ where E_0 is the infimum of the spectrum of H. This clearly implies the existence of a ground state. This theorem is well known [GJ1, BFS, AH] (although the proofs found in the literature seem to be more complicated). We show also the Mourre estimate for Pauli-Fierz Hamiltonians. Its proof mimicks the proof of its analog from the case of N-body Schrödinger operators. One of the key new ingredients is the induction with respect to the energy interval: in the nth step, the theorem is proven for the energy in $[E+(n-1)m, E+nm[$. Note that the proof breaks down if $m = 0$. An immediate consequence of the Mourre estimate is the local finiteness of the pure point spectrum away from the threshold set.

The remaining part of our paper is devoted to the scattering theory of Pauli-Fierz Hamiltonians. The first step of scattering theory for such Hamiltonians is the existence of the so-called *asymptotic fields*. They are defined as the limits on a dense domain of the usual fields in the so-called interaction picture:

$$a^{\sharp,+}(h) := \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} a^{\sharp}(h_t)\mathrm{e}^{-\mathrm{i}tH},$$

where $a^{\sharp}(h)$ equals either $a^*(h)$ – the creation operator – or $a(h)$ – the annihilation operator, and $h_t := \mathrm{e}^{-\mathrm{i}t\omega(k)}h$. The asymptotic creation and annihilation operators satisfy the canonical commutation relations (CCR). Let the Hilbert space $\mathcal{K}^+$ be defined as the space of the states annihilated by asymptotic annihilation operators $a^+(h)$. Physically, it can be understood as the space of asymptotic ("dressed") matter – it contains states with no asymptotically free bosons. Define $\mathcal{H}^+ := \mathcal{K}^+ \otimes \Gamma(\mathfrak{h})$ – the full asymptotic Hilbert space. Then, there is a natural definition of an isometric operator $\Omega^+ : \mathcal{H}^+ \to \mathcal{H}$ interwining the usual and the asymptotic fields:

$$\Omega^+ a^{\sharp}(h) = a^{\sharp,+}(h)\Omega^+.$$

The operator Ω^+ can be defined as a wave operator by the formula

$$\Omega^+ := \text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^+}, \tag{1.3}$$

where

$$H^+ = K^+ \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega(k))$$

is a (non-interacting) asymptotic Hamiltonian defined on $\mathcal{H}^+$ and $I : \mathcal{H}^+ \to \mathcal{H}$ is a certain naturally defined "identification operator".

Note that the above results about scattering theory for massive Pauli-Fierz Hamiltonians, possibly in a weaker form, can be extended to the mass zero case.

Using the positive mass assumption one can show that the operator Ω^+ is unitary. This means, in particular, that the representation of the CCR given by the asymptotic fields $a^+(h)$

is of the Fock type. Note that, in the case of a zero mass, depending on the assumptions on $v(k)$, the unitarity of Ω^+ may be violated, which means that the asymptotic fields may have non-Fock components. It may even happen that the space $\mathcal{K}^+$ is reduced to $\{0\}$.

The construction of asymptotic fields and of the wave operator use rather straightforward methods and has been essentially known for a long time. Up to technicalities related to the unboundedness of field operators, it follows by the so-called Cook's method. Very similar results, including the fact that the positivity of mass implies the Fock property, are contained in a series of papers by Høegh-Krohn [HK1, HK2, HK3].

After the asymptotic fields are defined, it is natural to ask how to characterize the space of asymptotic matter $\mathcal{K}^+$, and its analog for $t \to -\infty$, $\mathcal{K}^-$. A property, which is physically desirable is the equality

$$\mathcal{K}^- = \mathcal{K}^+. \tag{1.4}$$

This property implies in particular the unitarity of the scattering operator

$$S := \Omega^{+*}\Omega^-.$$

It is easy to show that

$$\mathrm{Ran}\mathbb{1}^{\mathrm{pp}}(H) \subset \mathcal{K}^- \cap \mathcal{K}^+,$$

where $\mathrm{Ran}\mathbb{1}^{\mathrm{pp}}(H)$ denotes the space of bound states of H. Thus, it is natural to expect that, if the matter system $\mathcal{K}$ is not to large, then

$$\mathcal{K}^+ = \mathcal{K}^- = \mathrm{Ran}\mathbb{1}^{\mathrm{pp}}(H). \tag{1.5}$$

Clearly, (1.5) implies (1.4). We call the property (1.5) *asymptotic completeness.* The remaining part of our paper is devoted to proving this property.

The eighties and the early nineties were a period when a substantial progress was reached in our understanding of scattering theory for N-body Schrödinger Hamiltonians. In papers [E, SigSof, Gr, De1, Ya] efficient techniques have been developed, which made it possible to prove asymptotic completeness for long-range systems with an arbitrary number of particles. A natural next step was to apply these techniques to Hamiltonians of quantum field theory. This was the idea behind the work of one of the authors [Ge], where asymptotic completeness for the spin-boson Hamiltonian with a particle number cut-off was proved.

In Section 6 we show a number of propagation estimates for Pauli-Fierz Hamiltonians. These estimates are very similar to the analogous estimates from the case of N-body Schrödinger Hamiltonians. This section can be viewed as a technical introduction to the next section, were more conceptual results will be given. Section 6 can be skipped on the first reading.

Section 7 is devoted to a proof of asymptotic completeness for massive Pauli-Fierz Hamiltonians. Most of the section is devoted to a proof of an intermediate result called *geometric asymptotic completeness.* In order to formulate this result one needs observables such as $\Gamma(q(\frac{x}{t}))$, with $q \in C_0^\infty(\mathbb{R}^d)$ and $q = 1$ in a neighborhood of zero, which localize in space. Using such observables, we construct a certain projection P_0^+ projecting onto the states that for a large time do not spread faster than $o(t)$. The precise statement of geometric asymptotic completeness is

$$\mathrm{Ran}P_0^+ = \mathcal{K}^+. \tag{1.6}$$

The proof of geometric asymptotic completeness has a number of ingredients known from N-body Schrödinger operators, such as propagation estimates and asymptotic observables. One of the main new ideas, is the use of certain natural operators $P_k(f_0, f_\infty)$. The operator $P_k(f_0, f_\infty)$ describes the states with exactly k bosons multiplied by f_∞, and the rest multiplied by f_0. Using asymptotic observables constructed with help of such operators, we construct mutually orthogonal projections P_k^+, which project onto the states with exactly k asymptotically free bosons. The projections P_k^+ form a partition of unity on the space $\mathcal{H}$, that is, their sum is the identity. We show that $\mathrm{Ran}P_k^+$ is the range of the wave operator Ω^+ restricted to k-particle states.

The reader familiar with the scattering theory of N-body systems, as described in [De1, DeGe], will note a very close analogy. In the proof of the asymptotic completeness of N-body Schrödinger Hamiltonians, one of the important steps is the following: using asymptotic observables one constructs certain projections $1\!\!1_{Z_a}(P^+)$ that form a partition of unity on the Hilbert space. Then one shows that $\mathrm{Ran}1\!\!1_{Z_a}(P^+)$ equals the range of the wave operator Ω_a^+.

The proof of geometric asymptotic completeness does not use the assumption of the compactness of the resolvent of K. This assumption is needed in Subsection 7.8, where we show asymptotic completeness itself. Here, the basic tool is the minimal velocity estimate, which is a consequence of the Mourre estimate. We show that states spreading not faster than $o(t)$ are exactly the bound states, in other words

$$\mathrm{Ran}P_0^+ = \mathrm{Ran}1\!\!1^{\mathrm{pp}}(H). \tag{1.7}$$

Now (1.6) and (1.7) imply asymptotic completeness (1.5). Note that all these arguments are very close to the arguments used in the scattering theory of N-body Schrödinger operators.

Our paper is essentially self-contained. In Section 2 we describe all the concepts related to Fock spaces that we need. We recall some basic constructions such as the operators $\Gamma(q)$ and $\mathrm{d}\Gamma(b)$ [BSZ, Sim, RS]. We introduce also a number of definitions that seem to be new in the literature. They were very useful in our paper and we think that they may find an application outside of our work. In particular, let us mention the operators $Q_k(f_0, f_\infty)$, which have very interesting properties playing an imporant role in our proof of geometric asymptotic completeness.

Physically, asymptotic completeness means that for large times states evolve according to a simpler evolution. In particular, it implies that the usual formalism of scattering theory involving a unitary scattering operator is justified. The scattering operator is one of the central objects of quantum field theory, usually introduced in a formal, perturbative way. Our article shows that, at least for a certain class of relatively simple but nontrivial models, the usual physical formalism is well-founded.

Let us mention another physical consequence of asymptotic completeness. Let us assume additionally that the interacting Hamiltonian has only one bound state, which can be shown in some cases, at least for small coupling (see [OY, BFS]). Then, as noted in [HuSp1], asymptotic completeness implies the property of *return to equilibrium.* This property plays an important role in statistical physics [BR].

We believe that our result is just one of initial steps of a mathematical study of scattering in quantum field theory. Quantum field theory is a vast subject with diverse models and various interesting problems [Frie, He, GJ2, BSZ, Ha, We]. ¿From the point of view of scattering theory one can distinguish certain natural classes of models. First of all, one should distinguish:
(1a) models with a localized interaction;
(1b) models with a translation invariant interaction.

Secondly one should make the following distinction:
(2a) models conserving the number of particles;
(2b) models changing the number of particles.
Clearly, models with the property (1b) or (2b) are more difficult than models with the property (1a) or (2a) respectively. Pauli-Fierz Hamiltonians are models with a localized interaction, but they do not conserve the number of particles – they are of type (1a,2b). We hope that the methods of our article can be extended to treat the scattering theory of other models of this type. For example, after minor modifications, one can extend our results to the interactions containing a term quadratic in the fields with a sufficiently small coupling constant. Likewise, instead of bosonic fields one can study fermionic fields. The extensively studied [Sim, HK3, GJ1, GJ2] $P(\phi)_2$ model with a spacial cutoff also belongs to the type (1a,2b) – it would be interesting to study asymptotic completeness also in this case.

Scattering theory for translation invariant models (1b) is more difficult. There exists however one case where this problem seems to be well understood – it is the class of models considered in [De2]. These models are of type (1b,2a), they are however quite special – they conserve the number of particles of each species and they are Galilei-covariant, which is also a severe restriction. In the case of these models, the Hilbert space can be split into sectors and within each sector they are described by an N-body Schrödinger Hamiltonian.

There exist also some partial results in the case of relativistic quantum field theory. The Haag-Ruelle theory (see [Ha] and references therein) and its continuation due to Buchholz and Fredenhagen [BF] allow us to define asymptotic fields in an axiomatic local quantum field theory. One can also show asymptotic completeness for low energies and small coupling constants in the $\lambda\phi_2^4$ model [CD, Ia].

A lot of research was devoted to Hamiltonians of quantum field theory in the sixties and the early seventies. Let us mention in particular the book by Friedrichs [Frie], which in a mathematically rigorous way described the perturbative approach to quantum field theory, papers of Høegh-Krohn [HK2, HK1, HK3], early papers on the constructive field theory (see [GJ1] and references in [GJ2, Sim]), papers of Fröhlich on translation-invariant models [Fro1, Fro2] and the work of Davies on the weak-coupling limit for Pauli-Fierz-type Hamiltonians [Da1, Da2]. It seems that in the late seventies and the eigthies there was a long period when little research on this subjects was performed (see however [A1, A2, OY, Ma, Sp1]). The Euclidean [GJ2, Sim] and the axiomatic [Ha] approaches replaced the Hamiltonian approach to quantum field theory. It was also a period of a considerable progress in the study of Schrödinger operators, especially the N-body Schrödinger operators [E, SigSof, Gr, De1, Ya, DeGe]. In the recent years one can see a renewed interest in Hamiltonians of quantum field theory, at least in the Pauli-Fierz Hamiltonians. Let us mention the paper of Huebner and Spohn [HuSp1] where wave operators for the spin-boson Hamiltonian were shown to exist and the problem of asymptotic completeness for such operators was discussed. Note in particular, that the formula (1.3) comes from this paper. Other results on the bound states and resonances of Pauli-Fierz Hamiltonians were given recently in [HuSp2, BFS, AH, JP1, JP2, JP3, Sp2, Sp3, Sk].

Acknowledgements. We would like to thank V. Bach, J. Fröhlich and H. Spohn for useful discussions. The work of Jan Dereziński is a part of the project nr 2 P03 029 08 financed by Komitet Badań Naukowych in the years 1995-97.

2 Basic constructions in bosonic Fock spaces

2.1 Introduction

In this section we describe various general constructions related to bosonic Fock spaces, which we will use in our paper. In Subsections 2.2– 2.8 we recall various well known objects and their properties, such as field operators, the operators $\mathrm{d}\Gamma$ and Γ. In the remaining part of the section we introduce concepts that seem not to belong to the standard tools used in the literature, but nevertheless we think that they can be useful outside of our work.

Among the constructions that we present let us mention the operators $Q_k(f)$ and $P_k(f)$, used to define certain partitions of unity on the Fock space $\Gamma(\mathfrak{h})$, which have very useful positivity properties. Their use is one of the key ideas of the proof of the geometric asymptotic completeness, presented in Section 7.

We also describe operators $\check{\Gamma}(j)$, which map the Fock space $\Gamma(\mathfrak{h})$ into the doubled Fock space $\Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})$. The operators $\check{\Gamma}(j)$ are easily defined using the usual functor Γ and the identification of the spaces $\Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$ and $\Gamma(\mathfrak{h} \oplus \mathfrak{h})$.

One of the main tools used in the "geometric approach to scattering theory" is calculating the so-called Heisenberg derivative. It is therefore useful to introduce certain operators $\mathrm{d}\Gamma(q,r)$, $\mathrm{d}Q_k(f,g)$ and $\mathrm{d}\check{\Gamma}(j,k)$, which arise when one computes the Heisenberg derivative of $\Gamma(q)$, $Q_k(f)$ and $\check{\Gamma}(j)$ respectively.

Subsections 2.8 and 2.9 are devoted to the operators $\Gamma(q)$ and $\mathrm{d}\Gamma(q,r)$. Subsections 2.10 and 2.11 are devoted to the operators $Q_k(f)$ and $\mathrm{d}Q_k(f,g)$. Subsections 2.13, 2.14 are devoted to the operators $\check{\Gamma}(j)$, $\mathrm{d}\check{\Gamma}(j,k)$. In our exposition, we tried to present the properties of these objects stressing their analogies.

2.2 Bosonic Fock spaces

Let $\mathfrak{h}$ be a Hilbert space, which we will call the 1-particle space. Let $\otimes_{\mathrm{s}}^n\mathfrak{h}$ denote the symmetric nth tensor power of $\mathfrak{h}$. Let $\mathcal{S}_n$ denote the orthogonal projection of $\otimes^n\mathfrak{h}$ onto $\otimes_{\mathrm{s}}^n\mathfrak{h}$. We define the Fock space over $\mathfrak{h}$ to be the direct sum

$$\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n\mathfrak{h}.$$

Ω will denote the vacuum vector – the vector $1 \in \mathbb{C} = \otimes_{\mathrm{s}}^0\mathfrak{h}$. The number operator N is defined as

$$N\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}} = n\mathbb{1}.$$

The space of finite particle vectors, for which $\mathbb{1}_{[n,+\infty]}(N)u = 0$ for some $n \in \mathbb{N}$, will be denoted by $\Gamma_{\mathrm{fin}}(\mathfrak{h})$.

2.3 Creation and annihilation operators

If $h \in \mathfrak{h}$, we define the creation operator $a^*(h)$ by setting

$$\begin{aligned} &a^*(h) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}), \\ &a^*(h)u := \sqrt{n+1}\mathcal{S}_{n+1} h \otimes u, \qquad u \in \otimes_{\mathrm{s}}^n\mathfrak{h}. \end{aligned}$$

$a(h)$ denotes the adjoint of $a^*(h)$, and is called the annihilation operator. Both $a^*(h)$ and $a(h)$ are defined on $\Gamma_{\text{fin}}(\mathfrak{h})$ and can be extended to densely defined closed operators on $\Gamma(\mathfrak{h})$. By writing $a^\sharp(h)$ we will mean both $a^*(h)$ and $a(h)$. Note the canonical commutation relations:

$$\begin{aligned}
&[a(h_1), a^*(h_2)] = (h_1|h_2)\mathbb{1}, \\
&[a(h_2), a(h_1)] = [a^*(h_2), a^*(h_1)] = 0.
\end{aligned}$$

It follows from the boundedness of $[a(h), a^*(h)]$ that $a(h)$ and $a^*(h)$ have the same domain.

Lemma 2.1 *i)*

$$\left\|(N+1)^p \prod_{i=1}^{n} a^\sharp(h_i)(N+1)^{-p-\frac{n}{2}}\right\| \le C_{n,p} \prod_{i=1}^{n} \|h_i\|,$$

ii) the map

$$\mathfrak{h}^n \ni (h_1, \dots, h_n) \mapsto (N+1)^p \prod_{i=1}^{n} a^\sharp(h_i)(N+1)^{-p-\frac{n}{2}} \in B(\Gamma(\mathfrak{h}))$$

is norm continuous.
iii) If $\mathrm{w}-\lim_{i\to\infty} h_{ij} = 0$*, and* $h_{ij} \in \mathfrak{h}$ *are uniformly bounded, then*

$$\text{s-}\lim_{i\to\infty} (N+1)^p \prod_{j=1}^{n} a(h_{ij})(N+1)^{-p-\frac{n}{2}} = 0.$$

2.4 Field operators

We define the field operator

$$\phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h)),\ h \in \mathfrak{h}.$$

The operators $\phi(h)$ are essentially selfadjoint on $\Gamma_{\text{fin}}(\mathfrak{h})$ and can be extended to self-adjoint operators on $\Gamma(\mathfrak{h})$. We have

$$\begin{aligned}
&a^*(h) = \tfrac{1}{\sqrt{2}}(\phi(h) - \mathrm{i}\phi(\mathrm{i}h)), \\
&a(h) = \tfrac{1}{\sqrt{2}}(\phi(h) + \mathrm{i}\phi(\mathrm{i}h)), \\
&[\phi(h_1), \phi(h_2)] = \mathrm{i}Im(h_1|h_2).
\end{aligned}$$

The following proposition is useful when one tries to reconstruct creation-annihilation operators from field operators.

Proposition 2.2 *If q, p are self-adjoint operators on a Hilbert space $\mathcal{H}$ satisfying $[q,p] = \mathrm{i}\mathbb{1}$ in the sense of forms on $\mathcal{D}(q) \cap \mathcal{D}(p)$, then the operators*

$$a^* := \frac{1}{\sqrt{2}}(q - \mathrm{i}p), \qquad a := \frac{1}{\sqrt{2}}(q + \mathrm{i}p)$$

defined on $\mathcal{D}(q) \cap \mathcal{D}(p)$ are closed.

Proof. We have

$$\tfrac{1}{2}(\|qu\|^2 + \|pu\|^2) = \|a^*u\|^2 - \tfrac{1}{2} = \|au\|^2 + \tfrac{1}{2}.$$

$\mathcal{D}(q)$ is complete with the norm $\|qu\|$ and $\mathcal{D}(p)$ is complete with the norm $\|pu\|$. Hence $\mathcal{D}(q) \cap \mathcal{D}(p)$ is complete with the norm $\sqrt{\|qu\|^2 + \|pu\|^2}$. $\square$

Lemma 2.3 *i)*

$$\|(N+1)^p \prod_{i=1}^{n} \phi(h_i)(N+1)^{-p-n/2}\| \leq C_{n,p} \prod_{i=1}^{n} \|h_i\|.$$

ii) The map

$$\mathfrak{h}^n \ni (h_1, \ldots, h_n) \mapsto (N+1)^p \prod_{i=1}^{n} \phi(h_i)(N+1)^{-p-n/2}$$

is continuous for the norm topology.

2.5 Weyl operators

We introduce also the Weyl operators:

$$W(h) := \mathrm{e}^{\mathrm{i}\phi(h)}.$$

Note the identities:

$$\begin{aligned} &[\phi(h), W(g)] = Im(g|h)W(g), \\ (2.1) \qquad &W(g)\phi(h)W(-g) = \phi(h) - Im(g|h), \\ &W(h)W(g) = \mathrm{e}^{-\mathrm{i}\frac{1}{2}Im(h|g)}W(h+g). \end{aligned}$$

Proposition 2.4 *i) For* $0 \leq \epsilon \leq 1$

$$\|(W(h) - \mathbb{1})u\| \leq C_\epsilon \||\phi(h)|^\epsilon u\|.$$

ii) the map

$$\mathbb{R} \ni s \mapsto W(sh)(N+1)^{-\frac{1}{2}}$$

is C^1 in the strong topology and the map

$$\mathbb{R} \ni s \mapsto W(sh)(N+1)^{-\frac{1}{2}-\epsilon}$$

is C^1 in the norm topology. More precisely,

$$\lim_{s\to 0} \sup_{\|h\|\leq C} s^{-1}\left\|(W(sh) - \mathbb{1} - \mathrm{i}s\phi(h))(N+1)^{-1/2-\epsilon}\right\| = 0.$$

iii)

$$\|(W(h_1) - W(h_2))u\| \leq C_\epsilon \|h_1 - h_2\|^\epsilon \Big((\|h_1\|^2 + \|h_2\|^2)^{\frac{\epsilon}{2}} \|u\| + \|(N+1)^{\frac{\epsilon}{2}} u\| \Big).$$

Proof. *i)* follows from the spectral theorem and the inequality

$$|\mathrm{e}^{\mathrm{i}s} - 1| \leq C_\epsilon |s|^\epsilon.$$

ii) follows from Lemma 2.3. To show *iii)* we note that

$$\begin{aligned} W(h_1) - W(h_2) &= W(h_1)(\mathbb{1} - \mathrm{e}^{-\frac{\mathrm{i}}{2}Im(h_1|h_2)}) \\ &\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}Im(h_1|h_2)} W(h_1)(\mathbb{1} - W(h_2 - h_1)). \end{aligned}$$

We note also that

$$|1 - \mathrm{e}^{-\frac{\mathrm{i}}{2}Im(h_1|h_2)}| \leq C_\epsilon |Im(h_1|h_2)|^\epsilon,$$

$$|Im(h_1|h_2)| \leq \frac{1}{\sqrt{2}} \|h_1 - h_2\| \sqrt{\|h_1\|^2 + \|h_2\|^2},$$

and by *i)*

$$\|(\mathbb{1} - W(h_2 - h_1))u\| \leq C_\epsilon \| |\phi(h_2 - h_1)|^\epsilon u\| \leq C_\epsilon \|h_2 - h_1\|^\epsilon \|(N+1)^{\frac{\epsilon}{2}} u\|.$$

□

2.6 Operator $\mathrm{d}\Gamma$

If b is an operator on $\mathfrak{h}$, we define the operator

$$\mathrm{d}\Gamma(b) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}),$$

$$\mathrm{d}\Gamma(b)\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}} := \sum_{j=1}^n \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{j-1} \otimes b \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{n-j}.$$

An important example is the number operator

$$N = \mathrm{d}\Gamma(1).$$

Lemma 2.5 *i) Heisenberg derivatives:*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\Gamma(b) = \mathrm{d}\Gamma(\tfrac{\mathrm{d}}{\mathrm{d}t}b),$$

$$[\mathrm{d}\Gamma(b_1), \mathrm{d}\Gamma(b_2)] = \mathrm{d}\Gamma([b_1, b_2]).$$

ii) Commutation properties:

$$[\mathrm{d}\Gamma(b), a^*(h)] = a^*(bh),$$

$$[\mathrm{d}\Gamma(b), a(h)] = -a(b^*h),$$

$$[\mathrm{d}\Gamma(b), \mathrm{i}\phi(h)] = \phi(\mathrm{i}bh), \ \textit{if } b = b^*$$

$$W(h)\mathrm{d}\Gamma(b)W(-h) = \mathrm{d}\Gamma(b) - \phi(\mathrm{i}bh) - \tfrac{1}{2}Re(bh|h) \ \textit{if } b = b^*.$$

iii) If $b_1 \leq b_2$, then $\mathrm{d}\Gamma(b_1) \leq \mathrm{d}\Gamma(b_2)$. Moreover,

$$\|N^{-\frac{1}{2}}\mathrm{d}\Gamma(b)u\| \leq \|\mathrm{d}\Gamma(b^*b)^{\frac{1}{2}}u\|.$$

2.7 Tensor product of Fock spaces

Let $\mathfrak{h}_i$, $i = 1, 2$ be Hilbert spaces. Let p_i be the projection of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ onto $\mathfrak{h}_i$, $i = 1, 2$. We define

$$U : \Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \to \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2),$$

by

$$\begin{aligned} &U\Omega = \Omega \otimes \Omega, \\ &Ua^\sharp(h) = \left(a^\sharp(p_1 h) \otimes \mathbb{1} + \mathbb{1} \otimes a^\sharp(p_2 h)\right) U, \ h \in \mathfrak{h}_1 \oplus \mathfrak{h}_2. \end{aligned} \tag{2.2}$$

Since the vectors $a^*(h_1) \cdots a^*(h_n)\Omega$ form a total family in $\Gamma(\mathfrak{h})$, and since U preserves the canonical commutation relations, we see that U extends as a unitary operator from $\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ to $\Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2)$. Moreover one has the following identity:

$$U \mathrm{d}\Gamma\left(\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}\right) = \left(\mathrm{d}\Gamma(b_1) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(b_2)\right) U. \tag{2.3}$$

It is easy to check that on $\otimes_{\mathrm{s}}^n(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$, U is given by

$$U\Big|_{\otimes_{\mathrm{s}}^n(\mathfrak{h}_1 \oplus \mathfrak{h}_2)} = \sum_{k=0}^{n} \sqrt{\frac{n!}{(n-k)!k!}}\, \underbrace{p_1 \otimes \cdots \otimes p_1}_{n-k} \otimes \underbrace{p_2 \otimes \cdots \otimes p_2}_{k}.$$

2.8 Functor Γ

Let $\mathfrak{h}_i, i = 1, 2$ be Hilbert spaces. Let $q : \mathfrak{h}_1 \mapsto \mathfrak{h}_2$ be a bounded linear operator. We define

$$\begin{aligned} &\Gamma(q) : \Gamma(\mathfrak{h}_1) \mapsto \Gamma(\mathfrak{h}_2) \\ &\Gamma(q)\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}_1} = q \otimes \cdots \otimes q. \end{aligned}$$

The Γ functor has the following properties:

Lemma 2.6 *i) Relationship with* $\mathrm{d}\Gamma$*: assume* $\mathfrak{h}_1 = \mathfrak{h}_2$*. Then*

$$\mathrm{e}^{\mathrm{d}\Gamma(b)} = \Gamma(\mathrm{e}^b).$$

ii) Intertwining properties:

$$\begin{aligned} \Gamma(q)a^*(h_1) &= a^*(qh_1)\Gamma(q), \quad h_1 \in \mathfrak{h}_1, \\ \Gamma(q)a(q^*h_2) &= a(h_2)\Gamma(q), \quad h_2 \in \mathfrak{h}_2. \end{aligned}$$

iii) Commutation properties: assume $\mathfrak{h}_1 = \mathfrak{h}_2$*. Then*

$$\begin{aligned} [a^*(h), \Gamma(q)] &= a^*((1-q)h)\Gamma(q), \\ [a(h), \Gamma(q)] &= -\Gamma(q)a((1-q^*)h). \end{aligned}$$

iv) If $\|q\| \leq 1$*, then*

$$\|\Gamma(q)\| \leq 1.$$

Let us note some additional properties in the isometric and unitary cases.

Lemma 2.7 *i) If q is isometric, that is $q^*q = 1$, then*

$$\Gamma(q)a^\sharp(h_1) = a^\sharp(qh_1)\Gamma(q),$$

$$\Gamma(q)\phi(h_1) = \phi(qh_1)\Gamma(q).$$

ii) If q is unitary, then

$$\Gamma(q)a^\sharp(h)\Gamma(q^{-1}) = a^\sharp(qh),$$

$$\Gamma(q)\phi(h)\Gamma(q^{-1}) = \phi(qh).$$

2.9 Operator $\mathrm{d}\Gamma(q,r)$

Let q, r be operators from $\mathfrak{h}_1$ to $\mathfrak{h}_2$. We define

$$\mathrm{d}\Gamma(q,r) : \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2),$$

$$\mathrm{d}\Gamma(q,r)\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}_1} = \sum_{j=1}^n \underbrace{q\otimes\cdots\otimes q}_{j-1}\otimes r\otimes \underbrace{q\otimes\cdots\otimes q}_{n-j}.$$

Lemma 2.8 *i) Relationship with $\mathrm{d}\Gamma$ and Γ:*

$$\mathrm{d}\Gamma(1,r) = \mathrm{d}\Gamma(r),$$

$$\mathrm{d}\Gamma(r,r) = N\Gamma(r).$$

If q is invertible, then

$$\mathrm{d}\Gamma(q,r) = \mathrm{d}\Gamma(rq^{-1})\Gamma(q) = \Gamma(q)\mathrm{d}\Gamma(q^{-1}r).$$

ii) Heisenberg derivatives of $\Gamma(q)$:

$$\mathrm{d}\Gamma(b_2)\Gamma(q) - \Gamma(q)\mathrm{d}\Gamma(b_1) = \mathrm{d}\Gamma(q, b_2q - qb_1),$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Gamma(q) = d\Gamma(q, \tfrac{\mathrm{d}}{\mathrm{d}t}q).$$

iii) Intertwing properties:

$$a(h_2)\mathrm{d}\Gamma(q,r) = \mathrm{d}\Gamma(q,r)a(q^*h_1) + \Gamma(q)a(r^*h_1),$$

$$\mathrm{d}\Gamma(q,r)a^*(h_1) = a^*(qh_1)\mathrm{d}\Gamma(q,r) + a^*(rh_1)\Gamma(q).$$

iv) Commutation properties: assume $\mathfrak{h}_1 = \mathfrak{h}_2$. Then

$$[a(h), \mathrm{d}\Gamma(q,r)] = -\mathrm{d}\Gamma(q,r)a((1-q^*)h) + \Gamma(q)a(r^*h),$$

$$[a^*(h), \mathrm{d}\Gamma(q,r)] = a^*((1-q)h)\mathrm{d}\Gamma(q,r) - a^*(rh)\Gamma(q).$$

v) If $\|q\| \leq 1$ then we have the following estimate:

$$|(u_2|\mathrm{d}\Gamma(q, r_2r_1)u_1)| \leq \|\mathrm{d}\Gamma(r_2r_2^*)^{\frac{1}{2}}u_2\|\|\mathrm{d}\Gamma(r_1^*r_1)^{\frac{1}{2}}u_1\|.$$

vi) If $\|q\| \leq 1$ then

$$\|N^{-\frac{1}{2}}\mathrm{d}\Gamma(q,r)u\| \leq \|\mathrm{d}\Gamma(r^*r)^{\frac{1}{2}}u\|.$$

Proof. Let us indicate the proof of parts *v)* and *vi)*, the other being elementary. For an operator r acting on $\mathfrak{h}$, we set

$$r_j := \underbrace{1\!\!1 \otimes \cdots \otimes 1\!\!1}_{j-1} \otimes r \otimes \underbrace{1\!\!1 \otimes \cdots \otimes 1\!\!1}_{n-j}, \text{ acting on } \otimes_{\mathrm{s}}^n \mathfrak{h}.$$

For $u_i \in \bigotimes_{\mathrm{s}}^n \mathfrak{h}_i$, we have:

$$|(u_2|\mathrm{d}\Gamma(q, r_2 r_1)u_1)| \leq \sum_{j=1}^{n} \|(r_2 r_2^*)_j^{\frac{1}{2}} u_2\| \|(r_1^* r_1)_j^{\frac{1}{2}} u_1\|,$$

since $\|q\| \leq 1$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{j=1}^{n} \|(r_2 r_2^*)_j^{\frac{1}{2}} u_2\| \|(r_1^* r_1)_j^{\frac{1}{2}} u_1\| &\leq \left(\sum_{j=1}^{n} (u_2|(r_2 r_2^*)_j u_2)\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} (u_1|(r_1^* r_1)_j u_1)\right)^{\frac{1}{2}} \\ &= \|\mathrm{d}\Gamma(r_2 r_2^*)^{\frac{1}{2}} u_2\| \|\mathrm{d}\Gamma(r_1^* r_1)^{\frac{1}{2}} u_1\|,\end{aligned}$$

which proves *v)*. To prove *vi)*, we have for $u \in \bigotimes_{\mathrm{s}}^n \mathfrak{h}$:

$$\|\mathrm{d}\Gamma(q, r)u\| \leq \sum_{j=1}^{n} \|r_j u\| \leq n^{\frac{1}{2}} \|\mathrm{d}\Gamma(r^* r)^{\frac{1}{2}} u\|,$$

again by the Cauchy-Schwarz inequality. □

2.10 Operators P_k and Q_k

Let f_0, f_∞ be operators from $\mathfrak{h}_1$ to $\mathfrak{h}_2$. Let $f := (f_0, f_\infty)$. We define the operators $P_k(f) = P_k(f_0, f_\infty)$ and $Q_k(f) = Q_k(f_0, f_\infty)$ for $k \in \mathbb{N}$ by setting

$$\begin{aligned}&P_k(f) : \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2), \\ &P_k(f)\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}_1} := \sum_{\sharp\{i|\epsilon_i = \infty\} = k} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_n}, \\ &Q_k(f) : \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2), \\ &Q_k(f)\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}_1} := \sum_{\sharp\{i|\epsilon_i = \infty\} \leq k} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_n},\end{aligned}$$

where $\epsilon_i = 0, \infty$. The following properties of $Q_k(f), P_k(f)$ can be verified by direct inspection.

Lemma 2.9 *i)*

$$\begin{aligned}&P_1(f) = \mathrm{d}\Gamma(f_0, f_\infty), \\ &Q_k(f) = \sum_{j=0}^{k} P_j(f), \qquad P_k(f) = Q_k(f) - Q_{k-1}(f), \\ &P_0(f) = Q_0(f) = \Gamma(f_0), \\ &P_k(qf) = \Gamma(q) P_k(f), \qquad Q_k(qf) = \Gamma(q) Q_k(f).\end{aligned}$$

ii) Intertwining properties (we set $Q_{-1}(f)=0$):

$$Q_k(f)a^*(h_1) = a^*(f_0h_1)Q_k(f) + a^*(f_\infty h_1)Q_{k-1}(f),$$
$$a(h_2)Q_k(f) = Q_k(f)a(f_0^*h_2) + Q_{k-1}(f)a(f_\infty^*h_2).$$

iii) Commutation properties: assume $\mathfrak{h}_1=\mathfrak{h}_2$. Then

$$[a(h), Q_k(f)] = -Q_k(f)a((1-f_0^*)h) + Q_{k-1}(f)a(f_\infty^*h),$$
$$[a^*(h), Q_k(f)] = a^*((1-f_0)h)Q_k(f) - a^*(f_\infty h)Q_{k-1}(f).$$

iv) Assume $\mathfrak{h}_1=\mathfrak{h}_2$. If $0\le f_0$, $0\le f_\infty$, $f_0+f_\infty\le 1$, then

$$0\le Q_k(f)\le \Gamma(f_0+f_\infty), \qquad 0\le P_k(f)\le\Gamma(f_0+f_\infty).$$

Proposition 2.10 *Let $f=(f_0,f_\infty)$ and $\tilde f=(\tilde f_0,\tilde f_\infty)$ and $\tilde f_0 f_\infty=0$. Then*

$$Q_l(\tilde f)P_k(f)=0, \qquad l<k, \tag{2.1}$$

$$Q_k(\tilde f)P_k(f) = P_k(\tilde f)P_k(f) = P_k(\tilde f_0 f_0, \tilde f_\infty f_\infty), \tag{2.2}$$

$$\begin{aligned} Q_l(\tilde f)Q_k(f) &= Q_l(\tilde f_0f_0, \tilde f_\infty(f_0+f_\infty)),\\ P_l(\tilde f)Q_k(f) &= P_l(\tilde f_0f_0, \tilde f_\infty(f_0+f_\infty)),\end{aligned} \qquad l\le k. \tag{2.3}$$

2.11 Operator $\mathrm{d}Q_k(f,g)$

For $f=(f_0,f_\infty)$ and $g=(g_0,g_\infty)$ we define

$$\begin{aligned}\mathrm{d}Q_k(f,g):\ &\ \Gamma(\mathfrak{h}_1)\to\Gamma(\mathfrak{h}_2),\\ \mathrm{d}Q_k(f,g)\Big|_{\bigotimes_s^n \mathfrak{h}_1} &:= \sum_{j=1}^n \sum_{\sharp\{i|\epsilon_i=\infty\}\le k} f_{\epsilon_1}\otimes\cdots\otimes f_{\epsilon_{j-1}}\otimes g_0\otimes f_{\epsilon_{j+1}}\otimes\cdots\otimes f_{\epsilon_n}\\ &+\sum_{j=1}^n \sum_{\sharp\{i|\epsilon_i=\infty\}\le k-1} f_{\epsilon_1}\otimes\cdots\otimes f_{\epsilon_{j-1}}\otimes g_\infty\otimes f_{\epsilon_{j+1}}\otimes\cdots\otimes f_{\epsilon_n}.\end{aligned}$$

Lemma 2.11 *i)*

$$\mathrm{d}Q_0(f,g) = \mathrm{d}\Gamma(f_0,g_0).$$

ii) Heisenberg derivatives of $Q_k(f)$:

$$\mathrm{d}\Gamma(d_2)Q_k(f) - Q_k(f)\mathrm{d}\Gamma(d_1) = \mathrm{d}Q_k(f, d_2f - fd_1),$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}Q_k(f) = \mathrm{d}Q_k(f,\tfrac{\mathrm{d}}{\mathrm{d}t}f).$$

iii) Intertwining properties:

$$\begin{aligned} a(h_2)\mathrm{d}Q_k(f,g) &= \mathrm{d}Q_k(f,g)a(f_0^*h_2) + \mathrm{d}Q_{k-1}(f,g)a(f_\infty^*h_2)\\ &+Q_k(f)a(g_0^*h_2) + Q_{k-1}(f)a(g_\infty^*h_2),\\ \mathrm{d}Q_k(f,k)a^*(h_1) &= a^*(f_0h_1)\mathrm{d}Q_k(f,g) + a^*(f_\infty h_1)\mathrm{d}Q_{k-1}(f,g)\\ &+a^*(g_0h_1)Q_k(f) + a^*(g_\infty h_1)Q_{k-1}(f).\end{aligned}$$

iv) Commutation properties: assume $\mathfrak{h}_1 = \mathfrak{h}_2$. *Then*

$$\begin{aligned}
[a(h), \mathrm{d}Q_k(f,g)] &= -\mathrm{d}Q_k(f,g)a((1-f_0^*)h) + \mathrm{d}Q_{k-1}(f,g)a(f_\infty^* h) \\
&\quad + Q_k(f)a(g_0^* h) + Q_{k-1}(f)a(g_\infty^* h), \\
[a^*(h), \mathrm{d}Q_k(f,g)] &= a^*(1-f_0)h)\mathrm{d}Q_k(f,g) - a^*(f_\infty h)\mathrm{d}Q_{k-1}(f,g) \\
&\quad - a^*(g_0 h)Q_k(f) - a^*(g_\infty h)Q_{k-1}(f).
\end{aligned}$$

v) If $\mathfrak{h}_1 = \mathfrak{h}_2$, $0 \leq f_0$, $0 \leq f_\infty$, $f_0 + f_\infty \leq 1$, g_0, g_∞ *are selfadjoint, then*

$$|(u_2|\mathrm{d}Q_k(f,g)u_1)| \leq \|\mathrm{d}\Gamma(|g_0|)^{\frac{1}{2}}u_2\|\|\mathrm{d}\Gamma(|g_0|)^{\frac{1}{2}}u_1\| + \|\mathrm{d}\Gamma(|g_\infty|)^{\frac{1}{2}}u_2\|\|\mathrm{d}\Gamma(|g_\infty|)^{\frac{1}{2}}u_1\|.$$

vi) If $\mathfrak{h}_1 = \mathfrak{h}_2$, $0 \leq f_0$, $0 \leq f_\infty$, $f_0 + f_\infty \leq 1$, *then we have the estimates*

$$\|N^{-\frac{1}{2}}\mathrm{d}Q_k(f,g)u\| \quad \leq \|\mathrm{d}\Gamma(g_0^* g_0 + g_\infty^* g_\infty)^{\frac{1}{2}}u\|.$$

Proof. As for Lemma 2.8, we content ourselves to indicate the proofs of parts *v) and vi)*, the rest of the lemma being easy to check. To prove *v)*, we write

$$\mathrm{d}Q_k(f,g) = \sum_{j=1}^n M_{j,0}g_{0,j} + M_{j,\infty}g_{\infty,j},$$

where

$$\begin{aligned}
M_{j,0} &= \sum_{\sharp\{i|\epsilon_i=\infty\}=k} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_{j-1}} \otimes 1\!\!1 \otimes f_{\epsilon_{j+1}} \otimes \cdots \otimes f_{\epsilon_n}, \\
M_{j,\infty} &= \sum_{\sharp\{i|\epsilon_i=\infty\}=k-1} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_{j-1}} \otimes 1\!\!1 \otimes f_{\epsilon_{j+1}} \otimes \cdots \otimes f_{\epsilon_n}.
\end{aligned}$$

Since $f_0 + f_\infty \leq 1$, we have $\|M_{j,0}\| \leq 1, \|M_{j,\infty}\| \leq 1$. Then we argue as in the proof of Lemma 2.8, writing $g_\epsilon = g_{2,\epsilon}g_{1,\epsilon}$ for $g_{1,\epsilon} = |g_\epsilon|^{\frac{1}{2}}$, $g_{2,\epsilon} = \mathrm{sgn} g_\epsilon |g_\epsilon|^{\frac{1}{2}}$. A similar argument gives the proof of *vi)*, following the proof of Lemma 2.8 *vi)*. □

2.12 Partitions of unity

In this subsection we further study the functors P_k, Q_k under the additional assumption

$$\mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}, \qquad f_0 + f_\infty = 1.$$

Lemma 2.12 *i) If* $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}$, $0 \leq f_0$, $0 \leq f_\infty$, $f_0 + f_\infty = 1$ *then the operators* $P_k(f)$ *form a partition of unity on* $\Gamma(\mathfrak{h})$:

$$\text{s-}\lim_{k\to\infty} Q_k(f) = 1\!\!1, \qquad \mathrm{s} - \sum_{k=0}^{\infty} P_k(f) = 1\!\!1.$$

ii) Intertwining properties:

$$a(h)Q_k(f) = Q_k(f)a(h) - P_k(f)a(f_\infty h) = Q_{k-1}(f)a(h) + P_k(f)a(f_0 h),$$

$$Q_k(f)a^*(h) = a^*(h)Q_k(f) - a^*(f_\infty h)P_k(f) = a^*(h)Q_{k-1}(f) + a^*(f_0 h)P_k(f).$$

iii) Commutation properties:

$$[a(h), Q_k(f)] = -P_k(f)a(f_\infty h),$$

$$[a^*(h), Q_k(f)] = a^*(f_\infty h)P_k(f).$$

Finally, the operators $P_k(f)$ and $Q_k(f)$ have other special properties, which will play an imporant role in our geometric analysis of scattering.

Proposition 2.13 *Let* $f_0 + f_\infty = 1$, $\tilde{f}_0 + \tilde{f}_\infty = 1$.
i) Let $\tilde{f}_0 f_\infty = 0$. *Then for* $l \leq k$

$$Q_l(\tilde{f})Q_k(f) = Q_l(\tilde{f}),$$

$$P_l(\tilde{f})Q_k(f) = P_l(\tilde{f}).$$

ii) If $0 \leq f_0 \leq \tilde{f}_0 \leq 1\!\!1$, *then*

$$Q_k(f) \leq Q_k(\tilde{f}).$$

Proof. *i)* follows from Prop. 2.10. Let us prove *ii)*. Note that if $f = (f_0, f_\infty)$ satisfies $f_0 + f_\infty = 1$, and depends on some parameter s then

$$\frac{\mathrm{d}}{\mathrm{d}s}f_0 = -\frac{\mathrm{d}}{\mathrm{d}s}f_\infty, \quad [b, f_0] = -[b, f_\infty].$$

We observe now that the operator $\mathrm{d}Q_k(f, g)$ under the condition

$$g_\infty = -g_0 \tag{2.4}$$

has a simpler form:

$$\mathrm{d}Q_k(f, g)\Big|_{\bigotimes_s^n \mathfrak{h}} = \sum_{j=1}^{n} \sum_{\sharp\{i|\epsilon_i=\infty\}=k} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_{j-1}} \otimes g_0 \otimes f_{\epsilon_{j+1}} \otimes \cdots \otimes f_{\epsilon_n} \tag{2.5}$$

Clearly, (2.5) is nonnegative if $f_0 \geq 0$, $f_\infty \geq 0$ and $g_0 \geq 0$. Now to prove *ii)*, we set

$$f^s := (1-s)f + s\tilde{f}, \; s \in [0, 1].$$

We have:

$$\frac{\mathrm{d}}{\mathrm{d}s}Q_k(f_0^s) = \mathrm{d}Q_k(f^s, (\tilde{f}_0 - f_0, f_0 - \tilde{f}_0)) \geq 0,$$

by (2.5). This completes the proof of *ii)*. □

2.13 Operator $\check{\Gamma}$

Along with the space $\Gamma(\mathfrak{h})$ we will consider the space $\Gamma(\mathfrak{h}\oplus\mathfrak{h}) \simeq \Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})$. We will use the notation

$$N_0 := N\otimes 1\!\!1, \qquad N_\infty := 1\!\!1\otimes N.$$

Let j_0, j_∞ be two operators on $\mathfrak{h}$. Set $j = (j_0, j_\infty)$. We identify j with the operator

$$\begin{aligned} &j : \mathfrak{h} \to \mathfrak{h}\oplus\mathfrak{h},\\ &jh := (j_0 h, j_\infty h). \end{aligned}$$

We have

$$\begin{aligned} &j^* : \mathfrak{h}\oplus\mathfrak{h} \to \mathfrak{h},\\ &j^*(h_0, h_\infty) = j_0^* h_0 + j_\infty^* h_\infty, \end{aligned}$$

and

$$j^* j = j_0^* j_0 + j_\infty^* j_\infty.$$

By second quantization, we obtain the map

$$\Gamma(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}\oplus\mathfrak{h}).$$

Let U denote the unitary operator identifying $\Gamma(\mathfrak{h}\oplus\mathfrak{h})$ with $\Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})$ introduced in Subsection 2.7. We define

$$\begin{aligned} &\check{\Gamma}(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h}),\\ &\check{\Gamma}(j) := U\Gamma(j). \end{aligned}$$

Another formula defining $\check{\Gamma}(j)$ is

$$\check{\Gamma}(j)\Pi_{i=1}^n a^*(h_i)\Omega := \Pi_{i=1}^n \left(a^*(j_0 h_i)\otimes 1\!\!1 + 1\!\!1\otimes a^*(j_\infty h_i)\right)\Omega\otimes\Omega, h_i \in \mathfrak{h}. \tag{2.6}$$

Finally, if we denote by I_k the natural isometry between $\bigotimes^n \mathfrak{h}$ and $\bigotimes^{n-k}\mathfrak{h}\otimes\bigotimes^k\mathfrak{h}$, then we have:

$$1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(j)\Big|_{\bigotimes_s^n \mathfrak{h}} = I_k\sqrt{\tfrac{n!}{(n-k)!k!}}\,\underbrace{j_0\otimes\cdots\otimes j_0}_{n-k}\otimes\underbrace{j_\infty\otimes\cdots\otimes j_\infty}_{k}.$$

Lemma 2.14 *i)*

$$\begin{aligned} &\check{\Gamma}(\tilde{j})^* 1\!\!1_{\{1,\dots,k\}}(N_\infty)\check{\Gamma}(j) = Q_k(\tilde{j}_0^* j_0, \tilde{j}_\infty^* j_\infty),\\ &\check{\Gamma}(\tilde{j})^* 1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(j) = P_k(\tilde{j}_0^* j_0, \tilde{j}_\infty^* j_\infty). \end{aligned}$$

ii) Intertwining properties:

$$\begin{aligned} &\check{\Gamma}(j)a^*(h) = \left(a^*(j_0 h)\otimes 1\!\!1 + 1\!\!1\otimes a^*(j_\infty h)\right)\check{\Gamma}(j),\\ &\check{\Gamma}(j)a(j_0^* h) = a(h)\otimes 1\!\!1\check{\Gamma}(j),\\ &\check{\Gamma}(j)a(j_\infty^* h) = 1\!\!1\otimes a(h)\check{\Gamma}(j). \end{aligned}$$

iii) Commutation properties:

$$\begin{aligned} &(a^*(h)\otimes 1\!\!1)\check{\Gamma}(j) - \check{\Gamma}(j)a^*(h) = (a^*((1-j_0)h)\otimes 1\!\!1 - 1\!\!1\otimes a^*(j_\infty h))\check{\Gamma}(j),\\ &(a(h)\otimes 1\!\!1)\check{\Gamma}(j) - \check{\Gamma}(j)a(h) = -\check{\Gamma}(j)a((1-j_0^*)h). \end{aligned}$$

iv) $\check{\Gamma}(j)$ *is bounded iff* $\|j_0^* j_0 + j_\infty^* j_\infty\| \leq 1$, *and then*

$$\|\check{\Gamma}(j)\| = 1.$$

Proof. *i)* is a direct computation. *ii)–iv)* follow from Subsects. 2.7, 2.8. □

Let us note some additional properties of $\check{\Gamma}$ in the isometric case.

Lemma 2.15 *Assume*

$$j_0^* j_0 + j_\infty^* j_\infty = 1. \tag{2.7}$$

(This assumption implies that j is isometric, that is $j^* j = 1$). Then

i)

$$\check{\Gamma}(j)^* \check{\Gamma}(j) = 1\!\!1.$$

ii) Intertwining properties:

$$\check{\Gamma}(j) a^\sharp(h) = \left(a^\sharp(j_0 h) \otimes 1\!\!1 + 1\!\!1 \otimes a^\sharp(j_\infty h)\right) \check{\Gamma}(j),$$

$$\check{\Gamma}(j) \phi(h) = \left(\phi(j_0 h) \otimes 1\!\!1 + 1\!\!1 \otimes \phi(j_\infty h)\right) \check{\Gamma}(j).$$

iii) Let b be an operator on $\mathfrak{h}$. Then

$$\mathrm{d}\Gamma(b) = \check{\Gamma}(j)^* \left(\mathrm{d}\Gamma(b) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(b)\right) \check{\Gamma}(j) + \tfrac{1}{2} \mathrm{d}\Gamma(\mathrm{ad}_{j_0}^2 b + \mathrm{ad}_{j_\infty}^2 b).$$

Proof. *i)* and *ii)* are direct consequences of Lemma 2.14. Property *iii)* is a kind of IMS localization formula which is shown by direct computation. □

2.14 Operator $\mathrm{d}\check{\Gamma}(j,k)$

Let $j = (j_0, j_\infty)$, $k = (k_0, k_\infty)$ be maps from $\mathfrak{h}$ to $\mathfrak{h} \oplus \mathfrak{h}$. Let U be the operator constructed in Subsect. 2.7. We set

$$\mathrm{d}\check{\Gamma}(j,k) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}),$$

$$\mathrm{d}\check{\Gamma}(j,k) := U \mathrm{d}\Gamma(j,k).$$

The operator $\mathrm{d}\check{\Gamma}(1,k) = U \mathrm{d}\Gamma(k)$ will be denoted simply by $\mathrm{d}\check{\Gamma}(k)$.

Lemma 2.16 *i) Heisenberg derivative of $\check{\Gamma}(j)$:*

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \check{\Gamma}(j) = \mathrm{d}\check{\Gamma}(j, \tfrac{\mathrm{d}}{\mathrm{d}t} j),$$

$$\left(\mathrm{d}\Gamma(b) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(b)\right) \check{\Gamma}(j) - \check{\Gamma}(j) \mathrm{d}\Gamma(b) = \mathrm{d}\check{\Gamma}(j, \check{\mathrm{ad}}_b(j)).$$

Here b is an operator on $\mathfrak{h}$ and

$$\check{\mathrm{ad}}_b(j) : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h},$$

$$\check{\mathrm{ad}}_b(j) h := ([b, j_0] h, [b, j_\infty] h).$$

ii) Intertwining properties:

$$a(h)\otimes 1\!\!1 \mathrm{d}\check{\Gamma}(j,k) = \mathrm{d}\check{\Gamma}(j,k)a(j_0^*h) + \check{\Gamma}(j)a(k_0^*h),$$

$$(a^*(j_0h)\otimes 1\!\!1 + 1\!\!1\otimes a^*(j_\infty h))\mathrm{d}\check{\Gamma}(j,k) + (a^*(k_0h)\otimes 1\!\!1 + 1\!\!1\otimes a^*(k_\infty h))\check{\Gamma}(j) = \mathrm{d}\check{\Gamma}(j,k)a^*(h).$$

iii) Commutation properties:

$$a(h)\otimes 1\!\!1 \mathrm{d}\check{\Gamma}(j,k) - \mathrm{d}\check{\Gamma}(j,k)a(h) = -\mathrm{d}\check{\Gamma}(j,k)a((1-j_0^*)h) + \check{\Gamma}(j)a(k_0^*h),$$

$$\begin{array}{rl} a^*(h)\otimes 1\!\!1 \mathrm{d}\check{\Gamma}(j,k) - \mathrm{d}\check{\Gamma}(j,k)a^*(h) & = (a^*((1-j_0)h)\otimes 1\!\!1 - 1\!\!1\otimes a^*(j_\infty h))\mathrm{d}\check{\Gamma}(j,k) \\ & \quad -(a^*(k_0h)\otimes 1\!\!1 + 1\!\!1\otimes a^*(k_\infty h))\check{\Gamma}(j). \end{array}$$

iv) If $j_0^*j_0 + j_\infty^*j_\infty \leq 1$, k_0, k_∞ are self-adjoint, we have the estimate:

$$\begin{array}{rl} |(u_2|\mathrm{d}\check{\Gamma}(j,k)u_1)| & \leq \|\mathrm{d}\Gamma(|k_0|)^{\frac{1}{2}}\otimes 1\!\!1 u_2\|\|\mathrm{d}\Gamma(|k_0|)^{\frac{1}{2}}u_1\| \\ & \quad +\|1\!\!1\otimes \mathrm{d}\Gamma(|k_\infty|)^{\frac{1}{2}}u_2\|\|\mathrm{d}\Gamma(|k_\infty|)^{\frac{1}{2}}u_1\|. \end{array}$$

v) If $j_0^*j_0 + j_\infty^*j_\infty \leq 1$, then

$$\|(N_0+N_\infty)^{-\frac{1}{2}}\mathrm{d}\check{\Gamma}(j,k)u\| \leq \|\mathrm{d}\Gamma(k_0^*k_0 + k_\infty^*k_\infty)^{\frac{1}{2}}u\|.$$

Proof. All statements follow directly from analogous statements in Lemma 2.8 and from the identities in Subsect. 2.7. The only point which deserve some care is *iv)*. To prove *iv)*, we write $k = k^0 + k^\infty$, where $k^0 = (k_0, 0), k^\infty = (0, k_\infty)$, and use Lemma 2.8 *v)*, writing k^0 as r_2r_1 with $r_2 = (|k_0|^{\frac{1}{2}}, 0)$, $r_1 = \mathrm{sgn}k_0|k_0|^{\frac{1}{2}}$, and k_∞ as r_2r_1 with $r_2 = (0, |k_\infty|^{\frac{1}{2}})$, $r_1 = \mathrm{sgn}k_\infty|k_\infty|^{\frac{1}{2}}$. $\Box$

2.15 Scattering identification operators

Let

$$\begin{array}{c} i : \mathfrak{h}\oplus\mathfrak{h} \to \mathfrak{h}, \\ (h_0, h_\infty) \mapsto h_0 + h_\infty. \end{array}$$

An important role in scattering theory is played by the following identification operator (see [HuSp1]):

$$I := \Gamma(i)U^* = \check{\Gamma}(i^*)^* : \Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}).$$

Note that since $\|i\| = \sqrt{2}$, the operator $\Gamma(i)$ is unbounded.

Another formula defining I is:

$$I \prod_{i=1}^{n} a^*(h_i)\Omega \otimes \prod_{i=1}^{p} a^*(g_i)\Omega := \prod_{i=1}^{p} a^*(g_i) \prod_{i=1}^{n} a^*(h_i)\Omega, \quad h_i, g_i \in \mathfrak{h}. \tag{2.8}$$

If $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$, then we can write still another formula for I:

$$Iu\otimes\psi = \frac{1}{(p!)^{\frac{1}{2}}}\int \psi(k_1,\cdots,k_p)a^*(k_1)\cdots a^*(k_p)u\mathrm{d}k, \quad u\in\Gamma(\mathfrak{h}),\ \psi\in\otimes_{\mathrm{s}}^p\mathfrak{h}. \tag{2.9}$$

We deduce from (2.8) that

$$I(N+1)^{-k/2}\otimes 1\!\!1 \text{ restricted to } \Gamma(\mathfrak{h})\otimes\otimes_{\mathrm{s}}^k\mathfrak{h} \text{ is bounded.} \tag{2.10}$$

Lemma 2.17 *Let b be an operator on $\mathfrak{h}$. Then*

$$i)\ \mathrm{d}\Gamma(b)I = I(\mathrm{d}\Gamma(b)\otimes 1\!\!1 + 1\!\!1\otimes \mathrm{d}\Gamma(b)),$$

$$ii)\ \phi(h)I - I(\phi(h)\otimes 1\!\!1) = \tfrac{1}{\sqrt{2}} I 1\!\!1 \otimes a(h), \qquad h\in\mathfrak{h}.$$

Proof. *i)* follows from Lemma 2.16 *i)*. *ii)* follows from Lemma 2.15 *ii)*. □

It is easy to construct a right inverse to the identification operator I. Let j_0, j_∞ be two operators on $\mathfrak{h}$ such that $0\leq j_0\leq 1, 0\leq j_\infty\leq 1$, and $j_0+j_\infty = 1$. Let $j=(j_0,j_\infty):\mathfrak{h}\to\mathfrak{h}\oplus\mathfrak{h}$, as in Subsect. 2.13. Clearly, $0\leq j^*j\leq 1$, hence $\|j\|\leq 1$, and therefore $\check{\Gamma}(j)$ is a bounded operator. We have $ij=1$, hence

$$I\check{\Gamma}(j) = 1\!\!1.$$

We also have

$$\begin{aligned} I1\!\!1_{\{1,\dots,k\}}(N_\infty)\check{\Gamma}(j) &= Q_k(j),\\ I1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(j) &= P_k(j). \end{aligned} \tag{2.11}$$

2.16 Space of additional degrees of freedom

In this subsection we fix some notation which will be used in the next section to define the interaction part of the Hamiltonian.

Suppose that $\mathcal{K}$ is a Hilbert space. If $v\in B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$, then we can define $a^*(v)$, $a(v)$, $\phi(v)$ as unbounded operators on $\mathcal{K}\otimes\Gamma(\mathfrak{h})$:

$$\begin{aligned} a^*(v)\Big|_{\mathcal{K}\otimes\bigotimes_{\rm s}^n\mathfrak{h}} &:= \sqrt{n+1}\Big(1\!\!1_{\mathcal{K}}\otimes\mathcal{S}_{n+1}\Big)\Big(v\otimes 1\!\!1_{\bigotimes_{\rm s}^n\mathfrak{h}}\Big),\\ a(v) &:= (a^*(v))^*,\\ \phi(v) &:= \tfrac{1}{\sqrt{2}}(a(v)+a^*(v)). \end{aligned}$$

They satisfy the estimates

$$\|a^\sharp(v)(N+1)^{-\frac{1}{2}}\|\leq\|v\|, \tag{2.12}$$

where $\|v\|$ is the norm of v in $B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$. Clearly, the condition $v\in B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$ is equivalent to

$$v^*v\in B(\mathcal{K}).$$

If $\mathfrak{h}=L^2(\mathbb{R}^d,\mathrm{d}k)$, then the operator v can be represented as a function $k\mapsto v(k)\in B(\mathcal{K})$ (defined a.e), and the condition $v\in B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$ is equivalent to

$$\int v^*(k)v(k)\mathrm{d}k\in B(\mathcal{K}). \tag{2.13}$$

(2.13) is implied in particular by

$$\int \|v(k)\|^2_{B(\mathcal{K})}\mathrm{d}k<\infty.$$

3 Pauli-Fierz Hamiltonians

3.1 Introduction

In this section we introduce the class of Hamiltonians that we will study in this paper and we give some examples. We also describe a number of related definitions which will be useful in our study, in particular the "extended Hamiltonian" H^{ext}. At the end of this section we prove some technical estimates concerning the Hamiltonian H.

3.2 Hamiltonian

Let $\mathcal{K}$ be a Hilbert space representing the degrees of freedom of the atomic system. The Hamiltonian describing the atomic system is denoted by K. We assume that K is selfadjoint on $\mathcal{D}(K) \subset \mathcal{K}$ and bounded below. A condition that will be sometimes imposed is

$$(H0) \qquad (K+\mathrm{i})^{-1} \text{ is compact.}$$

Its physical interpretation is that the atomic system is confined.

Let $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$ be the $1-$particle Hilbert space in the momentum representation and let $\Gamma(\mathfrak{h})$ be the bosonic Fock space over $\mathfrak{h}$, representing the field degrees of freedom. We will denote by k the momentum operator of multiplication by k on $L^2(\mathbb{R}^d, \mathrm{d}k)$, and by $x = \mathrm{i}\nabla_k$ the position operator on $L^2(\mathbb{R}^d, \mathrm{d}k)$. Let $\omega \in C(\mathbb{R}^d, \mathbb{R})$ be the boson dispersion relation. A general condition which will always be assumed is:

$$(H1) \qquad \begin{cases} \nabla\omega \in L^\infty(\mathbb{R}^d), \\ \nabla\omega(k) \neq 0 \text{ for } k \neq 0, \\ \lim_{|k|\to\infty}\omega(k) = +\infty, \\ \inf\omega(k) = \omega(0) =: m > 0. \end{cases}$$

The quantity $m = \inf\omega(k)$ is called the *boson rest mass* and plays a very important role in our analysis. The typical example is of course the relativistic dispersion relation $\omega(k) = (k^2+m^2)^{\frac{1}{2}}$.

We will sometimes need the following smoothness assumption:

$$(H2) \qquad |\partial_k^\alpha \omega(k)| \leq C_\alpha, \ |\alpha| \geq 1.$$

The Hamiltonian describing the field is equal to $\mathrm{d}\Gamma(\omega(k))$.

The Hilbert space of the interacting system is

$$\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h}).$$

The interaction between the atom and the boson field is described with a coupling operator v satisfying

$$(I1) \qquad v \in B(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}).$$

As we saw in Subsect. 2.16, condition *(I1)* is implied by the stronger condition

$$(I1)' \quad \int \|v(k)\|^2_{B(\mathcal{K})}\mathrm{d}k < \infty.$$

The interaction term is equal to:

$$V := \phi(v) = \frac{1}{\sqrt{2}}(a^*(v) + a(v)).$$

We consider the Hamiltonian

$$H := H_0 + V, \text{ acting on } \mathcal{H},$$

where

$$H_0 := K \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega(k)).$$

Proposition 3.1 *Assume hypotheses (H1) and (I1). Then V is H_0-bounded with the infinitesimal bound. Consequently H is selfadjoint on $\mathcal{D}(H_0)$ and bounded below.*

Proof. It follows from (2.12) and from the positivity of the mass that

$$\|a^\sharp(v)(H_0+1)^{-\frac{1}{2}}\| \leq C\|v\|. \tag{3.1}$$

This implies that V is H_0-bounded with the infinitesimal bound. □

To study the scattering theory for H, in particular to establish the existence of asymptotic fields, we will need to impose a stronger condition on the interaction v:

$$(SR) \qquad \|1\!\!1_{[R,\infty[}(|x|)v\|_{B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})} \leq CR^{-1-\mu}, \;\; \mu > 0.$$

This assumption is an analog of the short-range condition in non-relativistic scattering.

Note that without much additional work, essentially all our results could be proven under a somewhat weaker assumption

$$(SR') \qquad \|1\!\!1_{[R,\infty[}(|x|)(\mathrm{i}+K)^{-1}v\|_{B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})} \leq CR^{-1-\mu}, \;\; \mu > 0.$$

We will use the following notations for various Heisenberg derivatives:

$$\begin{aligned}
\mathbf{d}_0 &= \tfrac{\partial}{\partial t} + [\omega(k), \mathrm{i}\cdot], \text{ acting on } B(\mathfrak{h}),\\
\mathbf{D}_0 &= \tfrac{\partial}{\partial t} + [\mathrm{d}\Gamma(\omega(k)), \mathrm{i}\cdot], \text{ acting on } B(\Gamma(\mathfrak{h})),\\
\mathbf{D} &= \tfrac{\partial}{\partial t} + [H, \mathrm{i}\cdot], \text{ acting on } B(\mathcal{H}).
\end{aligned}$$

Note that we have

$$\mathbf{D}_0\mathrm{d}\Gamma(b) = \mathrm{d}\Gamma(\mathbf{d}_0 b).$$

3.3 Examples

Our first example is the spin-boson model, where the small system is simply a two components spin. We have then $\mathcal{K} = \mathbb{C}^2$, $K = \sigma_z$, and $v(k) = \sigma_x \otimes g(k)$, for a scalar function $g \in L^2(\mathbb{R}^d, \mathbb{C})$. Here $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. So the spin-boson Hamiltonian is given by

$$H = \sigma_z \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega) + \sigma_x \otimes \phi(g), \text{ acting on } \mathbb{C}^2 \otimes \Gamma(L^2(\mathbb{R}^d)).$$

Recently there has been a renewed interest in the scattering theory for the spin-boson model, in connection with the problem of radiative decay. In [HuSp2], Huebner and Spohn proved a Mourre estimate for the massive spin-boson model for a small coupling constant. In [HuSp1] the scattering theory for the spin -boson model is connected to the radiative decay problem and various questions are formulated.

Our second example is a simplified Hamiltonian of an atom iteracting with a massive relativistic bosonic field. (Note that a similar Hamiltonian was contained in the paper by Pauli and Fierz [PF], except that the field was electromagnetic, and hence massless). In this case

$$\mathcal{K} := L^2(\mathbb{R}^{3N}),$$

$$K = \sum_{j=1}^{N} \frac{1}{2} D_{x_j}^2 + \sum_{j=1}^{N} W(x_j) + \sum_{i<j} U(x_i - x_j).$$

Here W is the interaction between one electron and the nucleus and U the electron-electron interaction. If we assume that the potential W tends to $+\infty$ at infinity, ie that the atom is confined, then condition *(H0)* is satisfied.

The boson dispersion relation is $\omega(k) = (k^2 + m^2)^{\frac{1}{2}}$. The interaction V is given by

$$V = \sum_{j=1}^{N} \int (v(k, x_j) a^*(k) + \overline{v}(k, x_j) a(k)) \mathrm{d}k,$$

where x_j denotes the position of the jth electron and $v(k, x)$ is a function

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (k, x) \mapsto v(k, x) \in \mathbb{C}.$$

In order to satisfy condition (SR), we need to assume that

$$\text{(3.2)} \qquad \sup_x \left(\int_{|y|>R} |\hat{v}(y, x)|^2 \mathrm{d}y \right)^{\frac{1}{2}} \leq CR^{-1-\mu},$$

where $\hat{v}(y, x)$ denotes the Fourier transform of $v(k, x)$ with respect to the first variable. (3.2) is satisfied if for instance

$$v(k, x) = \rho(x) \mathrm{e}^{\mathrm{i}k \cdot x} \omega(k)^{-\frac{1}{2}} \chi(k),$$

where $\rho \in \mathcal{S}(\mathbb{R}^d)$ is a spacial cut-off and $\chi \in \mathcal{S}(\mathbb{R}^d)$ is an ultraviolet cutoff.

Note that if one uses the condition (SR'), then one can treat a somewhat more general, and perhaps even more physical, class of Hamiltonians. (SR') is implied by the following hypothesis:

$$\text{(3.3)} \qquad \begin{aligned} & W(x) \geq C_0 \langle x \rangle^{\sigma}, \quad \text{for some} \quad C_0 > 0, \ \sigma \geq 0, \\ & \sup_x \left(\int_{|y|>R} |\hat{v}(y, x)|^2 \langle x \rangle^{-2\sigma} \mathrm{d}y \right)^{\frac{1}{2}} \leq CR^{-1-\mu}. \end{aligned}$$

In particular, (3.3) is a consequence of the following conditions:

$$\text{(3.4)} \qquad \begin{aligned} & W(x) \geq C_0 \langle x \rangle^{1+\mu}, \quad \text{for some} \quad C_0 > 0, \\ & v(k, x) = \mathrm{e}^{\mathrm{i}kx} \omega(k)^{-\frac{1}{2}} \chi(k), \end{aligned}$$

where $\chi \in \mathcal{S}(\mathbb{R}^d)$ is an ultraviolet cutoff. Note that under the condition (3.4) the interaction V is translation invariant. Nevertheless, the atomic Hamiltonian K has to be confining, and hence it is not translation invariant.

For more discussion of similar models, their relationship with quantum electrodynamics and their validity the reader should consult [BFS].

3.4 Extended Hilbert space

Along with the space

$$\mathcal{H} = \mathcal{K} \otimes \Gamma(\mathfrak{h}),$$

we will use the "*extended space*"

$$\mathcal{H}^{\mathrm{ext}} := \mathcal{H} \otimes \Gamma(\mathfrak{h}) = \mathcal{K} \otimes \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}).$$

The extended Hilbert space is very convenient to set up the scattering theory for H. We will use the notation

$$N_0 := \mathbb{1} \otimes N \otimes \mathbb{1}, \quad N_\infty := \mathbb{1} \otimes \mathbb{1} \otimes N.$$

We will also need the "extended Hamiltonian" and the "extended free Hamiltonian"

$$\begin{aligned} H^{\mathrm{ext}} &:= H \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k)), \\ H_0^{\mathrm{ext}} &:= H_0 \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k)). \end{aligned}$$

It is useful to introduce the following asymmetric Heisenberg derivatives:

$$\begin{aligned} \check{\mathbf{d}}_0 f(t) \quad &:= \frac{\partial}{\partial t} f(t) + (\omega(k) \oplus \omega(k))\mathrm{i} f(t) - \mathrm{i} f(t)\omega(k), \\ &f(t) \in B(\mathfrak{h}, \mathfrak{h} \oplus \mathfrak{h}), \\ \check{\mathbf{D}}_0 F(t) \quad &:= \frac{\partial}{\partial t} F(t) + (\mathrm{d}\Gamma(\omega) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega))\mathrm{i} F(t) - \mathrm{i} F(t)\mathrm{d}\Gamma(\omega), \\ &F(t) \in B(\Gamma(\mathfrak{h}), \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})), \\ \check{\mathbf{D}} B(t) \quad &:= \frac{\partial}{\partial t} B(t) + H^{\mathrm{ext}}\mathrm{i} B(t) - \mathrm{i} B(t) H, \\ &B(t) \in B(\mathcal{H}, \mathcal{H}^{\mathrm{ext}}). \end{aligned}$$

Note that we have

$$\check{\mathbf{D}}_0 \mathrm{d}\Gamma(f) = \mathrm{d}\check{\Gamma}(\check{\mathbf{d}}_0 f).$$

For a selfadjoint operator A, we will denote by $\mathcal{H}_{\mathrm{comp}}(A)$ the subspace of vectors $u \in \mathcal{H}$ such that $u = \chi(A)u$, for some $\chi \in C_0^\infty(\mathbb{R})$. In particular, the space $\mathcal{H}_{\mathrm{comp}}(N)$ is the space $\Gamma_{\mathrm{fin}}(\mathfrak{h})$ of finite particle vectors.

3.5 Number-energy estimates

This subsection is devoted to some rather elementary bounds. Note that these bounds fail if the boson mass m is zero. They imply directly that in the estimates of Sect. 2, the factor $(N+1)$ can be replaced by $(H+\mathrm{i})$. This observation will be used often in the sequel.

Lemma 3.2 *Assume the hypotheses (H1), (I1).*
i) Then uniformly for z in a compact set of $\mathbb{C}$, we have

$$(N+1)^{-m}(z-H)^{-k}(N+1)^{m+k} \in O(|Imz|^{-C_{m,k}}).$$

ii) Let $\chi \in C_0^\infty(\mathbb{R})$. Then

$$\|N^m \chi(H) N^p\| < \infty, \quad n, p \in \mathbb{N}.$$

Proof. Clearly, $\mathrm{ad}_N^j H = \phi(\mathrm{i}^j v)$. By hypothesis *(I1)* $\phi(\mathrm{i}^j v)(H+\mathrm{i})^{-1}$ is bounded. We have

$$(H+z)^{-1}N^k = N(H+z)^{-1}N^{k-1} + (H+z)^{-1}\phi(\mathrm{i}v)(H+z)^{-1}N^{k-1}.$$

Moving repeatedly factors of N to the left, we get

$$(H+z)^{-1}N^k = N^k(H+z)^{-1} + \sum_{i=1}^{k} N^{k-l}(H+z)^{-1}B_l(z),$$

with $B_l(z) \in O(|Imz|^{-l})$. Therefore, using

$$N(H+z)^{-1} \in O(|Imz|^{-1})$$

we see that

$$(N+1)^{-k+1}(H+z)^{-1}(N+1)^k \in O(|Imz|^{-k}).$$

This implies *i)*. *ii)* follows directly from *i)* by writing

$$N^m\chi(H)N^p = \prod_{k=1}^{m} N^{m-k}(H+\mathrm{i})^{-1}N^{k-m-1}(H+\mathrm{i})^m\chi(H)(H+\mathrm{i})^p\prod_{k=1}^{p} N^{k-p-1}(H+\mathrm{i})^{-1}N^{p-k}.$$

□

We will use the following notation: for an operator $B(t) \in B(\mathcal{H})$ depending on some parameter t we will write

$$B(t) \in (N+1)^m O_N(t^p)$$
$$\text{if } \|(N+1)^{-m-k}B(t)(N+1)^k\| \leq C_k\langle t\rangle^p, \quad k \in \mathbb{Z}.$$

Likewise, for an operator $C(t) \in B(\mathcal{H}, \mathcal{H}^{\mathrm{ext}})$ we will write

$$C(t) \in \check{O}_N(t^p)(N+1)^m$$

$$\text{if } \|(N_0+N_\infty)^{-m-k}C(t)(N+1)^k\| \leq C_k\langle t\rangle^p, \quad k \in \mathbb{Z}.$$

Finally we will frequently use the following functional calculus formula (see [HS, DeGe]) for $\chi \in C_0^\infty(\mathbb{R})$:

$$\chi(A) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z)(z-A)^{-1}\mathrm{d}z \wedge \mathrm{d}\,\overline{z}, \tag{3.5}$$

where $\tilde{\chi} \in C_0^\infty(\mathbb{C})$ is an almost analytic extension of χ satisfying

$$\tilde{\chi}_{|\mathbb{R}} = \chi,$$
$$|\partial_{\overline{z}}\tilde{\chi}(z)| \leq C_n|Imz|^n, \quad n \in \mathbb{N}.$$

3.6 Commutator estimates

In this subsection we estimate commutators between some operators considered in Sect. 2 and functions of H and H^{ext}.

Lemma 3.3 *Let $f_0 \in C_0^\infty(\mathbb{R}^d)$, $f_\infty \in C^\infty(\mathbb{R}^d)$, $0 \le f_0$, $0 \le f_\infty$, $f_0 + f_\infty \le 1$, $f_0 = 1$ near 0 (and hence $f_\infty = 0$ near 0). Set $f := (f_0, f_\infty)$ and, for $R \ge 1$, $f^R = (f_0^R, f_\infty^R)$, where $f_0^R(x) = f_0(\frac{x}{R})$, $f_\infty^R(x) = f_\infty(\frac{x}{R})$. Assume hypotheses (H1), (H2). Let $\chi \in C_0^\infty(\mathbb{R})$. Then for $m \in \mathbb{N}$, one has*

$$N^m[\chi(H), Q_k(f^R)]\chi(H) \in \begin{cases} o(R^0) \text{ under } (I1), \\ O(R^{-1}) \text{ under } (SR). \end{cases} \tag{3.6}$$

Lemma 3.4 *Let $j_0 \in C_0^\infty(\mathbb{R}^d)$, $j_\infty \in C^\infty(\mathbb{R}^d)$, $0 \le j_0$, $0 \le j_\infty$, $j_0^2 + j_\infty^2 \le 1$, $j_0 = 1$ near 0 (and hence $j_\infty = 0$ near 0). Set $j := (j_0, j_\infty)$ and for $R \ge 1$ $j^R = (j_0^R, j_\infty^R)$. Assume hypotheses (H1), (H2).*

$$i)\ (H^{\mathrm{ext}} + \mathrm{i})^{-1}\check{\Gamma}(j^R) - \check{\Gamma}(j^R)(H + \mathrm{i})^{-1} \in \begin{cases} o(R^0) \text{ under } (I1), \\ O(R^{-1}) \text{ under } (SR). \end{cases}$$

ii) Let $\chi, \tilde{\chi} \in C_0^\infty(\mathbb{R})$. Then

$$(N_0 + N_\infty)^m \Big(\chi(H^{\mathrm{ext}})\check{\Gamma}(j^R) - \check{\Gamma}(j^R)\chi(H)\Big)\tilde{\chi}(H) \in \begin{cases} o(R^0) \text{ under } (I1), \\ O(R^{-1}) \text{ under } (SR). \end{cases}$$

Proof of Lemma 3.3. Let $m(R) := o(R^0)$, if hypothesis *(I1)* holds, and $m(R) := O(R^{-1})$, if hypothesis *(SR)* holds. Let us show the first estimate.

$$[H_0, Q_k(f^R)] = \mathrm{d}Q_k(f^R, [\omega(k), f^R]) \in O_N(R^{-1})(N + 1), \tag{3.7}$$

using Lemma 2.11 *vi)* and pseudodifferential calculus. Using then Lemma 2.9 *iii)*, and the fact that $0 \notin \operatorname{supp} f_\infty$, $0 \notin \operatorname{supp}(1 - f_0)$ we have

$$\begin{aligned} [V, Q_k(f^R)] &= \tfrac{1}{\sqrt{2}}\Big(a^*((1 - f_0^R)v)Q_k(f^R) - a^*(f_\infty^R v)Q_{k-1}(f^R) \\ &\quad -Q_k(f^R)a((1 - f_0^R)v) + Q_{k-1}(f^R)a(f_\infty^R v)\Big) \\ &\in n(R)(N + 1)^{\frac{1}{2}}, \end{aligned} \tag{3.8}$$

where $n(R) = o(R^0)$ under hypothesis *(I1)*, and $n(R) = O(R^{-1-\mu})$ under hypothesis *(SR)*. Hence

$$[H, Q_k(f^t)] \in m(R)(N + 1).$$

Next we use the functional calculus formula (3.5), which yields

$$\begin{aligned} &N^m[\chi(H), Q_k(f^t)]\chi(H) \\ &= \tfrac{\mathrm{i}}{2\pi} \int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z) N^m (z - H)^{-1}[H, Q_k(f^t)](z - H)^{-1}\chi(H)\mathrm{d}z \wedge \mathrm{d}\,\overline{z}. \end{aligned} \tag{3.9}$$

By Lemma 3.2,

$$N^m(z - H)^{-1}(N + 1)^{-m+1} \in O(|Imz|^{-C_m}), \tag{3.10}$$

$$\|(1 + N)^m\tilde{\chi}(H)\| \le C.$$

This shows that the integrand in (3.9) is bounded by $O(|Imz|^{-p})t^{-1}$ for some p, and completes the proof of the lemma. $\Box$

Proof of Lemma 3.4. Using Lemma 2.16 *i)*, we obtain

$$H_0^{\rm ext}\check{\Gamma}(j^R)-\check{\Gamma}(j^R)H_0=\mathrm{d}\check{\Gamma}(j^R,\check{\mathrm{ad}}_{\omega(k)}j^R)\in\check{O}_N(R^{-1})(N+1).$$

Likewise, using Lemma 2.14 *iii)*,

$$\begin{array}{rl}V\otimes 1\!\!1\check{\Gamma}(j^R)-\check{\Gamma}(j^R)V&=\frac{1}{\sqrt{2}}\Big(a^*((1-j_0^R)v\otimes 1\!\!1-1\!\!1\otimes a^*(j_\infty^R v)\check{\Gamma}(j^R)-\check{\Gamma}(j^R)a((1-j_0^R)v)\Big)\\ &\in\check{O}_N(n(R))(N+1)^{\frac12}.\end{array}\tag{3.11}$$

Therefore,

$$H^{\rm ext}\check{\Gamma}(j^R)-\check{\Gamma}(j^R)H\in\check{O}_N(m(R))(N+1).\tag{3.12}$$

This implies *i)*.

Using then formula (3.5), we have

$$\begin{array}{l}(N_0+N_\infty)^m\Big(\chi(H^{\rm ext})\check{\Gamma}(j^R)-\check{\Gamma}(j^R)\chi(H)\Big)\tilde{\chi}(H)\\ =\frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{\chi}(z)(N_0+N_\infty)^m(z-H^{\rm ext})^{-1}\Big(H^{\rm ext}\check{\Gamma}(j^R)-\check{\Gamma}(j^R)H\Big)(z-H)^{-1}\tilde{\chi}(H)\mathrm{d}z\wedge\mathrm{d}\,\overline{z}.\end{array}$$

By the same argument as in Lemma 3.2, we have

$$(N_0+N_\infty)^m(z-H^{\rm ext})^{-1}(N_0+N_\infty)^{-m+1}\in O(|Imz|^{-Cm}).$$

Then we argue as in the proof of Lemma 3.3. $\Box$

4 Spectral analysis of Pauli-Fierz Hamiltonians

4.1 Introduction

In this section we study the properties of the spectrum of H. The results of this section are fairly parallel to their analogs in the theory of N-body Schrödinger operators.

In Subsect. 4.2 we will show an analog of the HVZ theorem describing the essential spectrum of H. It will obviously imply the existence of a ground state of H. Note that in the massless case under certain additional assumptions, it is also possible to prove the existence of a ground state, but the result is deeper then (see [BFS, AH, Sp3]).

In Subsect. 4.3 we will prove the finiteness of the imbedded pure point spectrum outside of thresholds – a result that follows from an analog of the Mourre estimate, which we also prove in this subsection. Let us stress that the assumption that the boson mass is positive plays an important role in the results of Subsects 4.2, 4.3.

If A is an operator, then $\sigma(A)$ denotes its spectrum, $\sigma_{\rm pp}(A)$ its pure point spectrum and $\sigma_{\rm ess}(A)$ its essential spectrum. For a Borel subset $U\subset\mathbb{R}$ we use $1\!\!1_U(A)$ to denote the spectral projection of A onto U.

4.2 HVZ theorem and existence of a ground state

Let us state the main result of this subsection.

Theorem 4.1 *Assume hypotheses (H0), (H1), (I1). Then*

$$\sigma_{\mathrm{ess}}(H) = [\inf \sigma(H) + m, +\infty[.$$

Consequently, $\inf \sigma(H)$ *is a discrete eigenvalue of* H.

We will make use of the partitions of unity of Subsect. 2.12. Recall that to construct this partition we pick functions $j_0, j_\infty \in C^\infty(\mathbb{R}^d)$ with $0 \le j_0 \le 1, j_0 \in C_0^\infty(\mathbb{R}^d)$, $j_0 = 1$ near 0 and $j_0^2 + j_\infty^2 = 1$. For $R \ge 1$, j^R is defined as in Subsect. 3.6. We will also set $q^R = (j_0^R)^2$.

Lemma 4.2 *Assume hypotheses (H0), (H1) and (I1). Then the operator* $\Gamma(q^R)(H + \mathrm{i})^{-1}$ *is compact on* $\mathcal{H}$.

Proof. Since $\mathcal{D}(H) = \mathcal{D}(H_0)$, we see that it is enough to show the compactness of $\Gamma(q^R)(H_0 + \mathrm{i})^{-1}$. Since $1\!\!1_{[n,+\infty[}(N)(H_0 + \mathrm{i})^{-1}$ tends to 0 in norm when $n \to \infty$, it suffices to prove that $\Gamma(q^R)(H_0 + \mathrm{i})^{-1}$ is compact on every n-particle sector. But

$$\Gamma(q^R)(H_0 + \mathrm{i})^{-1}\Big|_{\mathcal{K}\otimes\bigotimes_{\mathrm{s}}^{n}\mathfrak{h}} = \Pi_{i=1}^{n} j_0^2(\tfrac{x_i}{R})(\mathrm{i} + K + \textstyle\sum_{i=1}^{n} \omega(D_i))^{-1}$$

is compact, using hypotheses *(H0)* and *(H1)*. $\square$

Proof of Theorem 4.1. We prove first the $\subset$ part of the theorem. Let $\chi \in C_0^\infty(]-\infty, \inf \sigma(H) + m[)$. Because of $\mathrm{supp}\chi$, we have:

$$\chi(H^{\mathrm{ext}}) = \chi(H^{\mathrm{ext}})1\!\!1_{\{0\}}(N_\infty).$$

Hence, using twice Lemma 3.4, we have

$$\begin{aligned}
\chi(H) &= \chi(H)\check{\Gamma}(j^R)^*\check{\Gamma}(j^R) = \check{\Gamma}(j^R)^*\chi(H^{\mathrm{ext}})\check{\Gamma}(j^R) + o(R^0) \\
&= \check{\Gamma}(j^R)^*\chi(H^{\mathrm{ext}})1\!\!1_{\{0\}}(N_\infty)\check{\Gamma}(j^R) + o(R^0) = \check{\Gamma}(j^R)^*1\!\!1_{\{0\}}(N_\infty)\check{\Gamma}(j^R)\chi(H) + o(R^0).
\end{aligned}$$

The operator $\check{\Gamma}(j^R)^*1\!\!1_{\{0\}}(N_\infty)\check{\Gamma}(j^R)\chi(H) = \Gamma(q^R)\chi(H)$ is compact by Lemma 4.2. Hence $\chi(H)$ is compact as a limit of compact operators.

Let us now prove the $\supset$ part of the theorem. Note that it follows from the $\subset$ part of the theorem that H admits a ground state. Let $\lambda > \inf \sigma(H) + m$. Let w be a ground state of H. Let $h \in C_0^\infty(\mathbb{R}^d)$ with $\int h(k)\mathrm{d}k = 1$ and let $x_0 \in \mathbb{R}^d, x_0 \neq 0$, $k_0 \in \mathbb{R}^d, k_0 \neq 0, \omega(k_0) = \lambda - \inf \sigma(H)$. Choose a sequence (R_j) such that $\lim_{j\to\infty} j^{-1}R_j = \infty$ and define $h_j \in C_0^\infty(\mathbb{R}^d)$ by setting

$$h_j(k) = j^{d/2}h(j(k - k_0))\mathrm{e}^{\mathrm{i}R_j\langle k, x_0\rangle}.$$

Then $\|h_j\| = 1$, $\mathrm{w} - \lim_{j\to\infty} h_j = 0$ and $\lim_{j\to\infty}(\omega(k) - \omega(k_0))h_j = 0$. Let

$$u_j := a^*(h_j)w.$$

We have $\lim_{j\to\infty} \|u_j\| = 1$ and, by Lemma 2.1 *iii)*, $\mathrm{w} - \lim_{j\to\infty} u_j = 0$. Now

$$\begin{aligned}
(H - \lambda)u_j &= a^*(h_j)(H - \lambda)w + a^*(\omega(k)h_j)w + (v|h_j)w \\
&= a^*\Big((\omega(k) - \omega(k_0))h_j\Big)w + (v|h_j)w \in o(j^0),
\end{aligned}$$

when $j \to \infty$. Since u_j tends weakly to 0, we have constructed a Weyl sequence for λ. $\square$

4.3 The Mourre estimate and local finiteness of point spectrum

Let $\tau := \sigma_{\rm pp}(H) + m\mathbb{N}^\times$, where $\mathbb{N}^\times$ are positive integers. Elements of τ will be called *thresholds*, in analogy with the case of $N-$particle Schrödinger operators.

Let a be the operator on $\mathfrak{h}$ defined as $a = \frac{1}{2}\left(\nabla\omega(k)\cdot D_k + D_k\cdot\nabla\omega(k)\right)$. Under hypothesis *(H2)* a is selfadjoint with domain $\mathcal{D}(a) := \{h \in \mathfrak{h} \mid ah \in \mathfrak{h}\}$.

We assume in this section that

$$(I2) \qquad \|av\|_{B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})} < \infty.$$

Let $A = 1\!\!1_{\mathcal{K}} \otimes {\rm d}\Gamma(a)$. Note that $[H, iA]$, defined as a quadratic form on $\mathcal{D}(A)\cap\mathcal{D}(H)$ equals

$$[H,{\rm i}A] = {\rm d}\Gamma(|\nabla\omega(k)|^2) + \phi({\rm i}av).$$

Moreover $\mathcal{D}(A)\cap\mathcal{D}(H)$ contains the space $\Gamma_{\rm fin}(\mathcal{S}(\mathbb{R}^d))$ which is a core for H. So $[H,{\rm i}A]$ extends as an operator bounded on $\mathcal{D}(H)$, similar to H, with $\omega(k)$ replaced by $|\nabla\omega(k)|^2$ and v by ${\rm i}av$. Finally using the fact that $\mathcal{D}(H) = \mathcal{D}(H_0)$, it is easy to check that $e^{i\alpha A}$ leaves $\mathcal{D}(H)$ invariant and that $\sup_{|\alpha|<1}\|He^{{\rm i}\alpha A}\psi\| < \infty$ for $\psi \in \mathcal{D}(H)$.

Consequently Lemma 3.4 applies to $[H,{\rm i}A]$. Another consequence of hypothesis *(I2)* is that $[H,{\rm i}A](H+{\rm i})^{-1}$ is bounded.

Theorem 4.3 *Assume the hypotheses (H0), (H1), (H2), (I1), (I2). Then*
i) Let $\lambda \in \mathbb{R}\backslash\tau$. Then there exists $\epsilon > 0, C_0 > 0$ and a compact operator K_0 such that

$$1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H,{\rm i}A]1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H) \geq C_0 1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H) + K_0.$$

ii) for all $[\lambda_1,\lambda_2]$ such that $[\lambda_1,\lambda_2]\cap\tau = \emptyset$, one has

$$\dim 1\!\!1^{\rm pp}_{[\lambda_1,\lambda_2]}(H) < \infty.$$

Consequently $\sigma_{\rm pp}(H)$ can accumulate only at τ, which is a closed countable set.
iii) Let $\lambda \in \mathbb{R}\backslash(\tau\cup\sigma_{\rm pp}(H))$. Then there exists $\epsilon > 0, C_0 > 0$ such that

$$1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H,{\rm i}A]1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H) \geq C_0 1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H).$$

Proof. The proof will have the same logical structure as in the case of $N-$particle Schrödinger operators. Let

$$\begin{aligned}
d(\lambda) &:= \inf^{\Omega^\perp}_{\{\sigma_{\rm pp}(H)+{\rm d}\Gamma(\omega(k))=\lambda\}} {\rm d}\Gamma(|\nabla\omega(k)|^2) \\
&= \inf\left\{\sum_{i=1}^n |\nabla\omega(k_i)|^2 \;\middle|\; \lambda_1 + \sum_{i=1}^n \omega(k_i) = \lambda,\quad n = 1,2,\ldots,\quad \lambda_1\in\sigma_{\rm pp}(H)\right\}, \\
\tilde{d}(\lambda) &:= \inf_{\{\sigma_{\rm pp}(H)+{\rm d}\Gamma(\omega(k))=\lambda\}} {\rm d}\Gamma(|\nabla\omega(k)|^2) \\
&= \inf\left\{\sum_{i=1}^n |\nabla\omega(k_i)|^2 \;\middle|\; \lambda_1 + \sum_{i=1}^n \omega(k_i) = \lambda,\quad n = 0,1,2,\ldots,\quad \lambda_1\in\sigma_{\rm pp}(H)\right\}.
\end{aligned}$$

The superscript $\Omega^\perp$ in the definition of $d(\lambda)$ means that one excludes the vacuum sector to compute the infimum. Let us note that

$$\tilde{d}(\lambda) := \begin{cases} d(\lambda),\ \lambda \notin \sigma_{\rm pp}(H), \\ 0,\ \lambda \in \sigma_{\rm pp}(H), \end{cases}$$

We introduce also "smeared out" versions of the functions $d(\lambda)$ and $\tilde{d}(\lambda)$. We set

$$\Delta_\lambda^\kappa := [\lambda-\kappa, \lambda+\kappa]$$

and

$$\begin{aligned} d^\kappa(\lambda) &:= \inf_{\mu \in \Delta_\lambda^\kappa} d(\mu), \\ \tilde{d}^\kappa(\lambda) &:= \inf_{\mu \in \Delta_\lambda^\kappa} \tilde{d}(\mu). \end{aligned}$$

Note that the following inequality holds

$$\inf^{\Omega^\perp} \left(\tilde{d}^\kappa(\lambda - \mathrm{d}\Gamma(\omega(k))) + \mathrm{d}\Gamma(|\nabla\omega(k)|^2) \right) \geq d^\kappa(\lambda). \tag{4.1}$$

or in other words, if $n = 1, 2, \ldots$ then

$$\tilde{d}^\kappa\left(\lambda - \sum_{i=1}^n \omega(k_i)\right) + \sum_{i=1}^n |\nabla\omega(k_i)|^2 \geq d^\kappa(\lambda).$$

We will use an induction with respect to $n \in \mathbb{N}$. Let us first list the statements that we will show. We put $E_0 := \inf \sigma(H)$.
$H_1(n)$: Let $\epsilon > 0$ and $\lambda \in [E_0, E_0 + nm[$. Then there exists a compact operator K_0, an interval $\Delta \ni \lambda$ such that

$$1\!\!1_\Delta(H)[H, \mathrm{i}A]1\!\!1_\Delta(H) \geq (d(\lambda) - \epsilon)1\!\!1_\Delta(H) + K_0.$$

$H_2(n)$: Let $\epsilon > 0$ and $\lambda \in [E_0, E_0 + nm[$. Then there exists an interval $\Delta \ni \lambda$ such that

$$1\!\!1_\Delta(H)[H, \mathrm{i}A]1\!\!1_\Delta(H) \geq (\tilde{d}(\lambda) - \epsilon)1\!\!1_\Delta(H).$$

$H_3(n)$: Let $\kappa > 0$, $\epsilon_0 > 0$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that for all $\lambda \in [E_0, E_0 + nm - \epsilon_0]$, one has

$$1\!\!1_{\Delta_\lambda^\delta}(H)[H, \mathrm{i}A]1\!\!1_{\Delta_\lambda^\delta}(H) \geq (\tilde{d}^\kappa(\lambda) - \epsilon)1\!\!1_{\Delta_\lambda^\delta}(H).$$

$S_1(n)$: τ is a closed countable set in $[E_0, E_0 + nm]$.
$S_2(n)$: for all $\lambda_1 \leq \lambda_2 \leq E_0 + nm$ with $[\lambda_1, \lambda_2] \cap \tau = \emptyset$, we have $\dim 1\!\!1^{\rm pp}_{[\lambda_1,\lambda_2]}(H) < \infty$.

For all $n \in \mathbb{N}$, we will describe the proof the following implications:

$$\begin{aligned} &H_1(n) \Rightarrow H_2(n), \\ &H_2(n) \Rightarrow H_3(n), \\ &H_1(n) \Rightarrow S_2(n), \\ &S_2(n-1) \Rightarrow S_1(n), \\ &S_1(n) \text{ and } H_3(n-1) \Rightarrow H_1(n). \end{aligned}$$

Note first that the statements $H_1(1)$ and $S_1(1)$ are immediate since the spectrum of H is discrete in $[E_0, E_0+m[$. Note also that the implication $S_2(n-1) \Rightarrow S_1(n)$ is obvious. The proofs of the implications $H_1(n) \Rightarrow H_2(n)$, $H_2(n) \Rightarrow H_3(n)$, $H_1(n) \Rightarrow S_2(n)$ are standard abstract arguments which adapt directly to the present setting (see eg [Mo] and [FH]). It remains to prove that $S_1(n)$ and $H_3(n-1) \Rightarrow H_1(n)$.

Using Lemma 3.4 for H and $[H, \mathrm{i}A]$, we write, for $\chi \in C_0^\infty([E_0, E_0+nm[)$,

$$
\begin{aligned}
\chi(H)[H,\mathrm{i}A]\chi(H) \quad &= \check{\Gamma}(j^R)^* 1\!\!1_{\{0\}}(N_\infty)\check{\Gamma}(j^R)\chi(H)[H,\mathrm{i}A]\chi(H) \\
&+\check{\Gamma}(j^R)^* 1\!\!1_{[1,\infty[}(N_\infty)\check{\Gamma}(j^R)\chi(H)[H,\mathrm{i}A]\chi(H) \\
&= \Gamma(q^R)\chi(H)[H,\mathrm{i}A]\chi(H) \\
&+\check{\Gamma}(j^R)^* 1\!\!1_{[1,\infty[}(N_\infty)\chi(H^{\mathrm{ext}})[H^{\mathrm{ext}},\mathrm{i}A]\chi(H^{\mathrm{ext}})\check{\Gamma}(j^R) + o(R^0).
\end{aligned} \tag{4.2}
$$

The first term of (4.2) is compact by Lemma 4.2. The second term in the r.h.s. of (4.2) we estimate by diagonalizing $\mathrm{d}\Gamma(\omega(k))$ and $\mathrm{d}\Gamma(|\nabla\omega(k)|^2)$ on the range of $1\!\!1_{[1,+\infty[}(N_\infty)$. Using the closedness of τ in $[E_0, E_0+nm]$, i.e the induction hypothesis $S_1(n)$, we see that

$$
d(\lambda) = \sup_{\kappa>0} d^\kappa(\lambda),
$$

for $\lambda \in [E_0, E_0+nm[$. So we may choose κ small enough so that $d^\kappa(\lambda) \geq d(\lambda) - \epsilon/3$. Next using $H_3(n-1)$ we choose δ such that for $\lambda \in [E_0, E_0+nm-\epsilon_0[$, we have

$$
\begin{aligned}
&1\!\!1_{\Delta_\lambda^\delta}(H+\mathrm{d}\Gamma(\omega(k)))\left([H,\mathrm{i}A]\otimes 1\!\!1 + 1\!\!1\otimes \mathrm{d}\Gamma(|\nabla\omega(k)|^2)\right) 1\!\!1_{\Delta_\lambda^\delta}(H+\mathrm{d}\Gamma(\omega(k)))1\!\!1_{[1,\infty[}(N_\infty) \\
&\geq 1\!\!1_{\Delta_\lambda^\delta}(H+\mathrm{d}\Gamma(\omega(k)))\left(\tilde{d}^\kappa(\lambda - \mathrm{d}\Gamma(\omega(k))) + \mathrm{d}\Gamma(|\nabla\omega(k)|^2) - \tfrac{\epsilon}{3}\right) 1\!\!1_{[1,\infty[}(N_\infty) \\
&\geq (d^\kappa(\lambda) - \tfrac{\epsilon}{3})1\!\!1_{\Delta_\lambda^\delta}(H+\mathrm{d}\Gamma(\omega(k)))1\!\!1_{[1,\infty[}(N_\infty) \\
&\geq (d(\lambda) - \tfrac{2\epsilon}{3})1\!\!1_{\Delta_\lambda^\delta}(H+\mathrm{d}\Gamma(\omega(k)))1\!\!1_{[1,\infty[}(N_\infty).
\end{aligned}
$$

Using again Lemmas 3.4 and 4.2, this yields, for $\mathrm{supp}\,\chi \subset [\lambda-\delta, \lambda+\delta]$,

$$
\chi(H)[H,\mathrm{i}A]\chi(H) \geq (d(\lambda) - 2\epsilon/3)\chi^2(H) + K_1 + o(R^0),
$$

where K_1 is compact. Picking R large enough, this proves $H_1(n)$. $\square$

5 Asymptotic fields and wave operators

5.1 Introduction

In this section we describe the existence of asymptotic fields. Using these fields we define wave operators. Results of this section follow easily by the Cook method and were well known for a long time (see for example [HK1, HK2]). They are the analog of the existence of the wave operators in non-relativistic scattering theory and serve as the conceptual basis for the scattering theory in QFT.

Note that most of the results, after minor modifications, hold even if the mass of the bosons is zero. The most important exception is the unitarity of the wave operator, which implies the Fock property of the asymptotic commutation relations.

5.2 Asymptotic fields

In all this section, we will assume the conditions *(H1)*, *(H2)*, *(I1)* and *(SR)*.

For $h \in \mathfrak{h}$ we set $h_t := \mathrm{e}^{-\mathrm{i}t\omega(k)}h$. We denote by $\mathfrak{h}_0 \subset \mathfrak{h}$ the space $C_0^\infty(\mathbb{R}^d \backslash \{0\})$.

Theorem 5.1 *i) For all $h \in \mathfrak{h}$ the strong limits*

$$W^+(h) := \text{s-}\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH} W(h_t) \mathrm{e}^{-\mathrm{i}tH} \tag{5.1}$$

exist. They are called the asymptotic Weyl operators. *For $h \in \mathfrak{h}_0$ the limit in (5.1) is a norm limit. For all $h \in \mathfrak{h}$ and $\epsilon > 0$ the asymptotic Weyl operators can be also defined using the norm limit:*

$$W^+(h)(\mathrm{i}+H)^{-\epsilon} = \lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH} W(h_t)(\mathrm{i}+H)^{-\epsilon} \mathrm{e}^{-\mathrm{i}tH}. \tag{5.2}$$

ii) The map

$$\mathfrak{h} \ni h \mapsto W^+(h) \tag{5.3}$$

is strongly continuous and the map

$$\mathfrak{h} \ni h \mapsto W^+(h)(\mathrm{i}+H)^{-\epsilon} \tag{5.4}$$

is norm continuous.

iii) The operators $W^+(h)$ satisfy the Weyl commutation relations:

$$W^+(h)W^+(g) = \mathrm{e}^{\mathrm{i}\frac{1}{2}Im(h|g)} W^+(h+g).$$

iv) The Hamiltonian preserves the asymptotic Weyl operators:

$$\mathrm{e}^{\mathrm{i}tH} W^+(h) \mathrm{e}^{-\mathrm{i}tH} = W^+(h_{-t}). \tag{5.5}$$

Proof. It follows from Lemma 2.7 *ii)* that

$$W(h_t) = \mathrm{e}^{-\mathrm{i}tH_0} W(h) \mathrm{e}^{\mathrm{i}tH_0},$$

which implies that, as a quadratic form on $\mathcal{D}(H_0)$, one has

$$\partial_t W(h_t) = -[H_0, \mathrm{i}W(h_t)]. \tag{5.6}$$

Using (5.6) and the fact that $\mathcal{D}(H) = \mathcal{D}(H_0)$, we have, as quadratic forms on $\mathcal{D}(H)$,

$$\partial_t \mathrm{e}^{\mathrm{i}tH} W(h_t) \mathrm{e}^{-\mathrm{i}tH} = \mathrm{i}\mathrm{e}^{\mathrm{i}tH} Im(h_t|v) W(h_t) \mathrm{e}^{-\mathrm{i}tH}.$$

Integrating this relation we obtain (first as a quadratic form identity on $\mathcal{D}(H_0)$, then by a simple argument, as an operator identity)

$$\mathrm{e}^{\mathrm{i}tH} W(h_t) \mathrm{e}^{-\mathrm{i}tH} - W(h) = \mathrm{i} \int_0^t \mathrm{e}^{\mathrm{i}sH} Im(h_s|v) W(h_s) \mathrm{e}^{-\mathrm{i}sH} \mathrm{d}s.$$

Using assumption *(SR)* and stationary phase arguments, we obtain that, for $h \in \mathfrak{h}_0$,

$$\|(h_t|v)\|_{B(\mathcal{K})} \leq C t^{-1-\mu}, \tag{5.7}$$

which proves the existence of the norm limit (5.1) for $h \in \mathfrak{h}_0$. For $h \in \mathfrak{h}$, let $h_n \in \mathfrak{h}_0$ such that $h = \lim_{n\to\infty} h_n$. Using the fact that $\|(N+1)^\epsilon(\mathrm{i}+H)^{-\epsilon}\| < \infty$ and Prop. 2.4 *iii)*, we have

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R}} \|\,(W(h_{n,t}) - W(h_t))\,(\mathrm{i}+H)^{-\epsilon}\| = 0.$$

This implies the existence of the norm limit (5.2) for all $h \in \mathfrak{h}$. Now (5.2) implies (5.1). This ends the proof of *i)* .

We have

$$\begin{aligned}\|\,(W^+(h) - W^+(g))\,\| &\le \lim_{t\to+\infty} \|\mathrm{e}^{\mathrm{i}tH}W(h_t)(H+\mathrm{i})^{-\epsilon}\mathrm{e}^{-\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH}W(g_t)(H+\mathrm{i})^{-\epsilon}\mathrm{e}^{-\mathrm{i}tH}\| \\ &= \lim_{t\to+\infty}\|\,(W(h_t) - W(g_t))\,(H+\mathrm{i})^{-\epsilon}\| \le C\|h-g\|^\epsilon,\end{aligned}$$

by Prop. 2.4 *iii)*, which implies the norm continuity of (5.4). This implies the strong continuity of (5.3) and completes the proof of *ii)*. Finally *iii)* and *iv)* are immediate. □

For two operators A_1, A_2 on a Hilbert space $\mathcal{H}$, we make the convention that $\mathcal{D}(A_1A_2) := \{u \in \mathcal{D}(A_2) | A_2u \in \mathcal{D}(A_1)\}$.

Theorem 5.2 *i) For any $h \in \mathfrak{h}$ there exists a selfadjoint operator $\phi^+(h)$, called the* asymptotic field*, such that*

$$W^+(h) = \mathrm{e}^{\mathrm{i}\phi^+(h)}.$$

ii) For $h_i \in \mathfrak{h}, 1 \le i \le n$, $\mathcal{D}((H+\mathrm{i})^{n/2}) \subset \mathcal{D}(\Pi_1^n\phi^+(h_i))$, and

$$\prod_{i=1}^n \phi^+(h_i)(H+\mathrm{i})^{-n/2} = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^n \phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}(H+\mathrm{i})^{-n/2}.$$

We have the bound

$$\|\Pi_1^n\phi^+(h_i)(H+\mathrm{i})^{-n/2}\| \le C_n\Pi_1^n\|h_i\|. \tag{5.8}$$

iii) the map

$$\mathfrak{h}^n \ni (h_1,\dots,h_n) \mapsto \Pi_1^n\phi^+(h_i)(H+\mathrm{i})^{-\frac{n}{2}} \in B(\mathcal{H})$$

is norm continuous.

iv) The operators $\phi^+(h)$ satisfy in the sense of quadratic forms on $\mathcal{D}(\phi^+(h_1)) \cap \mathcal{D}(\phi^+(h_2))$ the canonical commutation relations

$$[\phi^+(h_2), \phi^+(h_1)] = \mathrm{i}Im(h_2|h_1). \tag{5.9}$$

v)

$$\mathrm{e}^{\mathrm{i}tH}\phi^+(h)\mathrm{e}^{-\mathrm{i}tH} = \phi^+(h_{-t}).$$

vi) For $h \in \mathfrak{h}_0$, $\phi_+(h) - \phi(h)$ is bounded and

$$\lim_{t\to\infty}(\mathrm{e}^{\mathrm{i}tH}\phi(h_t)\mathrm{e}^{-\mathrm{i}tH} - \phi^+(h)) = 0.$$

Proof. By Thm. 5.1 *ii)*, $s \mapsto W^+(sh)$ is a strongly continuous unitary group. Thus the existence of $\phi^+(h)$ follows by Stone's theorem. This proves *i)*.

To prove *ii)*, let us first establish the existence of the norm limit

$$R(h_1,\dots,h_n) := \lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^n\phi(h_{i,t})(H+\mathrm{i})^{-n/2}\mathrm{e}^{-\mathrm{i}tH},\ h_i \in \mathfrak{h}. \tag{5.10}$$

The Heisenberg derivative $\mathbf{D}\phi(h_t)$ (first defined as a quadratic form on $\mathcal{D}(H)$, then as a bounded operator on $\mathcal{H}$) equals $Im(h_t|v)$. We deduce from this and Lemma 2.3 *i)* that similarly

$$\mathbf{D} \prod_{i=1}^{n} \phi(h_{i,t})(H+\mathrm{i})^{-n/2} = \sum_{j=1}^{n} Im(h_{j,t}|v) \prod_{i\neq j}^{n} \phi(h_{i,t})(H+\mathrm{i})^{-n/2}.$$

For $h \in \mathfrak{h}_0$, the existence of the limit (5.10) follows then from (5.7) and Lemma 2.3 *i)*. Moreover, for $h \in \mathfrak{h}_0$, we have

$$\begin{aligned}\phi^+(h) - \phi(h) &= \int_0^{+\infty} \mathrm{e}^{\mathrm{i}tH} Im(h_t|v)\mathrm{e}^{-\mathrm{i}tH}\mathrm{d}t,\\ \phi^+(h) - \mathrm{e}^{\mathrm{i}tH}\phi(h_t)\mathrm{e}^{-\mathrm{i}tH} &= \int_t^{+\infty} \mathrm{e}^{\mathrm{i}tH} Im(h_t|v)\mathrm{e}^{-\mathrm{i}tH}\mathrm{d}t,\end{aligned}$$

which proves *vi)*.

Next, for $h_i \in \mathfrak{h}$, we approximate h_i by sequences $h_{i,n} \in \mathfrak{h}_0$, and use Lemma 2.3 *ii)* to obtain the existence of (5.10).

Let us now prove *ii)* by induction on n. To prove the induction assumption for n we have to show that $\mathcal{D}((H+\mathrm{i})^{n/2}) \subset \mathcal{D}(\Pi_1^n \phi^+(h_i))$ and that $\Pi_1^n\phi^+(h_i)(H+\mathrm{i})^{-n/2} = R(h_1,\dots,h_n)$. Note that by Lemma 2.3 *i)* , we have $\|R(h_1,\dots,h_n)\| \leq C_n \Pi_{i=1}^n \|h_i\|$, which will then imply the bound (5.8). This amounts to prove that

$$R(h_1,\dots,h_n)u = \lim_{s\to 0} \tfrac{1}{s}\left(W^+(sh_1) - 1\!\!1\right) \prod_{i=2}^{n} \phi^+(h_i)(H+\mathrm{i})^{-n/2}u, \ u \in \mathcal{H}. \tag{5.11}$$

Note that by the induction assumption, we have $\mathcal{D}((H+\mathrm{i})^{n/2}) \subset \mathcal{D}(\Pi_2^n \phi^+(h_i))$ and

$$\prod_{i=2}^{n} \phi^+(h_i)(H+\mathrm{i})^{-n/2} = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} \prod_{i=2}^{n} \phi(h_{i,t})(H+\mathrm{i})^{-n/2}\mathrm{e}^{-\mathrm{i}tH}. \tag{5.12}$$

Using (5.12) and the fact that $\mathrm{e}^{\mathrm{i}tH}W(sh_{1,t})\mathrm{e}^{-\mathrm{i}tH}$ is uniformly bounded in t, we have

$$\begin{aligned}&\tfrac{1}{s}\left(W^+(sh_1) - 1\!\!1\right) \prod_{i=2}^{n} \phi^+(h_i)(H+\mathrm{i})^{-n/2}u\\ &= \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} \tfrac{1}{s}\left(W(sh_{1,t}) - 1\!\!1\right) \prod_{i=2}^{n} \phi(h_{i,t})(H+\mathrm{i})^{-n/2}\mathrm{e}^{-\mathrm{i}tH},\end{aligned}$$

so to prove (5.11), we have to check that

$$\lim_{s\to 0}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}R(s,t)\mathrm{e}^{-\mathrm{i}tH}u = 0, \tag{5.13}$$

for

$$R(s,t) := \left(\tfrac{1}{s}\left(W(sh_{1,t}) - 1\!\!1\right) \prod_{i=2}^{n} \phi(h_{i,t}) - \mathrm{i}\prod_{1}^{n}\phi(h_{i,t})\right)(H+\mathrm{i})^{-n/2}. \tag{5.14}$$

Using Prop. 2.4 *ii)* and Lemma 2.3 *i)*, we see that $R(s,t)$ is uniformly bounded in s,t. So it suffices to prove (5.13) for $u \in \mathcal{D}((H+\mathrm{i})^{\epsilon})$, $\epsilon > 0$. Again by Prop. 2.4 *ii)* and Lemma 2.3 *i)*, we obtain

$$\lim_{s\to 0}\sup_{t\in\mathbb{R}} \|R(s,t)(H+\mathrm{i})^{-\epsilon}\| = 0,$$

which proves (5.13) for $u \in \mathcal{D}((H+\mathrm{i})^{\epsilon})$ and hence for all $u \in \mathcal{H}$.

Property *iii)* follows from the existence of the norm limit in *ii)* and Lemma 2.3 *ii)* .

Next for $u_i \in \mathcal{D}(\phi^+(h_i)), i = 1, 2$ we set

$$f(t_2, t_1) := (W^+(-t_2h_2)u_2|W^+(t_1h_1)u_1) - (W^+(-t_1h_1)u_2|W^+(t_2h_2)u_1)\mathrm{e}^{\mathrm{i}t_1t_2Im(h_2|h_1)}.$$

We know that $f(t_2, t_1) = 0$. By Stone's theorem we are allowed to compute the derivative:

$$0 = \partial_{t_2}\partial_{t_1}f(0,0) = -(\phi^+(h_2)u_2|\phi^+(h_1)u_1) + (\phi^+(h_1)u_1|\phi^+(h_2)u_2) - \mathrm{i}Im(h_2|h_1)(u_2|u_1).$$

This proves *iv)*. Finally *v)* follows from Thm. 5.1 *iv)*. □

Theorem 5.3 *For any $h \in \mathfrak{h}$, the* asymptotic creation and annihilation operators *defined on $\mathcal{D}(a^{+\sharp}(h)) := \mathcal{D}(\phi^+(h)) \cap \mathcal{D}(\phi^+(\mathrm{i}h))$ by*

$$\begin{aligned} a^{+*}(h) &:= \tfrac{1}{\sqrt{2}}\left(\phi^+(h) - \mathrm{i}\phi^+(\mathrm{i}h)\right), \\ a^{+}(h) &:= \tfrac{1}{\sqrt{2}}\left(\phi^+(h) + \mathrm{i}\phi^+(\mathrm{i}h)\right). \end{aligned}$$

are closed.
ii) For $h_i \in \mathfrak{h}, 1 \le i \le n$, $\mathcal{D}((H+\mathrm{i})^{n/2}) \subset \mathcal{D}(\Pi_1^n a^{+\sharp}(h_i))$ and

$$\Pi_1^n a^{+\sharp}(h_i)(\mathrm{i}+H)^{-\frac{n}{2}} = \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\Pi_1^n a^{\sharp}(h_{i,t})(\mathrm{i}+H)^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH}.$$

iii) We have the bound

$$\|\Pi_1^n a^{+\sharp}(h_i)u\| \le C\Pi_1^n\|h_j\|\|(H+\mathrm{i})^{\frac{n}{2}}u\|,$$

and the map

$$\mathfrak{h}^n \ni (h_1, \dots, h_n) \mapsto \Pi_1^n a^{+,\sharp}(h_j)(\mathrm{i}+H)^{-\frac{n}{2}} \in B(\mathcal{H})$$

is norm continuous.
iv) The operators $a^{+\sharp}$ satisfy in the sense of forms on $\mathcal{D}((\mathrm{i}+H)^{\frac{1}{2}})$ the canonical commutation relations

$$\begin{aligned} [a^+(h_1), a^{+*}(h_2)] &= (h_1|h_2)\mathbb{1}, \\ [a^+(h_2), a^+(h_1)] &= [a^{+*}(h_2), a^{+*}(h_1)] = 0. \end{aligned}$$

v) One has

$$\mathrm{e}^{\mathrm{i}tH}a^{+\sharp}(h)\mathrm{e}^{-\mathrm{i}tH} = a^{+\sharp}(h_{-t}). \tag{5.15}$$

The following infinitesimal version of (5.15) is true for $h \in \mathcal{D}(\omega)$ in the sense of forms on $\mathcal{D}(H)$. It is known as the pullthrough formula*:*

$$\begin{aligned} a^{+*}(h)H &= Ha^{+*}(h) - a^{+*}(\omega h), \\ a^{+}(h)H &= Ha^{+}(h) + a^{+}(\omega h). \end{aligned} \tag{5.16}$$

vi) For $h \in \mathfrak{h}_0$, the operators $a^{+,\sharp}(h) - a^{\sharp}(h)$ are bounded and

$$\lim_{t\to\infty}(\mathrm{e}^{\mathrm{i}tH}a^{\sharp}(h_t)\mathrm{e}^{-\mathrm{i}tH} - a^{+,\sharp}(h)) = 0.$$

Proof. The closedness of $a^\sharp(h)$ follows from Prop. 2.2 and Thm. 5.2.

To prove the pullthrough formula we write in the sense of forms on $\mathcal{D}(H)$:

$$a^{+\sharp}(h)\left(\frac{\mathrm{e}^{-\mathrm{i}sH}-1\!\!1}{-\mathrm{i}s}\right)=\left(\frac{a^{+\sharp}(h_{-s})-a_{+\sharp}(h)}{-is}\right)+\left(\frac{\mathrm{e}^{-\mathrm{i}sH}-1\!\!1}{-\mathrm{i}s}\right)a^{+\sharp}(h).$$

Letting s tend to 0 and using *iii)*, we obtain (5.16). The other statements follow from analogous statements in Thm. 5.2. □

The following result is due to Høegh-Krohn [HK3].

Corollary 5.4 *For $h \in \mathfrak{h}$, one has:*

$$a^+(h)1\!\!1_{]-\infty,\lambda]}(H)\mathcal{H}\subset 1\!\!1_{]-\infty,\lambda-m]}(H)\mathcal{H}.$$

Proof. It follows from the spectral theorem that $u = 1\!\!1_{]-\infty,\lambda]}(H)u$ if and only if the function

$$\mathbb{R}\ni t\mapsto \mathrm{e}^{\mathrm{i}tH}u\in\mathcal{H}$$

has an analytic extension to $Imz < 0$ satisfying $\|\mathrm{e}^{\mathrm{i}zH}u\| \leq C\mathrm{e}^{|Imz|\lambda}$ for $Imz < 0$.

Let $u = 1\!\!1_{]-\infty,\lambda]}(H)u$. Using (5.15), we have:

$$\mathrm{e}^{\mathrm{i}tH}a^+(h)u=a^+(\mathrm{e}^{\mathrm{i}t\omega(k)}h)\mathrm{e}^{\mathrm{i}tH}u,\ t\in\mathbb{R}.$$

Since H and $\omega(k)$ are bounded below, the right hand side is analytic in $Imz < 0$, with an analytic extension equal to $a^+(\mathrm{e}^{-\mathrm{i}z\omega(k)}h)\mathrm{e}^{\mathrm{i}zH}u$ for $Imz < 0$ (remember that $a^+(h)$ is antilinear in h). Moreover using (3.1) one obtains

$$\begin{aligned}&\|a^+(\mathrm{e}^{-\mathrm{i}z\omega(k)}h)\mathrm{e}^{\mathrm{i}zH}u\|\leq\|a^+(\mathrm{e}^{-\mathrm{i}z\omega(k)}h)1\!\!1_{]-\infty,\lambda]}(H)\|\|\mathrm{e}^{\mathrm{i}zH}1\!\!1_{]-\infty,\lambda]}(H)u\|\\&\leq C\|\mathrm{e}^{-\mathrm{i}z\omega(k)}h\|\mathrm{e}^{|Imz|\lambda}\leq C\|h\|\mathrm{e}^{|Imz|(\lambda-m)}.\end{aligned}$$

This proves that $1\!\!1_{]-\infty,\lambda-m]}(H)a^+(h)u = a^+(h)u$ as claimed. □

5.3 Asymptotic spaces

We define the *asymptotic matter space* to be

$$\mathcal{K}^+:=\{u\in\mathcal{H}\mid a^+(h)u=0,\ h\in\mathfrak{h}\}.$$

The *asymptotic space* is defined as

$$\mathcal{H}^+:=\mathcal{K}^+\otimes\Gamma(\mathfrak{h}).$$

Proposition 5.5 *i) $\mathcal{K}^+$ is a closed $H-$invariant space.*
ii) $\mathcal{K}^+$ is included in the domain of

$$a^{+*}(h_1)\cdots a^{+*}(h_n)\ \textit{for all}\ h_1,\dots,h_n\in\mathfrak{h}.$$

iii)

$$\mathcal{H}_{\mathrm{pp}}(H)\subset\mathcal{K}^+.$$

Proof. $\mathcal{K}^+$ is obviously closed since $a^+(h)$ are closed operators. The fact that $\mathcal{K}^+$ is invariant under $\mathrm{e}^{-\mathrm{i}tH}$ follows from (5.15). Let us prove *ii)* by induction on n. Since $a^{+*}(h)$ and $a^+(h)$ have the same domain, *ii)* is true for $n = 1$. Assume that *ii)* is true for $n - 1$. By the remark above, it suffices to check that for $u \in \mathcal{K}^+$, $a^{+*}(h_2) \cdots a^{+*}(h_n)u \in \mathcal{D}(a^+(h_1))$. But this follows from the canonical commutation relations and the fact that $u \in \mathcal{K}^+$.

Now suppose that $Hu = Eu$. Then

$$\mathrm{e}^{\mathrm{i}tH} a(h_t) \mathrm{e}^{-\mathrm{i}tH} u = (E + \mathrm{i}) \mathrm{e}^{\mathrm{i}t(H-E)} a(h_t)(H + \mathrm{i})^{-1} u. \tag{5.17}$$

But $\mathrm{w} - \lim_{t\to\infty} h_t = 0$ and hence by Lemma 2.1 *iii)*

$$\text{s-}\lim_{t\to\infty} a(h_t)(H + \mathrm{i})^{-1} = 0.$$

Therefore, the limit of (5.17) is zero, which means that $a^+(h)u = 0$. This proves *iii)*. □

5.4 Wave operators

The *asymptotic matter Hamiltonian* and the *asymptotic Hamiltonian* are defined by the formulas

$$K^+ := H\Big|_{\mathcal{K}^+}, \qquad H^+ := K^+ \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega).$$

We also define

$$\begin{aligned} &\Omega^+ : \mathcal{H}^+ \to \mathcal{H}, \\ &\Omega^+ \psi \otimes a^*(h_1) \cdots a^*(h_n)\Omega := a^{+*}(h_1) \cdots a^{+*}(h_n)\psi, \quad h_1, \ldots, h_n \in \mathfrak{h}, \quad \psi \in \mathcal{K}^+. \end{aligned} \tag{5.18}$$

The map Ω^+ is called the *wave operator*. The following theorem is due to Høegh-Krohn [HK2].

Theorem 5.6 *Ω^+ is a unitary map from $\mathcal{H}^+$ to $\mathcal{H}$ such that:*

$$\begin{aligned} &a^{+\sharp}(h)\Omega^+ = \Omega^+ 1\!\!1 \otimes a^\sharp(h), \quad h \in \mathfrak{h}, \\ &H\Omega^+ = \Omega^+ H^+. \end{aligned}$$

Proof. Using the canonical commutation relations and the definition of $\mathcal{K}^+$, it is easy to see that Ω^+ is isometric. Moreover it follows from Thm. 5.3 *v)* that

$$\mathrm{e}^{\mathrm{i}tH}\Omega^+ = \Omega^+ \mathrm{e}^{\mathrm{i}tH^+}.$$

Let $u \in (\mathrm{Ran}\Omega^+)^\perp$. Since $\mathrm{Ran}\Omega^+$ is $H-$invariant, we may assume that $u = 1\!\!1_{]-\infty,\lambda]}(H)u$. By Thm. 5.3 *ii)*, u belongs to the domain of $\Pi_1^n a^+(h_i)$ for any $h_i \in \mathfrak{h}, 1 \le i \le n$.

By Corollary 5.4, if $nm > \lambda - \inf\sigma(H)$, then

$$a^+(h_1) \cdots a^+(h_n)u = 0, \ \forall h_1, \ldots, h_n \in \mathfrak{h}. \tag{5.19}$$

Let n_0 be the smallest positive integer with the property (5.19). This implies that $v = a^+(h_2) \cdots a^+(h_{n_0})u \in \mathcal{K}^+$. So we have

$$0 = (u | a^{+*}(h_2) \cdots a^{+*}(h_{n_0})v) = \|v\|^2.$$

Thus we have shown that

$$a^+(h_2) \cdots a^+(h_{n_0})u = 0, \ \forall h_2, \ldots, h_{n_0} \in \mathfrak{h},$$

which is a contradiction. □

5.5 Extended wave operator

Recall that in Subsect. 3.4 we introduced the extended Hilbert space and the extended Hamiltonian

$$\mathcal{H}^{\mathrm{ext}} = \mathcal{H} \otimes \Gamma(\mathfrak{h}), \qquad H^{\mathrm{ext}} = H \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega(k)).$$

Clearly, $\mathcal{H}^+$ is a subspace of $\mathcal{H}^{\mathrm{ext}}$ and

$$H^+ = H^{\mathrm{ext}}\Big|_{\mathcal{H}^+}.$$

Sometimes we will also need the "extended wave operator". Its domain can be chosen to be

$$\mathcal{D}(\Omega^{\mathrm{ext},+}) := \bigoplus_{n=0}^{\infty} \mathcal{D}((H+\mathrm{i})^{\frac{n}{2}}) \otimes \otimes_{\mathrm{s}}^{n} \mathfrak{h},$$

which is a dense subset of $\mathcal{H}^{\mathrm{ext}}$. Now we set

$$(5.20)\qquad \begin{array}{l} \Omega^{\mathrm{ext},+} : \mathcal{D}(\Omega^{\mathrm{ext},+}) \to \mathcal{H}, \\ \Omega^{\mathrm{ext},+}\psi \otimes a^*(h_1)\cdots a^*(h_n)\Omega := a^{+*}(h_1)\cdots a^{+*}(h_n)\psi, \qquad \psi \in \mathcal{D}((H+\mathrm{i})^{\frac{n}{2}}). \end{array}$$

Note that $\Omega^{\mathrm{ext},+}$ is an unbounded operator. Clearly,

$$(5.21)\qquad \Omega^{\mathrm{ext},+}\Big|_{\mathcal{H}^+} = \Omega^+.$$

We will sometimes treat Ω^+ as a partial isometry equal to zero on the orthogonal complement of $\mathcal{H}^+$ inside $\mathcal{H}^{\mathrm{ext}}$. We can then write the following identity:

$$(5.22)\qquad \Omega^+ = \Omega^{\mathrm{ext},+} 1\!\!1_{\mathcal{H}^+}.$$

5.6 Another construction of the wave operators

Recall that in Subsect. 2.15, we defined the (unbounded) identification operator $I : \Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h})$. By the same symbol we will denote the operator

$$1\!\!1_{\mathcal{K}} \otimes I : \mathcal{H}^{\mathrm{ext}} = \mathcal{K} \otimes \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) \to \mathcal{H} = \mathcal{K} \otimes \Gamma(\mathfrak{h}).$$

Theorem 5.7 *i) Let $w \in \mathcal{D}((H+\mathrm{i})^{k/2})$ with $w = 1\!\!1_{\{k\}}(N_\infty)w$. Then the limit*

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} w$$

exists and equals $\Omega^{\mathrm{ext},+}w$.
ii) Let $w \in \mathcal{H}_{\mathrm{comp}}(H^{\mathrm{ext}})$. Then the limit

$$(5.23)\qquad \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} w$$

exists and equals $\Omega^{\mathrm{ext},+}w$. In particular, $\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})$ is bounded for $\chi \in C_0^\infty(\mathbb{R})$.
iii) Let $w \in \mathcal{H}^+ \cap \mathcal{H}_{\mathrm{comp}}(H^{\mathrm{ext}})$. Then the limit

$$(5.24)\qquad \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^+} w$$

exists and equals Ω^+w.

Proof. Let us first show *i)*. Let $w \in \mathcal{D}((H+\mathrm{i})^{k/2})$ with $w = 1\!\!1_{\{k\}}(N_\infty)w$. Since by (2.10) $I(H+\mathrm{i})^{-k/2} \otimes 1\!\!1_{\{k\}}(N_\infty)$ is a bounded operator, it suffices to prove *i)* for $w = \psi \otimes \Pi_i^k a^*(h_i)\Omega$, $\psi \in \mathcal{D}((H+\mathrm{i})^{k/2})$, $h_i \in \mathfrak{h}$. It follows from property (2.8) of I that

$$\mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} \psi \otimes \Pi_1^k a^*(h_i)\Omega = \mathrm{e}^{\mathrm{i}tH} \Pi_1^k a^*(h_{i,t}) \mathrm{e}^{-\mathrm{i}tH}\psi.$$

i) follows then from Thm. 5.3 *ii)*.

To prove *ii)*, we observe that since the boson mass is positive, vectors in $\mathcal{H}_{\mathrm{comp}}(H^{\mathrm{ext}})$ are also in $\mathcal{H}_{\mathrm{comp}}(H)$ and in $\mathcal{H}_{\mathrm{comp}}(N_\infty)$. So *ii)* follows from *i)*. Finally *iii)* follows from *ii)* by (5.21). □

6 Propagation estimates

In this section, which serves as a technical preparation for Sect. 7, we collect various propagation estimates about the evolution $\mathrm{e}^{-\mathrm{i}tH}$ which will be used in the next section. They closely resemble propagation estimates used in the scattering theory for N-body Schrödinger operators, especially those introduced in [Gr].

In all this section we will assume the conditions *(H1)*, *(H2)*, *(I1)*, *(SR)*. Finally we mention that all the results of this section hold also for the dynamics $\mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}}$ with the obvious modifications.

6.1 Large velocity estimate

In this subsection we derive a standard large velocity estimate. It means that no boson can asymptotically propagate in the region $|x| > v_{\max}t$, where the maximal velocity $v_{\max}$ is equal to

$$v_{\max} := \sup_k |\nabla\omega(k)|.$$

Proposition 6.1 *Let $\chi \in C_0^\infty(\mathbb{R})$. For $R' > R > v_{\max}$, one has*

$$\int_1^\infty \left\| \mathrm{d}\Gamma\left(1\!\!1_{[R,R']}(\frac{|x|}{t})\right)^{\frac{1}{2}} \chi(H)\mathrm{e}^{-\mathrm{i}tH}u \right\|^2 \frac{\mathrm{d}t}{t} \le C\|u\|^2.$$

Proof. Let $F \in C^\infty(\mathbb{R})$ be a cutoff function equal to 1 near ∞, to 0 near the origin, with $F'(s) \ge 1\!\!1_{[R,R']}(s)$. Let

$$\begin{aligned} \Phi(t) &:= \chi(H)\mathrm{d}\Gamma\left(F(\tfrac{|x|}{t})\right)\chi(H), \\ b(t) &:= \mathbf{d}_0 F(\tfrac{x}{t}). \end{aligned}$$

By pseudodifferential calculus, and then taking into account the support of F', we obtain

$$\begin{aligned} b(t) &= F'(\tfrac{|x|}{t})\left(-\tfrac{|x|}{t} + \tfrac{x}{|x|t}\nabla\omega(k)\right) + O(t^{-2}) \\ &\le -\tfrac{C_0}{t}F'(\tfrac{|x|}{t}) + O(t^{-2}), \end{aligned}$$

for some $C_0 > 0$. Hence

$$\mathbf{D}_0\mathrm{d}\Gamma\left(F(\tfrac{|x|}{t})\right) = \mathrm{d}\Gamma\left(b(t)\right) \le -\tfrac{C_0}{t}\mathrm{d}\Gamma\left(F'(\tfrac{|x|}{t})\right) + C\tfrac{N}{t^2}.$$

Moreover,

$$[V, \mathrm{id}\Gamma\left(F(\tfrac{|x|}{t})\right)] = \phi(\mathrm{i}F(\tfrac{|x|}{t})f) \in O_N(t^{-1-\mu})(N+1)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned}\mathbf{D}\Phi(t) &= \chi(H)\Big(\mathbf{D}_0\mathrm{d}\Gamma(\tfrac{|x|}{t})\Big)\chi(H) + \chi(H)\Big[V, \mathrm{id}\Gamma(\tfrac{|x|}{t})\Big]\chi(H)\\ &\le -t^{-1}C_0\chi(H)\mathrm{d}\Gamma\Big(F'(\tfrac{|x|}{t})\Big)\chi(H) + O(t^{-1-\mu}).\end{aligned}$$

By Lemma A.1, we obtain the desired result. □

6.2 Phase space propagation estimate

This subsection is devoted to a more subtle propagation estimate. Its intuitive meaning is that along the evolution of an asymptotically free boson the instantaneous velocity $\nabla\omega(k)$ and the average velocity $\frac{x}{t}$ converge to each other as time goes to ∞.

Proposition 6.2 *Let $\chi \in C_0^\infty(\mathbb{R})$, $0 < c_0 < c_1$. Set*

$$\Theta_{[c_0,c_1]}(t) := \mathrm{d}\Gamma\left(\langle \tfrac{x}{t} - \nabla\omega(k), 1\!\!1_{[c_0,c_1]}(\tfrac{x}{t})(\tfrac{x}{t} - \nabla\omega(k))\rangle\right).$$

Then

$$\int_1^\infty \|\Theta_{[c_0,c_1]}(t)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^2 \frac{\mathrm{d}t}{t} \le C\|u\|^2.$$

Proof. We start by recalling a well-known construction, which can be viewed as a trivial version of the construction of the Graf vector field (see eg [Gr]). It is easy to see that there exists a function $R_0(x)$ such that

$$\begin{aligned}&R_0(x) = 0, \text{ for } |x| \le \tfrac{c_0}{2},\\ &R_0(x) = \tfrac{1}{2}x^2 + c, \text{ for } |x| \ge 2c_1,\\ &\nabla_x^2 R_0(x) \ge 1\!\!1_{[c_0,c_1]}(|x|).\end{aligned}$$

We fix the parameter $c_1 > v_{\max} + 1$, choose a constant $c_2 > c_1 + 1$ and consider

$$R(x) := F(|x|)R_0(x),$$

for $F(s) = 1, s \le c_1$, $F(s) = 0$, $s \ge c_2$. The function R satisfies now:

$$\begin{aligned}&\nabla_x^2 R(x) \ge 1\!\!1_{[c_0,c_1]}(|x|) - C1\!\!1_{[v_{\max}+2,c_2]},\\ &|\partial_x^\alpha R(x)| \le C_\alpha.\end{aligned} \tag{6.1}$$

It clearly suffices to prove Prop. 6.2 for $c_1 > v_{\max} + 1$, which we will assume in what follows. Let

$$b(t) := R(\tfrac{x}{t}) - \tfrac{1}{2}\left(\langle\nabla R(\tfrac{x}{t}), \tfrac{x}{t} - \nabla\omega(k)\rangle + \mathrm{hc}\right).$$

We consider the propagation observable

$$\Phi(t) = \chi(H)\mathrm{d}\Gamma\left(b(t)\right)\chi(H).$$

Using first pseudodifferential calculus, and then (6.1), we obtain

$$\begin{aligned}\mathbf{d}_0 b(t) &= \tfrac{1}{t}\langle \tfrac{x}{t} - \nabla\omega(k), \nabla^2 R(\tfrac{x}{t})\tfrac{x}{t} - \nabla\omega(k)\rangle + O(t^{-2}) \\ &\geq \tfrac{1}{t}\langle \tfrac{x}{t} - \nabla\omega(k), 1\!\!1_{[c_0,c_1]}(\tfrac{|x|}{t})\tfrac{x}{t} - \nabla\omega(k)\rangle \\ &\quad -\tfrac{C}{t} 1\!\!1_{[v_{\max}+2,c_2]}(\tfrac{|x|}{t}) + O(t^{-2}).\end{aligned}$$

This gives

$$\mathbf{D}_0\Phi(t) = \mathrm{d}\Gamma\left(\mathbf{d}_0 b(t)\right) \geq \tfrac{1}{t}\Theta_{[c_1,c_2]}(t) - \tfrac{C}{t}\mathrm{d}\Gamma\left(1\!\!1_{[v_{\max}+2,c_2]}(\tfrac{|x|}{t})\right) + O_N(t^{-2})(N+1). \tag{6.2}$$

Moreover, using pseudodifferential calculus and hypothesis *(SR)*, we have

$$\|b(t)v\| \leq C\|F(\tfrac{|x|}{t} \geq c_0)v\| + O(t^{-2}) \in O(t^{-1-\mu}).$$

Hence

$$[V, \mathrm{i}\mathrm{d}\Gamma\left(b(t)\right)] = \phi(\mathrm{i}b(t)v) \in O_N(t^{-1-\mu})(N+1)^{\frac{1}{2}}.$$

So we finally obtain

$$\begin{aligned}\mathbf{D}\Phi(t) &= \chi(H)\Big(\mathbf{D}_0\mathrm{d}\Gamma(b(t))\Big)\chi(H) + \chi(H)[\mathrm{i}V, \mathrm{d}\Gamma(b(t))]\chi(H) \\ &\geq \tfrac{1}{t}\chi(H)\Theta_{[c_1,c_2]}(t)\chi(H) + O(t^{-1-\mu}).\end{aligned}$$

Using Lemma A.1, we obtain the desired result. □

6.3 Improved phase-space propagation estimate

In this subsection, we will improve Prop. 6.2.

Proposition 6.3 *Let* $0 < c_0 < c_1$, $J \in C_0^\infty(\{c_0 < |x| < c_1\})$, $\chi \in C_0^\infty(\mathrm{I\!R})$. *Then for* $1 \leq i \leq d$

$$\int_1^{+\infty} \left\| \mathrm{d}\Gamma\left(\left|J\left(\frac{x}{t}\right)\left(\frac{x_i}{t} - \partial_i\omega(k)\right) + \mathrm{hc}\right|\right)^{\frac{1}{2}} \chi(H)\mathrm{e}^{-\mathrm{i}tH}u\right\|^2 \frac{\mathrm{d}t}{t} < C\|u\|^2.$$

Before starting the proof, we need some technical preparation.

Lemma 6.4 *Let* $A = (\frac{x}{t} - \nabla\omega(k))^2 + t^{-\delta}$, $\delta > 0$. *Let* $J, J_1, J_2 \in C_0^\infty(\mathrm{I\!R}^d)$ *with* $J_1 = 1$ *near* $\mathrm{supp}\, J$, $J_2 = 1$ *near* $\mathrm{supp}\, J_1$, $0 \leq J_2 \leq 1$. *Then*

i) $J(\frac{x}{t})A^{\frac{1}{2}} = O(1)$,

ii) $[A^{\frac{1}{2}}, J(\frac{x}{t})] = O(t^{\delta/2-1})$,

iii) $\mathbf{d}_0 A^{\frac{1}{2}} = -\frac{1}{t}A^{\frac{1}{2}} + O(t^{-1-\delta/2})$.

For $1 \leq i \leq d$ *and for* $\epsilon = \inf(\delta, 1-\delta/2)$

iv) $|J(\frac{x}{t})(\frac{x_i}{t} - \partial_i\omega(k)) + \mathrm{hc}| \leq CJ_2(\frac{x}{t})A^{\frac{1}{2}}J_2(\frac{x}{t}) + Ct^{-\epsilon/2}$,

v) $J(\frac{x}{t})(\frac{x_i}{t} - \partial_i\omega(k))A^{\frac{1}{2}}J_1(\frac{x}{t}) + \mathrm{hc} \leq C\left\langle(\frac{x}{t} - \nabla\omega(k)), J_2^2(\frac{x}{t})(\frac{x}{t} - \nabla\omega(k))\right\rangle + Ct^{-\epsilon}$.

Proof. *i)* is immediate using that $\|J(\frac{x}{t})A^{\frac{1}{2}}\| = \|J(\frac{x}{t})AJ(\frac{x}{t})\|^{\frac{1}{2}} \in O(1)$.

To prove *ii)* and *iii)*, we use the identities

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}t\omega(k)}A^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}t\omega(k)} &= (\tfrac{x^2}{t^2}+t^{-\delta})^{\frac{1}{2}} =: A_0, \\
\mathrm{e}^{\mathrm{i}t\omega(k)}J(\tfrac{x}{t})\mathrm{e}^{-it\omega(k)} &= J(v),
\end{aligned}$$

for $v := \frac{x}{t}+\nabla\omega(k)$. It is easy to check using pseudodifferential calculus that $[v, A_0] \in O(t^{-1+\delta/2})$. *ii)* follows then easily from the following functional calculus formula:

$$J(v) = (2\pi)^{-n}\int \hat{J}(\xi)\mathrm{e}^{\mathrm{i}\langle\xi,v\rangle}\mathrm{d}\xi.$$

To prove *iii)*, we notice that

$$\mathrm{e}^{\mathrm{i}t\omega(k)}\mathbf{d}_0A^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}t\omega(k)} = \frac{\mathrm{d}}{\mathrm{d}t}A_0 = -\frac{1}{t}A_0 + O(t^{-\delta/2-1}),$$

by a direct computation.

Let us now prove *iv)*. Set

$$\begin{aligned}
B_0 &:= J(\tfrac{x}{t})(\tfrac{x_i}{t} - \partial_i\omega(k)) + \mathrm{hc}, \\
B_2 &:= J_1(\tfrac{x}{t})A^{\frac{1}{2}}J_1(\tfrac{x}{t}).
\end{aligned}$$

By pseudodifferential calculus and *ii)* we have

$$\begin{aligned}
B_0^2 &= v_iJ^2(\tfrac{x}{t})v_i + O(t^{-1}) \le Cv_iJ_1^4(\tfrac{x}{t})v_i + Ct^{-1} \\
&= CJ_1^2(\tfrac{x}{t})v_i^2J_1^2(\tfrac{x}{t}) + O(t^{-1}) = CJ_1^2(\tfrac{x}{t})AJ_1^2(\tfrac{x}{t}) + O(t^{-\delta}) \\
&= CJ_1(\tfrac{x}{t})A^{\frac{1}{2}}J_1^2(\tfrac{x}{t})A^{\frac{1}{2}}J_1(\tfrac{x}{t}) + O(t^{-\epsilon}) = CB_2^2 + O(t^{-\epsilon}).
\end{aligned} \tag{6.3}$$

Since the function $\lambda \mapsto \lambda^{\frac{1}{2}}$ is matrix monotone (see [BR, Sect. 2.2.2]), we deduce from (6.3) that

$$|B_0| \le C(B_2^2 + t^{-\epsilon})^{\frac{1}{2}} \le CB_2 + Ct^{-\epsilon/2}.$$

which proves *iv)*.

To prove *v)*, we write using pseudodifferential calculus and *ii)*:

$$\begin{aligned}
&J(\tfrac{x}{t})(\tfrac{x_i}{t} - \partial_i\omega(k))A^{\frac{1}{2}}J_1(\tfrac{x}{t}) + \mathrm{hc} \\
&= v_iJ(\tfrac{x}{t})A^{\frac{1}{2}} + \mathrm{hc} + O(t^{-1+\delta/2}) = A^{\frac{1}{2}}v_iA^{-\frac{1}{2}}J(\tfrac{x}{t})A^{\frac{1}{2}} + \mathrm{hc} + O(t^{-1+\delta/2}) \\
&\le CA + Ct^{-1+\delta/2} \le C(\tfrac{x}{t} - \nabla\omega(k))^2 + Ct^{-\epsilon},
\end{aligned}$$

since $v_iA^{-\frac{1}{2}}$ is bounded. Next we use that $B_1 = J_2(\frac{x}{t})B_1J_2(\frac{x}{t}) + O(t^{-\infty})$ to obtain *v)*. □

Proof of Prop. 6.3. Let

$$b(t) = J(\tfrac{x}{t})A^{\frac{1}{2}}J(\tfrac{x}{t}),$$

where

$$A = (\tfrac{x}{t} - \nabla\omega(k))^2 + t^{-\delta},$$

and $J, J_1 \in C_0^\infty(\{c_0 \leq |x| \leq c_1\})$, $0 \leq J \leq 1$, $J = 1$ near $\mathrm{supp}\, J_1$. Let

$$\Phi(t) = -\chi(H)\mathrm{d}\Gamma(b(t))\chi(H).$$

Note that by Lemmas 6.4 *i)* and 3.2, $\Phi(t) \in O(1)$. Using Lemmas 6.4 *i)* and 3.2 and hypothesis *(SR)*, we have

$$[V, \mathrm{id}\Gamma(b(t))] = \phi(\mathrm{i}b(t)v) \in O_N(t^{-1-\mu})(N+1)^{\frac{1}{2}}. \tag{6.4}$$

Next, we have

$$\mathbf{D}_0\mathrm{d}\Gamma(b(t)) = \mathrm{d}\Gamma(\mathbf{d}_0 b(t)),$$

and

$$\mathbf{d}_0 b(t) = \left(\mathbf{d}_0 J(\tfrac{x}{t})\right) A^{\frac{1}{2}} J(\tfrac{x}{t}) + \mathrm{hc} + J(\tfrac{x}{t})\left(\mathbf{d}_0 A^{\frac{1}{2}}\right) J(\tfrac{x}{t}). \tag{6.5}$$

By Lemma 6.4 *iii)*

$$\mathbf{d}_0 A^{\frac{1}{2}} = -\frac{A^{\frac{1}{2}}}{t} + O(t^{-1-\delta/2}),$$

and by Lemma 6.4 *iv)*, we obtain, for some $C_0 > 0$,

$$-J(\tfrac{x}{t})\left(\mathbf{d}_0 A^{\frac{1}{2}}\right) J(\tfrac{x}{t}) \geq \tfrac{C_0}{t}|J_1(\tfrac{x}{t})(\tfrac{x_i}{t} - \partial_i\omega(k)) + \mathrm{hc}| - Ct^{-1-\epsilon}. \tag{6.6}$$

Next by pseudodifferential calculus, we have

$$\mathbf{d}_0 J(\tfrac{x}{t}) = -\tfrac{1}{t}\langle \nabla J(\tfrac{x}{t}), \tfrac{x}{t} - \nabla\omega(k)\rangle + \mathrm{hc} + O(t^{-2}),$$

which, by Lemma 6.4 *v)*, gives, for $J_2 \in C_0^\infty(\{c_0 \leq |x| \leq c_1\})$, $J_2 = 1$ near $\mathrm{supp}\, J$,

$$-\left(\mathbf{d}_0 J(\tfrac{x}{t})\right) A^{\frac{1}{2}} J(\tfrac{x}{t}) + \mathrm{hc} \geq -\tfrac{C}{t}\langle \tfrac{x}{t} - \nabla\omega(k), J_2(\tfrac{x}{t})(\tfrac{x}{t} - \nabla\omega(k))\rangle + O(t^{-1-\epsilon}). \tag{6.7}$$

Collecting (6.4), (6.6) and (6.7), we obtain finally, for some $\epsilon > 0$,

$$\begin{aligned} -\mathbf{D}\Phi(t) &= \chi(H)[V, \mathrm{id}\Gamma(b(t))]\chi(H) + \chi(H)\left(\mathbf{d}_0\mathrm{d}\Gamma(b(t))\chi(H)\right. \\ &\geq \tfrac{C_0}{t}\chi(H)\mathrm{d}\Gamma(|J_1(\tfrac{x}{t})(\tfrac{x_i}{t} - \partial_i\omega(k) + \mathrm{hc}|)\chi(H) \\ &\quad -\tfrac{C}{t}\chi(H)\mathrm{d}\Gamma(\langle \tfrac{x}{t} - \nabla\omega(k), J_2(\tfrac{x}{t})(\tfrac{x}{t} - \nabla\omega(k))\rangle)\chi(H) + O(t^{-1-\epsilon}). \end{aligned} \tag{6.8}$$

Since by Prop. 6.2 the second term in the right hand side of (6.8) is integrable along the evolution, we obtain the proposition. □

6.4 Minimal velocity estimate

This subsection is devoted to the proof of the minimal velocity estimate. It will use the Mourre estimate shown in Subsect. 4.3.

Proposition 6.5 *Assume additionally that (H0) holds. Let $\chi \in C_0^\infty(\mathbb{R})$ be supported in $\mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H))$. Then there exists $\epsilon > 0$ such that*

$$\int_1^{+\infty} \left\| \Gamma\left(1\!\!1_{[0,\epsilon]}\left(\frac{|x|}{t}\right)\right)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\right\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2.$$

Proof. Let us first prove the proposition for χ supported near an energy level $\lambda \in \mathbb{R}\backslash(\tau \cup \sigma_{\rm pp}(H))$. By Thm. 4.3, we will find $\chi \in C_0^\infty(\mathbb{R}^d)$ equal 1 near λ with $\operatorname{supp}\chi$ close enough to λ such that for some $C_0 > 0$

$$\chi(H)[H, \mathrm{i}A]\chi(H) \geq C_0\chi^2(H). \tag{6.9}$$

Let $\epsilon > 0$ be a number that will be fixed later on. Let $q \in C_0^\infty(\{|x| \leq 2\epsilon\})$ such that $0 \leq q \leq 1$, $q = 1$ near on $\{|x| \leq \epsilon\}$ and let $q^t = q(\frac{x}{t})$.

Let

$$\Phi(t) := \chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\chi(H).$$

Note that

$$\pm\Gamma(q^t)\frac{A}{t}\Gamma(q^t) \leq C\epsilon(N+1), \tag{6.10}$$

which, using Lemma 3.2, shows that $\Phi(t)$ is uniformly bounded. We compute its Heisenberg derivative:

$$\begin{aligned}\mathbf{D}\Phi(t) &= \chi(H)\mathrm{d}\Gamma(q^t, \mathbf{d}_0 q^t)\tfrac{A}{t}\Gamma(q^t)\chi(H) + \mathrm{hc}\\ &+\chi(H)[V, \mathrm{i}\Gamma(q^t)]\tfrac{A}{t}\Gamma(q^t)\chi(H) + \mathrm{hc}\\ &+t^{-1}\chi(H)\Gamma(q^t)[H, \mathrm{i}A]\Gamma(q^t)\chi(H)\\ &-t^{-1}\chi(H)\Gamma(q^t)\tfrac{A}{t}\Gamma(q^t)\chi(H)\\ &=: R_1(t) + R_2(t) + R_3(t) + R_4(t).\end{aligned}$$

By Lemma 2.9 *iii)* and condition *(SR)*, we have $[V, \mathrm{i}\Gamma(q^t)] \in O_N(t^{-1-\mu})(N+1)^{\frac{1}{2}}$, which implies that

$$\|R_2(t)\| \in O(t^{-1-\mu}). \tag{6.11}$$

Let us now consider $R_1(t)$. We have

$$\mathbf{d}_0 q^t = -\tfrac{1}{2t}\langle\tfrac{x}{t} - \nabla\omega(k), \nabla q(\tfrac{x}{t})\rangle + \mathrm{hc} + r^t =: \tfrac{1}{t}g^t + r^t,$$

where $r^t \in O(t^{-2})$. By Lemma 2.8 *vi)*, we have

$$\|\chi(H)\mathrm{d}\Gamma(q^t, r^t)\tfrac{A}{t}\Gamma(q^t)\chi(H)\| \in O(t^{-2}). \tag{6.12}$$

Next we set

$$B_1 := \chi(H)\mathrm{d}\Gamma(q^t, g^t)(N+1)^{-\frac{1}{2}}, \quad B_2^* := (N+1)^{\frac{1}{2}}\frac{A}{t}\Gamma(q^t)\chi(H),$$

and use the inequality

$$R_1(t) = t^{-1}B_1B_2^* + t^{-1}B_2B_1^* \geq -\epsilon_0^{-1}t^{-1}B_1B_1^* - \epsilon_0 t^{-1}B_2B_2^*. \tag{6.13}$$

We have

$$B_2B_2^* = \chi(H)\Gamma(q^t)\frac{A^2}{t^2}(N+1)\Gamma(q^t)\chi(H).$$

Introducing cutoff functions $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ and $\tilde{q} \in C_0^\infty(\mathbb{R}^d)$ with $\tilde{\chi}\chi = \chi, \tilde{q}q = q$, it is easy to check that

$$\begin{aligned}&\Gamma(q^t)\tfrac{A^2}{t^2}\Gamma(q^t) \leq C(N+1)\Gamma(q^t)^2(N+1),\\ &\chi(H)(N+1)^p\chi(H) \leq C\chi^2(H), \quad p \in \mathbb{N}.\end{aligned} \tag{6.14}$$

This gives using Lemma 3.3

$$
\begin{aligned}
-B_2B_2^* &\geq -C\chi(H)(N+1)^{3/2}\Gamma(q^t)^2(N+1)^{3/2}\chi(H) \\
&= -C\Gamma(q^t)\chi(H)(N+1)^3\chi(H)\Gamma(q^t)+O(t^{-1}) \\
&\geq -C_1\Gamma(q^t)\chi^2(H)\Gamma(q^t)-Ct^{-1} \\
&= -C_1\chi(H)\Gamma(q^t)^2\chi(H)-Ct^{-1}.
\end{aligned} \tag{6.15}
$$

Next we write

$$
B_1B_1^* = \chi(H)\mathrm{d}\Gamma(q^t,g^t)(N+1)^{-1}\mathrm{d}\Gamma(q^t,g^t)\chi(H),
$$

and use Lemma 2.8 *vi)* to obtain

$$
\begin{aligned}
|(u|B_1B_1^*u)| &= \|(N+1)^{-\frac{1}{2}}\mathrm{d}\Gamma(q^t,g^t)\chi(H)u\|^2 \\
&\leq \|\mathrm{d}\Gamma(g^{t*}g^t)^{\frac{1}{2}}\chi(H)u\|^2.
\end{aligned}
$$

Using Prop. 6.2, we obtain

$$
\int_1^\infty \|B_1\mathrm{e}^{-\mathrm{i}tH}u\|^2\frac{\mathrm{d}t}{t} \leq C\|u\|^2. \tag{6.16}
$$

Using Lemma 3.3, we have

$$
\begin{aligned}
R_3(t) &= t^{-1}\Gamma(q^t)\chi(H)[H,\mathrm{i}A]\chi(H)\Gamma(q^t)+O(t^{-2}) \\
&\geq C_0t^{-1}\Gamma(q^t)\chi^2(H)\Gamma(q^t)-Ct^{-2} \\
&\geq C_0t^{-1}\chi(H)\Gamma^2(q^t)\chi(H)-Ct^{-2}.
\end{aligned} \tag{6.17}
$$

On the other hand, we have by (6.10) and (6.14)

$$
\begin{aligned}
-R_4(t) &\leq C\tfrac{\epsilon}{t}\chi(H)(N+1)^{\frac{1}{2}}\Gamma^2(q^t)(N+1)^{\frac{1}{2}}\chi(H) \\
&\leq C\tfrac{\epsilon}{t}\Gamma(q^t)\chi(H)(N+1)\chi(H)\Gamma(q^t)+Ct^{-2} \\
&\leq C_2\tfrac{\epsilon}{t}\Gamma(q^t)\chi^2(H)\Gamma(q^t)+Ct^{-2} \\
&\leq C_2\tfrac{\epsilon}{t}\chi(H)\Gamma(q^t)^2\chi(H)+Ct^{-2}.
\end{aligned} \tag{6.18}
$$

Collecting (6.15), (6.17) and (6.18) we obtain

$$
\begin{aligned}
&-\epsilon_0t^{-1}B_2^*(t)B_2(t)+R_3(t)+R_4(t) \\
&\geq (-\epsilon_0C_1+C_0-\epsilon C_2)t^{-1}\chi(H)\Gamma(q^t)^2\chi(H)-Ct^{-2}.
\end{aligned} \tag{6.19}
$$

We pick now ϵ and ϵ_0 small enough so that $\tilde{C}_0 := -\epsilon_0C_1+C_0-\epsilon C_2>0$. Using (6.11), (6.16) and (6.19) we conclude that

$$
\mathbf{D}\Phi(t) \geq \frac{\tilde{C}_0}{t}\chi(H)\Gamma^2(q^t)\chi(H)-R(t)-Ct^{-1-\mu},
$$

where $R(t)$ is integrable along the evolution. By Lemma A.1, this proves the proposition for χ with support close enough to an energy level $\lambda \subset \mathbb{R}\backslash(\tau\cup\sigma_{\mathrm{pp}}(H))$. To prove the Proposition for all χ supported in $\mathbb{R}\backslash(\tau\cup\sigma_{\mathrm{pp}}(H))$ we use the argument in [DeGe, Prop. 4.4.7]. □

7 Asymptotic completeness

7.1 Introduction

In this subsection we describe the main results of this section (and of the whole article). They will be formulated in the following four theorems.

Theorem 7.1 *Assume hypotheses (H1), (H2), (I1) and (SR). Let $q \in C_0^\infty(\mathbb{R}^d)$, $0 \leq q \leq 1$, $q = 1$ near 0. Set $q^t(x) = q(\frac{x}{t})$. Then there exists*

$$\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}tH} =: \Gamma^+(q),$$

We have

$$\Gamma^+(q\tilde{q}) = \Gamma^+(q)\Gamma^+(\tilde{q}),$$
$$0 \leq \Gamma^+(q) \leq \Gamma^+(\tilde{q}) \leq 1\!\mathrm{l}, \qquad 0 \leq q \leq \tilde{q} \leq 1,$$
$$[H, \Gamma^+(q)] = 0.$$

The above theorem, or actually its generalization, will be proven in Thm. 7.5.

Using this theorem, we define

$$P_0^+ := \text{s-}\lim_{n\to\infty} \Gamma^+(q_n),$$

where $q_n \in C_0^\infty$ is a decreasing sequence of functions such that $q_n \searrow 1\!\mathrm{l}_{\{0\}}$.

Theorem 7.2 *P_0^+ does not depend on the choice of the sequence q_n. It satisfies*

$$(P_0^+)^2 = P_0^+, \qquad [H, P_0^+] = 0.$$

The first important result of this section is the following theorem, which we call *geometric asymptotic completeness.*

Theorem 7.3 *Assume hypotheses (H1), (H2), (I1) and (SR). Then the space of asymptotic matter is equal to the space of states living near the origin:*

$$\mathcal{K}^+ = \mathrm{Ran} P_0^+.$$

The second main result is the standard asymptotic completeness, which holds under the additional condition *(H0).*

Theorem 7.4 *Assume hypotheses (H0), (H1), (H2), (I1) and (SR). Then the space of asymptotic matter is equal to the space of bound states of H:*

$$\mathcal{K}^+ = \mathcal{H}_{\mathrm{pp}}(H).$$

7.2 An asymptotic partition of unity

Let $f_0 \in C_0^\infty(\mathrm{I\!R}^d)$, $f_\infty \in C^\infty(\mathrm{I\!R}^d)$, $0 \le f_0$, $0 \le f_\infty$, $f_0 + f_\infty \le 1$, $f_0 = 1$ near 0 (and hence $f_\infty = 0$ near 0). Set $f := (f_0, f_\infty)$. Set also $f^t = (f_0^t, f_\infty^t)$, where $f_0^t(x) = f_0(\frac{x}{t})$, $f_\infty^t(x) = f_\infty(\frac{x}{t})$.

Theorem 7.5 *i) The following limits exist:*

$$Q_k^+(f) := \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} Q_k(f^t)\mathrm{e}^{-\mathrm{i}tH},$$
$$P_k^+(f) = Q_k^+(f) - Q_{k-1}^+(f) = \text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} P_k(f^t)\mathrm{e}^{-\mathrm{i}tH}.$$

ii) For $q \in C^\infty(\mathrm{I\!R}^d)$ such that $0 \le q \le 1$, $\nabla q \in C_0^\infty(\mathrm{I\!R}^d)$, $q = 1$ on a neighborhood of zero and $q^t := q(\frac{x}{t})$, there exists

$$\Gamma^+(q) := \text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}tH}.$$

Clearly,

$$P_0^+(f) = Q_0^+(f) = \Gamma^+(f_0).$$

iii)

$$[Q_k^+(f), H] = 0.$$

iv) If $0 \le q \le 1$, $q = 1$ near 0, then

$$Q_k^+(fq) = Q_k^+(f)\Gamma^+(q),$$

where $qf = (qf_0, qf_\infty)$.
v)

$$0 \le Q_{k_1}^+(f) \le Q_{k_2}^+(f) \le \Gamma^+(f_0 + f_\infty), \quad k_1 \le k_2,$$
$$\text{s-}\lim_{k\to\infty} Q_k^+(f) = \Gamma^+(f_0 + f_\infty).$$
$$\|(H+\mathrm{i})^{-1}\left(Q_k^+(f) - \Gamma^+(f_0+f_\infty)\right)\| \le C(k+1)^{-1}.$$

If, moreover, $f_0 + f_\infty = 1$, then

$$\text{s-}\lim_{t\to\infty} Q_k^+(f) = 1\!\mathrm{l}, \quad \|(H+\mathrm{i})^{-1}\left(Q_k^+(f) - 1\!\mathrm{l}\right)\| \le C(k+1)^{-1}.$$

vi) If $\tilde{f} = (\tilde{f}_0, \tilde{f}_\infty)$ is another pair of functions satisfying the conditions stated at the beginning of this subsection, and moreover, $\tilde{f}_0 f_\infty = 0$, then

$$Q_k^+(\tilde{f})P_k^+(f) = P_k^+(\tilde{f})P_k^+(f) = P_k^+(\tilde{f}_0 f_0, \tilde{f}_\infty f_\infty).$$

Proof. Let us first prove *i)*. Using Lemma 3.3 for $m = 0$ and a density argument, it suffices to prove the existence of

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\chi(H)Q_k(f^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}.$$

We compute the Heisenberg derivative:

$$\chi(H)\mathbf{D}Q_k(f^t)\chi(H) = \chi(H)\mathrm{d}Q_k(f^t, \mathbf{d}_0 f^t)\chi(H) + \chi(H)[V, \mathrm{i}Q_k(f^t)]\chi(H),$$

by Lemma 2.11. From (3.8) and Lemma 3.2 we obtain

$$\|\chi(H)[V, \mathrm{i}Q_k(f^t)]\chi(H)\| \in O(t^{-1-\mu}). \tag{7.1}$$

Next we compute:

$$\mathbf{d}_0 f^t = \sum_1^d \frac{1}{t} g_i^t + r^t,$$

where

$$g_i^t = -\tfrac{1}{2}\left(\left(\tfrac{x_i}{t} - \partial_i\omega(k)\right)\partial_i f\left(\tfrac{x}{t}\right) + \mathrm{hc}\right) = (g_{i,0}^t, g_{i,\infty}^t),$$

and $r^t \in O(t^{-2})$. Using Lemma 2.11 *vi)* and Lemma 3.2, we obtain that

$$\|\chi(H)\mathrm{d}Q_k(f^t, r^t)\chi(H)\| \in O(t^{-2}). \tag{7.2}$$

On the other hand, using Lemma 2.11 *v)*, we have

$$\begin{aligned} |(u_2|\chi(H)\mathrm{d}Q_k(f^t, g_i^t)\chi(H)u_1)| \quad &\leq \|\mathrm{d}\Gamma(|g_{i,0}^t|)^{\frac{1}{2}}\chi(H)u_2\|\|\mathrm{d}\Gamma(|g_{i,0}^t|)^{\frac{1}{2}}\chi(H)u_1\| \\ &+\|\mathrm{d}\Gamma(|g_{i,\infty}^t|)^{\frac{1}{2}}\chi(H)u_2\|\|\mathrm{d}\Gamma(|g_{i,\infty}^t|)^{\frac{1}{2}}\chi(H)u_1\|. \end{aligned} \tag{7.3}$$

Hence the existence of the limits *i)* follows from (7.1)–(7.3), Proposition 6.3 and Lemma A.1.

ii) is obvious and *iii)* follows by Lemma 3.3. *iv)* follows from

$$Q_k(f^t q^t) = Q_k(f^t)\Gamma(q^t).$$

The first statement of *v)* follows from Lemma 2.9 *iv)*, and the second follows from the third. To see the third statement we first observe that

$$\|(N+1)^{-1}(Q_k(f^t) - \Gamma(f_0^t + f_\infty^t))\| \leq (k+1)^{-1},$$

which implies

$$\|(H+\mathrm{i})^{-1}(Q_k(f^t) - \Gamma(f_0^t + f_\infty^t))\| \leq \|(H+\mathrm{i})^{-1}(N+1)\|(k+1)^{-1}. \tag{7.4}$$

This completes the proof of *v)*. *vi)* follows from Prop. 2.10. □

An analogous theorem is true for the free Hamiltonian, but it is much easier. It follows within each n-particle sector by the stationary phase method. Note that in the free case one does not need to assume that the cutoff is one near zero.

Proposition 7.6 *Let* $f_0 \in C_0^\infty(\mathbb{R}^d), f_\infty \in C^\infty(\mathbb{R}^d)$ *with* $f_0 + f_\infty \leq 1$. *Then*

$$\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\omega(k))}Q_k(f^t)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\omega(k))} = Q_k(f(\nabla\omega(k))).$$

7.3 Asymptotic projections

In this section, using the monotonicity properties of the observables $Q_k(f^t)$ with respect to f_∞, we prove the existence of the limits

$$P_k^+(U) := \text{s-}\lim_{f_\infty \to 1\!\!1_U} P_k^+(f),$$

where U is an open or closed set in $\mathbb{R}^d \backslash \{0\}$. The range of $P_k(U)^+$ consists of the states with exactly k bosons with an asymptotic velocity in U. $P_k^+(U)$ are a mutually orthogonal family of projections with sum equal to one.

Let $U \subset \mathbb{R}^d$ be an open set with $0 \notin U$. Let $f_{\infty,n} \in C_0^\infty(\mathbb{R}^d)$ be a sequence of cutoff functions. We will say that $f_{\infty,n} \nearrow 1\!\!1_U$ if

$$0 \le f_{\infty,n} \le 1\!\!1_U, \quad f_{\infty,n} \le f_{\infty,n+1}, \quad f_{\infty,n+1} f_{\infty,n} = f_{\infty,n}, \quad f_{\infty,n} \to 1\!\!1_U \text{ pointwise.} \tag{7.5}$$

Similarly if $U \subset \mathbb{R}^d$ is a closed set with $0 \notin U$, we say that $f_{\infty,n} \searrow 1\!\!1_U$ if

$$1\!\!1_U \le f_{\infty,n} \le 1, \quad f_{\infty,n} \le f_{\infty,n+1}, \quad f_{\infty,n+1} f_{\infty,n} = f_{\infty,n+1}, \quad f_{\infty,n} \to 1\!\!1_U \text{ pointwise.} \tag{7.6}$$

For such sequences of cutoff functions and $f_n = (f_{0,n}, f_{\infty,n})$ with $f_{0,n} = 1 - f_{\infty,n}$, we denote by $P_{k,n}^+, Q_{k,n}^+$ the operators $P_k^+(f_n), Q_k^+(f_n)$.

Theorem 7.7 *i) Let $U \subset \mathbb{R}^d \backslash \{0\}$ be an open (resp. closed) set. Let f_n be a sequence of cutoff functions such that $f_{\infty,n} \nearrow 1\!\!1_U$ (resp. $f_{\infty,n} \searrow 1\!\!1_U$). Then the limits*

$$Q_k^+(U) := \text{s-}\lim_{n\to\infty} Q_{k,n}^+$$

exist and are independent of the sequence $f_{\infty,n}$. Moreover $\{Q_k^+(U)\}_{k\in\mathbb{N}}$ is an increasing family of projections such that

$$[Q_k(U)^+, H] = 0, \quad \text{s-}\lim_{k\to\infty} Q_k(U)^+ = 1\!\!1.$$

ii) The limits

$$P_k^+(U) := \text{s-}\lim_{n\to\infty} P_{k,n}^+$$

exist and are independent of the sequence f_n. Moreover, we have

$$P_k^+(U) = Q_k^+(U) - Q_{k-1}^+(U).$$

The family $\{P_k^+(U)\}_{k\in\mathbb{N}}$ is a family of mutually orthogonal projections such that

$$[P_k^+(U), H] = 0, \qquad \text{s} - \sum_{k=0}^{\infty} P_k^+(U) = 1\!\!1.$$

iii) If $f = (f_0, f_\infty)$ satisfies the hypotheses of Subsect. 7.2 and $f_0 + f_\infty = 1$, $\operatorname{supp} f_\infty \subset U$, then

$$Q_k^+(f) Q_k^+(U) = Q_k^+(U). \tag{7.7}$$

Of particular importance are the projections $Q_k^+(\mathbb{R}^d\backslash\{0\}), P_k^+(\mathbb{R}^d\backslash\{0\})$, which will be denoted simply by Q_k^+, P_k^+ in what follows. The range of P_k^+ corresponds to the states with exactly k asymptotically free bosons.

Proof of Thm. 7.7. It suffices to consider the case when U is an open set. The case of a closed set is similar. We deduce first from Prop. 2.13 that if $f_{\infty,n} \nearrow 1\!\!1_U$ in the sense of (7.5), we have

$$Q^+_{k,n+1} \leq Q^+_{k,n}, \quad Q^+_{l,m}Q^+_{k,n} = Q^+_{l,m}, \quad l \leq k, \quad n < m. \tag{7.8}$$

Using (7.8) and Lemma A.3, we obtain the existence of

$$Q_k^+(U) = \text{s-}\lim_{n\to\infty} Q^+_{k,n}$$

and hence of $P_k(U)^+$.

Let us check that $Q_k(U)^+$ is independent of the sequence $f_{\infty,n}$. Let $f_{\infty,n}, \tilde{f}_{\infty,n}$ be two sequences with $f_{\infty,n}, \tilde{f}_{\infty,n} \nearrow 1\!\!1_U$ in the sense of (7.5). Then there exists a sequence $m_n \in \mathbb{N}$ tending to ∞ such that $f_{\infty,n} \leq \tilde{f}_{\infty,m_n}$. This implies that

$$Q_k^+(f_n) \leq Q_k^+(\tilde{f}_{m_n}),$$

which shows that

$$\lim_{n\to\infty} Q_k^+(f_n) \leq \lim_{n\to\infty} Q_k^+(\tilde{f}_{m_n}) = \lim_{n\to\infty} Q_k^+(\tilde{f}_n).$$

Hence $Q_k^+(U)$ is independent of the sequence f_n.

We deduce from Thm. 7.5 *v)* that

$$0 \leq Q_k^+(U) \leq 1,$$

$$\|(H+\mathrm{i})^{-1}(Q_k(U)^+ - 1\!\!1)\| \leq C(k+1)^{-1},$$

which implies the strong convergence of $Q_k^+(U)$ to $1\!\!1$. The fact that $Q_k^+(U)$ commutes with H follows also from Thm. 7.5.

We deduce from (7.8) that

$$Q_l^+(U)Q_k^+(U) = Q_l^+(U), \; l \leq k.$$

This implies that $\{Q_k^+\}_{k\in\mathbb{N}}$ is an increasing family of projections. Finally *iii)* follows from Prop. 2.13. □

7.4 Asymptotic projections and asymptotic fields

In this subsection, we prove that applying an asymptotic annihilation operator $a^+(h)$ amounts to decrease the number of asymptotically free bosons by one. As an immediate consequence we obtain that the range of P_0^+ is included in the space of asymptotic matter $\mathcal{K}^+$.

Proposition 7.8 *Assume the hypotheses (H1), (H2), (I1), (SR). Then the following identities hold in the sense of quadratic forms on* $\mathcal{D}((H+\mathrm{i})^{\frac{1}{2}})$*:*

$$a^+(h)Q_k^+ = Q_{k-1}^+a^+(h), \quad h \in \mathfrak{h}.$$

$$a^+(h)P_k^+ = P_{k-1}^+a^+(h), \quad h \in \mathfrak{h}.$$

Proof. It is enough to prove the identity involving Q_k^+. By the continuity of $h \mapsto a^\sharp(h)(H+\mathrm{i})^{-\frac{1}{2}}$ it is enough to assume that $h \in \mathfrak{h}_0$. Let $f_{\infty,n} \nearrow 1\!\!1_{\mathbb{R}^d\setminus\{0\}}$ in the sense of (7.5) and $f_{0,n} = 1 - f_{\infty,n}$. We have

$$\begin{aligned}&(\mathrm{i}+H)^{-\frac{1}{2}}\Big(a^+(h)Q_k^+ - Q_{k-1}^+a^+(h)\Big)(\mathrm{i}+H)^{-\frac{1}{2}}\\ &= \lim_{n\to\infty}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}(\mathrm{i}+H)^{-\frac{1}{2}}\Big(a(h_t)Q_k(f_n^t) - Q_{k-1}(f_n^t)a(h_t)\Big)(\mathrm{i}+H)^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}tH}.\end{aligned} \tag{7.9}$$

By Lemma 2.12 *ii)*, we have

$$a(h_t)Q_k(f_n^t) - Q_{k-1}(f_n^t)a(h_t) = P_k(f_n^t)a(f_{0,n}^t h_t). \tag{7.10}$$

Since $h \in \mathfrak{h}_0$ and $h_t = \mathrm{e}^{-\mathrm{i}t\omega(k)}h$, we see by stationary phase arguments that, for $n \geq n_0$, $\|f_{0,n}^t h_t\| \in o(t^0)$. This gives

$$\|\Big(a(h_t)Q_k(f_n^t) - Q_{k-1}(f_n^t)a(h_t)\Big)(H+\mathrm{i})^{-\frac{1}{2}}\| \in o(t^0),$$

which implies that (7.9) is zero. □

Corollary 7.9 *Assume the hypotheses (H1), (H2), (I1), (SR). Then*

$$\mathrm{Ran}P_0^+ \subset \mathcal{K}^+.$$

Proof. Let $u \in \mathrm{Ran}P_0^+$ and let $u_n \in \mathcal{D}(H) \cap \mathrm{Ran}P_0^+$ be a sequence converging to u. By Prop. 7.8 we have $a^+(h)u_n = 0$, $h \in \mathfrak{h}$, and hence $u_n \in \mathcal{K}^+$. Since $\mathcal{K}^+$ is closed, $u \in \mathcal{K}^+$. □

7.5 Geometric inverse wave operators

Let $j_0 \in C_0^\infty(\mathbb{R}^d)$, $j_\infty \in C^\infty(\mathbb{R}^d)$, $0 \leq j_0$, $0 \leq j_\infty$, $j_0^2 + j_\infty^2 \leq 1$, $j_0 = 1$ near 0 (and hence $j_\infty = 0$ near 0). Set $j := (j_0, j_\infty)$. Set also $j^t = (j_0^t, j_\infty^t)$, where $j_0^t(x) = j_0(\frac{x}{t})$, $j_\infty^t(x) = j_\infty(\frac{x}{t})$.

As in Subsection 2.15, we identify the pair j^t with an operator $j^t : \mathfrak{h} \to \mathfrak{h}\oplus\mathfrak{h}$ and we introduce the operator $\check{\Gamma}(j^t) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$. We use the same notation $\check{\Gamma}(j^t)$ to denote the operator $1\!\!1_{\mathcal{K}} \otimes \check{\Gamma}(j^t) : \mathcal{H} = \mathcal{K} \otimes \Gamma(\mathfrak{h}) \to \mathcal{H}^{\mathrm{ext}} = \mathcal{K} \otimes \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$.

Theorem 7.10 *i) The following limits exist:*

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}}\check{\Gamma}(j^t)\mathrm{e}^{-\mathrm{i}tH}, \tag{7.11}$$

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\check{\Gamma}(j^t)^*\mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}}. \tag{7.12}$$

If we denote (7.11) by $W^+(j)$, then (7.12) equals $W^+(j)^*$. If we set

$$W_k^+(j) := 1\!\!1_{\{k\}}(N_\infty)W^+(j),$$

then

$$W_k^+(j) = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}}1\!\!1_k(N_\infty)\check{\Gamma}(j^t)\mathrm{e}^{-\mathrm{i}tH}.$$

ii) One has

$$W^+(j)(H+\mathrm{i})^{-1} = (H^{\mathrm{ext}}+\mathrm{i})^{-1}W^+(j),$$

$$W^+(j)\chi(H) = \chi(H^{\mathrm{ext}})W^+(j),\ \chi \in C_0^\infty(\mathbb{R}).$$

iii) Let $q_0, q_\infty \in C^\infty(\mathbb{R}^d)$, $\nabla q_0, \nabla q_\infty \in C_0^\infty(\mathbb{R}^d)$, $0 \leq q_0, q_\infty \leq 1$, $q_0 = 1$ *near 0. Set* $\tilde{j} := (\tilde{j}_0, \tilde{j}_\infty) := (q_0 j_0, q_\infty j_\infty)$. *Then*

$$\Gamma^+(q_0) \otimes \Gamma(q_\infty(\nabla\omega(k)))W^+(j) = W^+(\tilde{j}).$$

iv) Let $q \in C^\infty$, $\nabla q \in C_0^\infty$, $0 \leq q \leq 1$, $q = 1$ *near zero. Then*

$$W^+(j)\Gamma^+(q) = W^+(qj),$$

where $qj = (qj_0, qj_\infty)$.
v) Let $\tilde{j} = (\tilde{j}_0, \tilde{j}_\infty)$ *be another pair satisfying the conditions stated at the beginning of this subsection. (Note that* $0 \leq \tilde{j}_0 j_0, 0 \leq \tilde{j}_\infty j_\infty$, $\tilde{j}_0 j_0 + \tilde{j}_\infty j_\infty \leq 1$ *and* $\tilde{j}_0 j_0 = 1$ *near zero). Then*

$$W^+(\tilde{j})^* W^+(j) = \Gamma^+(\tilde{j}_0 j_0 + \tilde{j}_\infty j_\infty),$$

$$W_k^+(\tilde{j})^* W_k^+(j) = P_k^+(\tilde{j}_0 j_0, \tilde{j}_\infty j_\infty).$$

In particular, if $j_0^2 + j_\infty^2 = 1$, *then* $W^+(j)$ *is isometric.*
vi) Let $j_0 + j_\infty = 1$. *If* $u \in \mathcal{D}((H+\mathrm{i})^{k/2})$ *then* $W_k^+(j)u \in \mathcal{D}((H+\mathrm{i})^{k/2}) \otimes \bigotimes_{\mathrm{s}}^k \mathfrak{h} \subset \mathcal{D}(\Omega^{\mathrm{ext},+})$ *and*

$$\Omega^{\mathrm{ext},+}W_k^+(j)u = P_k^+(j)u.$$

If $u \in \mathcal{H}_{\mathrm{comp}}(H)$ *then* $W^+(j)u \subset \mathcal{D}(\Omega^{\mathrm{ext},+})$ *and*

$$\Omega^{\mathrm{ext},+}W^+(j)u = u.$$

Proof. Let us first prove the existence of the limit (7.11), the case of (7.12) being similar. Using Lemma 3.4 and a density argument, it suffices to prove the existence of

$$\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}}\chi(H^{\mathrm{ext}})\check{\Gamma}(j^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH},$$

for some $\chi \in C_0^\infty(\mathbb{R})$. We compute the asymmetric Heisenberg derivative

$$\begin{aligned}\chi(H^{\mathrm{ext}})\check{\mathbf{D}}\check{\Gamma}(j^t)\chi(H) \quad &= \chi(H^{\mathrm{ext}})\check{\mathbf{D}}_0\check{\Gamma}(j^t)\chi(H) \\ &\quad + \mathrm{i}\chi(H^{\mathrm{ext}})(V \otimes 1\!\mathrm{l}\check{\Gamma}(j^t) - \check{\Gamma}(j^t)V)\chi(H).\end{aligned}$$

¿From (3.11), we obtain

$$\|\chi(H^{\mathrm{ext}})(V \otimes 1\!\mathrm{l}\check{\Gamma}(j^t) - \check{\Gamma}(j^t)V)\chi(H)\| \in O(t^{-1-\mu}). \tag{7.13}$$

On the other hand by Lemma 2.16, we have $\check{\mathbf{D}}_0\check{\Gamma}(j^t) = \mathrm{d}\check{\Gamma}(j^t, \check{\mathbf{d}}_0 j^t)$, and, by pseudodifferential calculus,

$$\check{\mathbf{d}}_0^t j^t = \sum_1^d \frac{1}{t}k_i^t + r^t,$$

where

$$k_i^t = (k_{0,i}^t, k_{\infty,i}^t), \qquad k_{\epsilon,i}^t = -\frac{1}{2}((\frac{x_i}{t} - \partial_i\omega(k))\partial_i j_\epsilon(\frac{x}{t}) + \mathrm{hc}),$$

and $r^t \in O(t^{-2})$. Using Lemma 2.16 *v)* and Lemma 3.2, we obtain

$$\|\chi(H^{\mathrm{ext}})\mathrm{d}\check{\Gamma}(j^t, r^t)\chi(H)\| \in O(t^{-2}). \tag{7.14}$$

Using then Lemma 2.16 *iv)*, we obtain

$$\begin{aligned}&|(u_2|\chi(H^{\mathrm{ext}})\mathrm{d}\check{\Gamma}(j^t, k_i^t)\chi(H)u_1)|\\ &\leq \|(\mathrm{d}\Gamma(|k_{0,i}^t|)^{\frac{1}{2}} \otimes 1\!\!1)\chi(H^{\mathrm{ext}})u_2\|\|\mathrm{d}\Gamma(|k_{0,i}^t|)^{\frac{1}{2}}\chi(H)u_1\|\\ &+\|(\mathrm{d}\Gamma(|k_{\infty,i}^t|)^{\frac{1}{2}} \otimes 1\!\!1)\chi(H^{\mathrm{ext}})u_2\|\|\mathrm{d}\Gamma(|k_{\infty,i}^t|)^{\frac{1}{2}}\chi(H)u_1\|.\end{aligned} \tag{7.15}$$

Hence the existence of the limit (7.11) follows from (7.13)–(7.15), Proposition 6.3 and Lemma A.1.

ii) follows from Lemma 3.4. *iii)* follows from Prop. 7.6 and the fact that

$$\Gamma(q_0^t) \otimes \Gamma(q_\infty^t)\check{\Gamma}(j^t) = \check{\Gamma}(\tilde{j}^t).$$

iv) follows from

$$\check{\Gamma}(j^t)\Gamma(j^t) = \check{\Gamma}((jq)^t).$$

v) follows from

$$\begin{aligned}\check{\Gamma}^*(\tilde{j}^t)\check{\Gamma}(j^t) &= \Gamma(\tilde{j}_0^t j_0^t + \tilde{j}_\infty^t j_\infty^t),\\ \check{\Gamma}^*(\tilde{j}^t)1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(j^t) &= P_k(\tilde{j}_0^t j_0^t, \tilde{j}_\infty^t j_\infty^t).\end{aligned}$$

Up to technical details due to the unboundedness of I, *vi)* can be considered as a special case of *v)* with $\tilde{j} = (1,1)$. To prove *vi)* we note that it follows from *ii)* that $W^+(j)$ is bounded from $\mathcal{D}((H+\mathrm{i})^k)$ to $\mathcal{D}((H^{\mathrm{ext}}+\mathrm{i})^k)$ for $k \in \mathbb{N}$. This extends to all $k \in \mathbb{R}^+$ by interpolation. By Thm. 5.7 *i)*, we have for $w \in \mathcal{H} \otimes \bigotimes_{\mathrm{s}}^k \mathfrak{h}$

$$\Omega^{\mathrm{ext},+}(H+\mathrm{i})^{-k/2}w = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}}(H+\mathrm{i})^{-k/2}w.$$

Since by (2.10) $I(H+\mathrm{i})^{-k/2}1\!\!1_{\{k\}}(N_\infty)$ is a bounded operator, we can use the chain rule of the wave operators and write

$$\Omega^{\mathrm{ext},+}W_k^+(j)u = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I 1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(j^t)\mathrm{e}^{-\mathrm{i}tH}u = P_k^+(j)u,$$

by (2.11). Finally the second statement of *vi)* is an immediate consequence of the first. □

7.6 Asymptotic absolute continuity

In this subsection we will prove that if U is a closed set of Lebesgue measure 0 with $0 \notin U$, then $P_0(U)^+ = 1\!\!1$, which means that there are no bosons living asymptotically in U. This property will be needed in the next section to effectively decouple bosons close to the origin from bosons close to infinity.

Theorem 7.11 *Assume the hypotheses (H1), (H2), (I1), (SR). Let $U \subset \mathbb{R}^d \backslash \{0\}$ be a compact set of measure zero. Then*

$$P_0^+(U) = \mathbb{1}.$$

Proof. Let $j = (j_0, j_\infty)$ be as in Subsection 7.5. Assume additionally that $j_0^2 + j_\infty^2 = 1$ and $j_\infty = 1$ near U. Let $q_n \in C^\infty(\mathbb{R}^d \backslash \{0\})$, $0 \le q_n$, $q_n \nearrow \mathbb{R}^d \backslash U$ in the sense of (7.5). For large enough n we have $j_0 q_n = j_0$. Hence

$$\begin{aligned}\Gamma^+(q_n) &= \Gamma^+(q_n) W^+(j)^* W^+(j)\\ &= W^+(j_0, j_\infty q_n)^* W^+(j)\\ &= W^+(j)^* \mathbb{1} \otimes \Gamma(q_n(\nabla\omega(k))) W^+(j),\end{aligned}$$

by Thm. 7.10. But

$$\text{s-}\lim_{n\to\infty} q_n(\nabla\omega(k)) = \mathbb{1}.$$

Therefore,

$$\text{s-}\lim_{n\to\infty} \Gamma(q_n(\nabla\omega(k))) = \mathbb{1},$$

and

$$P_0^+(U) = \text{s-}\lim_{n\to\infty} \Gamma^+(q_n) = \mathbb{1}.$$

□

Note in parenthesis another result, which follows from exactly the same arguments.(This result will not be used in the proof of asymptotic completeness).

Proposition 7.12 *Assume hypotheses (H1), (H2), (I1), (SR). Let $U \subset \mathbb{R}^d \backslash \{0\}$ be an open or close set such that*

$$U \cap \nabla\omega(\mathbb{R}^d) = \emptyset.$$

Then

$$P_0^+(U) = \mathbb{1}.$$

7.7 Geometric asymptotic completeness

In this subsection we will show Theorem 7.3. It will follow from the following theorem, which gives an explicit construction of the inverse wave operator Ω^{+*} in terms of the geometric inverse wave operators.

Theorem 7.13 *Let $j_n = (j_{0,n}, j_{\infty,n})$ satisfy the conditions of Subsect. 7.5. Additionally, assume that $j_{0,n}^2 + j_{\infty,n}^2 = 1$ and $j_{\infty,n} \nearrow \mathbb{1}_{\mathbb{R}^d\backslash\{0\}}$ in the sense of (7.5). Then*

$$W^+ := \text{s-}\lim_{n\to\infty} \sum_{k=0}^{\infty} W_k^+(j_n) P_k^+$$

exists. One has

(7.16)
$$W^+ = \Omega^{+*},$$

and

$$\mathrm{Ran} W^+ = \mathcal{H}^+ = \mathrm{Ran} P_0^+ \otimes \Gamma(\mathfrak{h}).$$

Proof. Set

$$f_n = (f_{0,n}, f_{\infty,n}) := (j_{0,n}^2, j_{\infty,n}^2)$$

$$f_n^0 = (f_{0,n}^0, f_{\infty,n}^0) := (j_{0,n}, 1 - j_{0,n}),$$

$$f_n^\infty = (f_{0,n}^\infty, f_{\infty,n}^\infty) := (1 - j_{\infty,n}, j_{\infty,n}),$$

$$f_{n,m} = (f_{0,n,m}, f_{\infty,n,m}) = (j_{0,n}j_{0,m}, j_{\infty,n}j_{\infty,m}).$$

Let $m \leq n$ and $u \in \mathcal{H}$ such that $P_k^+ u = u$. By density we may assume that $u \in \mathcal{H}_{\rm comp}(H)$. Note that since $j_{\infty,n} \nearrow 1\!\mathrm{l}_{\mathbb{R}^d \backslash \{0\}}$ in the sense of (7.5), we have $j_{0,n}j_{\infty,m} = 0$, which by Thm. 7.5 *vi)* gives

(7.17)
$$P_k^+(f_{n,m}) = P_k^+(f_n^0)P_k^+(f_m^\infty).$$

Now we compute

$$\begin{aligned}\|W_k^+(j_n)u - W_k^+(j_m)u\|^2 &= \|W_k^+(j_n)u\|^2 + \|W_k^+(j_m)u\|^2 - 2Re(W_k^+(j_n)u|W_k^+(j_m)u)\\ &= (u|P_k^+(f_n)u) + (u|P_k^+(f_m)u) - 2Re(u|P_k^+(f_{n,m})u)\\ &= (u|P_k^+(f_n)u) + (u|P_k^+(f_m)u) - 2Re(P_k^+(f_m^\infty)u|P_k^+(f_n^0)u),\end{aligned}$$

where in the last step we used (7.17). Clearly,

$$f_{\infty,n}, f_{\infty,n}^0, f_{\infty,n}^\infty \nearrow \mathbb{R}^d \backslash \{0\}$$

in the sense of (7.5). Hence

$$\text{s-}\lim_{n\to\infty} P_k^+(f_n) = \text{s-}\lim_{n\to\infty} P_k^+(f_n^0) = \text{s-}\lim_{n\to\infty} P_k^+(f_n^\infty) = P_k^+.$$

Therefore, from $P_k^+u = u$ we see that

$$\text{s-}\lim_{n,m\to\infty} \|W_k^+(j_n)u - W_k^+(j_m)u\| = 0.$$

In other words, the sequence $W_k(j_n)u$ is Cauchy, and hence convergent.

Let us check that the limit

$$\text{s-}\lim_{n\to\infty} W_k^+(j_n)u =: W_k^+u$$

does not depend on the choice of the sequence j_n. In fact if $j_n, \tilde{j}_n$ are two sequences with $j_n, \tilde{j}_n \nearrow 1\!\mathrm{l}_{\mathbb{R}^d \backslash \{0\}}$, we can find a sequence m_n tending to ∞ such that $j_{0,n}\tilde{j}_{\infty,m_n} = 0$. Then we argue as above.

By Theorem 7.10 *ii)* we see that

$$W^+\chi(H) = \chi(H^{\rm ext})W^+.$$

If $q \in C_0^\infty(\mathbb{R}^d)$, $q = 1$ in a neighborhood of 0, $0 \leq q \leq 1$ and $qj_{0,n} = j_{0,n}$, then by Thm. 7.10 *iii)*

$$\Gamma^+(q) \otimes 1\!\mathrm{l} W_k^+(j_n) = W_k^+(j_n).$$

Therefore,

$$\Gamma^+(q) \otimes 1\!\mathrm{l} W^+ = W^+.$$

Hence

$$P_0^+ \otimes 1\!\!1 W^+ = W^+.$$

Thus

(7.18)
$$\mathrm{Ran} W^+ \subset \mathrm{Ran} P_0^+ \otimes \Gamma(\mathfrak{h}) \subset \mathcal{K}^+ \otimes \Gamma(\mathfrak{h}) = \mathcal{H}^+.$$

Let us now show (7.16). Let $u \in \mathcal{H}_{\mathrm{comp}}(H), u = P_k^+ u$. By Thm. 7.11, we can choose a sequence $\epsilon_n \searrow 0$ such that

(7.19)
$$\text{s-}\lim_{n\to\infty} P_0^+([\tfrac{1}{n} - \epsilon_n, \tfrac{1}{n} + \epsilon_n])u = u.$$

We can demand that the sequence $j_n = (j_{0,n}, j_{\infty,n})$ used to define W^+ satisfies additionally

$$\mathrm{supp}\, j_{0,n} \subset [0, \tfrac{1}{n} + \epsilon_n], \quad \mathrm{supp}\, j_{\infty,n} \subset [\tfrac{1}{n} - \epsilon_n, \infty[.$$

Note that $q_n := (j_{0,n} + j_{\infty,n})^{-1} \le 1$ and $q_n = 1$ outside of $[\frac{1}{n} - \epsilon_n, \frac{1}{n} + \epsilon_n]$. Hence, by (7.19)

$$\text{s-}\lim_{n\to\infty} \Gamma^+(q_n)u = u.$$

Set $\tilde{j}_n = (\tilde{j}_{0,n}, \tilde{j}_{\infty,n}) = (q_n j_{0,n}, q_n j_{\infty,n})$. Then $\tilde{j}_{0,n} + \tilde{j}_{\infty,n} = 1$. Hence, by Thm. 7.10 *vi)*, $W^+(\tilde{j}_n)u \in \mathcal{D}(\Omega^{\mathrm{ext},+})$ and

$$\Omega^{\mathrm{ext},+} W_k^+(\tilde{j}_n) = P_k^+(\tilde{j}_n).$$

Let $\chi \in C_0^\infty(\mathbb{R})$ such that $u = \chi(H)u$. Using the fact that $\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})$ is bounded, we have:

$$\begin{aligned}
u = P_k^+ u &= \lim_{n\to\infty} P_k^+(\tilde{j}_n)u \\
&= \lim_{n\to\infty} \Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}}) W_k^+(\tilde{j}_n)u \\
&= \Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}}) \lim_{n\to\infty} W_k^+(\tilde{j}_n)u \\
&= \Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}}) \lim_{n\to\infty} W_k^+(j_n)\Gamma^+(q_n)u \\
&= \Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}}) \lim_{n\to\infty} W_k^+(j_n)u \\
&= \Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}}) W^+ u = \Omega^{\mathrm{ext},+} W^+ u.
\end{aligned}$$

Hence

$$\Omega^{\mathrm{ext},+} W^+ u = u.$$

But by (5.22)

$$\Omega^{\mathrm{ext},+} 1\!\!1_{\mathcal{H}^+} = \Omega^+.$$

Therefore, by (7.18)

$$\Omega^+ W^+ = 1\!\!1_{\mathcal{H}}.$$

The fact that Ω^+ is unitary from $\mathcal{H}^+$ to $\mathcal{H}$ implies now

$$W^+ = \Omega^{+*}, \quad \mathrm{Ran} W^+ = \mathcal{H}^+.$$

Hence by (7.18) we obtain (7.16). $\square$

7.8 Asymptotic completeness

In this subsection, we will prove that $P_0^+ = 1\!\!1_{\rm pp}(H)$. Combined with Thm. 7.13, this will complete the proof of asymptotic completeness. Thm. 7.15 below will be a consequence of the Mourre estimate of Subsect.4.3. We first show that condition *(SR)* implies condition *(I2)*.

Lemma 7.14 *Hypothesis (SR) implies hypothesis (I2).*

Proof. Assume that *(SR)* holds. Let us prove that $\langle x\rangle v \in B(\mathcal{K}, \mathcal{K}\otimes \mathfrak{h})$ which implies *(I2)*. Equivalently we will show that

$$\sum_1^\infty n v_n \in B(\mathcal{K}, \mathcal{K}\otimes \mathfrak{h}). \tag{7.20}$$

for $v_n := 1\!\!1_{[n,n+1]}(\langle x\rangle)v$. If we set $w_n = 1\!\!1_{[n,+\infty[}(\langle x\rangle)v$, use $v_n = w_{n+1} - w_n$ and sum by parts in (7.20), we see that the convergence of (7.20) follows from *(SR)*. □

Theorem 7.15 *Assume the hypotheses (H0), (H1), (H2), (I1) and (SR). Then*

$$1\!\!1_{\rm pp}(H) = P_0^+.$$

Proof. By Proposition 5.5 and geometric asymptotic completeness we already know that

$$\mathcal{H}_{\rm pp}(H) \subset \mathcal{K}^+ = {\rm Ran} P_0^+.$$

Let us now prove that $P_0^+ \leq 1\!\!1_{\rm pp}(H)$. Let $\chi \in C_0^\infty(\mathbb{R}\backslash(\tau \cup \sigma_{\rm pp}(H)))$. Using an argument contained eg. in [DeGe, Prop. 4.4.8], we deduce from Prop. 6.5 in Sect. 6 that there exists $\epsilon > 0$ such that for $q \in C_0^\infty(\mathbb{R}^d)$ with $q(x) = 1$ for $|x| < \epsilon$ we have $\Gamma^+(q)\chi(H) = 0$. This implies that

$$P_0^+ \leq 1\!\!1_{\sigma_{\rm pp}\cup\tau}(H).$$

Since τ is a closed countable set and $\sigma_{\rm pp}(H)$ can accumulate only at τ, we see that $1\!\!1_{\rm pp}(H) = 1\!\!1_{\sigma_{\rm pp}\cup\tau}(H)$. This completes the proof of the theorem. □

A Appendix

The following lemma describes an argument commonly used to prove the so called propagation estimates (see [DeGe, Sect. 8.4] and references therein).

Lemma A.1 *Let H be a self-adjoint operator and $\mathbf{D}$ the corresponding Heisenberg derivative*

$$\mathbf{D} := \frac{\rm d}{{\rm d}t} + {\rm i}[H, \cdot].$$

Suppose that $\Phi(t)$ is a uniformly bounded family of self-adjoint operators. Suppose that there exist $C_0 > 0$ and operator valued functions $B(t)$ and $B_i(t)$, $i = 1, \dots, n$, such that

$$\begin{aligned} &\mathbf{D}\Phi(t) \geq C_0 B^*(t)B(t) - \sum_{i=1}^n B_i^*(t)B_i(t), \\ &\int_1^\infty \|B_i(t){\rm e}^{-{\rm i}tH}\phi\|^2 {\rm d}t \leq C\|\phi\|^2, \quad i = 1, \dots, n. \end{aligned}$$

Then there exists C_1 such that

$$\int_1^\infty \|B(t)\mathrm{e}^{-\mathrm{i}tH}\phi\|^2 \mathrm{d}t \leq C_1\|\phi\|^2. \tag{A.1}$$

Next we describe how one uses propagation estimates to prove the existence of asymptotic observables.

Lemma A.2 *Let H_1 and H_2 be two self-adjoint operators. Let ${}_2\mathbf{D}_1$ be the corresponding asymetric Heisenberg derivative:*

$$ {}_2\mathbf{D}_1\Phi(t) := \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + \mathrm{i}H_2\Phi(t) - \mathrm{i}\Phi(t)H_1.$$

Suppose that $\Phi(t)$ is a uniformly bounded function with values in self-adjoint operators. Let $\mathcal{D}_1 \subset \mathcal{H}$ be a dense subspace. Assume that

$$\begin{array}{l}
|(\psi_2|{}_2\mathbf{D}_1\Phi(t)\psi_1)| \leq \sum\limits_{i=1}^{n} \|B_{2i}(t)\psi_2\|\|B_{1i}(t)\psi_1\|, \\
\int\limits_1^\infty \|B_{2i}(t)\mathrm{e}^{-\mathrm{i}tH_2}\phi\|^2\mathrm{d}t \leq C\|\phi\|^2, \quad \phi\in\mathcal{H}, \quad i=1,\dots,n, \\
\int\limits_1^\infty \|B_{1i}(t)\mathrm{e}^{-\mathrm{i}tH_1}\phi\|^2\mathrm{d}t \leq C\|\phi\|^2, \quad \phi\in\mathcal{D}_1, \quad i=1,\dots,n.
\end{array}$$

Then the limit

$$\mathrm{s}\text{-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH_2}\Phi(t)\mathrm{e}^{-\mathrm{i}tH_1}$$

exists.

Finally, we describe a simple lemma about the convergence of positive operators.

Lemma A.3 *Let Q_n be a commuting sequence of selfadjoint operators such that:*

$$i)\ 0\leq Q_n\leq 1, \qquad Q_{n+1}\leq Q_n, \qquad Q_{n+1}Q_n = Q_{n+1},$$

or

$$ii)\ 0\leq Q_n\leq 1, \qquad Q_n\leq Q_{n+1}, \qquad Q_{n+1}Q_n = Q_n.$$

Then the limit

$$Q = \mathrm{s}\text{-}\lim_{n\to\infty} Q_n.$$

exists and is a projection.

Proof. Note that case *ii)* reduces to case *i)* by considering the operators $(1-Q_n)$, so it suffices to consider case *i)*.

Clearly we have

$$Q = \inf_n Q_n = \mathrm{w}-\lim_{n\to\infty} Q_n.$$

We use the identity

$$Q_nQ_m = Q_m, \text{ for } m>n,$$

and let m tend to ∞, which gives $Q_n Q = Q$. Letting then n tend to ∞ we get $Q^2 = Q$. Next we have $Q_n^2 \leq Q_n$ and $Q_{n+1} = Q_{n+1} Q_n \leq Q_n^2$, which gives

$$Q_{n+1} \leq Q_n^2 \leq Q_n.$$

Letting n tend to ∞, we get

$$Q = \mathrm{w} - \lim_{n\to\infty} Q_n^2.$$

Then we compute

$$\lim_{n\to\infty} \|(Q - Q_n)u\|^2 = \lim_{n\to\infty} \Big((Q_n^2 - Q)u|u\Big) = 0,$$

which proves that $Q = \text{s-}\lim_{n\to\infty} Q_n$. $\square$

References

[A1] Arai, A.: On a model of a harmonic oscillator coupled to a quantized, massless, scalar field I. J. Math. Phys. 22 (1981) 2539-2548

[A2] Arai, A.: Self-adjointness and spectrum of Hamiltonians in non-relativistic quantum electrodynamics, J. Math. Phys. 22 (1981) 534-537

[AH] Arai, A, Hirokawa, M: On the existence and uniqueness of ground states of the spin-boson Hamiltonian, preprint 1995

[BF] Buchholz, D., Fredenhagen, K.: Collision theory for massless bosons, Comm. Math. Phys. 52 (1977) 147-176

[BFS] V. Bach, J. Fröhlich, I. Sigal: Quantum electrodynamics of confined non-relativistic particles, Sonderforschungsbereich 288 preprint, 1996

[BR] Bratteli, O, Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics*, Springer Berlin 1981

[BSZ] Baez, J.C, Segal, I.E., Zhou, Z.: *Introduction to Algebraic and Constructive Quantum Field Theory*, Princeton University Press, Princeton 1992

[Bl] P. Blanchard: Discussion mathematique du modèle de Pauli et Fierz relatif à la catastrophe infrarouge, Comm. Math. Phys. 15 (1969) 156-172

[CT] Cohen-Tannoudji C., Dupont-Roc, J., Grynberg, G.: *Photons and Atoms– Introduction to Quantum Electrodynamics*, John Wiley, New York 1991

[CD] Combescure, M. Dunlop, F.: N-particle-irreducible functions in Euclidean quantum field theory, Ann. Phys. 122 (1979) 102-150

[Da1] Davies, E.B.: Markovian master equations, Comm. Math. Phys. 39 (1974) 91-110

[Da2] Davies, E.B.: Markovian master equations II, Math. Ann. 219 (1976) 147-158

[DeGe] Dereziński J., Gérard C.: *Scattering Theory of Classical and Quantum N-Particle Systems*, Texts and Monographs in Physics, Springer, 1997

[De1] Dereziński, J.: Asymptotic completeness for $N-$particle long-range quantum systems, Ann. of Math. **138** (1993) 427-476

[De2] Dereziński, J.: Asymptotic completeness in quantum field theory. A class of Galilei-covariant models, preprint Ecole Polytechnique, 1996, to appear in Rev. Math. Phys.

[E] Enss, V.: Long-range scattering of two- and three-body quantum systems, Journées "Equations aux dérivées partielles", Saint Jean de Monts, Juin 1989, Publications Ecole Polytechnique, Palaiseau 1989

[Frie] Friedrichs, K.O.: *Perturbation of Spectra in Hilbert Space*, AMS, Providence 1965

[Fro1] Fröhlich, J.: Existence of dressed one-electron states in a class of persistent models, Forschritte der Physik 22 (1974) 159-198

[Fro2] Fröhlich, J.: On the infrared problem in a model of scalar electrons and massless scalar bosons, Ann. Inst. Henri Poincaré A 19 (1973) 1-103

[FH] Froese, R., Herbst, I.: A new proof of the Mourre estimate, Duke Math. J. **49** (1982) 1075-1085

[Ge] Gérard, C.: Asymptotic completeness for the spin-boson model with a particle number cutoff, Rev. Math. Phys. 8 (1996) 549-589

[GJ1] Glimm. J., Jaffe, A.: The $\lambda(\phi^4)_2$ quantum field theory without cutoffs II. The field operators and the approximate vacuum, Ann. Math. 91 (1970) 362-401

[GJ2] Glimm. J., Jaffe, A.: *Quantum Physics. A Functional Integral Point of View,* sec. ed, 1987, Springer New York, Berlin, Heidelberg.

[Gr] Graf, G.M.: Asymptotic completeness for $N-$body short range quantum systems : A new proof, Comm. Math. Phys. **132** (1990) 73-101

[Ha] Haag, R. *Local Quantum Physics. Fields, Particles, Algebras*, Springer 1992, Berlin Heidelberg

[HS] Helffer, B., Sjöstrand, J.: Equation de Schrödinger avec champ magnétique et equation de Harper, Springer Lecture Notes in Physics **345** (1989) 118-197

[He] Hepp, H.: *Theorie de la renormalisation*, Berlin, Springer 1968

[HK1] Høegh-Krohn R.: Asymptotic limits in some models of quantum field theory, I J. Math. Phys. J. Math. Phys. 9 (1968) 2075-2079

[HK2] Høegh-Krohn R.: Boson fields under a general class of cut-off interactions, Comm. Math. Phys. 12 (1969) 216-225

[HK3] Høegh-Krohn, R,: On the scattering matrix for quantum fields, Comm. Math. Phys. 18 (1970) 109-117.

[HuSp1] Hübner, M., Spohn, H.: Radiative decay: nonperturbative approaches, Rev. Math. Phys 7 (1995) 363-387

[HuSp2] Hübner, M., Spohn, H.: Spectral properties of the spin-boson Hamiltonian, Ann. Inst. H. Poincare 62 (1995) 289-323

[Ia] Iagolnitzer, D.: Asymptotic completeness and multiparticle analysis in quantum field theories, Comm. Math. Phys. 110 (1987) 51

[JP1] Jaksic, V., Pillet, C.A.: On a model for quantum friction I: Fermi's golden rule at zero temperature, Ann. Inst. H. Poincare (1995) 62-47

[JP2] Jaksic, V., Pillet, C.A.: On a model for quantum friction II: Fermi's golden rule and dynamics at zero temperature, Comm. Math. Phys. 176 (1995) 176-619

[JP3] Jaksic, V., Pillet, C.A.: On a model for quantum friction III: Ergodic properties of the spin-boson system, to appear in Comm. Math. Phys.

[Ma] Massen, H.: Return to equilibrium for the solution of quantum Langevin equation, J. Stat. Phys. 34 (1984) 239-261

[Mo] Mourre, E.: Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys. 78 (1981) 391-408

[N] Nelson, E.: Interaction of nonrelativistic particles with a quantized scalar field, J. Math. Phys. 5 (1964) 1190-1997

[OY] Okamoto T., Yajima, K: Complex scaling technique in non-relativistic qed, Ann. Inst. H. Poincare 42 (1985) 311-327

[PF] Pauli W., Fierz, M.: Nuovo Cimento 15 (1938) 167

[RS] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics*, vol I and II, 1976, vol. III, 1979, vol. IV, 1978, Academic Press, London

[SigSof] Sigal, I.M., Soffer, A.: The $N-$particle scattering problem: asymptotic completeness for short-range quantum systems, Ann. of Math. **125** (1987) 35-108

[Sim] Simon, B.: *The $P(\phi)_2$ Euclidean (Quantum) Field Theory*, Princeton University Press, Princeton 1974

[Sk] Skibsted, E.: Spectral analysis of N-body systems coupled to a bosonic system, preprint, 1997

[Sp1] Spohn, H.: Ground states of the spin-boson hamiltonian, Comm. Math. Phys. **123** (1989) 277-304

[Sp2] Spohn, H.: Asymptotic completeness for Raleigh scattering, preprint 1996

[Sp3] Spohn, H.: Ground state of a quantum particle coupled to a scalar Bose field, preprint IHES 1997

[We] Weinberg, S. *The quantum theory of fields*, vol. I, Cambridge University Press, 1995

[Ya] Yafaev, D.R.: Radiation conditions and scattering theory for N-particle hamiltonians, Comm. Math. Phys. **154** (1993) 523-554