# PROPAGATORS ON CURVED SPACETIMES


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Consider a globally hyperbolic spacetime $(M, g_{\mu\nu})$.

The Klein–Gordon operator with electromagnetic potential $A_\mu$ and a scalar potential (mass squared) $Y$ is an operator acting on functions on $M$ given by

$$K := |g|^{-\frac{1}{4}}(x)\big(\mathrm{i}\partial_\mu + A_\mu(x)\big)g^{\mu\nu}|g|^{\frac{1}{2}}(x)\big(\mathrm{i}\partial_\nu + A_\nu(x)\big)|g|^{-\frac{1}{4}}(x)$$
$$+Y(x).$$

We say that $G$ is a bisolution of $K$ if

$$GK = KG = 0.$$

We say that $G$ is an inverse (Green's function or a fundamental solution) if

$$GK = KG = \mathbb{1}.$$

I will discuss how to define distinguished bisolutions and inverses. I will call them propagators. (This word is often used in this context in quantum field theory).

I will also discuss the problem of essential self-adjointness of the Klein-Gordon operator $K$ on $L^2(M)$ for curved spacetimes. (Note that $K$ is obviously Hermitian).

Note that the analogous problem of the essential self-adjointness of the Laplace-Beltrami operator has a positive answer for large classes of Riemannian manifolds.

For generic Lorentzian manifolds the problem of self-adjointness of $K$ seems rather difficult and is almost absent from mathematical literature.

On the other hand, in physical literature one can find many places where the authors tacitly assume that the Klein-Gordon operator is self-adjoint and write e.g.

$$\frac{1}{K} = -\mathrm{i} \int_0^\infty \mathrm{e}^{\mathrm{i}tK} \mathrm{d}t.$$

The method involving $\mathrm{e}^{\mathrm{i}tK}$ has a name: it is called the Fock-Schwinger proper time method.

Let me summarize what every student of QFT learns about propagators on the Minkowski space $\mathbb{R}^{1,d}$ for the free Klein-Gordon operator

$$K = p_\mu p^\mu + m^2,$$

where $p_\mu = -\mathrm{i}\partial_\mu$.

We have the following standard Green's functions:

the forward/backward or advanced/retarded propagator

$$G^{\pm} := \frac{1}{(p^2 + m^2 \mp \mathrm{i}0\,\mathrm{sgn} p^0)},$$

the Feynman/anti-Feynman propagator

$$G^{\mathrm{F}/\overline{\mathrm{F}}} := \frac{1}{(p^2 + m^2 \mp \mathrm{i}0)}.$$

The former have an obvious application to the Cauchy problem.

The Feynman propagator equals the expectation values of time-ordered products of fields and is used to evaluate Feynman diagrams.

We have the following standard bisolutions: and the positive/negative frequency bisolution

the Pauli-Jordan propagator

$$G^{\mathrm{PJ}} := \mathrm{sgn}(p^0)\delta(p^2 + m^2),$$

and the positive/negative frequency bisolution

$$G^{(+)/(-)} := \theta(\pm p^0)\delta(p^2 + m^2).$$

The former expresses commutation relations of fields, and hence it is often called the commutator function.

The positive frequency bisolution is the 2-point function of the vacuum state.

It is well known that

- the forward propagator $G^+$,
- the backward propagator $G^-$,
- the Pauli-Jordan propagator $G^{\mathrm{PJ}} := G^+ - G^-$.

are defined under very broad conditions on globally hyperbolic spaces. All of them have a causal support. We will jointly call them classical propagators.

We are however more interested in "non-classical propagators", typical for quantum field theory. They are less known to pure mathematicians and more difficult to define. They are

- the Feynman propagator $G^{\mathrm{F}}$,
- the anti-Feynman propagator $G^{\overline{\mathrm{F}}}$,
- the positive frequency bisolution $G^{(+)}$,
- the negative frequency bisolutions $G^{(-)}$.

There exists a well-known paper of Duistermat-Hörmander, which defined Feynman parametrices (a parametrix is an approximate inverse in appropriate sense).

There exists a large literature devoted to the so-called Hadamard states, which can be interpreted as bisolutons with approximately positive frequencies. These are however large classes of propagators. We would like to have distinguished choices.

It is helpful to introduce a time variable $t$, so that the spacetime is $M = \mathbb{R} \times \Sigma$. Assume that there are no time-space cross terms so that the metric can be written as

$$g_{00}(t,\vec{x})\mathrm{d}^2 t + g_{ij}(t,\vec{x})\mathrm{d}x^i \mathrm{d}x^j.$$

By conformal rescaling we can assume that $g_{00} = 1$, so that, setting $V := A^0$, we have

$$\begin{aligned} K &= (\mathrm{i}\partial_t + V)^2 + L, \\ L &= -|g|^{-\frac{1}{4}}(\mathrm{i}\partial_i + A_i)|g|^{\frac{1}{2}}g^{ij}(\mathrm{i}\partial_j + A_j)|g|^{-\frac{1}{4}} + Y. \end{aligned}$$

We rewrite the Klein-Gordon equation as a 1st order equation given by

$$\partial_t + \mathrm{i}B(t),$$

where

$$B(t) := \begin{pmatrix} W(t) & \mathbb{1} \\ L(t) & \overline{W}(t) \end{pmatrix},$$

$$W(t) := V(t) + \frac{\mathrm{i}}{4}|g|(t)^{-1}\partial_t|g|(t).$$

Denote by $U(t,t')$ the dynamics defined by $B(t)$, that is

$$\begin{aligned} \partial_t U(t,t') &= -\mathrm{i}B(t)U(t,t'), \\ U(t,t) &= \mathbb{1}. \end{aligned}$$

Note that if

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$$

is a bisolution/inverse of $\partial_t + \mathrm{i}B(t)$, then $E_{12}$ is a bisolution/inverse of $K$.

The classical propagators can be easily expressed in terms of the dynamics:

$$\begin{aligned}
E^{\mathrm{PJ}}(t,t') &:= U(t,t'), & E^{\mathrm{PJ}}_{12} &= G^{\mathrm{PJ}};\\
E^{+}(t,t') &:= \theta(t-t')\,U(t,t'), & E^{+}_{12} &= G^{+};\\
E^{-}(t,t') &:= -\theta(t'-t)\,U(t,t'), & E^{-}_{12} &= G^{-}.
\end{aligned}$$

We introduce the charge matrix

$$Q := \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} .$$

and the classical Hamiltonian

$$H(t) := QB(t) = \begin{pmatrix} L(t) & \overline{W}(t) \\ W(t) & \mathbb{1} \end{pmatrix} .$$

We will assume that $H(t)$ is positive and invertible.

Assume now for a moment that the problem is static, so that $L$, $V$, $B$, $H$ do not depend on time $t$. Clearly,

$$U(t, t') = \mathrm{e}^{-\mathrm{i}(t-t')B}.$$

The quadratic form $H$ defines the so-called energy scalar product. It is easy to see that $B$ is Hermitian in this product and has a gap in its spectrum around $0$. Let $\Pi^{(\pm)}$ be the projections onto the positive/negative part of the spectrum of $B$.

We define the Feynman and anti-Feynman inverse and the positive and negative frequency bisolutions on the level of $\partial_t + \mathrm{i}B(t)$:

$$E^{(\pm)}(t,t') := \pm \mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(\pm)},$$

$$E^{\mathrm{F}}(t,t') := \theta(t-t')\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(+)} - \theta(t'-t)\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(-)},$$

$$E^{\overline{\mathrm{F}}}(t,t') := \theta(t-t')\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(-)} - \theta(t'-t)\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(+)}.$$

They lead to corresponding propagators on the level of $K$:

$$G^{(\pm)} := E_{12}^{(\pm)},$$
$$G^{\mathrm{F}} := E_{12}^{\mathrm{F}},$$
$$G^{\overline{\mathrm{F}}} := E_{12}^{\overline{\mathrm{F}}}.$$

They satisfy the relations

$$G^{\mathrm{PJ}} = G^{(+)} - G^{(-)},$$
$$G^{\mathrm{F}} = G^{(+)} + G^{-} = G^{(-)} + G^{+},$$
$$G^{\overline{\mathrm{F}}} = -G^{(+)} + G^{+} = -G^{(-)} + G^{-}.$$

Nonclassical propagators are important in quantum field theory, and they are often called 2-point functions, because they are vacuum expectation values of free fields:

$$G^{(+)}(x,y) = \bigl(\Omega|\hat{\phi}(x)\hat{\phi}(y)\Omega\bigr),$$
$$G^{\mathrm{F}}(x,y) = \bigl(\Omega|\mathrm{T}\bigl(\hat{\phi}(x)\hat{\phi}(y)\bigr)\Omega\bigr).$$

$G^{\mathrm{F}}$ is used to evaluate Feynman diagrams.

It is easy to see that on a general spacetime the Klein-Gordon operator $K$ is Hermitian (symmetric) on $C_{\mathrm{c}}^{\infty}(M)$ in the sense of the Hilbert space $L^2(M)$. In the static case, using Nelson's Commutator Theorem one can show that it is essentially self-adjoint.

Theorem. For $s > \frac{1}{2}$, the operator $G^{\mathrm{F}}$ is bounded from the space $\langle t \rangle^{-s} L^2(M)$ to $\langle t \rangle^{s} L^2(M)$. Besides, in the sense of these spaces,

$$\mathrm{s}-\lim_{\epsilon \searrow 0} (K - \mathrm{i}\epsilon)^{-1} = G^{\mathrm{F}}.$$

Let $0 \leq \theta \leq \pi$. Suppose we replace the metric $g$ by

$$g_\theta := -\mathrm{e}^{-2\mathrm{i}\theta}\mathrm{d}t^2 + g_\Sigma$$

and the electric potential $V$ by $V_\theta := \mathrm{e}^{-\mathrm{i}\theta}V$. This replacement is called Wick rotation. The value $\theta = \frac{\pi}{2}$ corresponds to the Riemannian metric

$$g_{\pi/2} = \mathrm{d}t^2 + g_\Sigma.$$

The Wick rotated Klein-Gordon operator, which is elliptic and even invertible:

$$K_\theta = \mathrm{e}^{-\mathrm{i}2\theta}(\partial_t + \mathrm{i}V)^2 + L,$$

Theorem. For $s > \frac{1}{2}$, we have

$$\mathrm{s}-\lim_{\theta \searrow 0} K_\theta^{-1} = G^{\mathrm{F}},$$

in the sense of operators from $\langle t\rangle^{-s}L^2(M)$ to $\langle t\rangle^{s}L^2(M)$.

Can one generalize non-classical propagators to non-static spacetimes? We will assume that the spacetime is close to being static and for large times it approaches a static spacetime sufficiently fast.

One can define the incoming positive/negative frequency bisolution by cutting the phase space with the projections $\Pi_{-}^{(\pm)}$ onto the positive/negative part of the spectrum of $B(-\infty)$. $\Pi_{-}^{(+)}$ defines the vacuum state in the distant past given by a vector $\Omega_{-}$. It corresponds to a preparation of an experiment.

Analogously, one can define the outgoing positive/negative bisolutions by using the projections $\Pi_+^{(\pm)}$ onto the positive/negative part of the spectrum of $B(\infty)$. They correspond to the vacuum state in the remote future given by a vector $\Omega_+$. This vector is related to the future measurments.

The projection $\Pi_{-\infty}^{(+)}$ can be transported by the dynamics to any time $t$, obtaining the projection $\Pi_{-}^{(+)}(t)$. Similarly we obtain the projection $\Pi_{+}^{(-)}(t)$. We will say that the Klein-Gordon equation is asymptotically complementary if for some $t$ (and hence for all $t$) the subspaces

$$\mathrm{Ran}\, \Pi_{-}^{(+)}(t), \quad \mathrm{Ran}\, \Pi_{+}^{(-)}(t)$$

are complementary.

Assume that asymptotic complementarity holds. Define $\Pi_{\rm can}^{(+)}(t)$, $\Pi_{\rm can}^{(-)}(t)$ to be the unique pair of projections corresponding to the pair of spaces

$$\mathrm{Ran}\,\Pi_{-}^{(+)}(t), \quad \mathrm{Ran}\,\Pi_{+}^{(-)}(t)$$

The canonical Feynman propagator is defined as

$$\begin{aligned} E^{\rm F}(t_2,t_1) &:= \theta(t_2-t_1)U(t_2,t_1)\Pi_{\rm can}^{(+)}(t_1) \\ &\quad -\theta(t_1-t_2)U(t_2,t_1)\Pi_{\rm can}^{(-)}(t_1), \\ G^{\rm F} &:= E_{12}^{\rm F}. \end{aligned}$$

In a somewhat different setting, in the case of massless Klein-Gordon operator $G^{\mathrm{F}}_{\mathrm{can}}$ was considered before by A.Vasy et al. A similar construction can be found in a recent paper of Gerard-Wrochna.

Here is the physical meaning of the canonical Feynman propagator: it is the expectation value of the time-ordered product of fields between the in-vacuum and the out vacuum:

$$G^{\mathrm{F}}(x,y) = \frac{\big(\Omega_+|\mathrm{T}\big(\hat{\phi}(x)\hat{\phi}(y)\big)\Omega_-\big)}{\big(\Omega_+|\Omega_-\big)}.$$

Conjecture. For small enough, compactly supported perturbations of the static case, the following holds:

1. Asymptotic complementarity holds, so that we can define $G^{\mathrm{F}}$.
2. The Klein-Gordon operator $K$ is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(M)$.
3. In the sense of operators from $\langle t\rangle^{-s}L^2(M)$ to $\langle t\rangle^{s}L^2(M)$,

$$\mathrm{s-}\lim_{\epsilon\searrow 0}(K-\mathrm{i}\epsilon)^{-1}=G^{\mathrm{F}}.$$