

QUADRATIC HAMILTONIANS AND THEIR RENORMALIZATION

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Quadratic bosonic Hamiltonians are formally operators of the form

$$\hat{H} := \sum h_{ij} \hat{a}_i^* \hat{a}_j + \frac{1}{2} \sum g_{ij} \hat{a}_i^* \hat{a}_j^* + \frac{1}{2} \sum \bar{g}_{ij} \hat{a}_i \hat{a}_j + c.$$

(In the case of an infinite number of degrees of freedom c can be infinite!)

One can compute their infimum or **vacuum energy**

$$E := \inf \hat{H} = \frac{1}{4} \text{Tr} \left(\sqrt{B^2} - \sqrt{B_0^2} \right) + c.$$

where

$$B := \begin{bmatrix} h & -g \\ \bar{g} & -\bar{h} \end{bmatrix}, \quad B_0 := \begin{bmatrix} h & 0 \\ 0 & -\bar{h} \end{bmatrix}.$$

I would like to discuss two examples of quadratic Hamiltonians taken from QFT. They illustrate how nontrivial their theory can be:

1. neutral scalar field with position dependent mass,
2. charged scalar field in electromagnetic potential.

These models belong to **Local Quantum Physics**. Even though they are not translation invariant, their dynamics is **causal**.

Consider the **free classical neutral scalar field**

$$(-\square + m^2)\phi(x) = 0, \quad x \in \mathbb{R}^{1,3}.$$

Together with the conjugate fields $\pi(x) = \partial_t\phi(x)$ they have the Poisson brackets

$$\{\phi(\vec{x}), \phi(\vec{y})\} = \{\pi(\vec{x}), \pi(\vec{y})\} = 0,$$

$$\{\phi(\vec{x}), \pi(\vec{y})\} = \delta(\vec{x} - \vec{y}), \quad \vec{x} \in \mathbb{R}^3.$$

The Hamiltonian

$$H_0 := \int \left(\frac{1}{2} \pi^2(\vec{x}) + \frac{1}{2} (\vec{\partial} \phi(\vec{x}))^2 + \frac{1}{2} m^2 \phi^2(x) \right) d\vec{x}$$

generates the dynamics

$$\partial_t \phi(x) = \{ \phi(x), H_0 \}, \quad \partial_t \pi(x) = \{ \pi(x), H_0 \}.$$

In order to diagonalize the Hamiltonian, we set

$$\varepsilon(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$$

and we introduce the “normal modes”

$$a(k) := \int \frac{d\vec{x}}{\sqrt{(2\pi)^3}} e^{-i\vec{k}\vec{x}} \left(\sqrt{\frac{\varepsilon(\vec{k})}{2}} \phi(0, \vec{x}) + \frac{i}{\sqrt{2\varepsilon(\vec{k})}} \pi(0, \vec{x}) \right),$$

$$a^*(k) := \int \frac{d\vec{x}}{\sqrt{(2\pi)^3}} e^{i\vec{k}\vec{x}} \left(\sqrt{\frac{\varepsilon(\vec{k})}{2}} \phi(0, \vec{x}) - \frac{i}{\sqrt{2\varepsilon(\vec{k})}} \pi(0, \vec{x}) \right).$$

The normal modes diagonalize the Poisson relations and the Hamiltonian:

$$\{a(k), a(k')\} = \{a^*(k), a^*(k')\} = 0,$$

$$\{a(k), a^*(k')\} = -i\delta(\vec{k} - \vec{k}'),$$

$$H_0 = \int d\vec{k} \varepsilon(\vec{k}) a^*(k) a(k).$$

The fields can be expressed in terms of normal modes as

$$\phi(x) = \int \frac{d\vec{k}}{\sqrt{(2\pi)^3} \sqrt{2\varepsilon(\vec{k})}} \left(e^{ikx} a(k) + e^{-ikx} a^*(k) \right),$$

$$\pi(x) = \int \frac{d\vec{k} \sqrt{\varepsilon(\vec{k})}}{i \sqrt{(2\pi)^3} \sqrt{2}} \left(e^{ikx} a(k) - e^{-ikx} a^*(k) \right).$$

The **free quantum neutral scalar field**, denoted $\hat{\phi}(x)$, satisfies the hatted versions of the classical equations:

$$(-\square + m^2)\hat{\phi}(x) = 0, \quad \partial_t \hat{\phi}(x) = \hat{\pi}(x).$$

and the commutation relations

$$\begin{aligned} [\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] &= [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0, \\ [\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] &= i\delta(\vec{x} - \vec{y}). \end{aligned}$$

The free Hamiltonian is defined in the standard way:

$$\hat{H}_0^n := \int : \left(\frac{1}{2} \hat{\pi}^2(\vec{x}) + \frac{1}{2} (\vec{\partial} \hat{\phi}(\vec{x}))^2 + \frac{1}{2} m^2 \hat{\phi}^2(x) \right) : d\vec{x},$$

where the double dots denote the normal ordering.

One introduces $\hat{a}(k)$ and $\hat{a}^*(k)$, by the hatted classical relations. They diagonalize the Hamiltonian and the commutation relations:

$$\hat{H}_0^n = \int d\vec{k} \varepsilon(\vec{k}) \hat{a}^*(k) \hat{a}(k),$$
$$[\hat{a}(k), \hat{a}(k')] = [\hat{a}^*(k), \hat{a}^*(k')] = 0,$$
$$[\hat{a}(k), \hat{a}^*(k')] = \delta(\vec{k} - \vec{k}').$$

Consider the **classical field with a position dependent mass**:

$$(-\square + m^2)\phi(x) = -\kappa(\vec{x})\phi(x).$$

We assume that κ is a Schwartz function.

The classical Hamiltonian is

$$H := \int \left(\frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}(\vec{\partial}\phi(\vec{x}))^2 + \frac{1}{2}(m^2 + \kappa(\vec{x}))\phi^2(\vec{x}) \right) d\vec{x}.$$

We can rewrite the Hamiltonian in terms of normal modes

$$\begin{aligned} H = & \int d\vec{k} \varepsilon(\vec{k}) a^*(k) a(k) \\ & + \frac{1}{2} \int \frac{d\vec{k}_1 d\vec{k}_2 \kappa(\vec{k}_1 + \vec{k}_2)}{(2\pi)^3 \sqrt{2\varepsilon(\vec{k}_1)} \sqrt{2\varepsilon(\vec{k}_2)}} \\ & \times \left(a(-k_1) a(-k_2) + 2a^*(k_1) a(-k_2) + a^*(k_1) a^*(k_2) \right). \end{aligned}$$

The **quantum field with a position dependent mass** satisfies

$$(-\square + m^2)\hat{\phi}(x) = -\kappa(x)\hat{\phi}(x).$$

We assume that it coincides with the free field at time $t = 0$. We also would like to find a Hamiltonian \hat{H} such that

$$\hat{\phi}(t, \vec{x}) = e^{it\hat{H}} \hat{\phi}(\vec{x}) e^{-it\hat{H}}.$$

\hat{H} is defined up to an additive constant—we would like to fix a physically distinguished constant and obtain the vacuum energy.

Naively, the most obvious choice for \hat{H} is the normally ordered quantization of the classical Hamiltonian:

$$\hat{H}^n := \int : \left(\frac{1}{2} \hat{\pi}^2(\vec{x}) + \frac{1}{2} (\vec{\partial} \hat{\phi}(\vec{x}))^2 + \frac{1}{2} (m^2 + \kappa(\vec{x})) \hat{\phi}^2(\vec{x}) \right) : d\vec{x}$$

Expressed in terms of creation/annihilation operators it reads

$$\begin{aligned}\hat{H}^n &= \int d\vec{k} \varepsilon(\vec{k}) \hat{a}^*(k) \hat{a}(k) \\ &+ \frac{1}{2} \int \frac{d\vec{k}_1 d\vec{k}_2 \kappa(\vec{k}_1 + \vec{k}_2)}{(2\pi)^3 \sqrt{2\varepsilon(\vec{k}_1)} \sqrt{2\varepsilon(\vec{k}_2)}} \\ &\times \left(\hat{a}(-k_1) \hat{a}(-k_2) + 2\hat{a}^*(k_1) \hat{a}(-k_2) + \hat{a}^*(k_1) \hat{a}^*(k_2) \right).\end{aligned}$$

However, \hat{H}^n does not exist. This can be seen when we try to compute the infimum of \hat{H}^n : the 2nd order contribution to the energy E_2 given by the loop with 2 vertices diverges. However, the higher order terms are finite, and as a Hamiltonian implementing the dynamics we could take

$$\hat{H}^{2\text{ren}} := \hat{H}^n - E_2.$$

However, physically it is preferable to consider the renormalized Hamiltonian obtained by subtracting a “local counterterm”.

Using e.g. the Pauli-Villars method. we obtain the following renormalized expression for the 2nd order term:

$$E_2^{\text{ren}} := \int \pi^{\text{ren}}(\vec{k}^2) |\kappa(\vec{k})|^2 \frac{d\vec{k}}{(2\pi)^3}$$

$$= E_2 - C \int \kappa(\vec{x})^2 d\vec{x},$$

$$\pi^{\text{ren}}(\vec{k}^2) := \frac{1}{4(4\pi)^2} \left(\frac{\sqrt{\vec{k}^2 + 4m^2}}{\sqrt{\vec{k}^2}} \log \frac{\sqrt{\vec{k}^2 + 4m^2} + \sqrt{\vec{k}^2}}{\sqrt{\vec{k}^2 + 4m^2} - \sqrt{\vec{k}^2}} - 2 \right),$$

where C is an infinite counterterm.

We used $\pi^{\text{ren}}(0) = 0$ as the **renormalization condition**

The physically acceptable renormalized Hamiltonian can be formally written as

$$\begin{aligned}\hat{H}^{\text{ren}} &:= \hat{H}^{\text{n}} - C \int \kappa(\vec{x})^2 d\vec{x} \\ &= \hat{H}^{\text{n}} - E_2 + E_2^{\text{ren}}.\end{aligned}$$

\hat{H}^{ren} is a well defined self-adjoint operator (despite that \hat{H}^{n} is ill defined, and E_2, C are infinite). It is bounded from below and its infimum is

$$\begin{aligned}
 E^{\text{ren}} &= E_2^{\text{ren}} \\
 &+ \int \text{Tr} \frac{1}{(-\Delta + m^2 + \tau^2)} \kappa \frac{1}{(-\Delta + m^2 + \tau^2)} \kappa \\
 &\quad \times \frac{1}{(-\Delta + m^2 + \kappa + \tau^2)} \kappa \frac{1}{(-\Delta + m^2 + \tau^2)} \tau^2 \frac{d\tau}{2\pi}.
 \end{aligned}$$

Consider the **free charged scalar classical field** $\psi(x)$

$$(-\square + m^2)\psi(x) = 0.$$

$\psi^*(x)$ denotes its complex adjoint. The conjugate field is $\eta(x) := \partial_t\psi(x)$. The zero time Poisson brackets are

$$\begin{aligned}\{\psi(\vec{x}), \psi(\vec{y})\} &= \{\psi(\vec{x}), \eta(\vec{y})\} = \{\eta(\vec{x}), \eta(\vec{y})\} = 0, \\ \{\psi(\vec{x}), \eta^*(\vec{y})\} &= \{\psi^*(\vec{x}), \eta(\vec{y})\} = \delta(\vec{x} - \vec{y}).\end{aligned}$$

The Hamiltonian

$$H_0 = \int \left(\eta^*(\vec{x})\eta(\vec{x}) + \vec{\partial}\psi^*(\vec{x})\vec{\partial}\psi(\vec{x}) + m^2\psi^*(\vec{x})\psi(\vec{x}) \right) d\vec{x}.$$

generates the dynamics

$$\partial_t\psi(x) = \{\psi(x), H_0\}, \quad \partial_t\eta(x) = \{\eta(x), H_0\}.$$

One introduces normal modes:

$$a(p) = \int \left(\sqrt{\frac{\varepsilon(\vec{p})}{2}} \psi(0, \vec{x}) + \frac{i}{\sqrt{2\varepsilon(\vec{p})}} \eta(0, \vec{x}) \right) e^{-i\vec{p}\vec{x}} \frac{d\vec{x}}{\sqrt{(2\pi)^3}},$$

$$a^*(p) = \int \left(\sqrt{\frac{\varepsilon(\vec{p})}{2}} \psi^*(0, \vec{x}) - \frac{i}{\sqrt{2\varepsilon(\vec{p})}} \eta^*(0, \vec{x}) \right) e^{i\vec{p}\vec{x}} \frac{d\vec{x}}{\sqrt{(2\pi)^3}},$$

$$b(p) = \int \left(\sqrt{\frac{\varepsilon(\vec{p})}{2}} \psi^*(0, \vec{x}) + \frac{i}{\sqrt{2\varepsilon(\vec{p})}} \eta^*(0, \vec{x}) \right) e^{-i\vec{p}\vec{x}} \frac{d\vec{x}}{\sqrt{(2\pi)^3}},$$

$$b^*(p) = \int \left(\sqrt{\frac{\varepsilon(\vec{p})}{2}} \psi(0, \vec{x}) - \frac{i}{\sqrt{2\varepsilon(\vec{p})}} \eta(0, \vec{x}) \right) e^{i\vec{p}\vec{x}} \frac{d\vec{x}}{\sqrt{(2\pi)^3}}.$$

Normal modes diagonalize the Hamiltonian and the Poisson brackets

$$H_0 = \int d\vec{p} \varepsilon(\vec{p}) (a^*(p)a(p) + b^*(p)b(p)),$$
$$\{a(p), a^*(p')\} = \{b(p), b^*(p')\} = -i\delta(\vec{p} - \vec{p}'),$$

We can express fields in terms of normal modes:

$$\psi(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^3} \sqrt{2\varepsilon(\vec{p})}} \left(e^{ipx} a(p) + e^{-ipx} b^*(p) \right),$$

$$\eta(x) = \int \frac{d\vec{p} \sqrt{\varepsilon(\vec{p})}}{i \sqrt{(2\pi)^3} \sqrt{2}} \left(e^{ipx} a(p) - e^{-ipx} b^*(p) \right).$$

The **free quantum charged scalar field** is described by $\hat{\psi}(\vec{x}), \hat{\psi}^*(\vec{x})$. It satisfies

$$(-\square + m^2)\hat{\psi}(x) = 0, \quad \partial_t \hat{\psi}(x) = \hat{\eta}(x)$$

and has the commutation relations

$$\begin{aligned} [\hat{\psi}(\vec{x}), \hat{\psi}(\vec{y})] &= [\hat{\psi}(\vec{x}), \hat{\eta}(\vec{y})] = [\hat{\eta}(\vec{x}), \hat{\eta}(\vec{y})] = 0, \\ [\hat{\psi}(\vec{x}), \hat{\eta}^*(\vec{y})] &= [\hat{\psi}^*(\vec{x}), \hat{\eta}(\vec{y})] = i\delta(\vec{x} - \vec{y}). \end{aligned}$$

The free Hamiltonian is the normally ordered quantization of the classical Hamiltonian:

$$\hat{H}_0^n = \int : \left(\hat{\eta}^*(\vec{x}) \hat{\eta}(\vec{x}) + \vec{\partial} \hat{\psi}^*(\vec{x}) \vec{\partial} \hat{\psi}(\vec{x}) + m^2 \hat{\psi}^*(\vec{x}) \hat{\psi}(\vec{x}) \right) : d\vec{x}.$$

Introduce the creation/annihilation operators $\hat{a}^*(p)$, $\hat{a}(p)$, $\hat{b}^*(p)$, $\hat{b}(p)$. The Hamiltonian can be rewritten as

$$\hat{H}_0^n = \int d\vec{p} \varepsilon(\vec{p}) \left(\hat{a}^*(p) \hat{a}(p) + \hat{b}^*(p) \hat{b}(p) \right).$$

The **classical scalar field in an external electromagnetic potential** satisfies

$$\left(-(\partial_\mu + ieA_\mu(x))(\partial^\mu + ieA^\mu(x)) + m^2 \right) \psi(x) = 0.$$

The conjugate variable is $\eta(x) := \partial_t \psi(x) + ieA_0(x)\psi(x)$.

The Hamiltonian is

$$\begin{aligned} H = \int d\vec{x} & \left(\eta^*(\vec{x})\eta(\vec{x}) + ieA_0(\vec{x}) (\psi^*(\vec{x})\eta(\vec{x}) - \eta^*(\vec{x})\psi(\vec{x})) \right. \\ & \left. + (\partial_i - ieA_i(\vec{x}))\psi^*(\vec{x})(\partial_i + ieA_i(\vec{x}))\psi(\vec{x}) \right. \\ & \left. + m^2\psi^*(\vec{x})\psi(\vec{x}) \right) \end{aligned}$$

In terms of normal modes H has the form

$$\begin{aligned}
H = & \int d\vec{p} \varepsilon(\vec{p}) (a^*(p)a(p) + b^*(p)b(p)) \\
& + \frac{e}{2} \int \int \frac{d\vec{p}_1 d\vec{p}_2}{(2\pi)^3} \left(\sqrt{\frac{\varepsilon(\vec{p}_1)}{\varepsilon(\vec{p}_2)}} + \sqrt{\frac{\varepsilon(\vec{p}_2)}{\varepsilon(\vec{p}_1)}} \right) \\
& \times (A_0(\vec{p}_1 - \vec{p}_2) a^*(p_1) a(p_2) - A_0(-\vec{p}_1 + \vec{p}_2) b(p_1) b^*(p_2)) \\
& + \frac{e}{2} \int \int \frac{d\vec{p}_1 d\vec{p}_2}{(2\pi)^3} \left(\sqrt{\frac{\varepsilon(\vec{p}_1)}{\varepsilon(\vec{p}_2)}} - \sqrt{\frac{\varepsilon(\vec{p}_2)}{\varepsilon(\vec{p}_1)}} \right) \\
& \times (A_0(\vec{p}_1 + \vec{p}_2) a^*(p_1) b^*(p_2) - A_0(-\vec{p}_1 - \vec{p}_2) b(p_1) a(p_2)) \\
& + \frac{e}{2} \int \int \frac{d\vec{p}_1 d\vec{p}_2}{(2\pi)^3 \sqrt{\varepsilon(\vec{p}_1) \varepsilon(\vec{p}_2)}} (\vec{p}_1 + \vec{p}_2) \\
& \times \left(-\vec{A}(\vec{p}_1 - \vec{p}_2) a^*(p_1) a(p_2) + \vec{A}(-\vec{p}_1 + \vec{p}_2) b(p_1) b^*(p_2) \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{e}{2} \int \int \frac{d\vec{p}_1 d\vec{p}_2}{(2\pi)^3 \sqrt{\varepsilon(\vec{p}_1) \varepsilon(\vec{p}_2)}} (\vec{p}_1 - \vec{p}_2) \\
& \times \left(-\vec{A}(\vec{p}_1 + \vec{p}_2) a^*(p_1) b^*(p_2) + \vec{A}(-\vec{p}_1 - \vec{p}_2) b(p_1) a(p_2) \right) \\
& + \frac{e^2}{2} \int \int \frac{d\vec{p}_1 d\vec{p}_2}{(2\pi)^3 \sqrt{\varepsilon(\vec{p}_1)} \sqrt{\varepsilon(\vec{p}_2)}} \\
& \times \left(\vec{A}^2(\vec{p}_1 - \vec{p}_2) a^*(p_1) a(p_2) + \vec{A}^2(-\vec{p}_1 + \vec{p}_2) b(p_1) b^*(p_2) \right. \\
& \left. + \vec{A}^2(\vec{p}_1 + \vec{p}_2) a^*(p_1) b^*(p_2) + \vec{A}^2(-\vec{p}_1 - \vec{p}_2) b(p_1) a(p_2) \right).
\end{aligned}$$

Consider now the **quantum scalar field in an external electromagnetic potential**

$$\left(-(\partial_\mu + ieA_\mu(x))(\partial^\mu + ieA^\mu(x)) + m^2 \right) \hat{\psi}(x) = 0.$$

We ask whether there exists a Hamiltonian \hat{H} such that

$$\hat{\psi}(t, \vec{x}) = e^{it\hat{H}} \hat{\psi}(\vec{x}) e^{-it\hat{H}}.$$

First note that it is not natural to consider the normally ordered quantization of the classical Hamiltonian—it is not even gauge invariant (besides being ill-defined). The natural starting point for an analysis of the quantum Hamiltonian should be the symmetric (Weyl) quantization of the classical Hamiltonian: ill-defined as an operator, however formally gauge-invariant:

$$\begin{aligned} \hat{H}^{\text{W}} = & \int d\vec{x} \left(\hat{\eta}^*(\vec{x})\hat{\eta}(\vec{x}) + ieA_0(\vec{x}) (\hat{\psi}^*(\vec{x})\hat{\eta}(\vec{x}) - \hat{\eta}^*(\vec{x})\hat{\psi}(\vec{x})) \right. \\ & + (\partial_i - ieA_i(\vec{x}))\hat{\psi}^*(\vec{x})(\partial_i + ieA_i(\vec{x}))\hat{\psi}(\vec{x}) \\ & \left. + m^2\hat{\psi}^*(\vec{x})\hat{\psi}(\vec{x}) \right). \end{aligned}$$

We start from the formal expression for the infimum of the Weyl quadratic Hamiltonians and we expand it in e :

$$\begin{aligned} E^{\text{W}} &= \text{Tr} \sqrt{- (\vec{\partial} + ie\vec{A})^2 + m^2 - e^2 A_0^2} \\ &=: \sum_{n=1}^{\infty} e^{2n} E_{2n}. \end{aligned}$$

Note that only even powers of e appear—this goes under the name of the Furry Theorem.

Clearly, $E_0 = \text{Tr} \sqrt{-\vec{\partial}^2 + m^2}$ is infinite and should be dropped. The term E_2 is the sum of two diagrams: the loop with one 2-photon vertex and the loop with two 1-photon vertices. E_2 is infinite, however the next terms in the expansion are finite. Thus a possible finite expression for the vacuum energy could be

$$E^{2\text{ren}} = E^{\text{W}} - E_0 - e^2 E_2.$$

However, again, physically it is preferable to consider the renormalized vacuum energy obtained by subtracting a “local counterterm”. The Pauli-Villars method, leads to

$$\begin{aligned}
 E_2^{\text{ren}} &:= - \int \frac{d^4p}{2(2\pi)^4} \Pi^{\text{ren}}(p^2) \overline{F_{\mu\nu}(p)} F^{\mu\nu}(p) \\
 &= E_2 - C e^2 \int F_{\mu\nu}(\vec{x}) F^{\mu\nu}(\vec{x}) d\vec{x}, \\
 \Pi^{\text{ren}}(p^2) &:= \frac{e^2}{2 \cdot 3(4\pi)^2} \left(\frac{(p^2 + 4m^2)^{3/2}}{(p^2)^{3/2}} \log \frac{\sqrt{p^2 + 4m^2} + \sqrt{p^2}}{\sqrt{p^2 + 4m^2} - \sqrt{p^2}} \right. \\
 &\quad \left. - \frac{2}{3} - 2 \left(\frac{4m^2}{p^2} + 1 \right) \right).
 \end{aligned}$$

where C is an infinite counterterm.

Thus the physically acceptable formula for the vacuum energy is

$$E^{\text{ren}} = E^{\text{W}} - E_0 - E_2 + E_2^{\text{ren}}.$$

One can also try to use the corresponding renormalized Hamiltonian, formally written as

$$\begin{aligned}\hat{H}^{\text{ren}} &:= \hat{H}^{\text{W}} - E_0 - Ce^2 \int F_{\mu\nu}(\vec{x}) F^{\mu\nu}(\vec{x}) d\vec{x} \\ &= \hat{H}^{\text{W}} - E_0 - E_2 + E_2^{\text{ren}}.\end{aligned}$$

However, \hat{H}^{ren} exists only if $\vec{A} = 0$.