QUADRATIC HAMILTONIANS AND THEIR RENORMALIZATION

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Quadratic bosonic Hamiltonians are formally operators of the form

$$\hat{H} := \sum h_{ij}\hat{a}_i^*\hat{a}_j + \frac{1}{2}\sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \frac{1}{2}\sum \overline{g}_{ij}\hat{a}_i\hat{a}_j + c.$$

Special cases:

$c = \frac{1}{2}\sum h_{ii}$ corresponds to the Weyl quantization,

$c = 0$ corresponds to the normally ordered quantization.

We will see that other choices of c can be useful.

Note also that if the number of degrees of freedom is infinite, c can be infinite!

One can compute the infimum of $\hat{H}$ or the

vacuum energy

$$E := \inf \hat{H} = \frac{1}{4}\mathrm{Tr}\Big(\sqrt{B^2} - \sqrt{B_0^2}\Big) + c.$$

where

$$B := \begin{bmatrix} h & -g \\ \overline{g} & -\overline{h} \end{bmatrix}, \quad B_0 := \begin{bmatrix} h & 0 \\ 0 & -\overline{h} \end{bmatrix}.$$

I would like o discuss two examples of quadratic Hamiltonians taken from QFT. They illustrate how nontrivial their theory can be:

1. neutral scalar field with position dependent mass,
2. charged scalar field in electromagnetic potential.

These models belong to Local Quantum Physics. Even though they are not translation invariant, their dynamics is causal.

Consider the free classical neutral scalar field

$$(-\Box + m^2)\phi(x) = 0, \quad x \in \mathbb{R}^{1,3}.$$

Together with the conjugate fields $\pi(x) = \partial_t \phi(x)$ they have the Poisson brackets

$$\begin{aligned}\{\phi(\vec{x}), \phi(\vec{y})\} = \{\pi(\vec{x}), \pi(\vec{y})\} &= 0, \\ \{\phi(\vec{x}), \pi(\vec{y})\} &= \delta(\vec{x} - \vec{y}), \quad \vec{x} \in \mathbb{R}^3.\end{aligned}$$

The Hamiltonian

$$H_0 := \int \Big(\frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\phi(\vec{x})\big)^2 + \frac{1}{2}m^2\phi^2(x)\Big)\mathrm{d}\vec{x}$$

generates the dynamics

$$\partial_t\phi(x) = \{\phi(x), H_0\}, \quad \partial_t\pi(x) = \{\pi(x), H_0\}.$$

In order to diagonalize the Hamiltonian, we set

$$\varepsilon(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$$

and we introduce the "normal modes"

$$a(k) := \int \frac{\mathrm{d}\vec{x}}{\sqrt{(2\pi)^3}} \mathrm{e}^{-\mathrm{i}\vec{k}\vec{x}} \left(\sqrt{\frac{\varepsilon(\vec{k})}{2}} \phi(0,\vec{x}) + \frac{\mathrm{i}}{\sqrt{2\varepsilon(\vec{k})}} \pi(0,\vec{x}) \right),$$

$$a^*(k) := \int \frac{\mathrm{d}\vec{x}}{\sqrt{(2\pi)^3}} \mathrm{e}^{\mathrm{i}\vec{k}\vec{x}} \left(\sqrt{\frac{\varepsilon(\vec{k})}{2}} \phi(0,\vec{x}) - \frac{\mathrm{i}}{\sqrt{2\varepsilon(\vec{k})}} \pi(0,\vec{x}) \right).$$

The normal modes diagonalize the Poisson relations and the Hamiltonian:

$$\{a(k), a(k')\} = \{a^*(k), a^*(k')\} \; = \; 0,$$
$$\{a(k), a^*(k')\} \; = \; -\mathrm{i}\delta(\vec{k} - \vec{k}'),$$

$$H_0 \; = \; \int \mathrm{d}\vec{k}\varepsilon(\vec{k})a^*(k)a(k).$$

The fields can be expressed in terms of normal modes as

$$\phi(x) = \int \frac{\mathrm{d}\vec{k}}{\sqrt{(2\pi)^3}\sqrt{2\varepsilon(\vec{k})}} \left(\mathrm{e}^{\mathrm{i}kx} a(k) + \mathrm{e}^{-\mathrm{i}kx} a^*(k)\right),$$

$$\pi(x) = \int \frac{\mathrm{d}\vec{k}\sqrt{\varepsilon(\vec{k})}}{\mathrm{i}\sqrt{(2\pi)^3}\sqrt{2}} \left(\mathrm{e}^{\mathrm{i}kx} a(k) - \mathrm{e}^{-\mathrm{i}kx} a^*(k)\right).$$

The free quantum neutral scalar field, denoted $\hat{\phi}(x)$, satisfies the hatted versions of the classical equations:

$$(-\Box + m^2)\hat{\phi}(x) = 0, \quad \partial_t \hat{\phi}(x) = \hat{\pi}(x).$$

and the commutation relations

$$\begin{aligned} [\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] &= 0, \\ [\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] &= \mathrm{i}\delta(\vec{x} - \vec{y}). \end{aligned}$$

The free Hamiltonian is defined in the standard way:

$$\hat{H}_0^{\mathrm{n}} := \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\hat{\phi}(\vec{x})\big)^2 + \frac{1}{2}m^2\hat{\phi}^2(x)\Big): \mathrm{d}\vec{x},$$

where the double dots denote the normal ordering.

One introduces $\hat{a}(k)$ and $\hat{a}^*(k)$, by the hatted classical relations. They diagonalize the Hamiltonian and the commutation relations:

$$\begin{aligned} \hat{H}_0^{\mathrm{n}} &= \int \mathrm{d}\vec{k}\varepsilon(\vec{k})\hat{a}^*(k)\hat{a}(k), \\ [\hat{a}(k), \hat{a}(k')] = [\hat{a}^*(k), \hat{a}^*(k')] &= 0, \\ [\hat{a}(k), \hat{a}^*(k')] &= \delta(\vec{k} - \vec{k}'). \end{aligned}$$

Consider the classical field with a position dependent mass:

$$(-\Box + m^2)\phi(x) = -\kappa(\vec{x})\phi(x).$$

We assume that κ is a Schwartz function.

The classical Hamiltonian is

$$H := \int \Big(\frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\phi(\vec{x})\big)^2 + \frac{1}{2}\big(m^2 + \kappa(\vec{x})\big)\phi^2(\vec{x})\Big)\mathrm{d}\vec{x}.$$

We can rewrite the Hamiltonian in terms of normal modes

$$
\begin{aligned}
H &= \int \mathrm{d}\vec{k}\varepsilon(\vec{k})a^*(k)a(k) \\
&+\frac{1}{2}\int \frac{\mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\kappa(\vec{k}_1+\vec{k}_2)}{(2\pi)^3\sqrt{2\varepsilon(\vec{k}_1)}\sqrt{2\varepsilon(\vec{k}_2)}} \\
&\times\Big(a(-k_1)a(-k_2)+2a^*(k_1)a(-k_2)+a^*(k_1)a^*(k_2)\Big).
\end{aligned}
$$

The quantum field with a position dependent mass satisfies

$$(-\Box + m^2)\hat{\phi}(x) = -\kappa(x)\hat{\phi}(x).$$

We assume that it coincides with the free field at time $t = 0$. We also would like to find a Hamiltonian $\hat{H}$ such that

$$\hat{\phi}(t, \vec{x}) = \mathrm{e}^{\mathrm{i}t\hat{H}}\hat{\phi}(\vec{x})\mathrm{e}^{-\mathrm{i}t\hat{H}}.$$

$\hat{H}$ is defined up to an additive constant–we would like to fix a physically distinguished constant and obtain the vacuum energy.

Naively, the most obvious choice for $\hat{H}$ is the normally ordered quantization of the classical Hamiltonian:

$$\hat{H}^{\mathrm{n}} := \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\hat{\phi}(\vec{x})\big)^2 + \frac{1}{2}(m^2 + \kappa(\vec{x}))\hat{\phi}^2(\vec{x})\Big): \mathrm{d}\vec{x}$$

Expressed in terms of creation/annihilation operators it reads

$$\begin{aligned}\hat{H}^{\mathrm{n}} = & \int \mathrm{d}\vec{k}\varepsilon(\vec{k})\hat{a}^*(k)\hat{a}(k) \\ & + \frac{1}{2}\int \frac{\mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \kappa(\vec{k}_1 + \vec{k}_2)}{(2\pi)^3\sqrt{2\varepsilon(\vec{k}_1)}\sqrt{2\varepsilon(\vec{k}_2)}} \\ & \times\Big(\hat{a}(-k_1)\hat{a}(-k_2) + 2\hat{a}^*(k_1)\hat{a}(-k_2) + \hat{a}^*(k_1)\hat{a}^*(k_2)\Big).\end{aligned}$$

However, $\hat{H}^{\mathrm{n}}$ does not exist. This can be seen when we try to compute the infimum of $\hat{H}^{\mathrm{n}}$: the 2nd order contribution to the energy E_2 given by the loop with 2 vertices diverges. Fortunately, the higher order terms are finite, and as a Hamiltonian implementing the dynamics we could take

$$\hat{H}^{2\mathrm{ren}} := \hat{H}^{\mathrm{n}} - E_2.$$

Physically it is however preferable to make an additional finite renormalization, so that the counterterms are local.

Using e.g. the Pauli-Villars method we obtain the following renormalized expression for the 2nd order term:

$$\begin{aligned} E_2^{\mathrm{ren}} &:= \int \pi^{\mathrm{ren}}(\vec{k}^2) |\kappa(\vec{k})|^2 \frac{\mathrm{d}\vec{k}}{(2\pi)^3} \\ &= E_2 - C \int \kappa(\vec{x})^2 \mathrm{d}\vec{x}, \end{aligned}$$

$$\pi^{\mathrm{ren}}(\vec{k}^2) := \frac{1}{4(4\pi)^2} \left(\frac{\sqrt{\vec{k}^2 + 4m^2}}{\sqrt{\vec{k}^2}} \log \frac{\sqrt{\vec{k}^2 + 4m^2} + \sqrt{\vec{k}^2}}{\sqrt{\vec{k}^2 + 4m^2} - \sqrt{\vec{k}^2}} - 2 \right),$$

where C is an infinite counterterm.

We used $\pi^{\mathrm{ren}}(0) = 0$ as the renormalization condition.

The physically acceptable renormalized Hamiltonian can be formally written as

$$\begin{aligned}\hat{H}^{\mathrm{ren}} &:= \hat{H}^{\mathrm{n}} - C \int \kappa(\vec{x})^2 \mathrm{d}\vec{x} \\ &= \hat{H}^{\mathrm{n}} - E_2 + E_2^{\mathrm{ren}}.\end{aligned}$$

$\hat{H}^{\mathrm{ren}}$ is a well defined self-adjoint operator (despite that $\hat{H}^{\mathrm{n}}$ is ill defined, and E_2, C are infinite). It is bounded from below and its infimum is

$$\begin{aligned}
&E^{\mathrm{ren}} = E_2^{\mathrm{ren}} \\
&+ \int \mathrm{Tr} \frac{1}{(-\Delta + m^2 + \tau^2)} \kappa \frac{1}{(-\Delta + m^2 + \tau^2)} \kappa \\
&\qquad\qquad \times \frac{1}{(-\Delta + m^2 + \kappa + \tau^2)} \kappa \frac{1}{(-\Delta + m^2 + \tau^2)} \tau^2 \frac{\mathrm{d}\tau}{2\pi}.
\end{aligned}$$

Consider now a space-time dependent κ:

$$\mathbb{R}^{1,3} \ni (t, \vec{x}) \mapsto \kappa(t, \vec{x}).$$

We assume that κ is, say, a Schwartz function. The corresponding time-dependent renormalized Hamiltonian $\hat{H}^{\mathrm{ren}}(\kappa(t))$ generates the dynamics

$$U(\kappa, t_2, t_1) := \mathrm{Texp}\Big(-\mathrm{i} \int_{t_1}^{t_2} \hat{H}^{\mathrm{ren}}(\kappa(t))\mathrm{d}t \Big).$$

We can also introduce the scattering operator

$$S(\kappa) := \lim_{t \to \infty} \mathrm{e}^{\mathrm{i}t\hat{H}_0^{\mathrm{n}}} U(\kappa, t, -t) \mathrm{e}^{-\mathrm{i}t\hat{H}_0^{\mathrm{n}}}.$$

The scattering operator satisfies the Bogoliubov identity, which expresses the Einstein causality:
if $\mathrm{supp}\kappa_2$ is later than $\mathrm{supp}\kappa_1$, then

$$S(\kappa + \kappa_1 + \kappa_2) = S(\kappa + \kappa_2) S(\kappa)^{-1} S(\kappa + \kappa_1).$$

Consider the free charged scalar classical field $\psi(x)$

$$(-\Box + m^2)\psi(x) = 0.$$

$\psi^*(x)$ denotes its complex adjoint. The conjugate field is $\eta(x) := \partial_t \psi(x)$. The zero time Poisson brackets are

$$\{\psi(\vec{x}), \psi(\vec{y})\} = \{\psi(\vec{x}), \eta(\vec{y})\} = \{\eta(\vec{x}), \eta(\vec{y})\} = 0,$$
$$\{\psi(\vec{x}), \eta^*(\vec{y})\} = \{\psi^*(\vec{x}), \eta(\vec{y})\} = \delta(\vec{x} - \vec{y}).$$

The Hamiltonian

$$H_0 = \int \Big(\eta^*(\vec{x})\eta(\vec{x}) + \vec{\partial}\psi^*(\vec{x})\vec{\partial}\psi(\vec{x}) + m^2\psi^*(\vec{x})\psi(\vec{x}) \Big) \mathrm{d}\vec{x}.$$

generates the dynamics

$$\partial_t \psi(x) = \{\psi(x), H_0\}, \quad \partial_t \eta(x) = \{\eta(x), H_0\}.$$

One introduces normal modes:

$$
\begin{aligned}
a(p) &= \int \left(\sqrt{\frac{\varepsilon(\vec{p})}{2}} \psi(0,\vec{x}) + \frac{\mathrm{i}}{\sqrt{2\varepsilon(\vec{p})}} \eta(0,\vec{x}) \right) \mathrm{e}^{-\mathrm{i}\vec{p}\vec{x}} \frac{\mathrm{d}\vec{x}}{\sqrt{(2\pi)^3}}, \\
a^*(p) &= \int \left(\sqrt{\frac{\varepsilon(\vec{p})}{2}} \psi^*(0,\vec{x}) - \frac{\mathrm{i}}{\sqrt{2\varepsilon(\vec{p})}} \eta^*(0,\vec{x}) \right) \mathrm{e}^{\mathrm{i}\vec{p}\vec{x}} \frac{\mathrm{d}\vec{x}}{\sqrt{(2\pi)^3}}, \\
b(p) &= \int \left(\sqrt{\frac{\varepsilon(\vec{p})}{2}} \psi^*(0,\vec{x}) + \frac{\mathrm{i}}{\sqrt{2\varepsilon(\vec{p})}} \eta^*(0,\vec{x}) \right) \mathrm{e}^{-\mathrm{i}\vec{p}\vec{x}} \frac{\mathrm{d}\vec{x}}{\sqrt{(2\pi)^3}}, \\
b^*(p) &= \int \left(\sqrt{\frac{\varepsilon(\vec{p})}{2}} \psi(0,\vec{x}) - \frac{\mathrm{i}}{\sqrt{2\varepsilon(\vec{p})}} \eta(0,\vec{x}) \right) \mathrm{e}^{\mathrm{i}\vec{p}\vec{x}} \frac{\mathrm{d}\vec{x}}{\sqrt{(2\pi)^3}}.
\end{aligned}
$$

Normal modes diagonalize the Hamiltonian and the Poisson brackets

$$
\begin{aligned}
H_0 &= \int \mathrm{d}\vec{p}\,\varepsilon(\vec{p})\big(a^*(p)a(p) + b^*(p)b(p)\big), \\
\{a(p), a^*(p')\} &= \{b(p), b^*(p')\} = -\mathrm{i}\delta(\vec{p} - \vec{p}'),
\end{aligned}
$$

We can express fields in terms of normal modes:

$$\psi(x) = \int \frac{\mathrm{d}\vec{p}}{\sqrt{(2\pi)^3}\sqrt{2\varepsilon(\vec{p})}} \left(\mathrm{e}^{\mathrm{i}px} a(p) + \mathrm{e}^{-\mathrm{i}px} b^*(p) \right),$$

$$\eta(x) = \int \frac{\mathrm{d}\vec{p}\sqrt{\varepsilon(\vec{p})}}{\mathrm{i}\sqrt{(2\pi)^3}\sqrt{2}} \left(\mathrm{e}^{\mathrm{i}px} a(p) - \mathrm{e}^{-\mathrm{i}px} b^*(p) \right).$$

The free quantum charged scalar field is described by $\hat{\psi}(\vec{x}), \hat{\psi}^*(\vec{x})$. It satisfies

$$(-\Box + m^2)\hat{\psi}(x) = 0, \quad \partial_t\hat{\psi}(x) = \hat{\eta}(x)$$

and has the commutation relations

$$[\hat{\psi}(\vec{x}), \hat{\psi}(\vec{y})] = [\hat{\psi}(\vec{x}), \hat{\eta}(\vec{y})] = [\hat{\eta}(\vec{x}), \hat{\eta}(\vec{y})] = 0,$$
$$[\hat{\psi}(\vec{x}), \hat{\eta}^*(\vec{y})] = [\hat{\psi}^*(\vec{x}), \hat{\eta}(\vec{y})] = \mathrm{i}\delta(\vec{x} - \vec{y}).$$

The free Hamiltonian is the normally ordered quantization of the classical Hamiltonian:

$$\hat{H}_0^{\mathrm{n}} = \int :\Big(\hat{\eta}^*(\vec{x})\hat{\eta}(\vec{x}) + \vec{\partial}\hat{\psi}^*(\vec{x})\vec{\partial}\hat{\psi}(\vec{x}) + m^2\hat{\psi}^*(\vec{x})\hat{\psi}(\vec{x})\Big):\mathrm{d}\vec{x}.$$

Introduce the creation/annihilation operators $\hat{a}^*(p)$, $\hat{a}(p)$, $\hat{b}^*(p)$, $\hat{b}(p)$. The Hamiltonian can be rewritten as

$$\hat{H}_0^{\mathrm{n}} = \int \mathrm{d}\vec{p}\varepsilon(\vec{p})\big(\hat{a}^*(p)\hat{a}(p) + \hat{b}^*(p)\hat{b}(p)\big).$$

The classical scalar field in an external electromagnetic potential satisfies

$$\left(-(\partial_\mu + \mathrm{i}eA_\mu(x))(\partial^\mu + \mathrm{i}eA^\mu(x)) + m^2\right)\psi(x) = 0.$$

The conjugate variable is $\eta(x) := \partial_t\psi(x) + \mathrm{i}eA_0(x)\psi(x)$.

The Hamiltonian is

$$\begin{aligned} H = \int \mathrm{d}\vec{x}\Big(&\eta^*(\vec{x})\eta(\vec{x}) + \mathrm{i}eA_0(\vec{x})\big(\psi^*(\vec{x})\eta(\vec{x}) - \eta^*(\vec{x})\psi(\vec{x})\big) \\ &+(\partial_i - \mathrm{i}eA_i(\vec{x}))\psi^*(\vec{x})(\partial_i + \mathrm{i}eA_i(\vec{x}))\psi(\vec{x}) \\ &+m^2\psi^*(\vec{x})\psi(\vec{x})\Big) \end{aligned}$$

In terms of normal modes H has the form

$$
\begin{aligned}
H \;=\; & \int \mathrm{d}\vec{p}\varepsilon(\vec{p})\big(a^*(p)a(p)+b^*(p)b(p)\big) \\
& +\frac{e}{2}\int\int \frac{\mathrm{d}\vec{p}_1\mathrm{d}\vec{p}_2}{(2\pi)^3}\left(\sqrt{\frac{\varepsilon(\vec{p}_1)}{\varepsilon(\vec{p}_2)}}+\sqrt{\frac{\varepsilon(\vec{p}_2)}{\varepsilon(\vec{p}_1)}}\right) \\
& \times\left(A_0(\vec{p}_1-\vec{p}_2)a^*(p_1)a(p_2)-A_0(-\vec{p}_1+\vec{p}_2)b(p_1)b^*(p_2)\right) \\
& +\frac{e}{2}\int\int \frac{\mathrm{d}\vec{p}_1\mathrm{d}\vec{p}_2}{(2\pi)^3}\left(\sqrt{\frac{\varepsilon(\vec{p}_1)}{\varepsilon(\vec{p}_2)}}-\sqrt{\frac{\varepsilon(\vec{p}_2)}{\varepsilon(\vec{p}_1)}}\right) \\
& \times\left(A_0(\vec{p}_1+\vec{p}_2)a^*(p_1)b^*(p_2)-A_0(-\vec{p}_1-\vec{p}_2)b(p_1)a(p_2)\right) \\
& +\frac{e}{2}\int\int \frac{\mathrm{d}\vec{p}_1\mathrm{d}\vec{p}_2}{(2\pi)^3\sqrt{\varepsilon(\vec{p}_1)\varepsilon(\vec{p}_2)}}(\vec{p}_1+\vec{p}_2) \\
& \times\Big(-\vec{A}(\vec{p}_1-\vec{p}_2)a^*(p_1)a(p_2)+\vec{A}(-\vec{p}_1+\vec{p}_2)b(p_1)b^*(p_2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{e}{2}\int\int\frac{\mathrm{d}\vec{p}_1\mathrm{d}\vec{p}_2}{(2\pi)^3\sqrt{\varepsilon(\vec{p}_1)\varepsilon(\vec{p}_2)}}(\vec{p}_1-\vec{p}_2)\\
&\times\Big(-\vec{A}(\vec{p}_1+\vec{p}_2)a^*(p_1)b^*(p_2)+\vec{A}(-\vec{p}_1-\vec{p}_2)b(p_1)a(p_2)\Big)\\
&+\frac{e^2}{2}\int\int\frac{\mathrm{d}\vec{p}_1\mathrm{d}\vec{p}_2}{(2\pi)^3\sqrt{\varepsilon(\vec{p}_1)}\sqrt{\varepsilon(\vec{p}_2)}}\\
&\times\Big(\vec{A}^2(\vec{p}_1-\vec{p}_2)a^*(p_1)a(p_2)+\vec{A}^2(-\vec{p}_1+\vec{p}_2)b(p_1)b^*(p_2)\\
&+\vec{A}^2(\vec{p}_1+\vec{p}_2)a^*(p_1)b^*(p_2)+\vec{A}^2(-\vec{p}_1-\vec{p}_2)b(p_1)a(p_2)\Big).
\end{aligned}
$$

Consider now the quantum scalar field in an external electromagnetic potential

$$\left(-(\partial_\mu + \mathrm{i}eA_\mu(x))(\partial^\mu + \mathrm{i}eA^\mu(x)) + m^2\right)\hat{\psi}(x) = 0.$$

We ask whether there exists a Hamiltonian $\hat{H}$ such that

$$\hat{\psi}(t, \vec{x}) = \mathrm{e}^{\mathrm{i}t\hat{H}}\hat{\psi}(\vec{x})\mathrm{e}^{-\mathrm{i}t\hat{H}}.$$

First note that it is not natural to consider the normally ordered quantization of the classical Hamiltonian–it is not even gauge invariant (besides being ill-defined). The natural starting point for an analysis of the quantum Hamiltonian should be the symmetric (Weyl) quantization of the classical Hamiltonian: ill-defined as an operator, however formally gauge-invariant:

$$\begin{aligned}\hat{H}^{\mathrm{W}} = \int \mathrm{d}\vec{x}\Big(&\hat{\eta}^*(\vec{x})\hat{\eta}(\vec{x}) + \mathrm{i}eA_0(\vec{x})\big(\hat{\psi}^*(\vec{x})\hat{\eta}(\vec{x}) - \hat{\eta}^*(\vec{x})\hat{\psi}(\vec{x})\big)\\ &+(\partial_i - \mathrm{i}eA_i(\vec{x}))\hat{\psi}^*(\vec{x})(\partial_i + \mathrm{i}eA_i(\vec{x}))\hat{\psi}(\vec{x})\\ &+m^2\hat{\psi}^*(\vec{x})\hat{\psi}(\vec{x})\Big).\end{aligned}$$

We start from the formal expression for the infimum of the Weyl quadratic Hamiltonians and we expand it in e:

$$\begin{aligned} E^{\mathrm{W}} &= \mathrm{Tr}\sqrt{-(\vec{\partial}+\mathrm{i}e\vec{A})^2+m^2-e^2A_0^2} \\ &=: \sum_{n=1}^{\infty} e^{2n}E_{2n}. \end{aligned}$$

Note that only even powers of e appear–this goes under the name of the Furry Theorem.

Clearly, $E_0 = \mathrm{Tr}\sqrt{-\vec{\partial}^2 + m^2}$ is infinite and should be dropped. The term E_2 is the sum of two diagrams: the loop with one 2-photon vertex and the loop with two 1-photon vertices. E_2 is infinite, however the next terms in the expansion are finite. Thus a possible finite expression for the vacuum energy could be

$$E^{2\mathrm{ren}} = E^{\mathrm{w}} - E_0 - e^2 E_2.$$

However, again, physically it is preferable to consider the renormalized vacuum energy obtained by subtracting a "local counterterm". The Pauli-Villars method, leads to

$$
\begin{aligned}
E_2^{\mathrm{ren}} &:= -\int \frac{\mathrm{d}p}{2(2\pi)^4}\Pi^{\mathrm{ren}}(p^2)\overline{F_{\mu\nu}(p)}F^{\mu\nu}(p) \\
&= E_2 - Ce^2\int F_{\mu\nu}(\vec{x})F^{\mu\nu}(\vec{x})\mathrm{d}\vec{x},
\end{aligned}
$$

$$
\Pi^{\mathrm{ren}}(p^2) := \frac{e^2}{2\cdot 3(4\pi)^2}\Bigg(\frac{(p^2+4m^2)^{3/2}}{(p^2)^{3/2}}\log\frac{\sqrt{p^2+4m^2}+\sqrt{p^2}}{\sqrt{p^2+4m^2}-\sqrt{p^2}} - \frac{2}{3} - 2\Big(\frac{4m^2}{p^2}+1\Big)\Bigg).
$$

where C is an infinite counterterm.

Thus the physically acceptable formula for the vacuum energy is

$$E^{\mathrm{ren}} = E^{\mathrm{w}} - E_0 - E_2 + E_2^{\mathrm{ren}}.$$

One can also try to use the corresponding renormalized Hamiltonian, formally written as

$$\begin{aligned}\hat{H}^{\mathrm{ren}} &:= \hat{H}^{\mathrm{w}} - E_0 - Ce^2 \int F_{\mu\nu}(\vec{x}) F^{\mu\nu}(\vec{x}) \mathrm{d}\vec{x} \\ &= \hat{H}^{\mathrm{w}} - E_0 - E_2 + E_2^{\mathrm{ren}}.\end{aligned}$$

However, $\hat{H}^{\mathrm{ren}}$ exists only if $\vec{A} = 0$.

Consider now a space-time dependent A_μ:

$$\mathbb{R}^{1,3} \ni (t, \vec{x}) \mapsto A_\mu(t, \vec{x}).$$

We assume that it is a $C_{\rm c}^\infty$ function. Even though the time-dependent renormalized Hamiltonian $\hat{H}^{\rm ren}(A(t))$ usually does not exist, the corresponding evolution

$$U(\kappa, t_2, t_1) := {\rm Texp}\Big(-{\rm i} \int_{t_1}^{t_2} \hat{H}^{\rm ren}(A(t)){\rm d}t\Big)$$

is well defined if

$${\rm supp} A \subset]t_1, t_2[\times \mathbb{R}^3.$$

We can again introduce the scattering operator

$$S(A) := \lim_{t \to \infty} \mathrm{e}^{\mathrm{i}t\hat{H}_0^{\mathrm{n}}} U(A, t, -t) \mathrm{e}^{-\mathrm{i}t\hat{H}_0^{\mathrm{n}}},$$

which satisfies the Bogoliubov identity:

if $\mathrm{supp} A_2$ is later than $\mathrm{supp} A_1$, then

$$S(A + A_1 + A_2) = S(A + A_2) S(A)^{-1} S(A + A_1).$$