Conventional wisdom: If we make a localized perturbation to a thermal equilibrium state of a large quantum system, then its effects will disappear as time goes to infinity and the system will return to the equilibrium state.

This property goes under the name of

Return to Equilibrium.

Warning: This property may be not true if

- System has finite size (then it has discrete spectrum).
- System is nongeneric (e.g. has symmetries).
- Perturbation is infinitely extended (e.g. perturbation can change temperature T).

Usual description of a quantum system:

- **Hamiltonian** H;
- **evolution in Heisenberg picture given by**

$$\frac{\mathrm{d}}{\mathrm{d}t}A = \mathrm{i}[H, A] = \mathrm{i}L(A),$$

  where $L := [H, \cdot]$ **is sometimes called the Liouvillean;**
- **Gibbs state given by density matrix** $\mathrm{e}^{-\beta H}/\mathrm{Tr}\mathrm{e}^{-\beta H}$ **where** $\beta := \frac{1}{k_B T}$.

Return to equilibrium will never hold in such a system.

However, the return to equilibrium often is true for a dynamics on a W^***-algebra of observables corresponding to an infinitely extended system.**

Criterion: If the Liouvillean L has only absolutely continuous spectrum except for a simple eigenvalue at zero, then the return to equilibrium holds.

The effects of absolutely continuous part of spectrum "die away" as time goes to infinity because of

The Riemann-Lebesgue Lemma. Let $f \in L^1(\mathbb{R})$ and $\hat{f}$ its Fourier transform. Then

$$\lim_{t\to\infty} \hat{f}(t) = 0.$$

Simple example:

Free Bose gas at temperature T

If f is a test function, then $\phi(f)$ denotes the field smeared out with the function f.
ω denotes the Gibbs state at temperature T.

Confined case: box with periodic boundary conditions of size L:

$$\omega(\exp \mathrm{i}\phi(f)) = \exp\left(\frac{1}{2L^d}\sum_{\xi\in\mathbb{Z}^d/L}\frac{|f(\xi)|^2}{\mathrm{e}^{\beta|\xi|}-1}\right).$$

Return to equilibrium does not hold: the Liouvillean has discrete spectrum.

Infinitely extended case: We take thermodynamic limit $L\to\infty$:

$$\omega(\exp \mathrm{i}\phi(f)) = \exp\left(\frac{1}{2}\int\frac{|f(\xi)|^2}{\mathrm{e}^{\beta|\xi|}-1}\mathrm{d}\xi\right).$$

Return to equilibrium holds: The Liouvillean has absolutely continuous spectrum covering the whole real line and a simple eigenvalue corresponding to the KMS state.

More complicated example:

Small quantum system in contact with free Bose gas at temperature T

Small finite level system (a "quantum dot") described by a Hamiltonian K.

Noninteracting case: Small system and reservoir evolve independently.
Return to equilibrium does not hold: **Spectrum of the Liouvillean consists of absolutely continuous part, which is equal to real line and point spectrum consisting of differences of eigenvalues of K (transition energies of the small system)**

Interacting case: We add interaction that couples the small system and the Bose gas.
Return to equilibrium holds for a sufficiently small nonzero coupling constant, provided that the interaction is sufficiently "effective".
(Jakšić-Pillet, Jakšić-D., Bach-Fröhlich-Sigal, Merkli-Fröhlich).
This is because under the effect of interaction all eigenvalues of the Liouvillean turn to resonances except for a simple eigenvalue at zero.

Method of proof.

- KMS states are stable under a perturbation of dynamics *Araki, Jakšić-Pillet-D.*
- Fermi Golden Rule predicts that all other embedded eigenvalues are unstable.
- Using Positive Commutator Method we can show that spectrum is absolutely continuous (apart from the eigenvalue due to the KMS state).

Instability of embedded eigenvalues.

This is best explained using as a toy model the so-called Friedrichs Hamiltonian, (known also as the Wigner-Weisskopf atom).

It is an operator on $\mathbb{C} \oplus L^2(\mathbb{R})$ of the form

$$L := \begin{bmatrix} E_0 & \lambda \int v(k)(k|\mathrm{d}k \\ \lambda \int |k)\overline{v}(k)\mathrm{d}k & \int E(k)|k)(k|\mathrm{d}k \end{bmatrix},$$

where

(1) E_0 is the unperturbed eigenvalue;

(2) $E(k)$ is the unperturbed energy of the state $|k)$

(3) $v(k)$ is the strength of the off-diagonal perturbation.

E is an eigenvalue of the Friedrichs Hamiltonian L iff

$$\lambda^2 \int \frac{|v(k)|^2}{E - E(k)} \mathrm{d}k + E_0 = E$$

Therefore, if E is an eigenvalue, then
$|v(E)|^2 = 0$.

Hence if $|v(E_0)|^2 \neq 0$ ("interaction is effective") there are no eigeinvalues for small nonzero λ.

Fermi Golden Rule predicts that if $|v(E_0)|^2 \neq 0$, then the eigenvalue E_0 turns into a resonance with the decay rate $\lambda^2 \pi |v(E_0)|^2$.

Gibbs states.

Equilibrium states at temperature T in finite volume are Gibbs states given by the formula

$$\omega(A) = \mathrm{Tr} e^{-\beta H} A / \mathrm{Tr} e^{-\beta H}$$

Gibbs states are stable wrt a perturbation: if we replace H by $H+\lambda V$, then we have the Gibbs state

$$\omega_\lambda(A) = \mathrm{Tr} e^{-\beta(H+\lambda V)} A / \mathrm{Tr} e^{-\beta(H+\lambda V)}$$

KMS states

For systems in infinite volume Gibbs states are ill defined because $\mathrm{Tr}e^{-\beta H} = \infty$**.**

Equilibrium states at temperature T **are defined by the Kubo-Martin-Schwinger condition:**

$$\omega(AB) = \omega(B\tau_{\mathrm{i}\beta}(A)),$$

where τ_t **is the dynamics in the Heisenberg picture.**

KMS states are stable under a perturbation of dynamics *(Araki, Jakšić-Pillet-D).*

This violates the intuition saying that embedded eigenvalues are unstable under a perturbation!

Positive Commutator Method.

Theorem. Suppose that L and A are self-adjoint operators

(1) Assume and some technical conditions and

$$\mathrm{i}[A, L] \geq c_0 > 0.$$

Then L has no eigenvalues.

(2) Under additional technical conditions, L has only absolutely continuous spectrum (has no singular continuous spectrum) *(Mourre)*.

Proof of (1). Let Φ be an eigenvector of L with eigenvalue E. Then

$$\begin{aligned} \mathrm{i}(\Phi|[A, L]\Phi) &= \mathrm{i}(A\Phi|L\Phi) - \mathrm{i}(L\Phi|A\Phi) \\ &= \mathrm{i}E(A\Phi|\Phi) - \mathrm{i}E(\Phi|A\Phi) = 0. \end{aligned}$$

On the other hand, it has to be greater than $c_0\|\Phi\|^2 > 0$, which is a contradiction. $\square$

Physical intuition:

A is an observable that grows at a positive speed under the evolution generated by L:

$$\mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL} \geq A + c_0 t.$$

Bose gas at density ρ in the Araki-Woods representation

Consider the Fock space

$$\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)). \qquad (*)$$

We will write $a_{\mathrm{l}}(\xi)$, $a_{\mathrm{l}}^*(\xi)$, $a_{\mathrm{r}}(\xi)$, $a_{\mathrm{r}}^*(\xi)$ for the creation and annihilation operators corresponding to the left and right $L^2(\mathbb{R}^d)$ resp. We define the left/right Araki-Woods creation and annihillation opetators

$$\begin{aligned}
a_{\rho,\mathrm{l}}^*(\xi) &:= \sqrt{1+\rho(\xi)}a_{\mathrm{l}}^*(\xi) + \sqrt{\rho(\xi)}a_{\mathrm{r}}(\xi),\\
a_{\rho,\mathrm{l}}(\xi) &:= \sqrt{1+\rho(\xi)}a_{\mathrm{l}}(\xi) + \sqrt{\rho(\xi)}a_{\mathrm{r}}^*(\xi),\\
a_{\rho,\mathrm{r}}^*(\xi) &:= \sqrt{\rho(\xi)}a_{\mathrm{l}}(\xi) + \sqrt{1+\rho(\xi)}a_{\mathrm{r}}^*(\xi),\\
a_{\rho,\mathrm{r}}(\xi) &:= \sqrt{\rho(\xi)}a_{\mathrm{l}}^*(\xi) + \sqrt{1+\rho(\xi)}a_{\mathrm{r}}(\xi).
\end{aligned}$$

The left Araki-Woods algebra is denoted by $\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$ and defined as the W^*-algebra acting on the Hilbert space $(*)$ generated by the operators

$$\exp\left(\mathrm{i}\int (f(\xi)a_{\rho,\mathrm{l}}^*(\xi) + \overline{f}(\xi)a_{\rho,\mathrm{l}}(\xi))\mathrm{d}\xi\right).$$

Small quantum system in contact with Bose gas.

We denote by $\mathcal{K}$ the Hilbert space of the small quantum system. For simplicity, we assume that $\dim \mathcal{K} < \infty$. The W^*-algebra of the joint system consisting of the small system with the algebra of observables $\mathcal{B}(\mathcal{K})$ and the Bose gas at density ρ in its standard representation is

$$\mathcal{B}(\mathcal{K}) \otimes 1_{\overline{\mathcal{K}}} \otimes \mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}} \qquad (**)$$

acting on the Hilbert space $\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$.

Free Pauli-Fierz Liouvillean at density ρ

Suppose that K is a self-adjoint operator on $\mathcal{K}$ describing the Hamiltonian of the small system.
Let $|\xi|$ be the energy of the boson of momentum ξ.
Let $\mathbb{R}^d \ni \xi \mapsto v(\xi) \in \mathcal{B}(\mathcal{K})$ describe the coupling of the small system to the Bose gas. We assume that the Bose gas is at the density ρ.
Let $\lambda \in \mathbb{R}$.

The free Liouvillean equals

$$\begin{aligned} L_{\mathrm{fr}} &= K \otimes 1 \otimes 1 - 1 \otimes \overline{K} \otimes 1 \\ &+ 1 \otimes 1 \otimes \int |\xi| \big(a_{\mathrm{l}}^*(\xi) a_{\mathrm{l}}(\xi) - a_{\mathrm{r}}^*(\xi) a_{\mathrm{r}}(\xi) \big) \mathrm{d}\xi. \end{aligned}$$

Clearly, $\tau_{\mathrm{fr}}^t(A) := \mathrm{e}^{\mathrm{i}tL_\rho} A \mathrm{e}^{-\mathrm{i}tL_\rho}$ defines a dynamics on the W^* algebra $(**)$.
For thermal systems the density ρ is given by the Planck law

$$\rho(\xi) = (\mathrm{e}^{\beta|\xi|} - 1)^{-1}.$$

Full Pauli-Fierz Liouvillean at density ρ.

The interaction at density ρ:

$$Q_\rho = \int \big(v(\xi) \otimes 1 \otimes a^*_{\rho,\mathrm{l}}(\xi) + v^*(\xi) \otimes 1 \otimes a_{\rho,\mathrm{l}}(\xi)\big)\mathrm{d}\xi.$$

The interaction at density ρ composed with the modular conjugation:

$$JQ_\rho J = \int \big(1 \otimes \overline{v}(\xi) \otimes a^*_{\rho,\mathrm{r}}(\xi) + 1 \otimes \overline{v}^*(\xi) \otimes a_{\rho,\mathrm{r}}(\xi)\big)\mathrm{d}\xi.$$

The full Liouvillean

$$L_\rho := L_{\mathrm{fr}} + \lambda Q_\rho - \lambda J Q_\rho J.$$

Now $\tau^t_\rho(A) := \mathrm{e}^{\mathrm{i}tL_\rho} A \mathrm{e}^{-\mathrm{i}tL_\rho}$ **defines a dynamics on the** W^***-algebra** $(**)$.