Introduction to hypergeometric type functions

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1 Generalized hypergeometric equations and functions

1.1 Generalized hypergeometric series

For $a_1, \dots, a_k \in \mathbb{C}$, $c_1, \dots, c_m \in \mathbb{C} \backslash \{0, -1, -2, \dots\}$, we define the (generalized) hypergeometric series of type ${}_kF_m$:

$$ {}_kF_m(a_1, \dots, a_k; c_1, \dots, c_m; z) := \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_k)_j z^j}{(c_1)_j \cdots (c_m)_j j!}. \tag{1.1} $$

Notice that

1. if $m + 1 > k$, then (1.1) is convergent for $z \in \mathbb{C}$;
2. if $m + 1 = k$, then (1.1) is convergent for $|z| < 1$;
3. if $m + 1 < k$, then (1.1) is divergent (however sometimes we can give a meaning to the function ${}_kF_m$).

This follows by the d'Alembert criterion: if f_j is jth coefficient of (1.1), then

$$ \frac{f_{j+1}}{f_j} = \frac{(a_1 + j) \cdots (a_k + j)}{(c_1 + j) \cdots (c_m + j)}. $$

We can also use a different normalization:

$$ \begin{aligned} {}_k\mathbf{F}_m(a_1, \dots, a_k; c_1, \dots, c_m; z) &:= \frac{{}_kF_m(a_1, \dots, a_k; c_1, \dots, c_m; z)}{\Gamma(c_1) \cdots \Gamma(c_m)} \\ &= \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_k)_j z^j}{\Gamma(c_1 + j) \cdots \Gamma(c_m + j) j!}. \end{aligned} \tag{1.2} $$

Then we do not have to restrict the values of $c_1, \dots, c_m \in \mathbb{C}$. (If for some i $c_i \in \{0, -1, -2 \dots\}$, then $\mathbf{F}$ is zero).

1.2 Generalized hypergeometric equations

Theorem 1.1 *The function (1.1) solves the equation*

$$\Big((c_1 + z\partial_z)\cdots(c_m + z\partial_z)\partial_z - (a_1 + z\partial_z)\cdots(a_k + z\partial_z)\Big)F(z) = 0. \tag{1.3}$$

Proof. Write for brevity $F = \sum_{j=0}^{\infty} F_j z^j$. We check that

$$\begin{aligned}
(c_1 + z\partial_z)\cdots(c_k + z\partial_z)\partial_z F_j z^j &= \frac{(a_1)_j \cdots (a_k)_j z^{j-1}}{(c_1)_{j-1}\cdots(c_m)_{j-1}(j-1)!},\\
(a_1 + z\partial_z)\cdots(a_k + z\partial_z) F_{j-1} z^{j-1} &= \frac{(a_1)_j \cdots (a_k)_j z^{j-1}}{(c_1)_{j-1}\cdots(c_m)_{j-1}(j-1)!}.
\end{aligned}$$

□

Note that the equation (1.3) is of the order $\max(k, m+1)$. Below we list all equations and hypergeometric functions with equations of the order at most 2.

1.3 Hypergeometric function or ${}_2F_1$

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n.$$

The series is convergent for $|z| < 1$, it extends to a multivalued function on a covering of $\mathbb{C}\backslash\{0, 1\}$.

The function is a solution of the hypergeometric equation

$$\big(z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab\big)\, u(z) = 0$$

that is analytic around 0 and equals there 1.

1.4 Confluent function or ${}_1F_1$

$$F(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(c)_n} z^n.$$

The series is convergent for all $z \in \mathbb{C}$. It defines a solution analytic around 0 and equal there 1 of the confluent equation

$$(z\partial_z^2 + (c - z)\partial_z - a)u(z) = 0,$$

1.5 Function ${}_0F_1$

$$F(-;c;z) = F(c;z) = \sum_{n=0}^{\infty} \frac{1}{n!(c)_n} z^n.$$

The series is convergent for all $z \in \mathbb{C}$. It defines a solution analytic around 0 and equal there 1 of the ${}_0F_1$ equation (related to the Bessel equation)

$$(z\partial_z^2 + c\partial_z - 1)u(z) = 0.$$

1.6 ${}_2F_0$ function

This function has a branch point at zero. Hence it cannot be defined with a series around zero. It can be defined by other means on the covering of $\mathbb{C}\backslash\{0\}$. The (divergent) hypergeometric series gives its asymptotic expansion away from any wedge around the positive semiaxis.

$$F(a,b;-;z) \sim \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!} z^n$$

It solves the ${}_2F_0$ equation (related to the confluent equation)

$$\left(z^2\partial_z^2 + (-1 + (a+b+1)z)\partial_z + ab\right) u(z) = 0.$$

1.7 Power function ${}_1F_0$

$$F(a;-;z) = (1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n$$

The series is convergent for $|z| < 1$, it extends to a multivalued function on a covering of $\mathbb{C}\backslash\{1\}$. It is a solution of

$$((z-1)\partial_z - a)u(z) = 0.$$

1.8 Exponential function ${}_0F_0$

$$F(-;-;z) = \mathrm{e}^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

It solves

$$(\partial_z - 1)u(z) = 0.$$

2 2nd order differential equations in complex domain

In this section we will discuss a general theory of equations of the form

$$\left(b(z)\partial_z^2 + c(z)\partial_z + d(z)\right) u(z) = 0. \tag{2.4}$$

z will be a complex variable. The functions b, c, d will be usually holomorphic or at least meromorphic in an open set $\Omega \subset \mathbb{C}$.

Discussing an equation such as (2.4), we will often introduce an operator

$$A(z, \partial_z) := b(z)\partial_z^2 + c(z)\partial_z + d(z). \tag{2.5}$$

We will say that (2.4) is given by the operator (2.5). Indeed, u solves (2.4) iff u is in the kernel of (2.5).

By dividing (2.4) by $b(z)$ we obtain

$$\left(\partial_z^2 + \frac{c(z)}{b(z)}\partial_z + \frac{d(z)}{b(z)}\right) u(z) = 0. \tag{2.6}$$

Thus we can usually assume that $b(z)$ is 1 and consider

$$\big(\partial_z^2 + c(z)\partial_z + d(z)\big)\, u(z) = 0. \tag{2.7}$$

2.1 Wronskian

Let $u_1(z)$, $u_2(z)$ be a pair of functions. Their Wronskian is

$$W(u_1, u_2)(z) = W(z) := u_1(z)u_2'(z) - u_1'(z)u_2(z).$$

If they are solutions of (2.7), then the Wronskian satisfies

$$\big(\partial_z + c(z)\big)W(z) = 0.$$

If

$$\tilde{u}_1(z) = a_{11}u_1(z) + a_{12}u_2(z), \qquad \tilde{u}_2(z) = a_{21}u_1(z) + a_{22}u_2(z)$$

is another pair of solutions, then

$$W(\tilde{u}_1, \tilde{u}_2) = (a_{11}a_{22} - a_{12}a_{21})W(u_1, u_2).$$

2.2 Regular points

Definition 2.1 *We say that $z_0 \in \Omega$ is a* regular point *of (2.7) if $c(z)$ and $d(z)$ are analytic around z_0.*

Proposition 2.2 *Let $c(z)$, $d(z)$ be holomorphic in a connected and simply connected open subset $\Omega \subset \mathbb{C}$. Then the problem*

$$\begin{cases} \big(\partial_z^2 + c(z)\partial_z + d(z)\big)\, u(z) = 0 \\ u(z_0) = w_0, \ \partial_z u(z_0) = w_1, \end{cases} \tag{2.8}$$

has a unique solution in Ω.

Note that if b, c, d are holomorphic around z_0, then

$$\left(b(z)\partial_z^2 + c(z)\partial_z + d(z)\right) u(z) = 0 \tag{2.9}$$

is regular at z_0 iff $b(z_0) \neq 0$. Let us give the formula for the coefficients of the expansion

$$u(z) := \sum_{k=0}^{\infty} u_k z^k.$$

of (2.9).

$$\begin{cases} u_0 = w_0, \quad u_1 = w_1, \\ \sum\limits_{k=0}^{m} k(k-1)u_k b_{m-k} + \sum\limits_{k=0}^{m-1} k c_{m-k-1} u_k + \sum\limits_{k=0}^{m-2} d_{m-k-2} u_k = 0. \end{cases}$$

Definition 2.3 *Assume that $c(z)$, $d(z)$ are holomorphic for $|z| > R$. We say that ∞ is a regular point of (2.7) if after the change of coordinates $w = z^{-1}$ we obtain a regular point at 0.*

Consider (2.7). The change $w = z^{-1}$ and division by w^4 leads to

$$\Big(\partial_w^2 + (2w^{-1} - w^{-2}c(w^{-1}))\partial_w + w^{-4}d(w^{-1})\Big)u(w^{-1}) = 0.$$

Hence ∞ is a regular point if there exist (finite) limits

$$\lim_{z\to\infty}(2z - z^2 c(z)), \qquad \lim_{z\to\infty} z^4 d(z).$$

Theorem 2.4 *Let ∞ be a regular point of (2.7). Then for any w_0, w_1 there exists a unique solution of the problem*

$$\begin{cases} \left(\partial_z^2 + c(z)\partial_z + d(z)\right) u(z) = 0 \\ \lim\limits_{z\to\infty} u(z) = w_0, \ \lim\limits_{z\to\infty} (u(z) - w_0)z = w_1. \end{cases} \tag{2.10}$$

2.3 Regular-singular points

Definition 2.5 *We say that an equation*

$$\left(\partial_z^2 + c(z)\partial_z + d(z)\right) u(z) = 0 \tag{2.11}$$

has a regular-singular point *at $z_0 \in \Omega$, if $c(z)$ has at z_0 a pole of at most 1st order and $d(z)$ has at z_0 a pole of at most 2nd order.*

For simplicity, assume that $z_0 = 0$. We can rewrite the above equation as

$$\left(z^2\partial_z^2 + q(z)z\partial_z + r(z)\right) u(z) = 0. \tag{2.12}$$

0 is regular-singular iff q, r are analytic at 0.

Theorem 2.6 (Frobenius Method) *$z = 0$ is regular-singular iff q, r are holomorphic at $z = 0$. Assume that in addition q, r are holomorphic in an open connected simply connected set $\Omega \subset \mathbb{C}$ containing 0. Let $\lambda \in \mathbb{C}$ satisfy*

$$\begin{aligned}
&\lambda(\lambda-1)+\lambda q(0)+r(0)=0,\\
&(\lambda+m)(\lambda+m-1)+(\lambda+m)q(0)+r(0)\neq 0, \quad m=1,2,\ldots.
\end{aligned}$$

Then there exists a unique function $\tilde{u}(z)$ holomorphic in Ω, such that $u(z) := z^\lambda \tilde{u}(z)$ is a solution of the problem

$$\begin{cases}
\left(z^2\partial_z^2+q(z)z\partial_z+r(z)\right)u(z)=0,\\
\lim\limits_{z\to 0} z^{-\lambda}u(z)=1,
\end{cases} \tag{2.13}$$

Clearly, if p, q, r is analytic around 0, the equation

$$\left(p(z)z^2\partial_z^2+q(z)z\partial_z+r(z)\right)u(z)=0, \tag{2.14}$$

has a regular-singular point at 0 iff $p(0) \neq 0$. Let us give a recurrent formula for coefficients of

$$u(z) := \sum_{k=0}^{\infty} u_k z^{\lambda+k}$$

$$\begin{cases}
u_0 &= 1,\\
u_m &= -\left((\lambda+m)(\lambda+m-1)p_0+(\lambda+m)q_0+r_0\right)^{-1}\\
&\quad \times \sum\limits_{k=0}^{m-1}\big((\lambda+k)(\lambda+k-1)p_{m-k}+(\lambda+k)q_{m-k}+r_{m-k}\big)u_k.
\end{cases}$$

Thus, if we are looking for solutions (2.14), we should first find the roots λ_1, λ_2 of the so-called *indicial equation*

$$\lambda(\lambda-1)p(0)+\lambda q(0)+r(0)=0.$$

If $\lambda_1 - \lambda_2 \notin \mathbb{Z}$, then we can find two linearly independent solutions that behave at zero as z^{λ_1} and z^{λ_2}. If $\lambda_1 - \lambda_2 \in \mathbb{Z}$, then generally we can find only a solution behaving as z^{λ_1}, where $\lambda_1 - \lambda_2 \geq 0$.

Definition 2.7 *Assume that $q(z)$, $r(z)$ are holomorphic for $|z| > R$. We say that ∞ is a regular-singular point of (2.7) if after the change of coordinates $w = z^{-1}$ we obtain a regular-singular point at 0.*

Hence ∞ is a regular singular point of (2.11) if

$$\lim_{z\to\infty} zc(z), \quad \lim_{z\to\infty} z^2 d(z) \tag{2.15}$$

exist. Similarly, ∞ is a regular singular point of (2.12) if

$$\lim_{z\to\infty} q(z), \quad \lim_{z\to\infty} r(z) \tag{2.16}$$

exist.

Proposition 2.8 *Let $q(z)$, $r(z)$ be holomorphic in a connected simply connected open set $\Omega \subset \mathbb{C}$ containing $\{|z| > R\}$. Let $\lambda \in \mathbb{C}$ satisfy*

$$\lambda(\lambda+1) - \lambda q(\infty) + r(\infty) = 0,$$
$$(\lambda+m)(\lambda+m+1) - (\lambda+m)q(\infty) + r(\infty) \neq 0, \quad m = 1, 2, \dots.$$

Then there exists a unique function $\tilde{u}(z)$ holomorphic in Ω, such that $u(z) := z^{-\lambda}\tilde{u}(z)$ is a solution of

$$\begin{cases} \left(z^2\partial_z^2 + q(z)z\partial_z + r(z)\right) u(z) = 0, \\ \lim\limits_{z\to\infty} z^{\lambda}u(z) = 1. \end{cases} \tag{2.17}$$

Proposition 2.9 *Let*

$$\left(\partial_z^2 + \frac{q(z)}{(z-z_0)}\partial_z + \frac{r(z)}{(z-z_0)^2}\right) \tag{2.18}$$

have indices $\rho_0, \tilde{\rho}_0$ at z_0 and $\rho_\infty, \tilde{\rho}_\infty$ at ∞. Then

$$(z-z_0)^{\mu}\left(\partial_z^2 + \frac{q(z)}{(z-z_0)}\partial_z + \frac{r(z)}{(z-z_0)^2}\right)(z-z_0)^{-\mu} \tag{2.19}$$

has at z_0 indices $\rho_0 + \mu, \tilde{\rho}_0 + \mu$ and at ∞ indices $\rho_\infty - \mu, \tilde{\rho}_\infty - \mu$.

Proof. We can assume that $z_0 = 0$. We use $z^{\mu}\partial_z z^{-\mu} = \partial_z - \frac{\mu}{z}$. Then (2.19) is

$$\begin{aligned} &\left(\partial_z - \frac{\mu}{z}\right)^2 + \frac{q(z)}{z}\left(\partial_z - \frac{\mu}{z}\right) + \frac{r(z)}{z^2} \\ = \; &\partial_z^2 - 2\frac{\mu}{z}\partial_z + \frac{\mu+\mu^2}{z^2} + \frac{q(z)}{z}\partial_z - \frac{q(z)\mu}{z^2} + \frac{r(z)}{z^2} \\ = \; &\partial_z^2 + \frac{(-2\mu + q(z))}{z}\partial_z + \frac{\left(\mu + \mu^2 - \mu q(z) + r(z)\right)}{z^2}. \end{aligned}$$

Therefore, the indicial equation at 0 is

$$\lambda(\lambda-1) + \lambda(q(0) - 2\mu) + \mu + \mu^2 - q(0)\mu + r(0) \tag{2.20}$$
$$= (\lambda-\mu)(\lambda-\mu-1) + q(0)(\lambda-\mu) + r(0), \tag{2.21}$$

and the indicial equation at ∞ is

$$\lambda(\lambda+1) - \lambda(q(\infty) - 2\mu) + \mu + \mu^2 - q(\infty)\mu + r(\infty) \tag{2.22}$$
$$= (\lambda+\mu)(\lambda+\mu+1) - q(\infty)(\lambda+\mu) + r(\infty). \tag{2.23}$$

□

Theorem 2.10 *Suppose that we change the variables in the equations, considering a (holomorphic) map $y \mapsto z(y)$. Assume that y_0 is mapped at z_0 and $\frac{\partial z}{\partial y}(y_0) \neq 0$. Then the indices of the transformed equation coincide with the indices of the original equation.*

Proof. We will assume that the equation has the form (2.18) and $z_0 = y_0 = 0$. Let us compute the change of the differentiation operators:

$$\partial_z = \frac{1}{\frac{\partial z}{\partial y}} \partial_y, \tag{2.24}$$

$$\partial_z^2 = -\frac{\frac{\partial^2 z}{\partial y^2}}{\left(\frac{\partial z}{\partial y}\right)^3} \partial_y + \frac{1}{\left(\frac{\partial z}{\partial y}\right)^2} \partial_y^2. \tag{2.25}$$

Therefore,

$$\left(\partial_z^2 + \frac{q(z)}{z}\partial_z + \frac{r(z)}{z^2}\right) \tag{2.26}$$

$$= \frac{1}{\left(\frac{\partial z}{\partial y}\right)^2}\left(\partial_y^2 + \Big(\frac{q(z(y))\frac{\partial z(y)}{\partial y}}{z(y)} - \frac{\frac{\partial^2 z(y)}{\partial y^2}}{\frac{\partial z(y)}{\partial y}}\Big)\partial_y + \frac{\left(\frac{\partial z}{\partial y}\right)^2}{z(y)^2} r(z(y))\right) \tag{2.27}$$

$$= \frac{1}{\left(\frac{\partial z}{\partial y}\right)^2}\left(\partial_y^2 + \frac{\tilde q(y)}{y}\partial_y + \frac{\tilde r(y)}{y^2}\right) \tag{2.28}$$

Now it is easy to see that $\tilde q(0) = q(0)$ and $\tilde r(0) = r(0)$. □

2.4 Equations with two regular-singular points on the Riemann sphere

Example 2.11 *Every 2nd order equation that in $\mathbb{C} \cup \{\infty\}$ has only regular points except for two regular-singular points at 0 and ∞ has the form*

$$(z^2\partial_z^2 + qz\partial_z + r)u(z) = 0. \tag{2.29}$$

It is sometimes called the **homogeneous Euler equation***. Its indicial points are*

$$\begin{aligned} 0: &\qquad \lambda(\lambda-1) + q\lambda + r = 0, \\ \infty: &\qquad \lambda(\lambda+1) - q\lambda + r = 0. \end{aligned}$$

If $\rho, \tilde\rho$ are its indices at 0, then $-\rho, -\tilde\rho$ are its indices at ∞. Its solutions are z^ρ, $z^{\tilde\rho}$ if $\rho \neq \tilde\rho$ and z^ρ, $z^\rho \log z$ if $\rho = \tilde\rho$. The equation (2.29) can be rewritten as

$$(z^2\partial_z + (1-\rho-\tilde\rho)z\partial_z + \rho\tilde\rho)u(z) = 0.$$

Example 2.12 *Every 2nd order equation that in $\mathbb{C} \cup \{\infty\}$ has only regular points except for two regular-singular points at z_1 and z_2 has the form*

$$\left(\partial_z^2 + \left(g_1(z-z_1)^{-1} + g_2(z-z_2)^{-1}\right)\partial_z + h(z-z_1)^{-2}(z-z_2)^{-2}\right)u(z) = 0, \tag{2.30}$$

where $g_1 + g_2 = 2$. Its indicial equations are

$$z_1 : \quad \lambda(\lambda - 1) + g_1\lambda + h(z_1 - z_2)^{-2} = 0,$$

$$z_2 : \quad \lambda(\lambda - 1) + g_2\lambda + h(z_1 - z_2)^{-2} = 0.$$

If ρ, $\tilde{\rho}$ are indices at z_1, then $-\rho$, $-\tilde{\rho}$ are indices at z_2. Solutions have the form $(z - z_1)^\rho (z - z_2)^{-\rho}$, $(z - z_1)^{\tilde{\rho}} (z - z_2)^{-\tilde{\rho}}$, if $\rho \neq \tilde{\rho}$ and $(z - z_1)^\rho (z - z_2)^{-\rho}$, $(z - z_1)^\rho (z - z_2)^{-\rho} \log(z - z_1)(z - z_2)^{-1}$, if $\rho = \tilde{\rho}$.

Equation (2.30) can be rewritten as

$$\Big(\partial_z^2 + \Big((1 - \rho - \tilde{\rho})(z - z_1)^{-1} + (1 + \rho + \tilde{\rho})(z - z_2)^{-1}\Big)\partial_z$$
$$+\rho\tilde{\rho}(z_1 - z_2)^2 (z - z_1)^{-2}(z - z_2)^{-2}\Big)u(z) = 0.$$

3 Systems of 1st order equations

3.1 Regular points

This subsection can be skipped.

We will discuss differential equations

$$\partial_z v(z) = A(z)v(z). \tag{3.31}$$

where $A(z)$ is a matrix and $v(z) \in \mathbb{C}^n$.

Definition 3.1 *If $A(z)$ is analytic at z_0, then we say that z_0 is a regular point of (3.31).*

Theorem 3.2 *Let Ω be a connected simply connected open subset of $\mathbb{C}$. Let*

$$\Omega \ni z \mapsto A(z) = \begin{bmatrix} a_{11}(z) & \dots & a_{1n}(z) \\ & \dots & \\ a_{n1}(z) & \dots & a_{nn}(z) \end{bmatrix}$$

be a holomorphic function with values in $n \times n$ matrices and $w = \begin{bmatrix} w_1 \\ \dots \\ w_n \end{bmatrix} \in \mathbb{C}^n$.

Then there exists a unique holomorphic function $\Omega \ni z \mapsto v(z) = \begin{bmatrix} v_1(z) \\ \dots \\ v_n(z) \end{bmatrix} \in \mathbb{C}^n$ that solves the problem

$$\begin{cases} \frac{\mathrm{d}v(z)}{\mathrm{d}z} = A(z)v(z), \\ v(z_0) = w. \end{cases} \tag{3.32}$$

Proof. Let us first restrict ourselves to a disk $K(z_0, r)$ such that $K(z_0, r)^{\mathrm{cl}} \subset \Omega$. We can also assume that $z_0 = 0$.

Let

$$A(z) = \sum_{k=0}^{\infty} A_k z^k$$

Then the series

$$v(z) := \sum_{k=0}^{\infty} v_k z^k,$$

where

$$\begin{cases} v_0 = w, \\ v_{m+1} := \frac{1}{m+1} \sum\limits_{k=0}^{m} A_{m-k} v_k. \end{cases}$$

is the unique formal series solving (3.32).

Let us show that this series is convergent in $K(0, r)$. By the Cauchy inequality,

$$\|A_k\| \leq C r^{-k}.$$

If we set

$$\begin{cases} p_0 = \|w\| \\ p_{m+1} := \frac{1}{m+1} \sum\limits_{k=0}^{m} C r^{-m+k} p_k, \end{cases}$$

then we can show inductively that

$$\|v_m\| \leq p_m. \tag{3.33}$$

Indeed, we have

$$\|v_0\| = p_0.$$

Assume that

$$\|v_k\| \leq p_k, \;\; k = 0, \dots, m.$$

Then

$$\begin{aligned} \|v_{m+1}\| &\leq \frac{1}{m+1} \sum_{k=0}^{m} \|A_{m-k} v_k\| \\ &\leq \frac{1}{m+1} \sum_{k=0}^{m} \|A_{m-k}\| \|v_k\| \\ &\leq \frac{1}{m+1} \sum_{k=0}^{m} C r^{k-m} p_k = p_{m+1}. \end{aligned}$$

This proves (3.33).

If we subtract the formula

$$\begin{aligned} r(m+1) p_{m+1} &= \sum_{k=0}^{m} C r^{-m+k+1} p_k, \\ m p_m &= \sum_{k=0}^{m-1} C r^{-m+k+1} p_k, \end{aligned}$$

then we obtain

$$r(m+1)p_{m+1} = (Cr+m)p_m.$$

This immediately implies

$$\lim_{m\to\infty} \frac{p_{m+1}}{p_m} = r^{-1}.$$

Hence, by the d'Alembert criterion

$$\sum_{k=0}^{\infty} p_k z^k$$

is convergent in the disk $K(0,r)$. Therefore, so is

$$\sum_{k=0}^{\infty} v_k z^k$$

The above reasoning can be repeated for any disk contained in Ω. In this way, since Ω is connected, we can extend $v(z)$ to the whole Ω. Ω is simply connected, and therefore the resulting function will be univalued. □

Example 3.3

$$(\partial_z - 1)v(z) = 0, \quad v(0) = 1.$$

We set

$$v(z) = \sum_{n=0}^{\infty} v_n z^n.$$

We obtain a recurrence relation

$$nv_n = v_{n-1}.$$

Therefore,

$$v(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \ \ z \in \mathbb{C}.$$

Obviously, $v(z) = \mathrm{e}^z$.

Example 3.4 *Let* $\mu \in \mathbb{C}$, $z \neq -1$

$$\left(\partial_z - \mu(z+1)^{-1}\right) v(z) = 0, \quad v(0) = 1.$$

We set

$$v(z) = \sum_{n=0}^{\infty} v_n z^n.$$

We obtain a recurrence relation

$$nv_n = (\mu - n + 1)v_{n-1}.$$

Therefore,

$$v(z) = \sum_{n=0}^{\infty} \frac{\mu \ldots (\mu - n + 1) z^n}{n!}, \ \ |z| < 1.$$

Obviously, $v(z) = (1+z)^\mu$.

Proof of Thm 2.4. Define

$$v(z) := \begin{bmatrix} u(z) \\ u'(z) \end{bmatrix}, \quad w := \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$

and

$$A(z) := \begin{bmatrix} 0 & 1 \\ -d(z) & -c(z). \end{bmatrix}$$

Then (2.8) can be rewritten as

$$\begin{cases} \frac{\mathrm{d}v(z)}{\mathrm{d}z} = A(z)v(z), \\ v(z_0) = w. \end{cases}$$

We can apply Thm 3.2. □

Definition 3.5 *Assume that $A(z)$ is defined for $|z| > R$. We say that ∞ is a regular point of (3.31), if after the change of the variable $w = z^{-1}$ we obtain a reggular point at* 0.

Obviously, $\partial_z = -w^2\partial_w$. Hence, after the change of the variable (3.31) transforms into

$$\partial_w v(w^{-1}) = -w^{-2}A(w^{-1})v(w^{-1}).$$

Therefore, ∞ is a regular point iff there exists

$$\lim_{z\to\infty} z^2 A(z).$$

Theorem 3.6 *Let ∞ be a regular point of (3.34). Then for any $w \in \mathbb{C}^n$, there exists a unique holomorphic solution satisfying*

$$\begin{cases} \frac{\mathrm{d}v(z)}{\mathrm{d}z} = A(z)v(z), \\ \lim\limits_{z\to\infty} v(z) = w. \end{cases} \tag{3.34}$$

3.2 Regular-singular points

Definition 3.7 *We say that the equation*

$$\frac{\mathrm{d}v(z)}{\mathrm{d}z} = A(z)v(z) \tag{3.35}$$

has a regular-singular point *at z_0, if $A(z)$ has at z_0 a pole of at most 1st order.*

We can then rewrite (3.32) as

$$(z - z_0)\partial_z v(z) = B(z)v(z), \tag{3.36}$$

where $B(z)$ is holomorphic around z_0. The eigenvalues of the matrix $B(z_0)$ are called *indices of the singular point* z_0.

For simplicity, assume that $z_0 = 0$.

Theorem 3.8 (Frobenius method for systems of equations) *Let Ω be a connected simply connected open subset of $\mathbb{C}$ containing 0. Let*

$$\Omega \ni z \mapsto B(z) = \begin{bmatrix} b_{11}(z) & \dots & b_{1n}(z) \\ & \dots & \\ b_{n1}(z) & \dots & b_{nn}(z) \end{bmatrix}$$

be a holomorphic function with values in $n \times n$ matrices Let $w \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ satisfy

$$\begin{aligned} &(B(0) - \lambda)w = 0, \\ &\lambda + m \textit{ is not an eigenvalue of } B(0) \textit{ for } m = 1, 2, \dots. \end{aligned} \tag{3.37}$$

Then there exists a unique function $\tilde{v}(z)$ holomorphic on Ω such that $v(z) := z^\lambda \tilde{v}(z)$ solves the problem

$$\begin{cases} z\frac{\mathrm{d}v(z)}{\mathrm{d}z} = B(z)v(z), \\ \lim\limits_{z\to 0} z^{-\lambda} v(z) = w. \end{cases} \tag{3.38}$$

Proof. Let us first consider a disc $K(0, r)$ such that $K(0, r)^{\mathrm{cl}} \subset \Omega$.

Let

$$B(z) = \sum_{k=0}^{\infty} B_k z^k$$

Then the series

$$v(z) := z^\lambda \sum_{k=0}^{\infty} v_k z^k,$$

where

$$\begin{cases} v_0 = w \\ v_m := (\lambda + m - B_0)^{-1} \sum\limits_{k=0}^{m-1} B_{m-k} v_k. \end{cases}$$

is the unique formal series solving (3.38).

Let us show that this series is convergent in the disk $K(0, r)$. By the Cauchy inequality,

$$\|B_k\| \leq C r^{-k}.$$

If we set

$$\begin{cases} p_0 = \|w\| \\ p_m := \left\|(\lambda + m - B_0)^{-1}\right\| \sum\limits_{k=0}^{m-1} C r^{-m+k} p_k, \end{cases}$$

then we can show by induction that

$$\|v_m\| \leq p_m.$$

If we subtract the formulas

$$\begin{aligned} r\left\|(\lambda+m+1-B_0)^{-1}\right\|^{-1} p_{m+1} &= \sum_{k=0}^{m} C r^{-m+k} p_k, \\ \left\|(\lambda+m-B_0)^{-1}\right\|^{-1} p_m &= \sum_{k=0}^{m-1} C r^{-m+k} p_k, \end{aligned}$$

then we obtain

$$r\left\|(\lambda+m+1-B_0)^{-1}\right\|^{-1} p_{m+1} = \left(C + \left\|(\lambda+m-B_0)^{-1}\right\|^{-1}\right) p_m.$$

It is easy to see that

$$\lim_{m\to\infty} m\left\|(\lambda+m-B_0)^{-1}\right\| = 1.$$

Hence,

$$\lim_{m\to\infty} \frac{p_{m+1}}{p_m} = r^{-1}.$$

Thus by the d'Alembert criterion, the series that defines $\tilde{v}(z)$ is convergent in the disk $K(0,r)$.

Using Them 3.2 we can extend $\tilde{v}(z)$ to the whole Ω. $\square$

Example 3.9 *Let*

$$B = \begin{bmatrix} \lambda & & \dots & & \\ 1 & \lambda & \dots & & \\ & & \dots & & \\ & & \dots & \lambda & \\ & & \dots & 1 & \lambda \end{bmatrix}.$$

Consider the equation $z\partial_z v(z) = Bv(z)$. We obtain

$$\begin{aligned} z\partial_z v_1 &= \lambda v_1, \\ v_1 + z\partial_z v_2 &= \lambda v_2, \\ &\dots \\ v_{n-1} + z\partial_z v_n &= \lambda v_n. \end{aligned}$$

A basis of solution of this system is

$$\begin{bmatrix} 0 \\ \dots \\ 0 \\ z^\lambda \\ z^\lambda \log z \\ \dots \\ z^\lambda (\log z)^{m-1} \end{bmatrix}, \qquad m = 1, \dots, n.$$

Example 3.10 *The following equation has a regular-singular point at* 0*:*

$$\partial_z v(z) = (az^{-1} + b)v(z).$$

its solution is $v(z) = z^a \mathrm{e}^{bz}$

Proof of Thm 2.6 Define

$$v(z) := \begin{bmatrix} u(z) \\ zu'(z) \end{bmatrix}, \quad w := \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$$

and

$$B(z) := \begin{bmatrix} 0 & 1 \\ -c(z) & 1 - b(z) \end{bmatrix}.$$

We then have

$$B(z)v(z) = \begin{bmatrix} zu'(z) \\ -c(z)u(z) - b(z)zu'(z) + zu'(z) \end{bmatrix},$$

$$z\partial_z \begin{bmatrix} u(z) \\ zu'(z) \end{bmatrix} = \begin{bmatrix} zu'(z) \\ z^2u''(z) + zu'(z) \end{bmatrix},$$

$$z^{-\lambda}v(z) = \begin{bmatrix} \tilde{u}(z) \\ z\tilde{u}'(z) + \lambda\tilde{u}(z) \end{bmatrix}.$$

Hence (2.13) can be rewritten as

$$\begin{cases} z\frac{\mathrm{d}v(z)}{\mathrm{d}z} = B(z)v(z), \\ \lim\limits_{z\to 0} z^{-\lambda}v(z) = w. \end{cases}$$

We can apply Thm 3.8. □

Definition 3.11 *Assume that* $B(z)$ *is defined for* $|z| > R$*. We say that* ∞ *is a regular-singular point of (3.35), if after the change of the variable* $w = z^{-1}$ *we obtain a regular-singular point at* 0*.*

Thus (3.36) has a regular-singular point if $\lim\limits_{z\to\infty} B(z)$ exists.The eigenvalues of $-B(\infty)$ are called indices of ∞.

Theorem 3.12 *Let* Ω *be a connected simply connected subset of* $\mathbb{C}$ *containing* $\{|z| > R\}$*. Let*

$$\Omega \ni z \mapsto B(z) = \begin{bmatrix} a_{11}(z) & \dots & a_{1n}(z) \\ & \dots & \\ a_{n1}(z) & \dots & a_{nn}(z) \end{bmatrix}$$

be a holomorphic function with values in $n \times n$ *matrices. Let* $w \in \mathbb{C}^n$ *and* $\lambda \in \mathbb{C}$ *satisfy*

$$\begin{aligned} &(B(\infty) + \lambda)w = 0, \\ &\lambda + m \text{ is not an eigenvalue of } -B(\infty) \text{ for } m = 1, 2, \dots. \end{aligned} \tag{3.39}$$

Then there exists a unique function $\tilde{v}(z)$ holomorphic on Ω such that $v(z) := z^{-\lambda}\tilde{v}(z)$ solves

$$\begin{cases} z\frac{\mathrm{d}v(z)}{\mathrm{d}z} = B(z)v(z), \\ \lim\limits_{z\to\infty} z^{\lambda}v(z) = w. \end{cases} \tag{3.40}$$

Example 3.13 *Every 1st order equation on the Riemann sphere possessing only regular points except of regular-singular points at z_1, z_2 and ∞ has the form*

$$\partial_z v(z) = \Big(a_1(z-z_1)^{-1} + a_2(z-z_2)^{-1}\Big)v(z) \tag{3.41}$$

It has indices

$$z_1: \quad a_1, \qquad z_2: \quad a_2, \qquad \infty: \quad -a_1-a_2,$$

and a solution $(z-z_1)^{a_1}(z-z_2)^{a_2}$.

4 Hypergeometric equation

4.1 Riemann equations

Lemma 4.1 *Every 2nd order equation which on the Riemann sphere has only regular points except for 3 points at z_1, z_2 and ∞ is given by an operator of the form*

$$\begin{aligned} &\partial_z^2 + \Big(\frac{g_1}{z-z_1} + \frac{g_2}{z-z_2}\Big)\partial_z \\ &+\frac{h_1}{(z-z_1)^2} + \frac{h_2}{(z-z_2)^2} + \frac{k}{(z-z_1)(z-z_2)}. \end{aligned} \tag{4.42}$$

Proof. Consider

$$\partial_z^2 + c(z)\partial_z + d(z) \tag{4.43}$$

Clearly, if in $\mathbb{C}$ the only singular points are at z_1, z_2, and they are regular-singular, then

$$c(z) = c_{\mathrm{reg}}(z) + \frac{g_1}{z-z_1} + \frac{g_2}{z-z_2}, \tag{4.44}$$

$$d(z) = d_{\mathrm{reg}}(z) + \frac{h_1}{(z-z_1)^2} + \frac{h_2}{(z-z_2)^2} + \frac{k_1}{z-z_1} + \frac{k_2}{z-z_2}. \tag{4.45}$$

where c_{reg}, d_{reg} are entire functions. ∞ is a regular-singular point if the following limits also exist:

$$\lim_{z\to\infty} zc(z), \tag{4.46}$$

$$\lim_{z\to\infty} z^2 d(z). \tag{4.47}$$

(4.46) implies the existence of $\lim_{z\to\infty} zc_{\mathrm{reg}}(z)$. Thus, $zc_{\mathrm{reg}}(z)$ is a bounded entire function. By the Liouville Theorem, zc_{reg} is a constant. But c_{reg} is also an entire function. Hence $c_{\mathrm{reg}} = 0$

(4.47) implies the existence of a limit $\lim_{z\to\infty} zd(z)$, which in turn implies the existence of $\lim_{z\to\infty} zd_{\mathrm{reg}}(z)$. By the Liouville Theorem, zd_{reg} is a constant. But d_{reg} is also an entire function. Hence $d_{\mathrm{reg}} = 0$.

Using again (4.47), knowing that $d_{\mathrm{reg}} = 0$, we obtain $k_1 + k_2 = 0$. □

We can transform (4.42) further, obtaining

$$\partial_z^2 + \Big(\frac{g_1}{(z-z_1)} + \frac{g_2}{(z-z_2)}\Big)\partial_z + \frac{h_1(z_1-z_2)}{(z-z_1)^2(z-z_2)} + \frac{h_2(z_2-z_1)}{(z-z_2)^2(z-z_1)} + \frac{h}{(z-z_1)(z-z_2)}. \tag{4.48}$$

with $h = k - h_1 - h_2$.

Suppose that the indices are

$$\begin{array}{ll} z_1: & \rho_1, \tilde\rho_1, \\ z_2: & \rho_2, \tilde\rho_2, \\ \infty: & \rho_3, \tilde\rho_3. \end{array}$$

Lemma 4.2 *The sum of indices of (4.42) is* 1. *The Riemann operator expressed in terms of the indeces is*

$$\begin{aligned} &\mathcal{P}\begin{pmatrix} z_1 & z_2 & \infty & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde\rho_1 & \tilde\rho_2 & \tilde\rho_3 & \end{pmatrix} \\ = \ & \partial_z^2 - \left(\frac{\rho_1+\tilde\rho_1-1}{z-z_1} + \frac{\rho_2+\tilde\rho_2-1}{z-z_2}\right)\partial_z \\ & + \frac{\rho_1\tilde\rho_1(z_1-z_2)}{(z-z_1)^2(z-z_2)} + \frac{\rho_2\tilde\rho_2(z_2-z_1)}{(z-z_2)^2(z-z_1)} + \frac{\rho_3\tilde\rho_3}{(z-z_1)(z-z_2)} \end{aligned} \tag{4.49}$$

Proof. Its indicial equations are

$$\begin{array}{ll} z_1: & \lambda(\lambda-1) + g_1\lambda + h_1 = 0, \\ z_2: & \lambda(\lambda-1) + g_2\lambda + h_2 = 0, \\ \infty: & \lambda(\lambda+1) - (g_1+g_2)\lambda + h = 0. \end{array}$$

By the Vieta equations

$$\begin{aligned} -1 + g_1 &= -\rho_1 - \tilde\rho_1, \\ -1 + g_2 &= -\rho_2 - \tilde\rho_2, \\ 1 - g_1 - g_2 &= -\rho_\infty - \tilde\rho_\infty. \end{aligned}$$

We sum up these equations. □

It is easy to generalize (4.49) to an arbitrary triplet of points:

Theorem 4.3 *1. Suppose that we are given a 2nd order differential equation on the Riemann sphere having 3 singular points z_1, z_2, z_3, all of them regular singular points with the following indices*

$$\begin{array}{ll} z_1: & \rho_1, \tilde{\rho}_1, \\ z_2: & \rho_2, \tilde{\rho}_2, \\ z_3: & \rho_3, \tilde{\rho}_3. \end{array}$$

Then the following condition is satisfied:

$$\rho_1 + \tilde{\rho}_1 + \rho_2 + \tilde{\rho}_2 + \rho_3 + \tilde{\rho}_3 = 1. \tag{4.50}$$

Such an equation, normalized to have coefficient 1 at the 2nd derivative, is always equal to

$$\mathcal{P}\left(\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{array}\right)\phi(z) = 0, \tag{4.51}$$

where

$$\mathcal{P}\left(\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{array}\right) := \partial_z^2 - \left(\frac{\rho_1+\tilde{\rho}_1-1}{z-z_1} + \frac{\rho_2+\tilde{\rho}_2-1}{z-z_2} + \frac{\rho_3+\tilde{\rho}_3-1}{z-z_3}\right)\partial_z$$
$$+\frac{\rho_1\tilde{\rho}_1(z_1-z_2)(z_1-z_3)}{(z-z_1)^2(z-z_2)(z-z_3)} + \frac{\rho_2\tilde{\rho}_2(z_2-z_3)(z_2-z_1)}{(z-z_2)^2(z-z_3)(z-z_1)} + \frac{\rho_3\tilde{\rho}_3(z_3-z_1)(z_3-z_2)}{(z-z_3)^2(z-z_1)(z-z_2)}.$$

2. Let $z \mapsto w(z) = \frac{az+b}{cz+d}$. (Transformations of this form are called homographies or Möbius transformations). We can always assume that $ad-bc = 1$. Then

$$\mathcal{P}\left(\begin{array}{cccc} w(z_1) & w(z_2) & w(z_3) & \\ \rho_1 & \rho_2 & \rho_3 & w, \partial_w \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{array}\right) = (cz+d)^4\mathcal{P}\left(\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{array}\right),$$

3.

$$(z-z_1)^{-\lambda}(z-z_2)^{\lambda}\mathcal{P}\left(\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{array}\right)(z-z_1)^{\lambda}(z-z_2)^{-\lambda}$$
$$= \mathcal{P}\left(\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \rho_1-\lambda & \rho_2+\lambda & \rho_3 & z, \partial_z \\ \tilde{\rho}_1-\lambda & \tilde{\rho}_2+\lambda & \tilde{\rho}_3 & \end{array}\right).$$

Clearly, in all above formulas one of z_i can equal ∞, with an obvious meaning of various expressions.

4.2 Hypergeometric equation

By Thm 4.3 (2), we can assume that the points z_1, z_2, z_3 are any triplet of distinct points on the Riemann sphere. We choose them to be $0, 1, \infty$.

By Thm 4.3 (3), we can assume that ρ_1, ρ_2 are arbitrary numbers. We choose them to be both 0. The sum of remaining indices must be 1. Hence, we have 3 parameters left. We set
0, indices: 0, $1-c$;
1, indices: 0, $c-a-b$;
∞, indices: a, b. Thus

$$\begin{aligned} &\mathcal{P}\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z, \partial_z \\ 1-c & c-a-b & b & \end{pmatrix} \\ = \quad & \partial_z^2 - \Big(\frac{1-c-1}{z} + \frac{c-a-b-1}{(z-1)} \Big) \partial_z + \frac{ab}{z(z-1)}. \end{aligned} \tag{4.52}$$

Define

$$\mathcal{F}(a,b;c;z,\partial_z) \quad := \quad z(1-z)\mathcal{P}\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z, \partial_z \\ 1-c & c-a-b & b & \end{pmatrix} \tag{4.53}$$

$$= \quad z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab. \tag{4.54}$$

Rewrite the equation

$$\mathcal{F}(a,b;c;z,\partial_z)F(z) = 0$$

in the form

$$\big(z^2\partial_z^2 + (a+b+1)z\partial_z + ab\big)F(z) = (z\partial_z^2 + c\partial_z)F(z). \tag{4.55}$$

Substituting $F = \sum_{n=0}^\infty F_n z^n$ into (4.55) we obtain

$$\sum_{n=0}^\infty (n+a)(n+b)F_n z^n = \sum_{n=0}^\infty n(n+c-1)F_n z^{n-1}. \tag{4.56}$$

This leads to the recurrence relation

$$(n+a)(n+b)F_n = F_{n+1}(n+1)(n+c). \tag{4.57}$$

For $a \in \mathbb{C}$ we define

$$(a)_n := a(a+1)\cdots(a+n-1).$$

The solution analytic at 0 and equal there 1 is the *hypergeometric function*

$$F(a,b;c;z) = \sum_{j=0}^\infty \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!},$$

$F(a,b;c;z)$ is defined for $c \neq 0,-1,-2,\ldots$. Sometimes, it is more convenient to consider

$$\mathbf{F}(a,b;c;z) := \frac{F(a,b,c,z)}{\Gamma(c)} = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{\Gamma(c+j)} \frac{z^j}{j!}$$

defined for all a,b,c.

4.3 Solution $\sim z^{1-c}$ at 0

We have the identity

$$\begin{aligned} & z^{c-1}\mathcal{F}(a,b;c)z^{1-c} \\ = \quad & \mathcal{F}(b+1-c,a+1-c;2-c) \end{aligned}$$

Therefore, the solution of (4.54) behaving as z^{1-c} at zero is

$$z^{1-c}F(b+1-c,a+1-c;2-c;z) \tag{4.58}$$

4.4 Solutions having definite behaviors at 1

$w = 1-z$ is a substitution that exchanges 0 and 1:

$$\begin{aligned} & \mathcal{F}(a,b;c;z,\partial_z) := \\ = \quad & \mathcal{F}(a,b;a+b+1-c;w,\partial_w). \end{aligned}$$

Therefore, the solution analytic at 1 and having there the value 1 is

$$F(a,b;a+b+1-c;1-z).$$

There is also a solution behaving as $(1-z)^{c-a-b}$ at 1:

$$(1-z)^{c-a-b}F(-b+c,-a+c;1+c-a-b;1-z).$$

4.5 Solutions having definite behaviors at ∞

∞ is a regular-singular point with indices a,b. $w = z^{-1}$ is the substitution that exchanges 0 and ∞

$$\begin{aligned} & (-z)^{1+a}\mathcal{F}(a,b;c;z,\partial_z)(-z)^{-a} && (4.59) \\ = \quad & \mathcal{F}(a,a-c+1;a-b+1;w,\partial_w). && (4.60) \end{aligned}$$

Hence, the solution that behaves at ∞ as z^{-a} is

$$z^{-a}F(a,a-c+1;a-b+1;z^{-1}).$$

The second solution is obtained by exchanging a and b:

$$z^{-b}F(b-c+1,b;b-a+1;z^{-1}).$$

4.6 Identities

The following substitution does not move 0, and exchanges 1 and ∞: $z \mapsto w = \frac{z}{z-1}$. It leads to

$$\begin{aligned} & -(1-z)^{1+a}\mathcal{F}(a,b;c;z,\partial_z)(1-z)^{-a} \\ = \quad & \mathcal{F}\left(a, c-b; c; w, \partial_w\right) \end{aligned} \tag{4.61}$$

An analogous identity is obtained if we exchange a and b. This yields

$$\begin{aligned} & F(a,b;c;z) \\ & = (1-z)^{c-a-b}F\left(c-a, c-b; c; z\right) \\ & = (1-z)^{-a}F\left(a, c-b; c; \tfrac{z}{z-1}\right) \\ & = (1-z)^{-b}F\left(c-a, b; c; \tfrac{z}{z-1}\right). \end{aligned}$$

4.7 Integral representations

Theorem 4.4 *Let the curve* $[0,1] \ni \tau \overset{\gamma}{\mapsto} t(\tau)$ *satisfy*

$$t^{a-c+1}(1-t)^{c-b}(t-z)^{-a-1}\Big|_{t(0)}^{t(1)} = 0.$$

Then

$$\int_\gamma t^{a-c}(1-t)^{c-b-1}(t-z)^{-a}\mathrm{d}t \tag{4.62}$$

solves the hypergeometric equation.

Proof. We check that

$$\begin{aligned} & \left(z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab\right) t^{a-c}(1-t)^{c-b-1}(t-z)^{-a} \\ & = -a\partial_t t^{a-c+1}(1-t)^{c-b}(t-z)^{-a-1}. \end{aligned}$$

□

This implies the following representation of the hypergeometric function:

$$\int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-z)^{-a} \tag{4.63}$$

$$= \quad \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b;c;z), \qquad \mathrm{Re}(c-b) > 0,\ \mathrm{Re}b > 0. \tag{4.64}$$

Indeed, notice that (4.63) satisfies the assumptions of Thm 4.4, it is analytic around zero and at zero equals

$$\int_1^\infty t^{-c}(t-1)^{c-b-1}\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}.$$

Setting $z = 1$ in (4.63) we obtain

$$\int_1^\infty t^{a-c}(t-1)^{c-a-b-1}\mathrm{d}t = \frac{\Gamma(c-a-b)\Gamma(b)}{\Gamma(c-a)}. \tag{4.65}$$

Therefore,

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}c > \mathrm{Re}(a+b). \tag{4.66}$$

5 Confluent equation

5.1 ${}_1F_1$ equation as a limit of the hypergeometric equation

Let $a, c \in \mathbb{C}$. The *confluent* or the ${}_1F_1$ *equation* is given by the operator

$$\mathcal{F}(a; c; z, \partial_z) := z\partial_z^2 + (c - z)\partial_z - a. \tag{5.67}$$

The confluent equation is a limiting case of the hypergeometric equation:

$$\lim_{b\to\infty} \frac{1}{b}\mathcal{F}(a, b; c; z/b, \partial_{z/b}) = \mathcal{F}(a; c; z, \partial_z).$$

5.2 Confluent function

0 is a regular-singular point with indices 0, $1 - c$. The solution of the confluent equation analytic around 0 and equal 1 at 0 is

$$F(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(c)_n} z^n.$$

5.3 Solution $\sim z^{1-c}$ at 0

We have the identity

$$z^{c-1}\mathcal{F}(a; c)z^{1-c} \quad = \quad \mathcal{F}(1 + a - c; 2 - c).$$

Hence

$$z^{1-c}F(a - c + 1; 2 - c; z) = \sum_{n=0}^{\infty} \frac{(a - c + 1)_n}{n!(2 - c)_n} z^{1-c+n}.$$

is a solution of the confluent equation.

5.4 First Kummer's identity

Using $\mathrm{e}^{-z}\partial_z \mathrm{e}^z = \partial_z + 1$ we obtain the identity

$$\begin{aligned} & \mathrm{e}^{-z}(z\partial_z^2 + (c - z)\partial_z - a)\mathrm{e}^z \\ = \quad & z\partial_z^2 + (c + z)\partial_z + c - a. \end{aligned} \tag{5.68}$$

Substitute $z = -w$ and multiply by -1, obtaining

$$w\partial_w^2 + (c - w)\partial_w - c + a.$$

Thus

$$\mathrm{e}^{-z}\mathcal{F}(a; c; z, \partial_z)\mathrm{e}^z = \mathcal{F}(c - a; c; w, \partial_w).$$

Hence $\mathrm{e}^z F(c - a; c; -z)$ is a solution of the confluent equation analytic around 0 and equal 1 at 0. We obtain the identity

$$F(a; c; z) = \mathrm{e}^z F(c - a; c; -z). \tag{5.69}$$

5.5 Integral representations

If $[0,1] \ni \tau \mapsto \mathrm{s}(\tau) \in \Omega$ is a curve and f is a function on Ω, we introduce the notation

$$f\Big|_{\gamma(0)}^{\gamma(1)} := f(\gamma(1)) - f(\gamma(0)).$$

Theorem 5.1 *Let the curve γ satisfy*

$$\mathrm{e}^{zs} s^a (1-s)^{c-a}\Big|_{\gamma(0)}^{\gamma(1)} = 0. \tag{5.70}$$

Then

$$\int_\gamma \mathrm{e}^{zs} s^{a-1}(1-s)^{c-a-1}\mathrm{d}s \tag{5.71}$$

is a solution of the confluent equation

Proof.

$$\begin{aligned}
& (z\partial_z^2 + (c-z)\partial_z - a)\mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1} \\
= \; & z\mathrm{e}^{zs}s^{a+1}(1-s)^{c-a-1} + (c-z)\mathrm{e}^{zs}s^a(1-s)^{c-a-1} - a\mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1} \\
= \; & -z\mathrm{e}^{zs}s^a(1-s)^{c-a} - a\mathrm{e}^{zs}s^{a-1}(1-s)^{c-a} + (c-a)\mathrm{e}^{zs}s^a(1-s)^{c-a-1} \\
= \; & -\partial_s \mathrm{e}^{zs}s^a(1-s)^{c-a}.
\end{aligned}$$

□

Hence for $\mathrm{Re}a > 0$, $\mathrm{Re}(c-a) > 0$ we have

$$\int_0^1 \mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}F(a;c;z). \tag{5.72}$$

Indeed, the assumptions are satisfied, since the value of the function in (5.70) at 0 and 1 is zero. We obtain a solution of the confluent equation analytic around 0. We check that at zero it equals

$$\int_0^1 s^{a-1}(1-s)^{c-a-1}\mathrm{d}s = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}.$$

5.6 Laguerre polynomials

For $n = -a \in \{0,1,2,\dots\}$, $F(-n;c;z)$ is an nth degree polynomial. These are the so-called Laguerre polynomials. They can be represented as an integral with γ encircling 0:

$$\begin{aligned}
L_n^\alpha(z) &:= \frac{(1+\alpha)_n}{n!}F(-n;1+\alpha;z) \\
&= \frac{(-1)^n}{2\pi\mathrm{i}}\int_{[0^+]} \mathrm{e}^{tz}t^{-n-1}(1-t)^{\alpha+n}\mathrm{d}t.
\end{aligned}$$

5.7 The ${}_2F_0$ equation

Parallel to the ${}_1F_1$ equation we will consider the ${}_2F_0$ *equation*, given by the operator

$$\mathcal{F}(a,b;-;z,\partial_z) := z^2\partial_z^2 + (-1 + (1+a+b)z)\partial_z + ab, \tag{5.73}$$

where $a, b \in \mathbb{C}$. This equation is another limiting case of the hypergeometric equation:

$$\lim_{c\to\infty} \mathcal{F}(a,b;c;cz,\partial_{(cz)}) = -\mathcal{F}(a,b;-;z,\partial_z).$$

5.8 Point ∞ for the confluent equation

We have

$$\begin{aligned} & z^{a+1}(z\partial_z^2 + (c-z)\partial_z - a)z^{-a} \\ = \ & z^2\partial_z^2 + z(-2a + c - z)\partial_z + a(1+a-c). \quad (5.74) \\ = \ & z^2\partial_z^2 + z(1-a-b-z)\partial_z + ab, \quad (5.75) \end{aligned}$$

where we set $b := 1+a-c$. Substituting $w = -z^{-1}$ (with the inverse $z = -w^{-1}$), using $\partial_z = w^2\partial_w$, we obtain that (5.75) is

$$w^2\partial_w^2 + (-1 + (1+a+b)w)\partial_w + ab.$$

We thus obtained the ${}_2F_0$ equation. Note that 0 is an irregular singular point of this equation. Therefore, ∞ is an irregular singular point of the confluent equation.

If

$$(w^2\partial_w^2 + (-1 + (1+a+b)w)\partial_w + ab)g(w) = 0,$$

then

$$(z\partial_z^2 + (c-z)\partial_z - a)z^{-a}g(-z^{-1}) = 0. \tag{5.76}$$

Conversely, if

$$(z\partial_z^2 + (c-z)\partial_z - a)f(z) = 0,$$

then

$$(w^2\partial_w^2 + (-1 + (1+a+b)w)\partial_w + ab)w^{-a}f(-w^{-1}) = 0.$$

5.9 Asymptotic series

Let function f be defined on $K(z_0, r) \cap \{\alpha_1 < \arg(z - z_0) < \alpha_2\}$. We write

$$f(z) \sim \sum_{j=0}^{\infty} a_j (z - z_0)^j,$$

if for any n there exists C_n such that

$$\left| f(z) - \sum_{j=0}^{n} a_j (z - z_0)^j \right| \le C_n |z - z_0|^{n+1}.$$

Clearly, if $f(z)=\sum_{j=0}^{\infty}a_j(z-z_0)^j$ for $z\in K(z_0,r)$, then $f(z)\sim\sum_{j=0}^{\infty}a_j(z-z_0)^j$.

Example. For $-\frac{\pi}{2}+\epsilon<\arg z<\frac{\pi}{2}-\epsilon$

$$\mathrm{e}^{-\frac{1}{z}}\sim\sum_{j=0}^{\infty}0z^j.$$

Example. For $-\frac{\pi}{4}+\epsilon<\arg z<\frac{\pi}{4}-\epsilon$ and $-\frac{\pi}{4}+\epsilon<\arg -z<\frac{\pi}{4}-\epsilon$

$$\mathrm{e}^{-\frac{1}{z^2}}\sim\sum_{j=0}^{\infty}0z^j.$$

In particular, all derivatives of $\mathbb{R}\ni x\to\mathrm{e}^{-\frac{1}{x^2}}$ at zero are zero.

Example: Error Function.

$$\mathrm{Erf}(z):=\int_0^z\mathrm{e}^{-t^2}\mathrm{d}t.$$

Clearly, $\lim\limits_{\mathrm{Re}z\to\infty}\mathrm{Erf}(z)=\frac{1}{2}\sqrt{\pi}$. For $-\frac{\pi}{2}+\epsilon<\arg z<\frac{\pi}{2}-\epsilon$ we easily show by integration by parts that

$$\frac{1}{2}\sqrt{\pi}-\mathrm{Erf}(z)=\int_z^{\infty}\mathrm{e}^{-t^2}\mathrm{d}t\sim\frac{\mathrm{e}^{-z^2}}{2z}\Big(1+\sum_{k=1}^{\infty}(-1)^k\frac{1\cdot3\cdots(2k-1)}{(2z^2)^k}\Big).$$

5.10 ${}_2F_0$ function

We try to solve (5.76) with a power series

$$g(w)=\sum_{n=0}^{\infty}g_nw^n.$$

We obtain

$$\sum_{n=0}^{\infty}\big(n(n-1)g_nw^n-ng_nw^{n-1}+(1+a+b)ng_nw^n+abg_nw^n\big)=0$$

Hence

$$(n-1+a)(n-1+b)g_{n-1}=ng_n.$$

This gives the coefficients

$$g_n=\frac{(a)_n(b)_n}{n!}g_0$$

and leads to a divergent series.

Theorem 5.2 *Let a contour γ satisfy*

$$\mathrm{e}^{-t}t^{a}(1-wt)^{1-b}\Big|_{\gamma(0)}^{\gamma(1)}=0 \tag{5.77}$$

Then

$$\int_\gamma \mathrm{e}^{-t}t^{a-1}(1-wt)^{-b}\mathrm{d}t$$

is a solution of (5.76).

Proof. By Thm 5.1

$$\int_\gamma \mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s$$

is a solution of the confluent. Therefore,

$$w^{-a}\int_\gamma \mathrm{e}^{-sw^{-1}}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s,$$

for $b=1+a-c$ is a solution of (5.76). Next we substitute $t=\frac{s}{w}$. □

Fora $w\in\mathbb{C}\backslash[0,\infty[$, Re$a>0$ we define

$$F(a,b;-;w):=\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-t}t^{a-1}(1-wt)^{-b}\mathrm{d}t. \tag{5.78}$$

For other values of a we extend (5.78) by analytic continuation. We have the following asymptotic expansion:

$$F(a,b;-;w)\sim\sum_{n=0}^\infty\frac{(a)_n(b)_n}{n!}w^n.$$

More precisely, for any n, for $|\arg w|\geq\epsilon>0$,

$$\lim_{w\to0}w^{-n}\left(F(a,b;-;w)-\sum_{j=0}^n\frac{(a)_j(b)_j}{j!}w^j\right)=0.$$

To prove this, we use the formula

$$f(z)=\sum_{j=0}^{n-1}\frac{f^{(j)}(0)z^j}{j!}+z^n\int_0^1\frac{f^{(n)}(sz)n(1-s)^{n-1}}{n!}\mathrm{d}s,$$

which implies

$$(1-z)^{-b}=\sum_{j=0}^{n-1}\frac{(b)_jz^j}{j!}+\frac{(b)_nz^n}{n!}\int_0^1 n(1-s)^{n-1}(1-zs)^{-b-n}\mathrm{d}s.$$

Hence

$$\begin{aligned}
& F(a,b;-;w) \\
= \; & \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-t}t^{a-1}(1-wt)^{-b}\mathrm{d}t \\
= \; & \sum_{j=0}^{n-1}\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-t}t^{a-1}\frac{(b)_j w^j t^j}{j!}\mathrm{d}t \\
& +\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-t}t^{a-1}\frac{(b)_n w^n t^n}{n!}\int_0^1 (1-wts)^{-b-n}n(1-s)^{n-1}\mathrm{d}s \\
= \; & \sum_{j=0}^{n-1}\frac{(b)_j\Gamma(a+j)w^j}{\Gamma(a)j!} \\
& +\frac{w^n(b)_n}{\Gamma(a)n!}\int_0^1 n(1-s)^{n-1}\mathrm{d}s\int_0^\infty \mathrm{e}^{-t}t^{a-1+n}(1-wts)^{-b-n}\mathrm{d}t \\
= \; & \sum_{j=0}^{n-1}\frac{(b)_j(a)_j w^j}{j!} \\
& +\frac{w^n(b)_n(a)_n}{n!}\int_0^1 n(1-s)^{n-1}\mathrm{d}sF(a+n,b+n;-;ws).
\end{aligned}$$

5.11 Solutions of the confluent equation with definite behavior at ∞

Consider the analytic function on the upper halfplane given by

$$s \mapsto e^{zs}s^{a-1}(1-s)^{c-a-1},$$

where we use the principal branch of power functions. Assume that $\mathrm{Re}a > 0$, $\mathrm{Re}(c-a) > 0$. Remember that

$$F(z) = \int_0^1 \mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}F(a;c;z)$$

is one of solutions of the confluent equation. If $\mathrm{Im}z > 0$, then we can write the following solutions

$$\begin{aligned}
F_0(z) &= \int_0^{\mathrm{e}^{\mathrm{i}\phi}\infty} \mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s, \\
F_1(z) &= \int_1^{\mathrm{e}^{\mathrm{i}\phi}\infty} \mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s,
\end{aligned}$$

where $\phi \in]\frac{\pi}{2}-\arg z, \frac{3\pi}{2}-\arg z[$ guarantees that e^{zs} along the halfline where we integrate converges fast to zero (thus the appropriate condition is fulfilled). Notice that

$$F(z) + F_1(z) - F_0(z) = 0. \tag{5.79}$$

Substituting $s = -z^{-1}t$, where $t \in [0, \infty[$, for $\mathrm{Re}a > 0$ we obtain

$$\begin{aligned} F_0(z) &= \int_0^\infty \mathrm{e}^{-t}(-tz^{-1})^{a-1}(1+z^{-1}t)^{c-a-1}(-z^{-1})\mathrm{d}t \\ &= (-z)^{-a}\Gamma(a)F(a, a+1-c; -, -z^{-1}). \end{aligned}$$

Substituting $s = 1 - z^{-1}t$, where $t \in [0, \infty[$, for $\mathrm{Re}(c-a) > 0$ we can write

$$\begin{aligned} F_1(z) &= -\mathrm{e}^z \int_0^\infty \mathrm{e}^{-t}(1-z^{-1}t)^{a-1}z^{-c+a}t^{c-a-1}\mathrm{d}t \\ &= -\mathrm{e}^z z^{-c+a}\Gamma(c-a)F(c-a, 1-a; -, z^{-1}). \end{aligned}$$

By (5.79), we obtain

$$\frac{F(a;c;z)}{\Gamma(c)} = (-z)^{-a}\frac{F(a, a+1-c; -; -z^{-1})}{\Gamma(c-a)} + z^{-c+a}\frac{\mathrm{e}^z F(c-a, 1-a; -; z^{-1})}{\Gamma(a)}$$

5.12 Hydrogen atom

We transform the confluent operator

$$\begin{aligned} &\mathrm{e}^{-z/2}\left(z\partial_z^2 + (c-z)\partial_z - a\right)\mathrm{e}^{z/2} && (5.80) \\ =\ & z\partial_z^2 + c\partial_z + \frac{c}{2} - a - \frac{z}{4}; && (5.81) \end{aligned}$$

Next,

$$\begin{aligned} &z^{-(1-c)/2}\left(z\partial_z^2 + c\partial_z + \frac{c}{2} - a - \frac{z}{4}\right)z^{(1-c)/2} && (5.82) \\ =\ & z\partial_z^2 + \partial_z - \frac{z}{4} + \frac{c}{2} - a - \frac{(1-c)^2}{4z}. && (5.83) \end{aligned}$$

We divide (5.81) by z and substitute $z = 2w$. We obtain

$$\partial_w^2 + \frac{1}{w}\partial_w - 1 + (c-2a)\frac{1}{w} - \left(\frac{1-c}{2}\right)^2\frac{1}{w^2}, \tag{5.84}$$

We multiply (5.84) with w^2. We obtain

$$w^2\partial_w^2 + w\partial_w - w^2 + (c-2a)w - \left(\frac{1-c}{2}\right)^2, \tag{5.85}$$

or the equation for the radial wave function for the Coulomb potential in dimension 2 (which is easy to transform into an analogous equation in higher dimensions).

Therefore, if f satisfies the confluent equation, then $\mathrm{e}^{-w}w^{(-1+c)/2}f(2w)$ satisfies (5.85).

6 Dzeta function

6.1 Riemann's dzeta function

For $\mathrm{Re} s > 1$ Riemann's dzeta function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{6.86}$$

6.2 Prime numbers

Let $\mathrm{Re} s > 1$. We have

$$\zeta(s)(1 - 2^{-s}) = \frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \dots. \tag{6.87}$$

Likewise, if $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \dots$ are the prime numbers in the increasing order, then

$$\zeta(s)(1 - p_1^{-s}) \cdots (1 - p_n^{-s}) = \sum_{k \in X_n} \frac{1}{k^s}, \tag{6.88}$$

where X_n is the set of positive integers not divisible by $p_1, \dots, p_n$.

Proposition 6.1 *We have*

$$\prod_{n=j}^{\infty} (1 - p_j^{-s}) = \frac{1}{\zeta(s)}. \tag{6.89}$$

$\zeta \neq 0$ *for* $\mathrm{Re} s > 1$.

Proof. First note that the lhs of (6.89) is an absolutely convergent product, because

$$p_j^{-s} \leq j^{-s},$$

and $\sum j^{-s} < \infty$. By continuing (6.88) we obtain

$$\lim_{n \to \infty} \zeta(s) \prod_{j=1}^{n} (1 - p_j^{-s}) = 1, \tag{6.90}$$

which implies (6.89).

All the factors of (6.89) are nonzero. Hence (6.89) is nonzero. □

6.3 Holomorphic extension of dzeta function

Theorem 6.2 *For any s with* $\mathrm{Re} s > 1$ *we have*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} \mathrm{e}^{-x}}{1 - \mathrm{e}^{-x}} \mathrm{d}x. \tag{6.91}$$

ζ extends to a holomorphic function on $\mathbb{C}\backslash\{0\}$. It has the following integral representation valid for all s except for $s = 1, 2, \ldots$ (because of singularities of the Gamma function):

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi \mathrm{i}} \int_{]-\infty,0^+,-\infty[} \frac{z^{s-1}\mathrm{e}^z}{1-\mathrm{e}^z} \mathrm{d}z. \tag{6.92}$$

Proof. (6.91) follows by summing up

$$\int_0^\infty x^{s-1}\mathrm{e}^{-nx}\mathrm{d}x = \Gamma(s)\frac{1}{n^s}. \tag{6.93}$$

(6.92) follows by summing up

$$\frac{1}{2\pi\mathrm{i}} \int_{]-\infty,0^+,-\infty[} z^{s-1}\mathrm{e}^{nz}\mathrm{d}z = \frac{1}{\Gamma(1-s)n^s}. \tag{6.94}$$

(6.92) is holomorphic on the whole $\mathbb{C}$ except maybe at the singularities of $\Gamma(1-s)$, which are $1, 2, \ldots$. But we already know that ζ is holomorphic for $\mathrm{Re}s > 1$. Hence the only singularity can be at 1. □

Theorem 6.3

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s). \tag{6.95}$$

Proof.

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^\infty \int_n^\infty \frac{s}{x^{s+1}}\mathrm{d}x && (6.96)\\
&= \sum_{n=1}^\infty \int_n^{n+1} \frac{sn}{x^{s+1}}\mathrm{d}x && (6.97)\\
&= \int_1^\infty \frac{s}{x^s}\mathrm{d}x + \sum_{n=1}^\infty \int_n^{n+1} \frac{s(n-x)}{x^{s+1}}\mathrm{d}x. && (6.98)
\end{aligned}$$

Now,

$$\int_1^\infty \frac{s}{x^s}\mathrm{d}x = \frac{s}{s-1} = 1 + \frac{1}{s-1},$$

$$\lim_{s\searrow 1} \int_n^{n+1} \frac{s(n-x)}{x^{s+1}}\mathrm{d}x = \int_n^{n+1} \frac{n-x}{x^2}\mathrm{d}x = \frac{1}{n+1} - \log(n+1) + \log(n).$$

Therefore,

$$\begin{aligned}
&\lim_{s\searrow 0}\Big(\zeta(s) - \frac{1}{s-1}\Big) && (6.99)\\
=\ & 1 + \sum_{n=1}^\infty \Big(\frac{1}{n+1} - \log(n+1) + \log(n)\Big) && (6.100)\\
=\ & \sum_{n=1}^\infty \Big(\frac{1}{n} - \log(n+1) + \log(n)\Big) \quad = \quad \gamma. && (6.101)
\end{aligned}$$

$\square$

6.4 Bernoulli numbers

The Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{6.102}$$

The function

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \coth\left(\frac{x}{2}\right) \tag{6.103}$$

is even. Hence for odd n we have $B_n = 0$ except for $B_1 = -\frac{1}{2}$. Otherwise, $B_0 = 1$, $B_2 = \frac{1}{6}$, etc. We also have

$$\begin{aligned} x \coth(x) &= \sum_{k=0}^{\infty} B_{2k} \frac{(2x)^{2k}}{(2k)!}, & (6.104) \\ x \cot(x) &= \sum_{k=0}^{\infty} (-1)^k B_{2k} \frac{(2x)^{2k}}{(2k)!}. & (6.105) \end{aligned}$$

Theorem 6.4 *For positive even integers the dzeta function can be expressed in terms of Bernoulli numbers:*

$$\zeta(2k) = \frac{(-1)^{k+1} 2^{2k-1} \pi^{2k}}{(2k)!} B_{2k}. \tag{6.106}$$

For all negative integers the dzeta function can be expressed in terms of Bernoulli numbers:

$$\zeta(-n) = -\frac{-B_{n+1}}{n+1}. \tag{6.107}$$

(In particular, for even negative integers the dzeta function is zero).

Proof. To prove (6.106) we use

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}. \tag{6.108}$$

By (6.108),

$$\begin{aligned} x \cot x &= 1 + 2 \sum_{n=1}^{\infty} \frac{x^2}{x^2 - n^2 \pi^2} & (6.109) \\ &= 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^{2k}}{n^{2k} \pi^{2k}} & (6.110) \\ &= 1 - 2 \sum_{k=1}^{\infty} \frac{x^{2k}}{\pi^{2k}} \zeta(2k). & (6.111) \end{aligned}$$

(6.107) follows from (6.92), which for $s=-n$ can be rewritten as

$$\zeta(-n) = \frac{n!}{2\pi \mathrm{i}} \int_{[0^+]} \frac{z^{-1-n}}{\mathrm{e}^{-z}-1} \mathrm{d}z. \tag{6.112}$$

6.5 Riemann's reflection formula

Theorem 6.5

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin(\tfrac{\pi}{2}s) \zeta(1-s). \tag{6.113}$$

or equivalently

$$\zeta(s)\Gamma(s)\cos(\tfrac{\pi}{2}s) = 2^{s-1}\pi^s \zeta(1-s). \tag{6.114}$$

Proof. Assume that $\mathrm{Re} s < 0$. The function $\frac{z^{s-1}\mathrm{e}^z}{1-\mathrm{e}^z}$ has simple poles at $z \in \mathrm{i}2\pi\mathbb{Z}\backslash\{0\}$. We compute the residues:

$$\mathrm{Res} \frac{z^{s-1}\mathrm{e}^z}{1-\mathrm{e}^z}\Big|_{z=\mathrm{i}2\pi n} = -(\mathrm{i}2\pi n)^{s-1}. \tag{6.115}$$

Now for $n>0$,

$$\begin{aligned} & (\mathrm{i}2\pi n)^{s-1} + (\mathrm{i}2\pi(-n))^{s-1} & (6.116)\\ = \quad & (2\pi)^{s-1}(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(s-1)} + \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(s-1)})n^{s-1} & (6.117)\\ = \quad & 2^s\pi^{s-1}\sin\Big(\frac{\pi}{2}s\Big)n^{s-1}. & (6.118)\end{aligned}$$

On $\mathbb{C}\backslash]-\infty,0]$, treated as the domain of z^s, we consider the circle of radius $(2N+1)\pi$ and centered at 0. We treat it as a curve γ_N starting at $-(2N+1)\pi-\mathrm{i}0$ and ending at $-(2N+1)\pi+\mathrm{i}0$. Let $\delta_N := [-(2N+1)\pi-\mathrm{i}0, 0^+, -(2N+1)\pi+\mathrm{i}0]$. Then

$$\begin{aligned} & \frac{1}{2\pi\mathrm{i}}\int_{\gamma_N} \frac{z^{s-1}\mathrm{e}^z}{1-\mathrm{e}^z}\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\int_{\delta_N} \frac{z^{s-1}\mathrm{e}^z}{1-\mathrm{e}^z}\mathrm{d}z & (6.119)\\ = \quad & -\sum_{n=1}^{N} 2^s\pi^{s-1}\sin(\frac{\pi}{2}s)n^{s-1} & (6.120)\\ \to \quad & -2^s\pi^{s-1}\sin(\frac{\pi}{2}s)\zeta(1-s). & (6.121)\end{aligned}$$

But on γ_N

$$\Big|\frac{\mathrm{e}^z}{1-\mathrm{e}^z}\Big| < K \tag{6.122}$$

$$|z^{s-1}| < |(2N+1)\pi|^{\mathrm{Re}s-1} \tag{6.123}$$

Hence the first term of (6.119) converges to 0. Clearly, the second term of (6.119) converges to $-\zeta(s)$.

We pass from (6.113) to (6.114) by $\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$ and $\sin\pi s = 2\sin(\frac{\pi}{2}s)\cos(\frac{\pi}{2}s)$. □

Applying $\Gamma(s) = \pi^{-\frac12}2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+\frac12)$ and $\Gamma(\frac{s}{2}+\frac12)\Gamma(-\frac{s}{2}+\frac12) = \frac{\pi}{\cos(\frac{\pi}{2}s)}$ we obtain

$$\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{s-\frac12}\Gamma\Big(-\frac{s}{2}+\frac12\Big)\zeta(1-s). \tag{6.124}$$

If we introduce

$$\eta(s) := \pi^{-\frac{s}{2}}\Gamma\big(\tfrac{s}{2}\big)\zeta(s), \tag{6.125}$$

then (6.124) can be written in a symmetric way:

$$\eta(s) = \eta(1-s). \tag{6.126}$$

η has only two singularities: at 0 and 1. The following function is entire

$$\xi(s) := \frac12 s(s-1)\eta(s) = \frac12 s(s-1)\pi^{-\frac{s}{2}}\Gamma\big(\tfrac{s}{2}\big)\zeta(s), \tag{6.127}$$

and it also satisfies the symmetry

$$\xi(s) = \xi(1-s). \tag{6.128}$$

Note that

$$\log\big(\xi(s)\big) = \log\Big(\frac{s(-1+s)}{2}\Big) + \log\Big(\Gamma(\frac{s}{2})\Big) - \frac{s}{2}\log(\pi) + \log\big(\zeta(s)\big). \tag{6.129}$$

6.6 Poisson summation formula

In this subsection we adopt the following convention for Fourier transform:

$$\hat f(\xi) := \int f(x)\mathrm{e}^{-\mathrm{i}2\pi x\xi}\mathrm{d}x. \tag{6.130}$$

Putting 2π in the exonent is well adapted to periodic functions of period one: we then have nice formulas

$$\hat f_k = \int_{-\frac12}^{\frac12} f(x)\mathrm{e}^{-\mathrm{i}2\pi xk}\mathrm{d}x, \tag{6.131}$$

$$f(x) = \sum_k \hat f_k \mathrm{e}^{\mathrm{i}2\pi kx}. \tag{6.132}$$

Theorem 6.6 *If $f \in L^1$ and $\sum_j |\hat f(j)| < \infty$, then*

$$\sum_{k=-\infty}^{\infty} f(k) = \sum_{j=-\infty}^{\infty} \hat f(j). \tag{6.133}$$

Proof. Let $j \in \mathbb{Z}$.

$$\int_{-n-\frac12}^{n+\frac12} f(x)\mathrm{e}^{-\mathrm{i}2\pi xj}\mathrm{d}x = \sum_{k=-n}^{n}\int_{-\frac12}^{\frac12} f(x+k)\mathrm{e}^{-2\pi\mathrm{i}jx}\mathrm{d}x. \tag{6.134}$$

Define

$$g(x) := \lim_{n\to\infty} \sum_{k=-n}^{n} f(x+k), \tag{6.135}$$

which is a periodic integrable function with period 1. We have

$$\hat{f}(j) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x)\mathrm{e}^{-2\pi \mathrm{i} jx}\mathrm{d}x. \tag{6.136}$$

By the inversion of the Fourier transformation for periodic functions we obtain

$$g(x) = \sum_{j=-\infty}^{\infty} \hat{f}(j)\mathrm{e}^{2\pi \mathrm{i} jx}. \tag{6.137}$$

Setting $x = 0$ we obtain

$$\sum_{k=-\infty}^{\infty} f(k) = g(0) = \sum_{j=-\infty}^{\infty} \hat{f}(j). \tag{6.138}$$

□

6.7 2nd proof of Riemann's reflection formula

Define

$$\phi(x) := \sum_{n=1}^{\infty} \mathrm{e}^{-n^2\pi x}. \tag{6.139}$$

Note that

$$1 + 2\phi(x) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-n^2\pi x} = \theta_3(0, \mathrm{e}^{-x}), \tag{6.140}$$

$$\int \mathrm{e}^{-y^2\pi x}\mathrm{e}^{-\mathrm{i}2\pi y\xi}\mathrm{d}y = \frac{1}{\sqrt{x}}\mathrm{e}^{-\frac{\xi^2\pi}{x}}. \tag{6.141}$$

Hence the Poisson summation formula implies

$$1 + 2\phi(x) = \frac{1}{\sqrt{x}}\Big(1 + 2\phi\Big(\frac{1}{x}\Big)\Big) \tag{6.142}$$

Recall that

$$\eta(s) := \frac{\Gamma(\frac{s}{2})\zeta(s)}{\pi^{\frac{s}{2}}}. \tag{6.143}$$

Theorem 6.7 *We have the identities*

$$\begin{aligned}
\eta(s) &= \int_0^\infty \frac{\mathrm{d}x}{x}\phi(x)x^{\frac{s}{2}} && (6.144)\\
&= \int_1^\infty \frac{\mathrm{d}x}{x}\phi(x)\big(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\big) - \frac{1}{s(1-s)} && (6.145)\\
&= \int_1^\infty \frac{\mathrm{d}x}{x}\Big(\phi(x)x^{\frac{1}{4}} - \tfrac{1}{2}x^{-\frac{1}{4}}\Big)\big(x^{\frac{1}{2}(s-\frac{1}{2})} + x^{-\frac{1}{2}(s-\frac{1}{2})}\big) && (6.146)\\
&= \int_0^\infty \frac{1}{2}\frac{\mathrm{d}x}{x}\Big(\phi(x)x^{\frac{1}{4}} - \tfrac{1}{2}x^{-\frac{1}{4}}\Big)\big(x^{\frac{1}{2}(s-\frac{1}{2})} + x^{-\frac{1}{2}(s-\frac{1}{2})}\big) && (6.147)
\end{aligned}$$

The first is valid for $\mathrm{Re} s > 1$*, the second for all* s *and the two last for* $0 < \mathrm{Re} s < 1$.

In particular, by (6.145), η *extends analytically to the complex plane except for* $0, 1$.

Proof. For any $\mathrm{Re} s > 0$ we have

$$\frac{1}{n^s}\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big) = \int_0^\infty \frac{\mathrm{d}x}{x}\mathrm{e}^{-n^2\pi x}x^{\frac{s}{2}}. \tag{6.148}$$

For $\mathrm{Re} s > 1$ we can sum up (6.148) obtaining (6.144). Now,

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}x}{x}\phi(x)x^{\frac{s}{2}} &= \int_1^\infty \frac{\mathrm{d}x}{x}\phi(x)x^{\frac{s}{2}} + \int_1^\infty \frac{\mathrm{d}x}{x}\phi\Big(\frac{1}{x}\Big)x^{-\frac{s}{2}} \\
&= \int_1^\infty \frac{\mathrm{d}x}{x}\phi(x)\Big(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\Big) + \frac{1}{2}\int_1^\infty \frac{\mathrm{d}x}{x}\Big(x^{\frac{1-s}{2}} - x^{-\frac{s}{2}}\Big)
\end{aligned}$$

But

$$\frac{1}{2}\int_1^\infty \frac{\mathrm{d}x}{x}\Big(x^{\frac{1-s}{2}} - x^{-\frac{s}{2}}\Big) = \frac{1}{s-1} - \frac{1}{s} = -\frac{1}{s(1-s)}. \tag{6.149}$$

This implies (6.145) for $\mathrm{Re} s > 1$. But (6.145) is analytic except for $0, 1$. Hence the formula can be extended.

For $0 < \mathrm{Re} s < 1$,

$$-\frac{1}{2}\int_1^\infty \frac{\mathrm{d}x}{x}\Big(x^{\frac{s-1}{2}} + x^{-\frac{s}{2}}\Big) = -\frac{1}{1-s} - \frac{1}{s} = -\frac{1}{s(1-s)}. \tag{6.150}$$

Hence (6.145) implies (6.146)

(6.146) implies (6.147) by (6.142). □

Note that the function η satisfies the symmetries

$$\eta(s) = \eta(1-s), \tag{6.151}$$

$$\overline{\eta(s)} = \eta(\overline{s}). \tag{6.152}$$

The first follows from (6.145). The same symmetries are satisfied by

$$\xi(s) := \frac{1}{2}s(s-1)\eta(s).$$

6.8 Zeros of the dzeta function

Let $\frac{1}{2} + \mathrm{i}Z$ be the set of all zeros of the dzeta function away from $\mathbb{R}$. Z coincides with the set of zeros of η and of ξ. Note that $Z = \overline{Z} = -Z$. The Riemann hypothesis says that $Z \subset \mathbb{R}$.

Let $\frac{1}{2} + \mathrm{i}Z_+$ be the set of zeros of the dzeta function with positive imaginary part. Clearly, $Z = Z_+ \cup (-Z_+)$.

By the Hadamard Theorem,

$$\xi(s) = \prod_{\lambda\in Z}\Big(1-\frac{s}{\frac12+\mathrm{i}\lambda}\Big) \tag{6.153}$$

$$= \prod_{\lambda\in Z_+}\Big(1-\frac{s}{\frac12+\mathrm{i}\lambda}\Big)\Big(1-\frac{s}{\frac12-\mathrm{i}\lambda}\Big) \tag{6.154}$$

$$= \prod_{\lambda\in Z_+}\Big(1-\frac{s(1-s)}{\frac14+\lambda^2}\Big). \tag{6.155}$$

Hence

$$\log\big(\xi(s)\big) = \sum_{\lambda\in Z}\log\Big(1-\frac{s}{\frac12+\mathrm{i}\lambda}\Big). \tag{6.156}$$

6.9 Dzeta function as Mellin transform

Introduce

$$\pi(x) := \#\{\text{primes } \leq x\}, \tag{6.157}$$

$$\Pi(x) = \sum_{m=1}^{\infty}\frac{1}{m}\pi(x^{\frac{1}{m}}). \tag{6.158}$$

Then

$$\log\zeta(s) = -\sum_p \log(1-p^{-s}) \tag{6.159}$$

$$= -\sum_p\sum_{m=1}^{\infty}\frac{p^{-sm}}{m}, \tag{6.160}$$

$$= -\int x^{-s}\Pi'(x)\mathrm{d}x. \tag{6.161}$$

For any $a>0$, we can write Π' as the Fourier transform

$$\log\zeta(a+\mathrm{i}t) = -\int_0^\infty x^{-a-\mathrm{i}t}\Pi'(x)\mathrm{d}x \tag{6.162}$$

$$= -\int \mathrm{e}^{-au-\mathrm{i}tu}\Pi'(\mathrm{e}^u)\mathrm{e}^u\mathrm{d}u. \tag{6.163}$$

Inverting the Fourier transform we obtain

$$\Pi'(x) = -\frac{x^{a-1}}{2\pi}\int x^{\mathrm{i}t}\log\zeta(a+\mathrm{i}t)\mathrm{d}t. \tag{6.164}$$

We have for $\mathrm{Re}\lambda>0$

$$\int \mathrm{e}^{\mathrm{i}t\xi}\log(\mathrm{i}t+\lambda)\mathrm{d}t = -2\pi\mathrm{i}\frac{\mathrm{e}^{-\xi\lambda}}{|\xi|_+}. \tag{6.165}$$

Hence, for $\mathrm{Re}\lambda > a$,

$$-\frac{x^{a-1}}{2\pi}\int x^{\mathrm{i}t}\log(\mathrm{i}t+a-\lambda)\mathrm{d}t = \frac{x^{\lambda-1}}{|\log x|_+}. \tag{6.166}$$

Using now (6.156), Riemann derived the formula

$$\Pi(x) = \mathrm{Li}(x) - \sum_{\lambda\in Z_+}\mathrm{Li}(x^\lambda) - \log(2) + \int_x^\infty \frac{\mathrm{d}t}{t(t^2-1)\log(t)}. \tag{6.167}$$

Here

$$\mathrm{Li}(x) := \int_0^x \frac{\mathrm{d}t}{\log(t)}. \tag{6.168}$$

Introduce the Mellin transformation:

$$(\mathcal{M}f)(t) := \int_0^\infty x^{-\frac{1}{2}-\mathrm{i}t}f(x)\mathrm{d}x, \tag{6.169}$$

$$(\mathcal{M}^{-1}g)(x) := \frac{1}{2\pi}\int_{-\infty}^\infty x^{-\frac{1}{2}+\mathrm{i}t}g(t)\mathrm{d}t. \tag{6.170}$$

6.10 The Hurwitz dzeta function

For $\mathrm{Re}s > 0$ and $a \notin \{\dots,-2,-1,0\}$, we define

$$\zeta(s,a) := \sum_{n=1}^\infty \frac{1}{(a+n)^s}. \tag{6.171}$$

Theorem 6.8 *For any s, a with $\mathrm{Re}s > 0$ and $\mathrm{Re}a > 0$, we have*

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}\mathrm{e}^{-ax}}{1-\mathrm{e}^{-x}}\mathrm{d}x. \tag{6.172}$$

$s \mapsto \zeta(s,a)$ extends to a holomorphic function on $\mathbb{C}\backslash\{0\}$. It has the following integral representation valid for all s except for $s = 0, 1, 2, \dots$:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi\mathrm{i}}\int_{]-\infty,0^+,-\infty[}\frac{z^{s-1}\mathrm{e}^{az}}{1-\mathrm{e}^z}\mathrm{d}z. \tag{6.173}$$

For $s \approx 0$ we have

$$\zeta(s) = \frac{1}{s-1} + O(s^0). \tag{6.174}$$

6.11 The Hurwitz identity

Theorem 6.9 *Let $0 < \mathrm{Re}a \leq 1$. Then*

$$\zeta(s,a) = 2^s\pi^{s-1}\Gamma(1-s)\sum_{n=1}^\infty \frac{\sin(2\pi na + \frac{\pi}{2}s)}{n^{1-s}}. \tag{6.175}$$