

# Introduction to hypergeometric type functions

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## 1 Generalized hypergeometric equations and functions

### 1.1 Generalized hypergeometric series

For  $a_1, \dots, a_k \in \mathbb{C}$ ,  $c_1, \dots, c_m \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ , we define the (generalized) hypergeometric series of type  ${}_kF_m$ :

$${}_kF_m(a_1, \dots, a_k; c_1, \dots, c_m; z) := \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_k)_j z^j}{(c_1)_j \cdots (c_m)_j j!}. \quad (1.1)$$

Notice that

1. if  $m + 1 > k$ , then (1.1) is convergent for  $z \in \mathbb{C}$ ;
2. if  $m + 1 = k$ , then (1.1) is convergent for  $|z| < 1$ ;
3. if  $m + 1 < k$ , then (1.1) is divergent (however sometimes we can give a meaning to the function  ${}_kF_m$ ).

This follows by the d'Alembert criterion: if  $f_j$  is  $j$ th coefficient of (1.1), then

$$\frac{f_{j+1}}{f_j} = \frac{(a_1 + j) \cdots (a_k + j)}{(c_1 + j) \cdots (c_m + j)}.$$

We can also use a different normalization:

$$\begin{aligned} {}_k\mathbf{F}_m(a_1, \dots, a_k; c_1, \dots, c_m; z) &:= \frac{{}_kF_m(a_1, \dots, a_k; c_1, \dots, c_m; z)}{\Gamma(c_1) \cdots \Gamma(c_m)} \\ &= \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_k)_j z^j}{\Gamma(c_1 + j) \cdots \Gamma(c_m + j) j!}. \end{aligned} \quad (1.2)$$

Then we do not have to restrict the values of  $c_1, \dots, c_m \in \mathbb{C}$ . (If for some  $i$   $c_i \in \{0, -1, -2, \dots\}$ , then  $\mathbf{F}$  is zero).

## 1.2 Generalized hypergeometric equations

**Theorem 1.1** *The function (1.1) solves the equation*

$$\left( (c_1 + z\partial_z) \cdots (c_m + z\partial_z) \partial_z - (a_1 + z\partial_z) \cdots (a_k + z\partial_z) \right) F(z) = 0. \quad (1.3)$$

**Proof.** Write for brevity  $F = \sum_{j=0}^{\infty} F_j z^j$ . We check that

$$\begin{aligned} (c_1 + z\partial_z) \cdots (c_k + z\partial_z) \partial_z F_j z^j &= \frac{(a_1)_j \cdots (a_k)_j z^{j-1}}{(c_1)_{j-1} \cdots (c_m)_{j-1} (j-1)!}, \\ (a_1 + z\partial_z) \cdots (a_k + z\partial_z) F_{j-1} z^{j-1} &= \frac{(a_1)_j \cdots (a_k)_j z^{j-1}}{(c_1)_{j-1} \cdots (c_m)_{j-1} (j-1)!}. \end{aligned}$$

□

Note that the equation (1.3) is of the order  $\max(k, m+1)$ . Below we list all equations and hypergeometric functions with equations of the order at most 2.

### 1.3 Hypergeometric function or ${}_2F_1$

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n.$$

The series is convergent for  $|z| < 1$ , it extends to a multivalued function on a covering of  $\mathbb{C} \setminus \{0, 1\}$ .

The function is a solution of the hypergeometric equation

$$(z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab) u(z) = 0$$

that is analytic around 0 and equals there 1.

### 1.4 Confluent function or ${}_1F_1$

$$F(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n! (c)_n} z^n.$$

The series is convergent for all  $z \in \mathbb{C}$ . It defines a solution analytic around 0 and equal there 1 of the confluent equation

$$(z\partial_z^2 + (c-z)\partial_z - a)u(z) = 0,$$

### 1.5 Function ${}_0F_1$

$$F(-; c; z) = F(c; z) = \sum_{n=0}^{\infty} \frac{1}{n!(c)_n} z^n.$$

The series is convergent for all  $z \in \mathbb{C}$ . It defines a solution analytic around 0 and equal there 1 of the  ${}_0F_1$  equation (related to the Bessel equation)

$$(z\partial_z^2 + c\partial_z - 1)u(z) = 0.$$

### 1.6 ${}_2F_0$ function

This function has a branch point at zero. Hence it cannot be defined with a series around zero. It can be defined by other means on the covering of  $\mathbb{C} \setminus \{0\}$ . The (divergent) hypergeometric series gives its asymptotic expansion away from any wedge around the positive semiaxis.

$$F(a, b; -; z) \sim \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!} z^n$$

It solves the  ${}_2F_0$  equation (related to the confluent equation)

$$(z^2\partial_z^2 + (-1 + (a + b + 1)z)\partial_z + ab) u(z) = 0.$$

### 1.7 Power function ${}_1F_0$

$$F(a; -; z) = (1 - z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n$$

The series is convergent for  $|z| < 1$ , it extends to a multivalued function on a covering of  $\mathbb{C} \setminus \{1\}$ . It is a solution of

$$((z - 1)\partial_z - a)u(z) = 0.$$

### 1.8 Exponential function ${}_0F_0$

$$F(-; -; z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

It solves

$$(\partial_z - 1)u(z) = 0.$$

## 2 2nd order differential equations in complex domain

In this section we will discuss a general theory of equations of the form

$$(b(z)\partial_z^2 + c(z)\partial_z + d(z)) u(z) = 0. \tag{2.4}$$

$z$  will be a complex variable. The functions  $b, c, d$  will be usually holomorphic or at least meromorphic in an open set  $\Omega \subset \mathbb{C}$ .

Discussing an equation such as (2.4), we will often introduce an operator

$$A(z, \partial_z) := b(z)\partial_z^2 + c(z)\partial_z + d(z). \quad (2.5)$$

We will say that (2.4) is given by the operator (2.5). Indeed,  $u$  solves (2.4) iff  $u$  is in the kernel of (2.5).

By dividing (2.4) by  $b(z)$  we obtain

$$\left( \partial_z^2 + \frac{c(z)}{b(z)}\partial_z + \frac{d(z)}{b(z)} \right) u(z) = 0. \quad (2.6)$$

Thus we can usually assume that  $b(z)$  is 1 and consider

$$(\partial_z^2 + c(z)\partial_z + d(z)) u(z) = 0. \quad (2.7)$$

## 2.1 Wronskian

Let  $u_1(z), u_2(z)$  be a pair of functions. Their Wronskian is

$$W(u_1, u_2)(z) = W(z) := u_1(z)u_2'(z) - u_1'(z)u_2(z).$$

If they are solutions of (2.7), then the Wronskian satisfies

$$(\partial_z + c(z))W(z) = 0.$$

If

$$\tilde{u}_1(z) = a_{11}u_1(z) + a_{12}u_2(z), \quad \tilde{u}_2(z) = a_{21}u_1(z) + a_{22}u_2(z)$$

is another pair of solutions, then

$$W(\tilde{u}_1, \tilde{u}_2) = (a_{11}a_{22} - a_{12}a_{21})W(u_1, u_2).$$

## 2.2 Regular points

**Definition 2.1** We say that  $z_0 \in \Omega$  is a regular point of (2.7) if  $c(z)$  and  $d(z)$  are analytic around  $z_0$ .

**Proposition 2.2** Let  $c(z), d(z)$  be holomorphic in a connected and simply connected open subset  $\Omega \subset \mathbb{C}$ . Then the problem

$$\begin{cases} (\partial_z^2 + c(z)\partial_z + d(z)) u(z) = 0 \\ u(z_0) = w_0, \quad \partial_z u(z_0) = w_1, \end{cases} \quad (2.8)$$

has a unique solution in  $\Omega$ .

Note that if  $b, c, d$  are holomorphic around  $z_0$ , then

$$(b(z)\partial_z^2 + c(z)\partial_z + d(z)) u(z) = 0 \quad (2.9)$$

is regular at  $z_0$  iff  $b(z_0) \neq 0$ . Let us give the formula for the coefficients of the expansion

$$u(z) := \sum_{k=0}^{\infty} u_k z^k.$$

of (2.9).

$$\begin{cases} u_0 = w_0, & u_1 = w_1, \\ \sum_{k=0}^m k(k-1)u_k b_{m-k} + \sum_{k=0}^{m-1} k c_{m-k-1} u_k + \sum_{k=0}^{m-2} d_{m-k-2} u_k = 0. \end{cases}$$

**Definition 2.3** Assume that  $c(z), d(z)$  are holomorphic for  $|z| > R$ . We say that  $\infty$  is a regular point of (2.7) if after the change of coordinates  $w = z^{-1}$  we obtain a regular point at 0.

Consider (2.7). The change  $w = z^{-1}$  and division by  $w^4$  leads to

$$\left( \partial_w^2 + (2w^{-1} - w^{-2}c(w^{-1}))\partial_w + w^{-4}d(w^{-1}) \right) u(w^{-1}) = 0.$$

Hence  $\infty$  is a regular point if there exist (finite) limits

$$\lim_{z \rightarrow \infty} (2z - z^2 c(z)), \quad \lim_{z \rightarrow \infty} z^4 d(z).$$

**Theorem 2.4** Let  $\infty$  be a regular point of (2.7). Then for any  $w_0, w_1$  there exists a unique solution of the problem

$$\begin{cases} (\partial_z^2 + c(z)\partial_z + d(z)) u(z) = 0 \\ \lim_{z \rightarrow \infty} u(z) = w_0, \quad \lim_{z \rightarrow \infty} (u(z) - w_0)z = w_1. \end{cases} \quad (2.10)$$

### 2.3 Regular-singular points

**Definition 2.5** We say that an equation

$$(\partial_z^2 + c(z)\partial_z + d(z)) u(z) = 0 \quad (2.11)$$

has a regular-singular point at  $z_0 \in \Omega$ , if  $c(z)$  has at  $z_0$  a pole of at most 1st order and  $d(z)$  has at  $z_0$  a pole of at most 2nd order.

For simplicity, assume that  $z_0 = 0$ . We can rewrite the above equation as

$$(z^2 \partial_z^2 + q(z)z \partial_z + r(z)) u(z) = 0. \quad (2.12)$$

0 is regular-singular iff  $q, r$  are analytic at 0.

**Theorem 2.6 (Frobenius Method)**  $z = 0$  is regular-singular iff  $q, r$  are holomorphic at  $z = 0$ . Assume that in addition  $q, r$  are holomorphic in an open connected simply connected set  $\Omega \subset \mathbb{C}$  containing 0. Let  $\lambda \in \mathbb{C}$  satisfy

$$\begin{aligned}\lambda(\lambda - 1) + \lambda q(0) + r(0) &= 0, \\ (\lambda + m)(\lambda + m - 1) + (\lambda + m)q(0) + r(0) &\neq 0, \quad m = 1, 2, \dots\end{aligned}$$

Then there exists a unique function  $\tilde{u}(z)$  holomorphic in  $\Omega$ , such that  $u(z) := z^\lambda \tilde{u}(z)$  is a solution of the problem

$$\begin{cases} (z^2 \partial_z^2 + q(z)z \partial_z + r(z)) u(z) = 0, \\ \lim_{z \rightarrow 0} z^{-\lambda} u(z) = 1, \end{cases} \quad (2.13)$$

Clearly, if  $p, q, r$  is analytic around 0, the equation

$$(p(z)z^2 \partial_z^2 + q(z)z \partial_z + r(z)) u(z) = 0, \quad (2.14)$$

has a regular-singular point at 0 iff  $p(0) \neq 0$ . Let us give a recurrent formula for coefficients of

$$u(z) := \sum_{k=0}^{\infty} u_k z^{\lambda+k}$$

$$\begin{cases} u_0 = 1, \\ u_m = -((\lambda + m)(\lambda + m - 1)p_0 + (\lambda + m)q_0 + r_0)^{-1} \\ \quad \times \sum_{k=0}^{m-1} ((\lambda + k)(\lambda + k - 1)p_{m-k} + (\lambda + k)q_{m-k} + r_{m-k})u_k. \end{cases}$$

Thus, if we are looking for solutions (2.14), we should first find the roots  $\lambda_1, \lambda_2$  of the so-called *indicial equation*

$$\lambda(\lambda - 1)p(0) + \lambda q(0) + r(0) = 0.$$

If  $\lambda_1 - \lambda_2 \notin \mathbb{Z}$ , then we can find two linearly independent solutions that behave at zero as  $z^{\lambda_1}$  and  $z^{\lambda_2}$ . If  $\lambda_1 - \lambda_2 \in \mathbb{Z}$ , then generally we can find only a solution behaving as  $z^{\lambda_1}$ , where  $\lambda_1 - \lambda_2 \geq 0$ .

**Definition 2.7** Assume that  $q(z), r(z)$  are holomorphic for  $|z| > R$ . We say that  $\infty$  is a regular-singular point of (2.7) if after the change of coordinates  $w = z^{-1}$  we obtain a regular-singular point at 0.

Hence  $\infty$  is a regular singular point of (2.11) if

$$\lim_{z \rightarrow \infty} zc(z), \quad \lim_{z \rightarrow \infty} z^2 d(z) \quad (2.15)$$

exist. Similarly,  $\infty$  is a regular singular point of (2.12) if

$$\lim_{z \rightarrow \infty} q(z), \quad \lim_{z \rightarrow \infty} r(z) \quad (2.16)$$

exist.

**Proposition 2.8** *Let  $q(z), r(z)$  be holomorphic in a connected simply connected open set  $\Omega \subset \mathbb{C}$  containing  $\{|z| > R\}$ . Let  $\lambda \in \mathbb{C}$  satisfy*

$$\lambda(\lambda + 1) - \lambda q(\infty) + r(\infty) = 0,$$

$$(\lambda + m)(\lambda + m + 1) - (\lambda + m)q(\infty) + r(\infty) \neq 0, \quad m = 1, 2, \dots$$

*Then there exists a unique function  $\tilde{u}(z)$  holomorphic in  $\Omega$ , such that  $u(z) := z^{-\lambda}\tilde{u}(z)$  is a solution of*

$$\begin{cases} (z^2 \partial_z^2 + q(z)z \partial_z + r(z)) u(z) = 0, \\ \lim_{z \rightarrow \infty} z^\lambda u(z) = 1. \end{cases} \quad (2.17)$$

**Proposition 2.9** *Let*

$$\left( \partial_z^2 + \frac{q(z)}{(z - z_0)} \partial_z + \frac{r(z)}{(z - z_0)^2} \right) \quad (2.18)$$

*have indices  $\rho_0, \tilde{\rho}_0$  at  $z_0$  and  $\rho_\infty, \tilde{\rho}_\infty$  at  $\infty$ . Then*

$$(z - z_0)^\mu \left( \partial_z^2 + \frac{q(z)}{(z - z_0)} \partial_z + \frac{r(z)}{(z - z_0)^2} \right) (z - z_0)^{-\mu} \quad (2.19)$$

*has at  $z_0$  indices  $\rho_0 + \mu, \tilde{\rho}_0 + \mu$  and at  $\infty$  indices  $\rho_\infty - \mu, \tilde{\rho}_\infty - \mu$ .*

**Proof.** We can assume that  $z_0 = 0$ . We use  $z^\mu \partial_z z^{-\mu} = \partial_z - \frac{\mu}{z}$ . Then (2.19) is

$$\begin{aligned} & \left( \partial_z - \frac{\mu}{z} \right)^2 + \frac{q(z)}{z} \left( \partial_z - \frac{\mu}{z} \right) + \frac{r(z)}{z^2} \\ &= \partial_z^2 - 2\frac{\mu}{z} \partial_z + \frac{\mu + \mu^2}{z^2} + \frac{q(z)}{z} \partial_z - \frac{q(z)\mu}{z^2} + \frac{r(z)}{z^2} \\ &= \partial_z^2 + \frac{(-2\mu + q(z))}{z} \partial_z + \frac{(\mu + \mu^2 - \mu q(z) + r(z))}{z^2}. \end{aligned}$$

Therefore, the indicial equation at 0 is

$$\lambda(\lambda - 1) + \lambda(q(0) - 2\mu) + \mu + \mu^2 - q(0)\mu + r(0) \quad (2.20)$$

$$= (\lambda - \mu)(\lambda - \mu - 1) + q(0)(\lambda - \mu) + r(0), \quad (2.21)$$

and the indicial equation at  $\infty$  is

$$\lambda(\lambda + 1) - \lambda(q(\infty) - 2\mu) + \mu + \mu^2 - q(\infty)\mu + r(\infty) \quad (2.22)$$

$$= (\lambda + \mu)(\lambda + \mu + 1) - q(\infty)(\lambda + \mu) + r(\infty). \quad (2.23)$$

□

**Theorem 2.10** *Suppose that we change the variables in the equations, considering a (holomorphic) map  $y \mapsto z(y)$ . Assume that  $y_0$  is mapped at  $z_0$  and  $\frac{\partial z}{\partial y}(y_0) \neq 0$ . Then the indices of the transformed equation coincide with the indices of the original equation.*

**Proof.** We will assume that the equation has the form (2.18) and  $z_0 = y_0 = 0$ . Let us compute the change of the differentiation operators:

$$\partial_z = \frac{1}{\frac{\partial z}{\partial y}} \partial_y, \quad (2.24)$$

$$\partial_z^2 = -\frac{\frac{\partial^2 z}{\partial y^2}}{\left(\frac{\partial z}{\partial y}\right)^3} \partial_y + \frac{1}{\left(\frac{\partial z}{\partial y}\right)^2} \partial_y^2. \quad (2.25)$$

Therefore,

$$\left( \partial_z^2 + \frac{q(z)}{z} \partial_z + \frac{r(z)}{z^2} \right) \quad (2.26)$$

$$= \frac{1}{\left(\frac{\partial z}{\partial y}\right)^2} \left( \partial_y^2 + \left( \frac{q(z(y)) \frac{\partial z(y)}{\partial y}}{z(y)} - \frac{\frac{\partial^2 z(y)}{\partial y^2}}{\frac{\partial z(y)}{\partial y}} \right) \partial_y + \frac{\left(\frac{\partial z}{\partial y}\right)^2}{z(y)^2} r(z(y)) \right) \quad (2.27)$$

$$= \frac{1}{\left(\frac{\partial z}{\partial y}\right)^2} \left( \partial_y^2 + \frac{\tilde{q}(y)}{y} \partial_y + \frac{\tilde{r}(y)}{y^2} \right) \quad (2.28)$$

Now it is easy to see that  $\tilde{q}(0) = q(0)$  and  $\tilde{r}(0) = r(0)$ .  $\square$

## 2.4 Equations with two regular-singular points on the Riemann sphere

**Example 2.11** *Every 2nd order equation that in  $\mathbb{C} \cup \{\infty\}$  has only regular points except for two regular-singular points at 0 and  $\infty$  has the form*

$$(z^2 \partial_z^2 + qz \partial_z + r)u(z) = 0. \quad (2.29)$$

*It is sometimes called the **homogeneous Euler equation**. Its indicial points are*

$$0: \quad \lambda(\lambda - 1) + q\lambda + r = 0,$$

$$\infty: \quad \lambda(\lambda + 1) - q\lambda + r = 0.$$

*If  $\rho, \tilde{\rho}$  are its indices at 0, then  $-\rho, -\tilde{\rho}$  are its indices at  $\infty$ . Its solutions are  $z^\rho, z^{\tilde{\rho}}$  if  $\rho \neq \tilde{\rho}$  and  $z^\rho, z^\rho \log z$  if  $\rho = \tilde{\rho}$ . The equation (2.29) can be rewritten as*

$$(z^2 \partial_z + (1 - \rho - \tilde{\rho})z \partial_z + \rho \tilde{\rho})u(z) = 0.$$

**Example 2.12** *Every 2nd order equation that in  $\mathbb{C} \cup \{\infty\}$  has only regular points except for two regular-singular points at  $z_1$  and  $z_2$  has the form*

$$\left( \partial_z^2 + \left( g_1(z - z_1)^{-1} + g_2(z - z_2)^{-1} \right) \partial_z + h(z - z_1)^{-2} (z - z_2)^{-2} \right) u(z) = 0, \quad (2.30)$$



where  $g_1 + g_2 = 2$ . Its indicial equations are

$$z_1 : \quad \lambda(\lambda - 1) + g_1\lambda + h(z_1 - z_2)^{-2} = 0,$$

$$z_2 : \quad \lambda(\lambda - 1) + g_2\lambda + h(z_1 - z_2)^{-2} = 0.$$

If  $\rho, \tilde{\rho}$  are indices at  $z_1$ , then  $-\rho, -\tilde{\rho}$  are indices at  $z_2$ . Solutions have the form  $(z - z_1)^\rho(z - z_2)^{-\rho}$ ,  $(z - z_1)^{\tilde{\rho}}(z - z_2)^{-\tilde{\rho}}$ , if  $\rho \neq \tilde{\rho}$  and  $(z - z_1)^\rho(z - z_2)^{-\rho}$ ,  $(z - z_1)^\rho(z - z_2)^{-\rho} \log(z - z_1)(z - z_2)^{-1}$ , if  $\rho = \tilde{\rho}$ .

Equation (2.30) can be rewritten as

$$\begin{aligned} & \left( \partial_z^2 + \left( (1 - \rho - \tilde{\rho})(z - z_1)^{-1} + (1 + \rho + \tilde{\rho})(z - z_2)^{-1} \right) \partial_z \right. \\ & \left. + \rho\tilde{\rho}(z_1 - z_2)^2(z - z_1)^{-2}(z - z_2)^{-2} \right) u(z) = 0. \end{aligned}$$

### 3 Systems 1st order equations

#### 3.1 Regular points

This subsection can be skipped.

We will discuss differential equations

$$\partial_z v(z) = A(z)v(z). \quad (3.31)$$

where  $A(z)$  is a matrix and  $v(z) \in \mathbb{C}^n$ .

**Definition 3.1** If  $A(z)$  is analytic at  $z_0$ , then we say that  $z_0$  is a regular point of (3.31).

**Theorem 3.2** Let  $\Omega$  be a connected simply connected open subset of  $\mathbb{C}$ . Let

$$\Omega \ni z \mapsto A(z) = \begin{bmatrix} a_{11}(z) & \dots & a_{1n}(z) \\ \dots & \dots & \dots \\ a_{n1}(z) & \dots & a_{nn}(z) \end{bmatrix}$$

be a holomorphic function with values in  $n \times n$  matrices and  $w = \begin{bmatrix} w_1 \\ \dots \\ w_n \end{bmatrix} \in \mathbb{C}^n$ .

Then there exists a unique holomorphic function  $\Omega \ni z \mapsto v(z) = \begin{bmatrix} v_1(z) \\ \dots \\ v_n(z) \end{bmatrix} \in \mathbb{C}^n$  that solves the problem

$$\begin{cases} \frac{dv(z)}{dz} = A(z)v(z), \\ v(z_0) = w. \end{cases} \quad (3.32)$$

**Proof.** Let us first restrict ourselves to a disk  $K(z_0, r)$  such that  $K(z_0, r)^{\text{cl}} \subset \Omega$ . We can also assume that  $z_0 = 0$ .

Let

$$A(z) = \sum_{k=0}^{\infty} A_k z^k$$

Then the series

$$v(z) := \sum_{k=0}^{\infty} v_k z^k,$$

where

$$\begin{cases} v_0 = w, \\ v_{m+1} := \frac{1}{m+1} \sum_{k=0}^m A_{m-k} v_k. \end{cases}$$

is the unique formal series solving (3.32).

Let us show that this series is convergent in  $K(0, r)$ . By the Cauchy inequality,

$$\|A_k\| \leq C r^{-k}.$$

If we set

$$\begin{cases} p_0 = \|w\| \\ p_{m+1} := \frac{1}{m+1} \sum_{k=0}^m C r^{-m+k} p_k, \end{cases}$$

then we can show inductively that

$$\|v_m\| \leq p_m. \tag{3.33}$$

Indeed, we have

$$\|v_0\| = p_0.$$

Assume that

$$\|v_k\| \leq p_k, \quad k = 0, \dots, m.$$

Then

$$\begin{aligned} \|v_{m+1}\| &\leq \frac{1}{m+1} \sum_{k=0}^m \|A_{m-k} v_k\| \\ &\leq \frac{1}{m+1} \sum_{k=0}^m \|A_{m-k}\| \|v_k\| \\ &\leq \frac{1}{m+1} \sum_{k=0}^m C r^{k-m} p_k = p_{m+1}. \end{aligned}$$

This proves (3.33).

If we subtract the formula

$$\begin{aligned} r(m+1)p_{m+1} &= \sum_{k=0}^m C r^{-m+k+1} p_k, \\ mp_m &= \sum_{k=0}^{m-1} C r^{-m+k+1} p_k, \end{aligned}$$

then we obtain

$$r(m+1)p_{m+1} = (Cr+m)p_m.$$

This immediately implies

$$\lim_{m \rightarrow \infty} \frac{p_{m+1}}{p_m} = r^{-1}.$$

Hence, by the d'Alembert criterion

$$\sum_{k=0}^{\infty} p_k z^k$$

is convergent in the disk  $K(0, r)$ . Therefore, so is

$$\sum_{k=0}^{\infty} v_k z^k$$

The above reasoning can be repeated for any disk contained in  $\Omega$ . In this way, since  $\Omega$  is connected, we can extend  $v(z)$  to the whole  $\Omega$ .  $\Omega$  is simply connected, and therefore the resulting function will be univalued.  $\square$

**Example 3.3**

$$(\partial_z - 1)v(z) = 0, \quad v(0) = 1.$$

We set

$$v(z) = \sum_{n=0}^{\infty} v_n z^n.$$

We obtain a recurrence relation

$$nv_n = v_{n-1}.$$

Therefore,

$$v(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}.$$

Obviously,  $v(z) = e^z$ .

**Example 3.4** Let  $\mu \in \mathbb{C}$ ,  $z \neq -1$

$$(\partial_z - \mu(z+1)^{-1})v(z) = 0, \quad v(0) = 1.$$

We set

$$v(z) = \sum_{n=0}^{\infty} v_n z^n.$$

We obtain a recurrence relation

$$nv_n = (\mu - n + 1)v_{n-1}.$$

Therefore,

$$v(z) = \sum_{n=0}^{\infty} \frac{\mu \dots (\mu - n + 1) z^n}{n!}, \quad |z| < 1.$$

Obviously,  $v(z) = (1+z)^\mu$ .

**Proof of Thm 2.4.** Define

$$v(z) := \begin{bmatrix} u(z) \\ u'(z) \end{bmatrix}, \quad w := \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$

and

$$A(z) := \begin{bmatrix} 0 & 1 \\ -d(z) & -c(z) \end{bmatrix}$$

Then (2.8) can be rewritten as

$$\begin{cases} \frac{dv(z)}{dz} = A(z)v(z), \\ v(z_0) = w. \end{cases}$$

We can apply Thm 3.2. □

**Definition 3.5** Assume that  $A(z)$  is defined for  $|z| > R$ . We say that  $\infty$  is a regular point of (3.31), if after the change of the variable  $w = z^{-1}$  we obtain a regular point at 0.

Obviously,  $\partial_z = -w^2 \partial_w$ . Hence, after the change of the variable (3.31) transforms into

$$\partial_w v(w^{-1}) = -w^{-2} A(w^{-1}) v(w^{-1}).$$

Therefore,  $\infty$  is a regular point iff there exists

$$\lim_{z \rightarrow \infty} z^2 A(z).$$

**Theorem 3.6** Let  $\infty$  be a regular point of (3.34). Then for any  $w \in \mathbb{C}^n$ , there exists a unique holomorphic solution satisfying

$$\begin{cases} \frac{dv(z)}{dz} = A(z)v(z), \\ \lim_{z \rightarrow \infty} v(z) = w. \end{cases} \quad (3.34)$$

## 3.2 Regular-singular points

**Definition 3.7** We say that the equation

$$\frac{dv(z)}{dz} = A(z)v(z) \quad (3.35)$$

has a regular-singular point at  $z_0$ , if  $A(z)$  has at  $z_0$  a pole of at most 1st order.

We can then rewrite (3.32) as

$$(z - z_0) \partial_z v(z) = B(z)v(z), \quad (3.36)$$

where  $B(z)$  is holomorphic around  $z_0$ . The eigenvalues of the matrix  $B(z_0)$  are called *indices of the singular point*  $z_0$ .

For simplicity, assume that  $z_0 = 0$ .

**Theorem 3.8 (Frobenius method for systems of equations)** *Let  $\Omega$  be a connected simply connected open subset of  $\mathbb{C}$  containing 0. Let*

$$\Omega \ni z \mapsto B(z) = \begin{bmatrix} b_{11}(z) & \dots & b_{1n}(z) \\ \vdots & \ddots & \vdots \\ b_{n1}(z) & \dots & b_{nn}(z) \end{bmatrix}$$

*be a holomorphic function with values in  $n \times n$  matrices. Let  $w \in \mathbb{C}^n$  and  $\lambda \in \mathbb{C}$  satisfy*

$$\begin{aligned} (B(0) - \lambda)w &= 0, \\ \lambda + m &\text{ is not an eigenvalue of } B(0) \text{ for } m = 1, 2, \dots \end{aligned} \tag{3.37}$$

*Then there exists a unique function  $\tilde{v}(z)$  holomorphic on  $\Omega$  such that  $v(z) := z^\lambda \tilde{v}(z)$  solves the problem*

$$\begin{cases} z \frac{dv(z)}{dz} = B(z)v(z), \\ \lim_{z \rightarrow 0} z^{-\lambda} v(z) = w. \end{cases} \tag{3.38}$$

**Proof.** Let us first consider a disc  $K(0, r)$  such that  $K(0, r)^{\text{cl}} \subset \Omega$ .

Let

$$B(z) = \sum_{k=0}^{\infty} B_k z^k$$

Then the series

$$v(z) := z^\lambda \sum_{k=0}^{\infty} v_k z^k,$$

where

$$\begin{cases} v_0 = w \\ v_m := (\lambda + m - B_0)^{-1} \sum_{k=0}^{m-1} B_{m-k} v_k. \end{cases}$$

is the unique formal series solving (3.38).

Let us show that this series is convergent in the disk  $K(0, r)$ . By the Cauchy inequality,

$$\|B_k\| \leq Cr^{-k}.$$

If we set

$$\begin{cases} p_0 = \|w\| \\ p_m := \|(\lambda + m - B_0)^{-1}\| \sum_{k=0}^{m-1} Cr^{-m+k} p_k, \end{cases}$$

then we can show by induction that

$$\|v_m\| \leq p_m.$$

If we subtract the formulas

$$\begin{aligned} r \left\| (\lambda + m + 1 - B_0)^{-1} \right\|^{-1} p_{m+1} &= \sum_{k=0}^m C r^{-m+k} p_k, \\ \left\| (\lambda + m - B_0)^{-1} \right\|^{-1} p_m &= \sum_{k=0}^{m-1} C r^{-m+k} p_k, \end{aligned}$$

then we obtain

$$r \left\| (\lambda + m + 1 - B_0)^{-1} \right\|^{-1} p_{m+1} = \left( C + \left\| (\lambda + m - B_0)^{-1} \right\|^{-1} \right) p_m.$$

It is easy to see that

$$\lim_{m \rightarrow \infty} m \left\| (\lambda + m - B_0)^{-1} \right\| = 1.$$

Hence,

$$\lim_{m \rightarrow \infty} \frac{p_{m+1}}{p_m} = r^{-1}.$$

Thus by the d'Alembert criterion, the series that defines  $\tilde{v}(z)$  is convergent in the disk  $K(0, r)$ .

Using Them 3.2 we can extend  $\tilde{v}(z)$  to the whole  $\Omega$ .  $\square$

**Example 3.9** *Let*

$$B = \begin{bmatrix} \lambda & \dots & & & \\ 1 & \lambda & \dots & & \\ & & \dots & & \\ & & & \lambda & \\ & & & & 1 & \lambda \end{bmatrix}.$$

*Consider the equation  $z\partial_z v(z) = Bv(z)$ . We obtain*

$$\begin{aligned} z\partial_z v_1 &= \lambda v_1, \\ v_1 + z\partial_z v_2 &= \lambda v_2, \\ &\dots \\ v_{n-1} + z\partial_z v_n &= \lambda v_n. \end{aligned}$$

*A basis of solution of this system is*

$$\begin{bmatrix} 0 \\ \dots \\ 0 \\ z^\lambda \\ z^\lambda \log z \\ \dots \\ z^\lambda (\log z)^{m-1} \end{bmatrix}, \quad m = 1, \dots, n.$$

**Example 3.10** The following equation has a regular-singular point at 0:

$$\partial_z v(z) = (az^{-1} + b)v(z).$$

its solution is  $v(z) = z^a e^{bz}$

**Proof of Thm 2.6** Define

$$v(z) := \begin{bmatrix} u(z) \\ zu'(z) \end{bmatrix}, \quad w := \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$$

and

$$B(z) := \begin{bmatrix} 0 & 1 \\ -c(z) & 1 - b(z) \end{bmatrix}.$$

We then have

$$\begin{aligned} B(z)v(z) &= \begin{bmatrix} zu'(z) \\ -c(z)u(z) - b(z)zu'(z) + zu'(z) \end{bmatrix}, \\ z\partial_z \begin{bmatrix} u(z) \\ zu'(z) \end{bmatrix} &= \begin{bmatrix} zu'(z) \\ z^2u''(z) + zu'(z) \end{bmatrix}, \\ z^{-\lambda}v(z) &= \begin{bmatrix} \tilde{u}(z) \\ z\tilde{u}'(z) + \lambda\tilde{u}(z) \end{bmatrix}. \end{aligned}$$

Hence (2.13) can be rewritten as

$$\begin{cases} z \frac{dv(z)}{dz} = B(z)v(z), \\ \lim_{z \rightarrow 0} z^{-\lambda}v(z) = w. \end{cases}$$

We can apply Thm 3.8. □

**Definition 3.11** Assume that  $B(z)$  is defined for  $|z| > R$ . We say that  $\infty$  is a regular-singular point of (3.35), if after the change of the variable  $w = z^{-1}$  we obtain a regular-singular point at 0.

Thus (3.36) has a regular-singular point if  $\lim_{z \rightarrow \infty} B(z)$  exists. The eigenvalues of  $-B(\infty)$  are called indices of  $\infty$ .

**Theorem 3.12** Let  $\Omega$  be a connected simply connected subset of  $\mathbb{C}$  containing  $\{|z| > R\}$ . Let

$$\Omega \ni z \mapsto B(z) = \begin{bmatrix} a_{11}(z) & \dots & a_{1n}(z) \\ & \dots & \\ a_{n1}(z) & \dots & a_{nn}(z) \end{bmatrix}$$

be a holomorphic function with values in  $n \times n$  matrices. Let  $w \in \mathbb{C}^n$  and  $\lambda \in \mathbb{C}$  satisfy

$$\begin{aligned} (B(\infty) + \lambda)w &= 0, \\ \lambda + m &\text{ is not an eigenvalue of } -B(\infty) \text{ for } m = 1, 2, \dots \end{aligned} \tag{3.39}$$

Then there exists a unique function  $\tilde{v}(z)$  holomorphic on  $\Omega$  such that  $v(z) := z^{-\lambda}\tilde{v}(z)$  solves

$$\begin{cases} z \frac{dv(z)}{dz} = B(z)v(z), \\ \lim_{z \rightarrow \infty} z^\lambda v(z) = w. \end{cases} \quad (3.40)$$

**Example 3.13** Every 1st order equation on the Riemann sphere possessing only regular points except of regular-singular points at  $z_1, z_2$  and  $\infty$  has the form

$$\partial_z v(z) = \left( a_1(z - z_1)^{-1} + a_2(z - z_2)^{-1} \right) v(z) \quad (3.41)$$

It has indices

$$z_1 : a_1, \quad z_2 : a_2, \quad \infty : -a_1 - a_2,$$

and a solution  $(z - z_1)^{a_1}(z - z_2)^{a_2}$ .

## 4 Hypergeometric equation

### 4.1 Riemann equations

**Lemma 4.1** Every 2nd order equation which on the Riemann sphere has only regular points except for 3 points at  $z_1, z_2$  and  $\infty$  is given by an operator of the form

$$\begin{aligned} & \partial_z^2 + \left( g_1(z - z_1)^{-1} + g_2(z - z_2)^{-1} \right) \partial_z \\ & + h_1(z - z_1)^{-2} + h_2(z - z_2)^{-2} + k(z - z_1)^{-1}(z - z_2)^{-1}. \end{aligned} \quad (4.42)$$

**Proof.** Consider

$$\partial_z^2 + c(z)\partial_z + d(z) \quad (4.43)$$

Clearly, if in  $\mathbb{C}$  the only singular points are at  $z_1, z_2$ , and they are regular-singular, then

$$c(z) = c_{\text{reg}}(z) + \frac{g_1}{z - z_1} + \frac{g_2}{z - z_2}, \quad (4.44)$$

$$d(z) = d_{\text{reg}}(z) + \frac{h_1}{(z - z_1)^2} + \frac{h_2}{(z - z_2)^2} + \frac{k_1}{z - z_1} + \frac{k_2}{z - z_2}. \quad (4.45)$$

where  $c_{\text{reg}}, d_{\text{reg}}$  are entire functions.  $\infty$  is a regular-singular point if the following limits also exist:

$$\lim_{z \rightarrow \infty} zc(z), \quad (4.46)$$

$$\lim_{z \rightarrow \infty} z^2 d(z). \quad (4.47)$$

(4.46) implies the existence of  $\lim_{z \rightarrow \infty} zc_{\text{reg}}(z)$ . Thus,  $zc_{\text{reg}}(z)$  is a bounded entire function. By the Liouville Theorem,  $zc_{\text{reg}}$  is a constant. But  $c_{\text{reg}}$  is also an entire function. Hence  $c_{\text{reg}} = 0$



(4.47) implies the existence of a limit  $\lim_{z \rightarrow \infty} zd(z)$ , which in turn implies the existence of  $\lim_{z \rightarrow \infty} zd_{\text{reg}}(z)$ . By the Liouville Theorem,  $zd_{\text{reg}}$  is a constant. But  $d_{\text{reg}}$  is also an entire function. Hence  $d_{\text{reg}} = 0$ .

Using again (4.47), knowing that  $d_{\text{reg}} = 0$ , we obtain  $k_1 + k_2 = 0$ .  $\square$

Let us transform (4.42) further, obtaining

$$\begin{aligned} & \partial_z^2 + \left( \frac{g_1}{(z - z_1)} + \frac{g_2}{(z - z_2)} \right) \partial_z \\ & + \frac{h_1(z_1 - z_2)}{(z - z_1)^2(z - z_2)} + \frac{h_2(z_2 - z_1)}{(z - z_2)^2(z - z_1)} + \frac{h}{(z - z_1)(z - z_2)}. \end{aligned} \quad (4.48)$$

with  $h = k - h_1 - h_2$ .

Suppose that the indices are

$$\begin{aligned} z_1 & : \quad \rho_1, \tilde{\rho}_1, \\ z_2 & : \quad \rho_2, \tilde{\rho}_2, \\ \infty & : \quad \rho_3, \tilde{\rho}_3. \end{aligned}$$

**Lemma 4.2** *The sum of indices of (4.42) is 1. The Riemann operator expressed in terms of the indices is*

$$\begin{aligned} & \mathcal{P} \begin{pmatrix} z_1 & z_2 & \infty & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix} \\ & = \partial_z^2 - \left( \frac{\rho_1 + \tilde{\rho}_1 - 1}{z - z_1} + \frac{\rho_2 + \tilde{\rho}_2 - 1}{z - z_2} \right) \partial_z \\ & + \frac{\rho_1 \tilde{\rho}_1 (z_1 - z_2)}{(z - z_1)^2 (z - z_2)} + \frac{\rho_2 \tilde{\rho}_2 (z_2 - z_1)}{(z - z_2)^2 (z - z_1)} + \frac{\rho_3 \tilde{\rho}_3}{(z - z_1)(z - z_2)} \end{aligned} \quad (4.49)$$

**Proof.** Its indicial equations are

$$\begin{aligned} z_1 & : \quad \lambda(\lambda - 1) + g_1 \lambda + h_1 = 0, \\ z_2 & : \quad \lambda(\lambda - 1) + g_2 \lambda + h_2 = 0, \\ \infty & : \quad \lambda(\lambda + 1) - (g_1 + g_2) \lambda + h = 0. \end{aligned}$$

By the Vieta equations

$$\begin{aligned} -1 + g_1 & = -\rho_1 - \tilde{\rho}_1, \\ -1 + g_2 & = -\rho_2 - \tilde{\rho}_2, \\ 1 - g_1 - g_2 & = -\rho_\infty - \tilde{\rho}_\infty. \end{aligned}$$

We sum up these equations.  $\square$

It is easy to generalize (4.49) to an arbitrary triple of points:

**Theorem 4.3** 1. Suppose that we are given a 2nd order differential equation on the Riemann sphere having 3 singular points  $z_1, z_2, z_3$ , all of them regular singular points with the following indices

$$\begin{aligned} z_1 &: \rho_1, \tilde{\rho}_1, \\ z_2 &: \rho_2, \tilde{\rho}_2, \\ z_3 &: \rho_3, \tilde{\rho}_3. \end{aligned}$$

Then the following condition is satisfied:

$$\rho_1 + \tilde{\rho}_1 + \rho_2 + \tilde{\rho}_2 + \rho_3 + \tilde{\rho}_3 = 1. \quad (4.50)$$

Such an equation, normalized to have coefficient 1 at the 2nd derivative, is always equal to

$$\mathcal{P} \begin{pmatrix} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix} \phi(z) = 0, \quad (4.51)$$

where

$$\begin{aligned} \mathcal{P} \begin{pmatrix} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix} &:= \partial_z^2 - \left( \frac{\rho_1 + \tilde{\rho}_1 - 1}{z - z_1} + \frac{\rho_2 + \tilde{\rho}_2 - 1}{z - z_2} + \frac{\rho_3 + \tilde{\rho}_3 - 1}{z - z_3} \right) \partial_z \\ &+ \frac{\rho_1 \tilde{\rho}_1 (z_1 - z_2)(z_1 - z_3)}{(z - z_1)^2 (z - z_2)(z - z_3)} + \frac{\rho_2 \tilde{\rho}_2 (z_2 - z_3)(z_2 - z_1)}{(z - z_2)^2 (z - z_3)(z - z_1)} + \frac{\rho_3 \tilde{\rho}_3 (z_3 - z_1)(z_3 - z_2)}{(z - z_3)^2 (z - z_1)(z - z_2)}. \end{aligned}$$

2. Let  $z \mapsto w(z) = \frac{az+b}{cz+d}$ . (Transformations of this form are called homographies or Möbius transformations). We can always assume that  $ad - bc = 1$ . Then

$$\mathcal{P} \begin{pmatrix} w(z_1) & w(z_2) & w(z_3) & \\ \rho_1 & \rho_2 & \rho_3 & w, \partial_w \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix} = (cz+d)^4 \mathcal{P} \begin{pmatrix} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix},$$

3.

$$\begin{aligned} &(z - z_1)^{-\lambda} (z - z_2)^{\lambda} \mathcal{P} \begin{pmatrix} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix} (z - z_1)^{\lambda} (z - z_2)^{-\lambda} \\ &= \mathcal{P} \begin{pmatrix} z_1 & z_2 & z_3 & \\ \rho_1 - \lambda & \rho_2 + \lambda & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 - \lambda & \tilde{\rho}_2 + \lambda & \tilde{\rho}_3 & \end{pmatrix}. \end{aligned}$$

Clearly, in all above formulas one of  $z_i$  can equal  $\infty$ , with an obvious meaning of various expressions.

## 4.2 Hypergeometric equation

By Thm 4.3 (2), we can assume that the points  $z_1, z_2, z_3$  are any triple of distinct points on the Riemann sphere. We choose them to be  $0, 1, \infty$ .

By Thm 4.3 (3), we can assume that  $\rho_1, \rho_2$  are arbitrary numbers. We choose them to be both 0. The sum of remaining indices must be 1. Hence, we have 3 parameters left. We set

0, indices:  $0, 1 - c$ ;

1, indices:  $0, c - a - b$ ;

$\infty$ , indices:  $a, b$ . Thus

$$\begin{aligned} & \mathcal{P} \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1 - c & c - a - b & b \end{pmatrix} z, \partial_z \\ &= \partial_z^2 - \left( \frac{1 - c - 1}{z} + \frac{c - a - b - 1}{z - 1} \right) \partial_z + \frac{ab}{z(z - 1)}. \end{aligned} \quad (4.52)$$

Define

$$\mathcal{F}(a, b; c; z, \partial_z) := z(1 - z) \mathcal{P} \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1 - c & c - a - b & b \end{pmatrix} z, \partial_z \quad (4.53)$$

$$= z(1 - z) \partial_z^2 + (c - (a + b + 1)z) \partial_z - ab. \quad (4.54)$$

Rewrite the equation

$$\mathcal{F}(a, b; c; z, \partial_z) F(z) = 0$$

in the form

$$(z^2 \partial_z^2 + (a + b + 1)z \partial_z + ab) F(z) = (z \partial_z^2 + c \partial_z) F(z). \quad (4.55)$$

Substituting  $F = \sum_{n=0}^{\infty} F_n z^n$  into (4.55) we obtain

$$\sum_{n=0}^{\infty} (n + a)(n + b) F_n z^n = \sum_{n=0}^{\infty} n(n + c - 1) F_n z^{n-1}. \quad (4.56)$$

This leads to the recurrence relation

$$(n + a)(n + b) F_n = F_{n+1} (n + 1)(n + c). \quad (4.57)$$

For  $a \in \mathbb{C}$  we define

$$(a)_n := a(a + 1) \cdots (a + n - 1).$$

The solution analytic at 0 and equal there 1 is the *hypergeometric function*

$$F(a, b; c; z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!},$$

$F(a, b; c; z)$  is defined for  $c \neq 0, -1, -2, \dots$ . Sometimes, it is more convenient to consider

$$\mathbf{F}(a, b; c; z) := \frac{F(a, b, c, z)}{\Gamma(c)} = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{\Gamma(c+j)} \frac{z^j}{j!}$$

defined for all  $a, b, c$ .

### 4.3 Solution $\sim z^{1-c}$ at 0

We have the identity

$$\begin{aligned} & z^{c-1} \mathcal{F}(a, b; c) z^{1-c} \\ &= \mathcal{F}(b+1-c, a+1-c; 2-c) \end{aligned}$$

Therefore, the solution of (4.54) behaving as  $z^{1-c}$  at zero is

$$z^{1-c} F(b+1-c, a+1-c; 2-c; z) \quad (4.58)$$

### 4.4 Solutions having definite behaviors at 1

$w = 1 - z$  is a substitution that exchanges 0 and 1:

$$\begin{aligned} & \mathcal{F}(a, b; c; z, \partial_z) := \\ &= \mathcal{F}(a, b; a+b+1-c; w, \partial_w). \end{aligned}$$

Therefore, the solution analytic at 1 and having there the value 1 is

$$F(a, b; a+b+1-c; 1-z).$$

There is also a solution behaving as  $(1-z)^{c-a-b}$  at 1:

$$(1-z)^{c-a-b} F(-b+c, -a+c; 1+c-a-b; 1-z).$$

### 4.5 Solutions having definite behaviors at $\infty$

$\infty$  is a regular-singular point with indices  $a, b$ .  $w = z^{-1}$  is the substitution that exchanges 0 and  $\infty$

$$\begin{aligned} & (-z)^{1+a} \mathcal{F}(a, b; c; z, \partial_z) (-z)^{-a} \\ &= \mathcal{F}(a, a-c+1; a-b+1; w, \partial_w). \end{aligned} \quad \begin{array}{l} (4.59) \\ (4.60) \end{array}$$

Hence, the solution that behaves at  $\infty$  as  $z^{-a}$  is

$$z^{-a} F(a, a-c+1; a-b+1; z^{-1}).$$

The second solution is obtained by exchanging  $a$  and  $b$ :

$$z^{-b} F(b-c+1, b; b-a+1; z^{-1}).$$

## 4.6 Identities

The following substitution does not move 0, and exchanges 1 and  $\infty$ :  $z \mapsto w = \frac{z}{z-1}$ . It leads to

$$\begin{aligned} & -(1-z)^{1+a} \mathcal{F}(a, b; c; z, \partial_z) (1-z)^{-a} \\ &= \mathcal{F}(a, c-b; c; w, \partial_w) \end{aligned} \quad (4.61)$$

An analogous identity is obtained if we exchange  $a$  and  $b$ . This yields

$$\begin{aligned} & F(a, b; c; z) \\ &= (1-z)^{c-a-b} F(c-a, c-b; c; z) \\ &= (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right) \\ &= (1-z)^{-b} F\left(c-a, b; c; \frac{z}{z-1}\right). \end{aligned}$$

## 4.7 Integral representations

**Theorem 4.4** *Let the curve  $[0, 1] \ni \tau \xrightarrow{\gamma} t(\tau)$  satisfy*

$$t^{a-c+1} (1-t)^{c-b} (t-z)^{-a-1} \Big|_{t(0)}^{t(1)} = 0.$$

Then

$$\int_{\gamma} t^{a-c} (1-t)^{c-b-1} (t-z)^{-a} dt \quad (4.62)$$

solves the hypergeometric equation.

**Proof.** We check that

$$\begin{aligned} & (z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab) t^{a-c} (1-t)^{c-b-1} (t-z)^{-a} \\ &= -a\partial_t t^{a-c+1} (1-t)^{c-b} (t-z)^{-a-1}. \end{aligned}$$

□

This implies the following representation of the hypergeometric function:

$$\begin{aligned} & \int_1^{\infty} t^{a-c} (t-1)^{c-b-1} (t-z)^{-a} dt \quad (4.63) \\ &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b; c; z), \quad \operatorname{Re}(c-b) > 0, \operatorname{Re}b > 0. \end{aligned} \quad (4.64)$$

Indeed, notice that (4.63) satisfies the assumptions of Thm 4.4, it is analytic around zero and at zero equals

$$\int_1^{\infty} t^{-c} (t-1)^{c-b-1} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}.$$

Setting  $z = 1$  in (4.63) we obtain

$$\int_1^\infty t^{a-c}(t-1)^{c-a-b-1}dt = \frac{\Gamma(c-a-b)\Gamma(b)}{\Gamma(c-a)}. \quad (4.65)$$

Therefore,

$$F(a, b; c; 1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \text{Re } c > \text{Re}(a+b). \quad (4.66)$$

## 5 Confluent equation

### 5.1 ${}_1F_1$ equation as a limit of the hypergeometric equation

Let  $a, c \in \mathbb{C}$ . The *confluent* or the  ${}_1F_1$  equation is given by the operator

$$\mathcal{F}(a; c; z, \partial_z) := z\partial_z^2 + (c - z)\partial_z - a. \quad (5.67)$$

The confluent equation is a limiting case of the hypergeometric equation:

$$\lim_{b \rightarrow \infty} \frac{1}{b} \mathcal{F}(a, b; c; z/b, \partial_{z/b}) = \mathcal{F}(a; c; z, \partial_z).$$

### 5.2 Confluent function

0 is a regular-singular point with indices 0,  $1 - c$ . The solution of the confluent equation analytic around 0 and equal 1 at 0 is

$$F(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(c)_n} z^n.$$

### 5.3 Solution $\sim z^{1-c}$ at 0

We have the identity

$$z^{c-1} \mathcal{F}(a; c) z^{1-c} = \mathcal{F}(1 + a - c; 2 - c).$$

Hence

$$z^{1-c} F(a - c + 1; 2 - c; z) = \sum_{n=0}^{\infty} \frac{(a - c + 1)_n}{n!(2 - c)_n} z^{1-c+n}.$$

is a solution of the confluent equation.

### 5.4 First Kummer's identity

Using  $e^{-z} \partial_z e^z = \partial_z + 1$  we obtain the identity

$$\begin{aligned} & e^{-z} (z\partial_z^2 + (c - z)\partial_z - a) e^z \\ &= z\partial_z^2 + (c + z)\partial_z + c - a. \end{aligned} \quad (5.68)$$

Substitute  $z = -w$  and multiply by  $-1$ , obtaining

$$w\partial_w^2 + (c - w)\partial_w - c + a.$$

Thus

$$e^{-z} \mathcal{F}(a; c; z, \partial_z) e^z = \mathcal{F}(c - a; c; w, \partial_w).$$

Hence  $e^z F(c - a; c; -z)$  is a solution of the confluent equation analytic around 0 and equal 1 at 0. We obtain the identity

$$F(a; c; z) = e^z F(c - a; c; -z). \quad (5.69)$$

## 5.5 Integral representations

If  $[0, 1] \ni \tau \mapsto s(\tau) \in \Omega$  is a curve and  $f$  is a function on  $\Omega$ , we introduce the notation

$$f \Big|_{\gamma(0)}^{\gamma(1)} := f(\gamma(1)) - f(\gamma(0)).$$

**Theorem 5.1** *Let the curve  $\gamma$  satisfy*

$$e^{zs} s^a (1-s)^{c-a} \Big|_{\gamma(0)}^{\gamma(1)} = 0. \quad (5.70)$$

Then

$$\int_{\gamma} e^{zs} s^{a-1} (1-s)^{c-a-1} ds \quad (5.71)$$

is a solution of the confluent equation

**Proof.**

$$\begin{aligned} & (z\partial_z^2 + (c-z)\partial_z - a)e^{zs} s^{a-1} (1-s)^{c-a-1} \\ = & ze^{zs} s^{a+1} (1-s)^{c-a-1} + (c-z)e^{zs} s^a (1-s)^{c-a-1} - ae^{zs} s^{a-1} (1-s)^{c-a-1} \\ = & -ze^{zs} s^a (1-s)^{c-a} - ae^{zs} s^{a-1} (1-s)^{c-a} + (c-a)e^{zs} s^a (1-s)^{c-a-1} \\ = & -\partial_s e^{zs} s^a (1-s)^{c-a}. \end{aligned}$$

□

Hence for  $\operatorname{Re} a > 0$ ,  $\operatorname{Re}(c-a) > 0$  we have

$$\int_0^1 e^{zs} s^{a-1} (1-s)^{c-a-1} ds = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F(a; c; z). \quad (5.72)$$

Indeed, the assumptions are satisfied, since the value of the function in (5.70) at 0 and 1 is zero. We obtain a solution of the confluent equation analytic around 0. We check that at zero it equals

$$\int_0^1 s^{a-1} (1-s)^{c-a-1} ds = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}.$$

## 5.6 Laguerre polynomials

For  $n = -a \in \{0, 1, 2, \dots\}$ ,  $F(-n; c; z)$  is an  $n$ th degree polynomial. These are the so-called Laguerre polynomials. They can be represented as an integral with  $\gamma$  encircling 0:

$$\begin{aligned} L_n^\alpha(z) & := \frac{(1+\alpha)_n}{n!} F(-n; 1+\alpha; z) \\ & = \frac{(-1)^n}{2\pi i} \int_{[0^+]} e^{tz} t^{-n-1} (1-t)^{\alpha+n} dt. \end{aligned}$$



## 5.7 The ${}_2F_0$ equation

Parallel to the  ${}_1F_1$  equation we will consider the  ${}_2F_0$  equation, given by the operator

$$\mathcal{F}(a, b; -; z, \partial_z) := z^2 \partial_z^2 + (-1 + (1 + a + b)z) \partial_z + ab, \quad (5.73)$$

where  $a, b \in \mathbb{C}$ . This equation is another limiting case of the hypergeometric equation:

$$\lim_{c \rightarrow \infty} \mathcal{F}(a, b; c; cz, \partial_{(cz)}) = -\mathcal{F}(a, b; -; z, \partial_z).$$

## 5.8 Point $\infty$ for the confluent equation

We have

$$\begin{aligned} & z^{a+1} (z \partial_z^2 + (c - z) \partial_z - a) z^{-a} \\ &= z^2 \partial_z^2 + z(-2a + c - z) \partial_z + a(1 + a - c). \end{aligned} \quad (5.74)$$

$$= z^2 \partial_z^2 + z(1 - a - b - z) \partial_z + ab, \quad (5.75)$$

where we set  $b := 1 + a - c$ . Substituting  $w = -z^{-1}$  (with the inverse  $z = -w^{-1}$ ), using  $\partial_z = w^2 \partial_w$ , we obtain that (5.75) is

$$w^2 \partial_w^2 + (-1 + (1 + a + b)w) \partial_w + ab.$$

We thus obtained the  ${}_2F_0$  equation. Note that 0 is an irregular singular point of this equation. Therefore,  $\infty$  is an irregular singular point of the confluent equation.

If

$$(w^2 \partial_w^2 + (-1 + (1 + a + b)w) \partial_w + ab)g(w) = 0,$$

then

$$(z \partial_z^2 + (c - z) \partial_z - a) z^{-a} g(-z^{-1}) = 0. \quad (5.76)$$

Conversely, if

$$(z \partial_z^2 + (c - z) \partial_z - a)f(z) = 0,$$

then

$$(w^2 \partial_w^2 + (-1 + (1 + a + b)w) \partial_w + ab)w^{-a} f(-w^{-1}) = 0.$$

## 5.9 Asymptotic series

Let function  $f$  be defined on  $K(z_0, r) \cap \{\alpha_1 < \arg(z - z_0) < \alpha_2\}$ . We write

$$f(z) \sim \sum_{j=0}^{\infty} a_j (z - z_0)^j,$$

if for any  $n$  there exists  $C_n$  such that

$$\left| f(z) - \sum_{j=0}^n a_j (z - z_0)^j \right| \leq C_n |z - z_0|^{n+1}.$$

Clearly, if  $f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j$  for  $z \in K(z_0, r)$ , then  $f(z) \sim \sum_{j=0}^{\infty} a_j(z-z_0)^j$ .

**Example.** For  $-\frac{\pi}{2} + \epsilon < \arg z < \frac{\pi}{2} - \epsilon$

$$e^{-\frac{1}{z}} \sim \sum_{j=0}^{\infty} 0z^j.$$

**Example.** For  $-\frac{\pi}{4} + \epsilon < \arg z < \frac{\pi}{4} - \epsilon$  and  $-\frac{\pi}{4} + \epsilon < \arg -z < \frac{\pi}{4} - \epsilon$

$$e^{-\frac{1}{z^2}} \sim \sum_{j=0}^{\infty} 0z^j.$$

In particular, all derivatives of  $\mathbb{R} \ni x \rightarrow e^{-\frac{1}{x^2}}$  at zero are zero.

**Example: Error Function.**

$$\operatorname{Erf}(z) := \int_0^z e^{-t^2} dt.$$

Clearly,  $\lim_{\operatorname{Re} z \rightarrow \infty} \operatorname{Erf}(z) = \frac{1}{2}\sqrt{\pi}$ . For  $-\frac{\pi}{2} + \epsilon < \arg z < \frac{\pi}{2} - \epsilon$  we easily show by integration by parts that

$$\frac{1}{2}\sqrt{\pi} - \operatorname{Erf}(z) = \int_z^{\infty} e^{-t^2} dt \sim \frac{e^{-z^2}}{2z} \left( 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{(2z^2)^k} \right).$$

## 5.10 ${}_2F_0$ function

We try to solve (5.76) with a power series

$$g(w) = \sum_{n=0}^{\infty} g_n w^n.$$

We obtain

$$\sum_{n=0}^{\infty} (n(n-1)g_n w^n - n g_n w^{n-1} + (1+a+b)n g_n w^n + a b g_n w^n) = 0$$

Hence

$$(n-1+a)(n-1+b)g_{n-1} = n g_n.$$

This gives the coefficients

$$g_n = \frac{(a)_n (b)_n}{n!} g_0$$

and leads to a divergent series.

**Theorem 5.2** *Let a contour  $\gamma$  satisfy*

$$e^{-t}t^a(1-wt)^{1-b}\Big|_{\gamma(0)}^{\gamma(1)} = 0 \quad (5.77)$$

*Then*

$$\int_{\gamma} e^{-t}t^{a-1}(1-wt)^{-b}dt$$

*is a solution of (5.76).*

**Proof.** By Thm 5.1

$$\int_{\gamma} e^{zs}s^{a-1}(1-s)^{c-a-1}ds$$

is a solution of the confluent. Therefore,

$$w^{-a} \int_{\gamma} e^{-sw^{-1}}s^{a-1}(1-s)^{c-a-1}ds,$$

for  $b = 1 + a - c$  is a solution of (5.76). Next we substitute  $t = \frac{s}{w}$ .  $\square$

For  $w \in \mathbb{C} \setminus [0, \infty[$ ,  $\text{Re } a > 0$  we define

$$F(a, b; -; w) := \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-t}t^{a-1}(1-wt)^{-b}dt. \quad (5.78)$$

For other values of  $a$  we extend (5.78) by analytic continuation. We have the following asymptotic expansion:

$$F(a, b; -; w) \sim \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!} w^n.$$

More precisely, for any  $n$ , for  $|\arg w| \geq \epsilon > 0$ ,

$$\lim_{w \rightarrow 0} w^{-n} \left( F(a, b; -; w) - \sum_{j=0}^n \frac{(a)_j (b)_j}{j!} w^j \right) = 0.$$

To prove this, we use the formula

$$f(z) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)z^j}{j!} + z^n \int_0^1 \frac{f^{(n)}(sz)n(1-s)^{n-1}}{n!} ds,$$

which implies

$$(1-z)^{-b} = \sum_{j=0}^{n-1} \frac{(b)_j z^j}{j!} + \frac{(b)_n z^n}{n!} \int_0^1 n(1-s)^{n-1}(1-zs)^{-b-n} ds.$$

Hence

$$\begin{aligned}
& F(a, b; -; w) \\
&= \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} (1-wt)^{-b} dt \\
&= \sum_{j=0}^{n-1} \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} \frac{(b)_j w^j t^j}{j!} dt \\
&\quad + \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} \frac{(b)_n w^n t^n}{n!} \int_0^1 (1-wts)^{-b-n} n(1-s)^{n-1} ds \\
&= \sum_{j=0}^{n-1} \frac{(b)_j \Gamma(a+j) w^j}{\Gamma(a) j!} \\
&\quad + \frac{w^n (b)_n}{\Gamma(a) n!} \int_0^1 n(1-s)^{n-1} ds \int_0^\infty e^{-t} t^{a-1+n} (1-wts)^{-b-n} dt \\
&= \sum_{j=0}^{n-1} \frac{(b)_j (a)_j w^j}{j!} \\
&\quad + \frac{w^n (b)_n (a)_n}{n!} \int_0^1 n(1-s)^{n-1} ds F(a+n, b+n; -; ws).
\end{aligned}$$

### 5.11 Solutions of the confluent equation with definite behavior at $\infty$

Consider the analytic function on the upper halfplane given by

$$s \mapsto e^{zs} s^{a-1} (1-s)^{c-a-1},$$

where we use the principal branch of power functions. Assume that  $\operatorname{Re} a > 0$ ,  $\operatorname{Re}(c-a) > 0$ . Remember that

$$F(z) = \int_0^1 e^{zs} s^{a-1} (1-s)^{c-a-1} ds = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F(a; c; z)$$

is one of solutions of the confluent equation. If  $\operatorname{Im} z > 0$ , then we can write the following solutions

$$\begin{aligned}
F_0(z) &= \int_0^{e^{i\phi}\infty} e^{zs} s^{a-1} (1-s)^{c-a-1} ds, \\
F_1(z) &= \int_1^{e^{i\phi}\infty} e^{zs} s^{a-1} (1-s)^{c-a-1} ds,
\end{aligned}$$

where  $\phi \in ]\frac{\pi}{2} - \arg z, \frac{3\pi}{2} - \arg z[$  guarantees that  $e^{zs}$  along the halfline where we integrate converges fast to zero (thus the appropriate condition is fulfilled). Notice that

$$F(z) + F_1(z) - F_0(z) = 0. \quad (5.79)$$

Substituting  $s = -z^{-1}t$ , where  $t \in [0, \infty[$ , for  $\text{Re} a > 0$  we obtain

$$\begin{aligned} F_0(z) &= \int_0^\infty e^{-t} (-tz^{-1})^{a-1} (1+z^{-1}t)^{c-a-1} (-z^{-1}) dt \\ &= (-z)^{-a} \Gamma(a) F(a, a+1-c; -, -z^{-1}). \end{aligned}$$

Substituting  $s = 1 - z^{-1}t$ , where  $t \in [0, \infty[$ , for  $\text{Re}(c-a) > 0$  we can write

$$\begin{aligned} F_1(z) &= -e^z \int_0^\infty e^{-t} (1-z^{-1}t)^{a-1} z^{-c+a} t^{c-a-1} dt \\ &= -e^z z^{-c+a} \Gamma(c-a) F(c-a, 1-a; -, z^{-1}). \end{aligned}$$

By (5.79), we obtain

$$\frac{F(a; c; z)}{\Gamma(c)} = (-z)^{-a} \frac{F(a, a+1-c; -, -z^{-1})}{\Gamma(c-a)} + z^{-c+a} \frac{e^z F(c-a, 1-a; -, z^{-1})}{\Gamma(a)}$$

## 5.12 Hydrogen atom

We transform the confluent operator

$$e^{-z/2} (z\partial_z^2 + (c-z)\partial_z - a) e^{z/2} \quad (5.80)$$

$$= z\partial_z^2 + c\partial_z + \frac{c}{2} - a - \frac{z}{4}; \quad (5.81)$$

Next,

$$z^{-(1-c)/2} \left( z\partial_z^2 + c\partial_z + \frac{c}{2} - a - \frac{z}{4} \right) z^{(1-c)/2} \quad (5.82)$$

$$= z\partial_z^2 + \partial_z - \frac{z}{4} + \frac{c}{2} - a - \frac{(1-c)^2}{4z}. \quad (5.83)$$

We divide (5.81) by  $z$  and substitute  $z = 2w$ . We obtain

$$\partial_w^2 + \frac{1}{w}\partial_w - 1 + (c-2a)\frac{1}{w} - \left(\frac{1-c}{2}\right)^2 \frac{1}{w^2}, \quad (5.84)$$

We multiply (5.84) with  $w^2$ . We obtain

$$w^2\partial_w^2 + w\partial_w - w^2 + (c-2a)w - \left(\frac{1-c}{2}\right)^2, \quad (5.85)$$

or the equation for the radial wave function for the Coulomb potential in dimension 2 (which is easy to transform into an analogous equation in higher dimensions).

Therefore, if  $f$  satisfies the confluent equation, then  $e^{-w} w^{(-1+c)/2} f(2w)$  satisfies (5.85).

## 6 Dzeta function

### 6.1 Riemann's dzeta function

For  $\text{Res} > 1$  Riemann's dzeta function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (6.86)$$

### 6.2 Prime numbers

Let  $\text{Res} > 1$ . We have

$$\zeta(s)(1 - 2^{-s}) = \frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \dots \quad (6.87)$$

Likewise, if  $p_1 = 2, p_2 = 3, p_3 = 5, \dots$  are the prime numbers in the increasing order, then

$$\zeta(s)(1 - p_1^{-s}) \cdots (1 - p_n^{-s}) = \sum_{k \in X_n} \frac{1}{k^s}, \quad (6.88)$$

where  $X_n$  is the set of positive integers not divisible by  $p_1, \dots, p_n$ . Therefore,

$$\lim_{n \rightarrow \infty} \zeta(s) \prod_{j=1}^n (1 - p_j^{-s}) = 1. \quad (6.89)$$

Hence we have the following identity

$$\prod_{n=j}^{\infty} (1 - p_j^{-s}) = \frac{1}{\zeta(s)}. \quad (6.90)$$

**Proposition 6.1**  $\zeta \neq 0$  for  $\text{Res} > 1$ .

**Proof.** (6.90) is an absolutely convergent product, because

$$p_j^{-s} \leq j^{-s},$$

and  $\sum j^{-s} < \infty$ . All its factors are nonzero. Hence (6.90) is nonzero.  $\square$

### 6.3 Holomorphic extension of dzeta function

**Theorem 6.2** For any  $s$  with  $\text{Res} > 1$  we have

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-x}}{1 - e^{-x}} dx. \quad (6.91)$$

$\zeta$  extends to a holomorphic function on  $\mathbb{C} \setminus \{0\}$ . It has the following integral representation valid for all  $s$  except for  $s = 1, 2, \dots$  (because of singularities of the Gamma function):

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{]-\infty, 0^+, -\infty[} \frac{z^{s-1} e^z}{1 - e^z} dz. \quad (6.92)$$

**Proof.** (6.91) follows by summing up

$$\int_0^{\infty} x^{s-1} e^{-nx} dx = \Gamma(s) \frac{1}{n^s}. \quad (6.93)$$

(6.92) follows by summing up

$$\frac{1}{2\pi i} \int_{]-\infty, 0^+, -\infty[} z^{s-1} e^{nz} dz = \frac{1}{\Gamma(1-s)n^s}. \quad (6.94)$$

(6.92) is holomorphic on the whole  $\mathbb{C}$  except maybe at the singularities of  $\Gamma(1-s)$ , which are  $1, 2, \dots$ . But we already know that  $\zeta$  is holomorphic for  $\text{Res} > 1$ . Hence the only singularity can be at 1.  $\square$

**Theorem 6.3**

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s). \quad (6.95)$$

**Proof.**

$$\zeta(s) = \sum_{n=1}^{\infty} \int_n^{\infty} \frac{1}{x^{s+1}} dx \quad (6.96)$$

$$= \sum_{n=1}^{\infty} \int_n^{n+1} \frac{n}{x^{s+1}} dx \quad (6.97)$$

$$= s \int_1^{\infty} \frac{1}{x^s} dx + \sum_{n=1}^{\infty} \int_n^{n+1} \frac{n-x}{x^{s+1}} dx. \quad (6.98)$$

Now, setting  $s = 1$ , we get

$$\int_n^{n+1} \frac{n-x}{x^2} dx = \frac{1}{n+1} - \log(n+1) + \log(n). \quad (6.99)$$

Therefore,

$$\lim_{s \searrow 0} \left( \zeta(s) - \frac{1}{s-1} \right) \quad (6.100)$$

$$= \lim_{s \searrow 0} \left( \frac{s}{s-1} + \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \log(n+1) + \log(n) \right) - \frac{1}{s-1} \right) \quad (6.101)$$

$$= \lim_{s \searrow 0} \sum_{n=1}^{\infty} \left( \frac{1}{n} - \log(n+1) + \log(n) \right) = \gamma. \quad (6.102)$$

$\square$

## 6.4 Bernoulli numbers

The Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \quad (6.103)$$

The function

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \coth\left(\frac{x}{2}\right) \quad (6.104)$$

is even. Hence for odd  $n$  we have  $B_n = 0$  except for  $B_1 = -\frac{1}{2}$ . Otherwise,  $B_0 = 1$ ,  $B_2 = \frac{1}{6}$ , etc. We also have

$$x \coth(x) = \sum_{k=0}^{\infty} B_{2k} \frac{(2x)^{2k}}{(2k)!}, \quad (6.105)$$

$$x \cot(x) = \sum_{k=0}^{\infty} (-1)^k B_{2k} \frac{(2x)^{2k}}{(2k)!}. \quad (6.106)$$

**Theorem 6.4** *For positive even integers the dzeta function can be expressed in terms of Bernoulli numbers:*

$$\zeta(2k) = \frac{(-1)^{k+1} 2^{2k-1} \pi^{2k}}{(2k)!} B_{2k}. \quad (6.107)$$

*For all negative integers the dzeta function can be expressed in terms of Bernoulli numbers:*

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}. \quad (6.108)$$

*(In particular, for even negative integers the dzeta function is zero).*

**Proof.** To prove (6.107) we use

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}. \quad (6.109)$$

By (6.109),

$$x \cot x = 1 + 2 \sum_{n=1}^{\infty} \frac{x^2}{x^2 - n^2 \pi^2} \quad (6.110)$$

$$= 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^{2k}}{n^{2k} \pi^{2k}} \quad (6.111)$$

$$= 1 - 2 \sum_{k=1}^{\infty} \frac{x^{2k}}{\pi^{2k}} \zeta(2k). \quad (6.112)$$

(6.108) follows from (6.92), which for  $s = -n$  can be rewritten as

$$\zeta(-n) = \frac{n!}{2\pi i} \int_{[0^+]} \frac{z^{-1-n}}{e^{-z} - 1} dz. \quad (6.113)$$



## 6.5 Riemann's reflection formula

### Theorem 6.5

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{\pi}{2}s\right) \zeta(1-s). \quad (6.114)$$

or equivalently

$$\zeta(s) \Gamma(s) \cos\left(\frac{\pi}{2}s\right) = 2^{s-1} \pi^s \zeta(1-s). \quad (6.115)$$

**Proof.** Assume that  $\text{Re } s < 0$ . The function  $\frac{z^{s-1}e^z}{1-e^z}$  has simple poles at  $z \in i2\pi\mathbb{Z} \setminus \{0\}$ . We compute the residues:

$$\text{Res} \frac{z^{s-1}e^z}{1-e^z} \Big|_{z=i2\pi n} = -(i2\pi n)^{s-1}. \quad (6.116)$$

Now for  $n > 0$ ,

$$(i2\pi n)^{s-1} + (i2\pi(-n))^{s-1} \quad (6.117)$$

$$= (2\pi)^{s-1} (e^{i\frac{\pi}{2}(s-1)} + e^{-i\frac{\pi}{2}(s-1)}) n^{s-1} \quad (6.118)$$

$$= 2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right) n^{s-1}. \quad (6.119)$$

On  $\mathbb{C} \setminus ]-\infty, 0]$ , treated as the domain of  $z^s$ , we consider the circle of radius  $(2N+1)\pi$  and centered at 0. We treat it as a curve  $\gamma_N$  starting at  $-(2N+1)\pi - i0$  and ending at  $-(2N+1)\pi + i0$ . Let  $\delta_N := [-(2N+1)\pi - i0, 0^+, -(2N+1)\pi + i0]$ . Then

$$\frac{1}{2\pi i} \int_{\gamma_N} \frac{z^{s-1}e^z}{1-e^z} dz - \frac{1}{2\pi i} \int_{\delta_N} \frac{z^{s-1}e^z}{1-e^z} dz \quad (6.120)$$

$$= - \sum_{n=1}^N 2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right) n^{s-1} \quad (6.121)$$

$$\rightarrow -2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right) \zeta(1-s). \quad (6.122)$$

But on  $\gamma_N$

$$\left| \frac{e^z}{1-e^z} \right| < K \quad (6.123)$$

$$|z^{s-1}| < |(2N+1)\pi|^{\text{Re } s - 1} \quad (6.124)$$

Hence the first term of (6.120) converges to 0. Clearly, the second term of (6.120) converges to  $-\zeta(s)$ .

We pass from (6.114) to (6.115) by  $\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$  and  $\sin \pi s = 2 \sin\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{2}s\right)$ .  $\square$

Applying  $\Gamma(s) = \pi^{-\frac{1}{2}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2} + \frac{1}{2}\right)$  and  $\Gamma\left(\frac{s}{2} + \frac{1}{2}\right) \Gamma\left(-\frac{s}{2} + \frac{1}{2}\right) = \frac{\pi}{\cos\left(\frac{\pi}{2}s\right)}$  we obtain

$$\Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{s-\frac{1}{2}} \Gamma\left(-\frac{s}{2} + \frac{1}{2}\right) \zeta(1-s). \quad (6.125)$$

If we introduce

$$\eta(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s), \quad (6.126)$$

then (6.125) can be written in a symmetric way:

$$\eta(s) = \eta(1 - s). \quad (6.127)$$

$\eta$  has only two singularities: at 0 and 1. The following function is entire

$$\xi(s) := \frac{1}{2} s(s-1) \eta(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s), \quad (6.128)$$

and it also satisfies the symmetry

$$\xi(s) = \xi(1 - s). \quad (6.129)$$

Note that

$$\log(\xi(s)) = \log\left(\frac{s(-1+s)}{2}\right) + \log\left(\Gamma\left(\frac{s}{2}\right)\right) - \frac{s}{2} \log(\pi) + \log(\zeta(s)). \quad (6.130)$$

## 6.6 Poisson summation formula

In this subsection we adopt the following convention for Fourier transform:

$$\hat{f}(\xi) := \int f(x) e^{-i2\pi x \xi}. \quad (6.131)$$

**Theorem 6.6** *If  $f \in L^1$  and  $\sum_j |\hat{f}(j)| < \infty$ , then*

$$\sum_{k=-\infty}^{\infty} f(k) = \sum_{j=-\infty}^{\infty} \hat{f}(j). \quad (6.132)$$

**Proof.** Let  $j \in \mathbb{Z}$ .

$$\int_{-n-\frac{1}{2}}^{n+\frac{1}{2}} f(x) e^{-i2\pi x j} dx = \sum_{k=-n}^n \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+k) e^{-2\pi i j x} dx. \quad (6.133)$$

Define

$$g(x) := \lim_{n \rightarrow \infty} \sum_{k=-n}^n f(x+k), \quad (6.134)$$

which is a periodic integrable function with period 1. We have

$$\hat{f}(j) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x) e^{-2\pi i j x} dx. \quad (6.135)$$

By the inversion of the Fourier transformation for periodic functions we obtain

$$g(x) = \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{2\pi i j x}. \quad (6.136)$$

Setting  $x = 0$  we obtain

$$\sum_{k=-\infty}^{\infty} f(k) = g(0) = \sum_{j=-\infty}^{\infty} \hat{f}(j). \quad (6.137)$$

□

## 6.7 2nd proof of Riemann's reflection formula

Define

$$\phi(x) := \sum_{n=1}^{\infty} e^{-n^2 \pi x}. \quad (6.138)$$

Note that

$$1 + 2\phi(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x} = \theta_3(0, e^{-x}), \quad (6.139)$$

$$\int e^{-y^2 \pi x} e^{-i2\pi y \xi} dy = \frac{1}{\sqrt{x}} e^{-\frac{\xi^2 \pi}{x}}. \quad (6.140)$$

Hence the Poisson summation formula implies

$$1 + 2\phi(x) = \frac{1}{\sqrt{x}} \left( 1 + 2\phi\left(\frac{1}{x}\right) \right) \quad (6.141)$$

Clearly,

$$0 \leq \phi(x) \leq ce^{-x}. \quad (6.142)$$

Recall that

$$\eta(s) := \frac{\Gamma(\frac{s}{2})\zeta(s)}{\pi^{\frac{s}{2}}}. \quad (6.143)$$

**Theorem 6.7** *We have the identities*

$$\eta(s) = \int_0^{\infty} \frac{dx}{x} \phi(x) x^{\frac{s}{2}} \quad (6.144)$$

$$= \int_1^{\infty} \frac{dx}{x} \phi(x) \left( x^{\frac{s}{2}} + x^{\frac{1-s}{2}} \right) - \frac{1}{s(1-s)} \quad (6.145)$$

$$= \int_1^{\infty} \frac{dx}{x} \left( \phi(x) x^{\frac{1}{4}} - \frac{1}{2} x^{-\frac{1}{4}} \right) \left( x^{\frac{1}{2}(s-\frac{1}{2})} + x^{-\frac{1}{2}(s-\frac{1}{2})} \right) \quad (6.146)$$

$$= \int_0^{\infty} \frac{1}{2} \frac{dx}{x} \left( \phi(x) x^{\frac{1}{4}} - \frac{1}{2} x^{-\frac{1}{4}} \right) \left( x^{\frac{1}{2}(s-\frac{1}{2})} + x^{-\frac{1}{2}(s-\frac{1}{2})} \right) \quad (6.147)$$

*The first is valid for  $\text{Res} > 0$ , the second for all  $s$  and the two last for  $0 < \text{Res} < 1$ .*

*In particular, by (6.145),  $\eta$  extends analytically to the complex plane except for  $0, 1$  and satisfies the symmetry  $\eta(s) = \eta(1-s)$ .*

**Proof.**

$$\frac{1}{n^s} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) = \int_0^\infty \frac{dx}{x} e^{-n^2 \pi x} x^{\frac{s}{2}}. \quad (6.148)$$

This implies (6.144). Now, for  $\text{Res} > 1$ ,

$$\begin{aligned} \int_0^\infty \frac{dx}{x} \phi(x) x^{\frac{s}{2}} &= \int_1^\infty \frac{dx}{x} \phi(x) x^{\frac{s}{2}} + \int_1^\infty \frac{dx}{x} \phi\left(\frac{1}{x}\right) x^{-\frac{s}{2}} \\ &= \int_1^\infty \frac{dx}{x} \phi(x) \left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right) + \frac{1}{2} \int_1^\infty \frac{dx}{x} \left(x^{\frac{1-s}{2}} - x^{-\frac{s}{2}}\right) \end{aligned}$$

But

$$\frac{1}{2} \int_1^\infty \frac{dx}{x} \left(x^{\frac{1-s}{2}} - x^{-\frac{s}{2}}\right) = \frac{1}{s-1} - \frac{1}{s} = -\frac{1}{s(1-s)}. \quad (6.149)$$

This implies (6.145).

For  $0 < \text{Res} < 1$ ,

$$-\frac{1}{2} \int_1^\infty \frac{dx}{x} \left(x^{\frac{s-1}{2}} + x^{-\frac{s}{2}}\right) = -\frac{1}{1-s} - \frac{1}{s} = -\frac{1}{s(1-s)}. \quad (6.150)$$

Hence (6.145) implies (6.146)

(6.146) implies (6.147) by (6.141).  $\square$

## 6.8 Zeros of the dzeta function

Let  $\frac{1}{2} + iZ$  be the set of all zeros of the dzeta function away from  $\mathbb{R}$ .  $Z$  coincides with the set of zeros of  $\xi$ . Note that  $Z = \bar{Z} = -Z$ . The Riemann hypothesis says that  $Z \subset \mathbb{R}$ .

Let  $\frac{1}{2} + iZ_+$  be the set of zeros of the dzeta function with positive imaginary part. Clearly,  $Z = Z_+ \cup (-Z_+)$ .

By the Hadamard Theorem,

$$\xi(s) = \prod_{\lambda \in Z} \left(1 - \frac{s}{\frac{1}{2} + i\lambda}\right) \quad (6.151)$$

$$= \prod_{\lambda \in Z_+} \left(1 - \frac{s}{\frac{1}{2} + i\lambda}\right) \left(1 - \frac{s}{\frac{1}{2} - i\lambda}\right) \quad (6.152)$$

$$= \prod_{\lambda \in Z_+} \left(1 - \frac{s(1-s)}{\frac{1}{4} + \lambda^2}\right). \quad (6.153)$$

Hence

$$\log(\xi(s)) = \sum_{\lambda \in Z} \log\left(1 - \frac{s}{\frac{1}{2} + i\lambda}\right). \quad (6.154)$$

## 6.9 Dzeta function as Mellin transform

Introduce

$$\pi(x) := \#\{\text{primes } \leq x\}, \quad (6.155)$$

$$\Pi(x) = \sum_{m=1}^{\infty} \frac{1}{m} \pi\left(x \frac{1}{m}\right). \quad (6.156)$$

Then

$$\log \zeta(s) = - \sum_p \log(1 - p^{-s}) \quad (6.157)$$

$$= - \sum_p \sum_{m=1}^{\infty} \frac{p^{-sm}}{m}, \quad (6.158)$$

$$= - \int x^{-s} \Pi'(x) dx. \quad (6.159)$$

For any  $a > 0$ , we can write  $\Pi'$  as the Fourier transform

$$\log \zeta(a + it) = - \int_0^{\infty} x^{-a-it} \Pi'(x) dx \quad (6.160)$$

$$= - \int e^{-au-itu} \Pi'(e^u) e^u du. \quad (6.161)$$

Inverting the Fourier transform we obtain

$$\Pi'(x) = - \frac{x^{a-1}}{2\pi} \int x^{it} \log \zeta(a + it) dt. \quad (6.162)$$

We have for  $\text{Re} \lambda > 0$

$$\int e^{it\xi} \log(it + \lambda) dt = -2\pi i \frac{e^{-\xi\lambda}}{|\xi|_+}. \quad (6.163)$$

Hence, for  $\text{Re} \lambda > a$ ,

$$- \frac{x^{a-1}}{2\pi} \int x^{it} \log(it + a - \lambda) dt = \frac{x^{\lambda-1}}{|\log x|_+}. \quad (6.164)$$

Using now (6.154), Riemann derived the formula

$$\Pi(x) = \text{Li}(x) - \sum_{\lambda \in \mathbb{Z}_+} \text{Li}(x^\lambda) - \log(2) + \int_x^{\infty} \frac{dt}{t(t^2 - 1) \log(t)}. \quad (6.165)$$

Here

$$\text{Li}(x) := \int_0^x \frac{dt}{\log(t)}. \quad (6.166)$$

Introduce the Mellin transformation:

$$(\mathcal{M}f)(t) := \int_0^{\infty} x^{-\frac{1}{2}-it} f(x) dx, \quad (6.167)$$

$$(\mathcal{M}^{-1}g)(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{-\frac{1}{2}+it} g(t) dt. \quad (6.168)$$

### 6.10 The Hurwitz dzeta function

For  $\operatorname{Re} s > 0$  and  $a \notin \{\dots, -2, -1, 0\}$ , we define

$$\zeta(s, a) := \sum_{n=1}^{\infty} \frac{1}{(a+n)^s}. \quad (6.169)$$

**Theorem 6.8** *For any  $s, a$  with  $\operatorname{Re} s > 0$  and  $\operatorname{Re} a > 0$ , we have*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-ax}}{1 - e^{-x}} dx. \quad (6.170)$$

$s \mapsto \zeta(s, a)$  extends to a holomorphic function on  $\mathbb{C} \setminus \{0\}$ . It has the following integral representation valid for all  $s$  except for  $s = 0, 1, 2, \dots$ :

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{]-\infty, 0^+, -\infty[} \frac{z^{s-1} e^{az}}{1 - e^z} dz. \quad (6.171)$$

For  $s \approx 0$  we have

$$\zeta(s) = \frac{1}{s-1} + O(s^0). \quad (6.172)$$

### 6.11 The Hurwitz identity

**Theorem 6.9** *Let  $0 < \operatorname{Re} a \leq 1$ . Then*

$$\zeta(s, a) = 2^s \pi^{s-1} \Gamma(1-s) \sum_{n=1}^{\infty} \frac{\sin(2\pi na + \frac{\pi}{2}s)}{n^{1-s}}. \quad (6.173)$$