EXCITATION SPECTRUM OF INTERACTING BOSONS IN THE MEAN FIELD INFINITE VOLUME LIMIT

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We show that low lying excitation spectrum of N -body bosonic Schrödinger Hamiltonians with repulsive interaction is approximately given by the Bogoliubov approximation. We consider the limit $N \to \infty$, weak coupling and large density. We allow for an arbitrarily large size of a box provided that it does not grow too fast with N.

We start with a potential that is a real function v on $\mathbb{R}^d$ such that $v(\mathrm{x}) = v(-\mathrm{x})$ and

$$v \in L^1(\mathbb{R}^d), \quad \hat{v} \in L^1(\mathbb{R}^d),$$

$$v(\mathrm{x}) \geq 0, \ \ \mathrm{x} \in \mathbb{R}^d, \qquad \hat{v}(\mathrm{p}) \geq 0, \ \ \mathrm{p} \in \mathbb{R}^d.$$

Then we replace the original v by the periodized potential

$$v^L(\mathrm{x}) = \frac{1}{L^d} \sum_{\mathrm{p} \in (2\pi/L)\mathbb{Z}^d} \mathrm{e}^{\mathrm{ipx}} \hat{v}(\mathrm{p}),$$

which is well defined on the torus $[-L/2, L/2[^d$.

We use the symmetric N-particle Hilbert space

$$L^2_{\mathrm{s}}\Big(\big([-L/2, L/2[^d\big)^N\Big)$$

and the periodic boundary conditions indicated by L.

Momentum

$$P_N^L := -\sum_{i=1}^{N} \mathrm{i}\partial^L_{\mathrm{x}_i}.$$

Hamiltonian

$$H_N^L = -\sum_{i=1}^{N} \Delta_i^L + \frac{L^d}{N} \sum_{1 \le i < j \le N} v^L(\mathrm{x}_i - \mathrm{x}_j).$$

In the sequel, we drop the superscript L.

Note that $\operatorname{spec} P_N = \frac{2\pi}{L}\mathbb{Z}^d$ and $H_N P_N = P_N H_N$. Hence

$$H_N = \bigoplus_{\mathrm{k} \in \operatorname{spec} P_N} H_N(\mathrm{k}).$$

We can define the energy-momentum spectrum

$$\operatorname{spec}(H_N, P_N).$$

We will denote by E_N the ground state energy of H_N. By the excitation spectrum we will mean

$$\operatorname{spec}(H_N - E_N, P_N).$$

We introduce the Bogoliubov energy

$$E_{\mathrm{Bog}} := -\frac{1}{2} \sum_{\mathrm{p} \in \frac{2\pi}{L}\mathbb{Z}^d \setminus \{0\}} \left(|\mathrm{p}|^2 + \hat{v}(\mathrm{p}) - |\mathrm{p}| \sqrt{|\mathrm{p}|^2 + 2\hat{v}(\mathrm{p})} \right)$$

and the Bogoliubov dispersion relation

$$e_{\mathrm{p}} = |\mathrm{p}| \sqrt{|\mathrm{p}|^2 + 2\hat{v}(\mathrm{p})}.$$

Bogoliubov Hamiltonian

$$H_{\mathrm{Bog}} := E_{\mathrm{Bog}} + \sum_{\mathrm{p}\neq 0} e_{\mathrm{p}} a_{\mathrm{p}}^{\dagger} a_{\mathrm{p}},$$

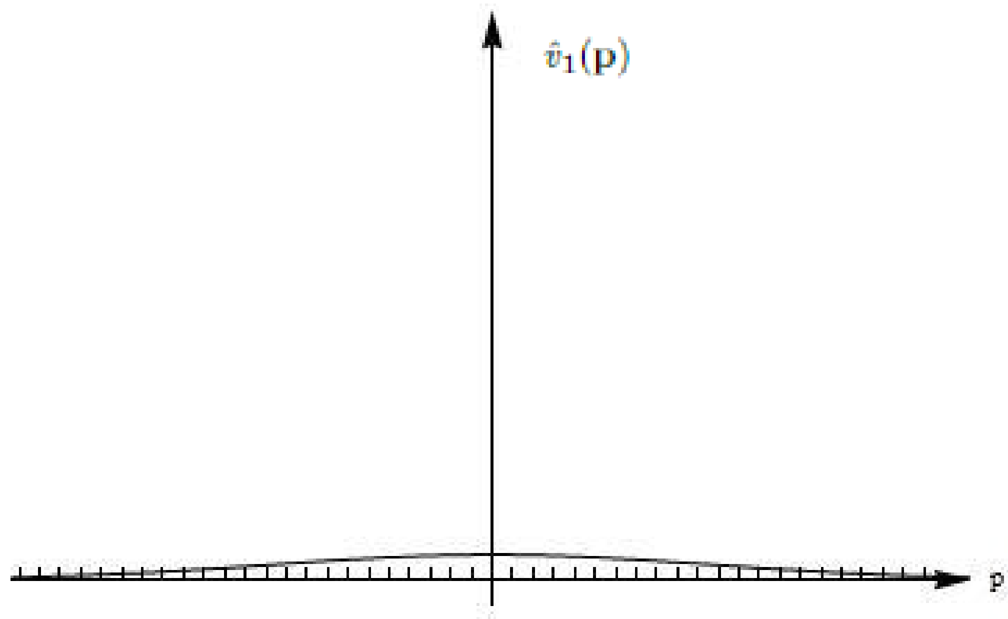
Bogoliubov momentum

$$P_{\mathrm{Bog}} := \sum_{\mathrm{p}\neq 0} \mathrm{p}\, a_{\mathrm{p}}^{\dagger} a_{\mathrm{p}},$$

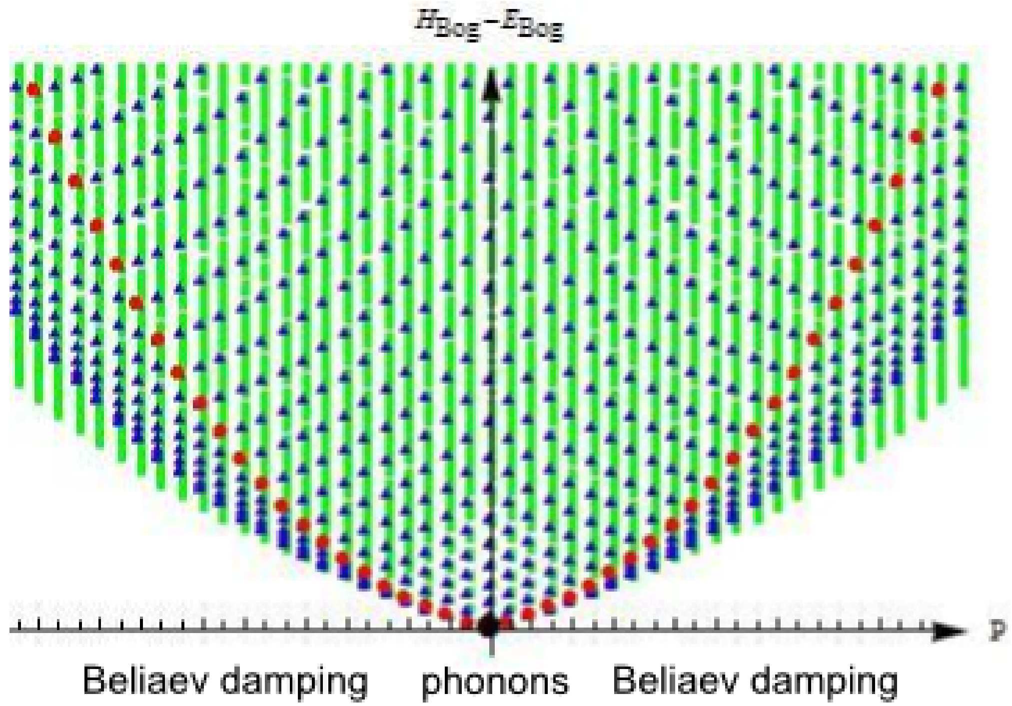
Clearly, $H_{\mathrm{Bog}} P_{\mathrm{Bog}} = P_{\mathrm{Bog}} H_{\mathrm{Bog}}$.

Above, $a_{\mathrm{p}}^{\dagger}$ and a_{p} are bosonic creation/annihilation operators on the bosonic Fock space $\Gamma_{\mathrm{s}}\Big(L^2\big(\mathrm{spec}\,(P_N)\backslash 0\big)\Big)$.

We would like to show that the excitation spectrum of H_N is well approximated by the excitation spectrum of the Bogoliubov Hamiltonian. In the examples below we ilustrate that the latter has a special shape involving a positive critical velocity, which according to the Landau criterion is responsible for superfluidity.

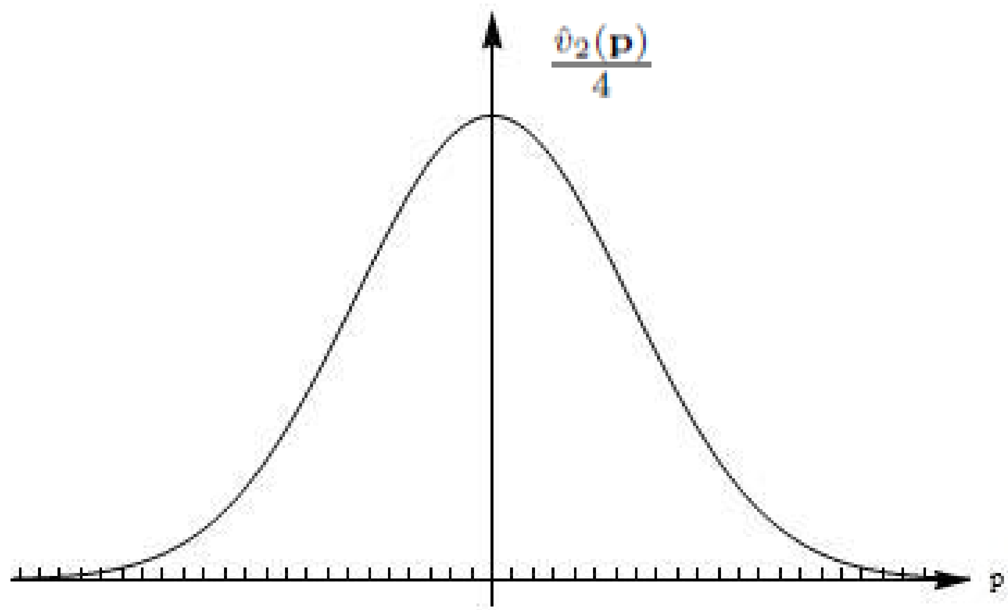


$$\hat{v}_1(\mathrm{p}) = \frac{\mathrm{e}^{-\mathrm{p}^2/5}}{10}$$

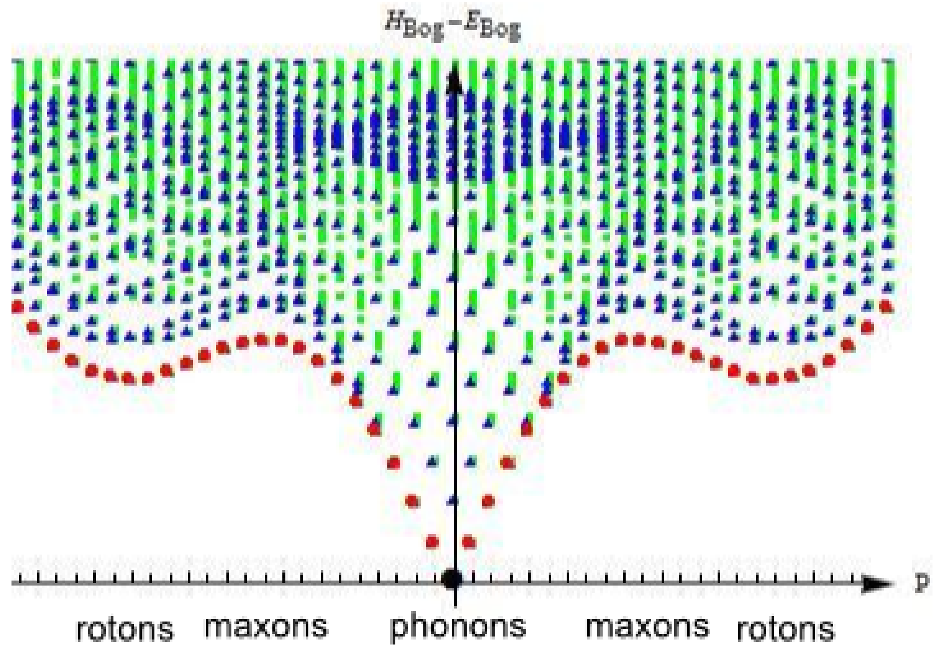


Excitation spectrum of 1-dimensional homogeneous Bose gas

with potential v_1 in the Bogoliubov approximation.



$$\hat{v}_2(\mathrm{p}) = \frac{15\mathrm{e}^{-\mathrm{p}^2/2}}{2}$$



Excitation spectrum of 1-dimensional homogeneous Bose gas

with potential v_2 in the Bogoliubov approximation.

Let A be a bounded from below self-adjoint operator with only discrete spectrum. We define $\overrightarrow{\mathrm{sp}}(A) := (a_1, a_2, \dots)$, where $a_1, a_2, \dots$ are the eigenvalues of A in the increasing order. If $\dim \mathcal{H} = n$, then we set $a_{n+1} = a_{n+2} = \cdots = \infty$.

Excitation energies of the N-body Hamiltonian.

If $\mathrm{p} \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{0\}$, set

$$\big(K_N^1(\mathrm{p}), K_N^2(\mathrm{p}), \dots\big) := \overrightarrow{\mathrm{sp}}\big(H_N(\mathrm{p}) - E_N\big).$$

The lowest eigenvalue of $H_N(0) - E_N$ is 0 by general arguments.

Set

$$\big(0, K_N^1(0), K_N^2(0), \dots\big) := \overrightarrow{\mathrm{sp}}\big(H_N(0) - E_N\big).$$

Bogoliubov excitation energies.

If $\mathrm{p} \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{0\}$, set

$$\big(K^1_{\mathrm{Bog}}(\mathrm{p}), K^2_{\mathrm{Bog}}(\mathrm{p}), \dots\big) := \overrightarrow{\mathrm{sp}}\big(H^L_{\mathrm{Bog}}(\mathrm{p}) - E^L_{\mathrm{Bog}}\big).$$

The lowest eigenvalue of $H_{\mathrm{Bog}}(0) - E_{\mathrm{Bog}}$ is obviously 0. Set

$$\big(0, K^1_{\mathrm{Bog}}(0), K^2_{\mathrm{Bog}}(0), \dots\big) := \overrightarrow{\mathrm{sp}}\big(H^L_{\mathrm{Bog}}(0) - E^L_{\mathrm{Bog}}\big).$$

For any $\mathrm{p} \in \frac{2\pi}{L}\mathbb{Z}^d$ the Bogoliubov excitation energies are given by

$$\Big\{ \sum_{i=1}^{j} e_{\mathrm{k}_i} \ : \ \mathrm{k}_1, \ldots, \mathrm{k}_j \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{0\},\ \mathrm{k}_1 + \cdots + \mathrm{k}_j = \mathrm{p},\ \ j = 1, 2, \ldots \Big\},$$

in the increasing order.

Upper bound Let $c > 0$. Then there exists C such that if $L^{2d+2} \leq cN$, then

$$E_N \geq \frac{1}{2}\hat{v}(0)(N-1) + E_{\mathrm{Bog}} - CN^{-1/2}L^{2d+3};$$

If in addition $K_N^j(\mathrm{p}) \leq cNL^{-d-2}$, then

$$\begin{aligned} E_N + K_N^j(\mathrm{p}) \ \geq \ & \frac{1}{2}\hat{v}(0)(N-1) + E_{\mathrm{Bog}} + K_{\mathrm{Bog}}^j(\mathrm{p}) \\ & -CN^{-1/2}L^{d/2+3}\big(K_N^j(\mathrm{p}) + L^d\big)^{3/2}. \end{aligned}$$

Lower bound. Let $c > 0$. Then there exists $c_1 > 0$ and C such that if $L^{2d+1} \le cN$, $L^{d+1} \le c_1 N$, then

$$E_N \le \frac{1}{2}\hat{v}(0)(N-1) + E_{\mathrm{Bog}} + CN^{-1/2}L^{2d+3/2};$$

If in addition $K^j_{\mathrm{Bog}}(\mathrm{p}) \le cNL^{-d-2}$ and $K^j_{\mathrm{Bog}}(\mathrm{p}) \le c_1 NL^{-2}$, then

$$\begin{aligned} E_N + K^j_N(\mathrm{p}) \ \le \ & \frac{1}{2}\hat{v}(0)(N-1) + E_{\mathrm{Bog}} + K^j_{\mathrm{Bog}}(\mathrm{p}) \\ & + CN^{-1/2}L^{d/2+3}(K^j_{\mathrm{Bog}}(\mathrm{p}) + L^{d-1})^{3/2}. \end{aligned}$$

Special case of this theorem with $L = 1$ was proven by R. Seiringer. Mimicking his proof gives big error terms for large L: they are of the order $N^{-1/2}\exp(L^{d/2})$. To get better error estimates we need to use additional ideas.

Bosonic Fock space

$$\mathcal{H} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N = \Gamma_{\mathrm{s}}\Big(l^2\big(\frac{2\pi}{L}\mathbb{Z}^d\big)\Big).$$

Hamiltonian in second quantized notation

$$H := \bigoplus_{N=0}^{\infty} H_N = \sum_{\mathrm{p}} \mathrm{p}^2 a_{\mathrm{p}}^\dagger a_{\mathrm{p}} + \frac{1}{2N} \sum_{\mathrm{p,q,k}} \hat{v}(\mathrm{k}) a_{\mathrm{p+k}}^\dagger a_{\mathrm{q-k}}^\dagger a_{\mathrm{q}} a_{\mathrm{p}}.$$

Number of particles in condensate $N_0 = a_0^\dagger a_0$.

Number of particles outside of condensate $N^> = \sum_{\mathrm{p}\neq 0} a_{\mathrm{p}}^\dagger a_{\mathrm{p}}$.

The exponential property of Fock spaces gives

$$\mathcal{H} \simeq \Gamma_{\mathrm{s}}(\mathbb{C}) \otimes \Gamma_{\mathrm{s}}\Big(l^2\big(\frac{2\pi}{L}\mathbb{Z}^d\backslash\{0\}\big)\Big).$$

Embed the space of zero modes $\Gamma_{\mathrm{s}}(\mathbb{C}) = l^2(\{0, 1, \dots\})$ in a larger space $l^2(\mathbb{Z})$. Thus we obtain the extended Hilbert space

$$\mathcal{H}^{\mathsf{ext}} := l^2(\mathbb{Z}) \otimes \Gamma_{\mathrm{s}}\Big(l^2\big(\frac{2\pi}{L}\mathbb{Z}^d\backslash\{0\}\big)\Big).$$

The operator N_0 extends to an operator $N_0^{\rm ext}$ satisfying

$$\mathcal{H} = \mathrm{Ran}\mathbb{1}_{[0,\infty[}(N_0^{\rm ext}).$$

If $N \in \mathbb{Z}$, we will write $\mathcal{H}_N^{\rm ext}$ for the subspace of $\mathcal{H}^{\rm ext}$ corresponding to $N^{>} + N_0^{\rm ext} = N$.

We have also a unitary operator

$$U|n_0\rangle \otimes \Psi^{>} = |n_0 - 1\rangle \otimes \Psi^{>}.$$

We now define for $\mathrm{p} \neq 0$ the following operator on $\mathcal{H}^{\text{ext}}$:

$$b_{\mathrm{p}} := a_{\mathrm{p}} U^{\dagger}.$$

Operators b_{p} and $b_{\mathrm{q}}^{\dagger}$ satisfy the same CCR as a_{p} and $a_{\mathrm{q}}^{\dagger}$.

Estimating Hamiltonian on $\mathcal{H}_N$

$$
\begin{aligned}
H_{N,\epsilon} \;:=\; & \frac{1}{2}\hat{v}(0)(N-1) + \sum_{\mathrm{p}\neq 0}\big(|\mathrm{p}|^2 + \hat{v}(\mathrm{p})\big)a_{\mathrm{p}}^\dagger a_{\mathrm{p}} \\
& + \frac{1}{2N}\sum_{\mathrm{p}\neq 0}\hat{v}(\mathrm{p})\Big(a_0^\dagger a_0^\dagger a_{\mathrm{p}} a_{-\mathrm{p}} + a_{\mathrm{p}}^\dagger a_{-\mathrm{p}}^\dagger a_0 a_0\Big) \\
& - \frac{1}{N}\sum_{\mathrm{p}\neq 0}\big(\hat{v}(\mathrm{p}) + \frac{\hat{v}(0)}{2}\big)a_{\mathrm{p}}^\dagger a_{\mathrm{p}} N^{>} + \frac{\hat{v}(0)}{2N}N^{>} \\
+ \frac{\epsilon}{N}\sum_{\mathrm{p}\neq 0} & \big(\hat{v}(\mathrm{p}) + \hat{v}(0)\big)a_{\mathrm{p}}^\dagger a_{\mathrm{p}} N_0 + +(1+\epsilon^{-1})\frac{1}{2N}v(0)L^d N^{>}(N^{>}-1)
\end{aligned}
$$

$$
H_N \geq H_{N,-\epsilon}, \quad 0 < \epsilon \leq 1; \qquad H_N \leq H_{N,\epsilon}, \quad 0 < \epsilon.
$$

Extended estimating Hamiltonian on $\mathcal{H}_N^{\mathrm{ext}}$

$$
\begin{aligned}
H_{N,\epsilon}^{\mathrm{ext}} \;:=\; & \frac{1}{2}\hat{v}(0)(N-1) + \sum_{\mathrm{p}\neq 0}\big(|\mathrm{p}|^2 + \hat{v}(\mathrm{p})\big) b_{\mathrm{p}}^{\dagger} b_{\mathrm{p}} \\
& + \frac{1}{2}\sum_{\mathrm{p}\neq 0}\hat{v}(\mathrm{p})\Big(\frac{\sqrt{(N_0^{\mathrm{ext}}-1)N_0^{\mathrm{ext}}}}{N} b_{\mathrm{p}} b_{-\mathrm{p}} + \mathrm{hc}\Big) \\
& - \frac{1}{N}\sum_{\mathrm{p}\neq 0}\big(\hat{v}(\mathrm{p}) + \frac{\hat{v}(0)}{2}\big) b_{\mathrm{p}}^{\dagger} b_{\mathrm{p}} N^{>} + \frac{\hat{v}(0)}{2N} N^{>} \\
& + \frac{\epsilon}{N}\sum_{\mathrm{p}\neq 0}\big(\hat{v}(\mathrm{p}) + \hat{v}(0)\big) b_{\mathrm{p}}^{\dagger} b_{\mathrm{p}} N_0^{\mathrm{ext}} \\
& + (1+\epsilon^{-1})\frac{1}{2N} v(0) L^d N^{>}(N^{>}-1).
\end{aligned}
$$

$H_{N,\epsilon}^{\mathrm{ext}}$ preserves $\mathcal{H}_N$ and restricted to $\mathcal{H}_N$ coincides with $H_{N,\epsilon}$.

$$\sum_{\mathrm{p}\neq 0}\Big(|\mathrm{p}|^2+\hat{v}(\mathrm{p})\Big)b_{\mathrm{p}}^{\dagger}b_{\mathrm{p}}+\frac{1}{2}\sum_{\mathrm{p}\neq 0}\hat{v}(\mathrm{p})\Big(b_{\mathrm{p}}b_{-\mathrm{p}}+b_{\mathrm{p}}^{\dagger}b_{-\mathrm{p}}^{\dagger}\Big).$$

preserves $\mathcal{H}_N^{\mathrm{ext}}$. Its restriction to $\mathcal{H}_N^{\mathrm{ext}}$ will be denoted $H_{\mathrm{Bog},N}$. Applying an appropriate Bogoliubov transformation we see that $H_{\mathrm{Bog},N}$ is unitarily equivalent to H_{Bog}, which we introduced before.

$$
\begin{aligned}
H^{\mathrm{ext}}_{N,\epsilon} &= \frac{1}{2}\hat{v}(0)(N-1) + H_{\mathrm{Bog},N} + R_{N,\epsilon},\\
R_{N,\epsilon} &:= \frac{1}{2}\sum_{\mathrm{p}\neq 0}\hat{v}(\mathrm{p})\Big(\Big(\frac{\sqrt{(N_0^{\mathrm{ext}}-1)N_0^{\mathrm{ext}}}}{N}-1\Big)b_{\mathrm{p}}b_{-\mathrm{p}} + \mathrm{hc}\Big)\\
&\quad -\frac{1}{N}\sum_{\mathrm{p}\neq 0}\Big(\hat{v}(\mathrm{p}) + \frac{\hat{v}(0)}{2}\Big)b_{\mathrm{p}}^{\dagger}\mathrm{b}_{\mathrm{p}}N^{>} + \frac{\hat{v}(0)}{2N}N^{>}\\
&+\frac{\epsilon}{N}\sum_{\mathrm{p}\neq 0}\Big(\hat{v}(\mathrm{p}) + \hat{v}(0)\Big)b_{\mathrm{p}}^{\dagger}b_{\mathrm{p}}N_0^{\mathrm{ext}} + (1+\epsilon^{-1})\frac{1}{2N}v(0)L^d N^{>}(N^{>}-1).
\end{aligned}
$$

Consequence of the min-max principle:

$$A \leq B \text{ implies } \overrightarrow{\mathrm{sp}}(A) \leq \overrightarrow{\mathrm{sp}}(B).$$

Rayleigh-Ritz principle:

$$\overrightarrow{\mathrm{sp}}(A) \leq \overrightarrow{\mathrm{sp}}\left(P_{\mathcal{K}} A P_{\mathcal{K}}\Big|_{\mathcal{K}}\right).$$

Proof of lower bound

For brevity set

$$\mathbb{1}_\kappa^N := \mathbb{1}_{[0,\kappa]}(H_N - E_N).$$

For $0 < \epsilon \leq 1$,

$$\mathbb{1}_\kappa^N H_N \mathbb{1}_\kappa^N \geq \mathbb{1}_\kappa^N \left(\frac{1}{2}\hat{v}(0)(N-1) + H_{\mathrm{Bog},N} + R_{N,-\epsilon} \right) \mathbb{1}_\kappa^N.$$

Hence,

$$\overrightarrow{\mathrm{sp}}\left(\mathbb{1}_\kappa^N H_N \mathbb{1}_\kappa^N \right) \geq \frac{1}{2}\hat{v}(0)(N-1) + \overrightarrow{\mathrm{sp}}\left(H_{\mathrm{Bog}} \right) - \|R_{N,-\epsilon}\|.$$

Suppose now that G is a smooth nonnegative function on $[0,\infty[$ such that

$$G(s) = \begin{cases} 1, & \text{if } s \in [0, \frac{1}{3}] \\ 0, & \text{if } s \in [1, \infty[. \end{cases}$$

For brevity, we set $\mathbb{1}_\kappa^{\mathrm{Bog}} := \mathbb{1}_{[0,\kappa]}(H_{\mathrm{Bog},N} - E_{\mathrm{Bog}})$. We define

$$Z_\kappa := \left(\mathbb{1}_\kappa^{\mathrm{Bog}} G(N^>/N)^2 \mathbb{1}_\kappa^{\mathrm{Bog}}\right)^{-1/2} \mathbb{1}_\kappa^{\mathrm{Bog}} G(N^>/N).$$

Clearly, Z_κ is a partial isometry with initial space $\mathrm{Ran}(G(N^>/N)\mathbb{1}_\kappa^{\mathrm{Bog}})$ and final space $\mathrm{Ran}(\mathbb{1}_\kappa^{\mathrm{Bog}})$.

$$\overrightarrow{\mathrm{sp}} H_N \ \leq \ \overrightarrow{\mathrm{sp}} \left(Z_\kappa^\dagger Z_\kappa H_N Z_\kappa^\dagger Z_\kappa \Big|_{\mathrm{Ran} Z_\kappa^\dagger} \right) = \overrightarrow{\mathrm{sp}} \left(Z_\kappa H_N Z_\kappa^\dagger \Big|_{\mathrm{Ran} \mathbb{1}_\kappa^{\mathrm{Bog}}} \right).$$

$$\begin{aligned} Z_\kappa H_N Z_\kappa^\dagger \ &\leq \ Z_\kappa H_{N,\epsilon} Z_\kappa^\dagger \\ &= \ \frac{1}{2}\hat{v}(0)(N-1)\mathbb{1}_\kappa^{\mathrm{Bog}} + H_{\mathrm{Bog}} \mathbb{1}_\kappa^{\mathrm{Bog}} \\ &\quad + Z_\kappa (H_{\mathrm{Bog}} - E_{\mathrm{Bog}}) Z_\kappa^\dagger - (H_{\mathrm{Bog}} - E_{\mathrm{Bog}}) \mathbb{1}_\kappa^{\mathrm{Bog}} \\ &\quad + Z_\kappa R_{N,\epsilon} Z_\kappa^\dagger. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\overrightarrow{\mathrm{sp}}(H_N) &\leq Z_\kappa H_{N,\epsilon} Z_\kappa^\dagger \\
&= \frac{1}{2}\hat{v}(0)(N-1) + \overrightarrow{\mathrm{sp}}\Big(H_{\mathrm{Bog}} \mathbb{1}_\kappa^{\mathrm{Bog}}\Big) \\
&\quad + \Big\| Z_\kappa (H_{\mathrm{Bog}} - E_{\mathrm{Bog}}) Z_\kappa^\dagger - (H_{\mathrm{Bog}} - E_{\mathrm{Bog}}) \mathbb{1}_\kappa^{\mathrm{Bog}} \Big\| \\
&\quad + \Big\| Z_\kappa R_{N,\epsilon} Z_\kappa^\dagger \Big\| .
\end{aligned}
$$