Elements of general topology

Jan Dereziński

Department of Mathematical Methods in Physics
Warsaw University
Hoża 74, 00-682, Warszawa, Poland

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Contents

1 General concepts

1.1 Open and closed sets

$(X, \mathcal{T})$ is a topological space iff X is a set and $\mathcal{T} \subset 2^X$ satisfies

1. $\emptyset, X \in \mathcal{T}$;
2. $A_i \in \mathcal{T},\ i \in I, \Rightarrow \bigcup\limits_{i \in I} A_i \in \mathcal{T}$;
3. $A_1, \ldots, A_n \in \mathcal{T} \ \Rightarrow \bigcap\limits_{i=1}^{n} A_i \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called sets open in X. We will call $\mathcal{T}$ "a topology".
A set $A \subset X$ is called closed in X iff $X \backslash A$ is open.
If $\mathcal{T}$, $\mathcal{S}$ are topologies on X, then we say that $\mathcal{T}$ is weaker than $\mathcal{S}$ iff $\mathcal{T} \subset \mathcal{S}$.
If $\mathcal{T} = 2^X$, then we say that X is a discrete space.

1.2 Basis and subbasis of a topology

Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subset 2^X$. We say that $\mathcal{B}$ is a basis of a topology $\mathcal{T}$ iff $\mathcal{T}$ equals the family of unions of elements of $\mathcal{B}$.

Theorem 1.1 *Let $\mathcal{H} \subset 2^X$. Then the following conditions are equivalent:*

1. *the family of finite intersections of elements of $\mathcal{H}$ is a basis of $\mathcal{T}$;*
2. *$\mathcal{T}$ is the least topology containing $\mathcal{H}$.*
3. *$\mathcal{T} = \cap \mathcal{T}'$ where $\mathcal{T}'$ runs over all topologies containing $\mathcal{H}$*

If $\mathcal{H}$ satisfies one of the above conditions, then it is called a subbasis of $\mathcal{T}$.

1.3 Basic constructions

If $Y \subset X$, and $(X, \mathcal{T})$ is a topological space, then define the relative topology on Y:

$$\{A \cap Y \ : \ A \in \mathcal{T}\}.$$

If $(X_i, \mathcal{T}_i)_{i \in I}$ is a family of disjoint topological spaces, then $\bigcup\limits_{i \in I} X_i$ is equipped with a natural topology

$$\{\bigcup_{i \in I} A_i \ : \ A_i \in \mathcal{T}_i\}.$$

If $(X_i, \mathcal{T}_i)_{i \in I}$ is a family of topological spaces, then we define the product topology on $\mathop{\times}\limits_{i \in I} X_i$ as the topology with a subbasis

$$\{\bigcap_{j=1}^{n} \pi_i^{-1}(U_i) \ : \ U_i \in \mathcal{T}_i,\ i \in I\},$$

where $\pi_i : X \to X_i$ are the canonical projections.

1.4 Neighborhoods

Let $x \in X$. We say that $A \subset X$ is a neighborhood of x, if there exist an open U such that $x \in U \subset A$. We say that $\mathcal{V}_x \subset 2^X$ is a basis of neighborhoods iff all element of $\mathcal{V}_x$ are neighboroods of x and for any neighborhood U of x there exists $V \in \mathcal{V}_x$ such that $V \subset U$.

1.5 δ-open and σ-closed sets

Definition 1.2 *δ-open sets are countable unions of open sets.*
σ-closed sets are countable unions of closed sets.

The complement of a σ-closed set is a δ-open set and vice versa.

1.6 Convergence

The set $(I, \leq)$ with a relation is called a directed set iff

1. $i \leq j,\ j \leq k \Rightarrow i \leq k$,
2. $i \leq j,\ j \leq i \Rightarrow i = j$,
3. for any i, j there exists k such that $i \leq k$ and $j \leq k$.

A net in X is a directed set $(I, \leq)$ together with a map $I \ni i \mapsto x_i \in X$. We will write $(x_i)_{i \in I}$.
We say that $(x_j)_{j \in J}$ is a subnet of $(x_i)_{i \in I}$ iff

1. $J \subset I$,
2. if $i \in I$, then there exists $j \in J$ such that $i \leq j$,
3. $J \ni j \mapsto x_j \in X$ is the restriction of $I \ni i \mapsto x_i \in X$ to J.

We say that a net $(x_i)_{i \in I}$ is convergent to $x \in X$, iff for any neighborhood U of x there exists $i_U \in I$ such that if $i \geq i_U$, then $x_i \in U$.

Theorem 1.3 *If $(x_i)_{i \in I}$ is a net convergent to x, then any its subnet is convergent to x.*

1.7 Interior and closure

Theorem 1.4 *Let $A, B \subset X$. Then the following conditions are equivalent:*

1. *B is the smallest closed set containing A;*
2. *$B = \cap B'$ where B' runs over all closed sets containing A;*
3. *B is the set of all limits of convergent nets in A.*
4. *Let $x \in X$. Then $x \in B$ iff for any neighborhood U of x we have $U \cap A \neq \emptyset$.*
5. *For any $x \in X$ choose a basis of neighborhoods of x, denoted $\mathcal{V}_x$. Then $x \in B$ iff for any $V \in \mathcal{V}_x$ we have $V \cap A \neq \emptyset$.*

If the above conditions are satisfied then we say that B is the closure of A and we write $B = A^{\mathrm{cl}}$.

Theorem 1.5 *Let $A, C \subset X$. Then the following conditions are equivalent:*

1. *C is the largest open set containing A;*

2. *$C = \cup C'$ where C' runs over all open sets contained in A;*

3. *Let $x \in X$. Then $x \in C$ iff for any net $(x_i)_{i\in I}$ in X convergent to x there exists $i_0 \in I$ such that for $i \geq i_0$ we have $x_i \in A$;*

4. *Let $x \in X$. Then $x \in C$ iff there exists a neighborhood U of x such that $U \subset A$;*

5. *For any $x \in X$ choose a basis of neighborhoods of x, denoted $\mathcal{V}_x$. Then $x \in C$ iff there exists a $V \in \mathcal{V}_x$ such that $V \subset A$.*

Clearly, $A^\circ = (X \backslash A^{\mathrm{cl}})^{\mathrm{cl}}$

Example 1.6 *Consider the index set I and the spaces $X_i := \{0, 1\}$ with the discrete topology. Consider $X := \underset{i\in I}{\times} X_i$ and let $A \subset X$. Elements of X can be labelled by subsets of I: in fact, if $J \subset I$, then $x_i^J := 1$ for $i \in J$ and $x_i^J = 0$ otherwise. Let X_{fin} be the set of sequences x^J with a finite J. Then the closure of X^{fin} equals X. In fact, if $J \subset I$ then we take the directed set 2^J_{fin} of finite subsets of J, then $(x^K)_{K\in 2^J_{\mathrm{fin}}}$ converges to x^J.*

1.8 Accumulation points of a net

Theorem 1.7 *Let $(x_i)_{\in I}$ be a net and $x \in X$. Then the following conditions are equivalent:*

1. *For any neighborhood U of x and $i \in I$ there exists $j \geq i$ such that $x_j \in U$;*

2. *There exists a basis of neighborhoods $\mathcal{V}_x$ such that for any $U \in \mathcal{V}_x$ and $i \in I$ there exists $j \geq i$ such that $x_j \in U$.*

3. *There exists a subnet $(x_j)_{j\in J}$ convergent to x;*

4. *x belongs to*
$$\bigcap_{i\in I} \{x_j \ : \ j \geq i\}^{\mathrm{cl}}.$$

If x satisfies the above conditions, then we say that it is an accumulation point of the net $(x_i)_{i\in I}$.

1.9 Continuity

Let X, Y be topological spaces and $f : X \to Y$.

Theorem 1.8 *Let $x_0 \in X$. The following conditions are equivalent*

1. *For any neighborhood V of $f(x_0)$, $f^{-1}(V)$ is a neighborhood of x_0;*

2. *There exists basis $\mathcal{V}_{x_0}$ of neighborhoods of x_0 and $\mathcal{W}_{f(x_0)}$ such that for any $W \in \mathcal{W}_{f(x_0)}$ there exists $V \in \mathcal{V}_{x_0}$*

3. *For any net (x_i) converging to x_0, $(f(x_i))$ converges to $f(x_0)$.*

We say that f is continuous at x_0 iff the conditions of the above theorem hold.

Theorem 1.9 *The following conditions are equivalent*

1. *The function f is continuous at every $x \in X$;*

2. *For any open set $V \in Y$, $f^{-1}(V)$ is open.*

3. *For any closed set $V \in Y$, $f^{-1}(V)$ is closed.*

We say that f is continuous iff the conditions of the above theorem hold.
The set of continuous functions from X to Y is denoted $C(X, Y)$. We will write

$$C(X) := C(X, \mathbb{C}), \quad C_{\mathrm{sa}}(X) := C(X, \mathbb{R}), \quad C_{+}(X) := C(X, [0, \infty[).$$

Let X, Y be topological spaces. Let $p : X \to Y$ be a continuous map. Then $p^{\#} : C(Y) \to C(X)$ defined by

$$(p^{\#} f)(y) := f(p(y)),$$

is a unital homomorphism of commutative algebras.

Theorem 1.10 *$p^{\#}$ is injective $\Leftarrow$ p has a dense image.*

2 Separation axioms

2.1 Hausdorff spaces

Let X be a topological space.

Theorem 2.1 *The following conditions are equivalent*

1. *For any distinct $x_1, x_2 \in X$ there exist disjoint open sets A_1, A_2 such that $x_1 \in A_1$ and $x_2 \in A_2$;*

2. *Every the net converges to at most one point.*

3. *The diagonal $\{(x, x) \ : \ x \in X\}$ is closed in $X \times X$*

If X satisfies the above conditions, then it is called a Hausdorff space (or a T_2 space).

Theorem 2.2 *Let X be a Hausdorff space and $Y \subset X$. Then Y is a Hausdorff space.*

Theorem 2.3 *Let $(X_i)_{i \in I}$ be a family of disjoint topological spaces. then $\underset{i \in I}{\cup} X_i$ is a Hausdorff space iff all X_i are Hausdorff spaces.*

Theorem 2.4 *Let $(X_i)_{i \in I}$ be a family of topological spaces. then $\underset{i \in I}{\times} X_i$ is a Hausdorff space iff all X_i are Hausdorff spaces.*

2.2 Tikhonov spaces

Let X be a Hausdorff space. It is called a Tikhonov space iff for any closed $C \subset X$ and $x \in X \backslash C$ there exists $f \in C(X)$ uch that $f(x) = 0$ and $f = 1$ on C.

Theorem 2.5 *Let X be a Tikhonov space and $Y \subset X$. Then Y is a Tikhonov space (or a $T_{3\frac{1}{2}}$ space).*

Theorem 2.6 *Let $(X_i)_{i \in I}$ be a family of disjoint topological spaces. then $\underset{i \in I}{\cup} X_i$ is a Tikhonov space iff all X_i are Tikhonov spaces.*

Theorem 2.7 *Let $(X_i)_{i \in I}$ be a family of topological spaces. then $\underset{i \in I}{\times} X_i$ is a Tikhonov space iff all X_i are Tikhonov spaces.*

Theorem 2.8 *Let X, Y be Tikhonov spaces and $p : X \to Y$ continuous. Then*

1. *$p^{\#}$ is surjective $\Rightarrow$ p is injective.*
2. *$p^{\#}$ is injective iff p is surjective.*

Proof. 1.$\Rightarrow$ Let p be not injective. Let $x_1, x_2 \in Y$ with $p(x_1) = p(x_2)$. Then $(p^{\#}f)(x_1) = f(p(x_1)) = f(p(x_2)) = (p^{\#}f)(x_2)$. But there exist $g \in C(X)$ such that $g(x_1) \neq g(x_2)$. Hence $g \notin p^{\#}(C(Y))$ and thus $p^{\#}$ is not surjective.

2.$\Rightarrow$ Let $p(X)$ be not dense. Let $y_0 \in Y \backslash p(X)^{\mathrm{cl}}$. Then there exists $f \in C(Y)$ such that $f = 0$ on $p(X)^{\mathrm{cl}}$ and $f(y_0) \neq 0$. Thus $p^{\#}f = 0$. Hence $p^{\#}$ is not injective.

2.$\Leftarrow$ follows from theorem 1.10. □

2.3 Normal spaces

Theorem 2.9 *Let X be a Hausdorff space. The following conditions are equivalent:*

1. *For any disjoint closed sets $C_1, C_2 \subset X$ there exist disjoint open $U_1, U_2 \subset X$ such that*
$$C_1 \subset U_1, \ C_2 \subset U_2;$$
2. *Let $C_1, \ldots, C_n \subset X$ be disjoint and closed. Then there exist open disjoint $U_1, \ldots, U_n$ such that $C_j \subset U_j$, $j = 1, \ldots, n$;*
3. *Let $C, D \subset X$ closed disjoint. Then there exists $f \in C_+(X)$ such that $f = 0$ on C and $f = 1$ on D.*
4. *Let $A_1, \ldots, A_n \subset X$ be open and $C \subset \bigcup_{j=1}^n A_j$ closed. Then there exist $h_j \in C_+(X)$, $j = 1, \ldots, n$ such that $\mathrm{supp} h_j \subset A_j$ and*
$$\sum_{j=1}^n h_j = 1 \ \text{on}\ C.$$

Definition 2.10 *A Hausdorff space satisfying the above conditions is called a normal space (or a T_4 space).*

Remark 2.11 *The implication 1.$\Rightarrow$2. is called* **Urysohn's lemma**.

Proof. 3.$\Rightarrow$ 4. We construct the sequence $k_j \in C(X)$ such that

$$\begin{aligned}
&k_1 = 1 \text{ on } C, \ \mathrm{supp} k_1 \subset \textstyle\bigcup_{j=1}^n A_j, \\
&\ldots, \\
&k_{j+1} = 1 \text{ on } \mathrm{supp} k_j \backslash A_j, \ \mathrm{supp} k_{j+1} \subset \textstyle\bigcup_{i=j+1}^n A_i, \\
&\ldots, \\
&k_{n+1} = 0.
\end{aligned}$$

We put
$$h_j = k_j(1 - k_{j+1}).$$
We see that
$$\mathrm{supp} h_j \subset A_j.$$
We prove by induction that
$$h_1 + \ldots + h_j + k_{j+1} = 1 \text{ on } C.$$
□

Theorem 2.12 (Tietze) *If X is normal, $C \subset X$ is a closed subset and $f : C \to \mathbb{R}$ is continuous, then there exists a continuous $\tilde{f} : X \to \mathbb{R}$, which is an extension of f.*

3 Compactness

3.1 Compact spaces

Let X be a topological space and $Y \subset X$. $\mathcal{S} \subset 2^X$ is called a covering of Y if $Y \subset \bigcup_{U \in \mathcal{S}} U$. $\mathcal{S}$ is called open if all elements of $\mathcal{S}$ are open. $\mathcal{S}_0$ is called a subcovering $\mathcal{S}$ if it is a covering contained in $\mathcal{S}$.

A family $\mathcal{R} \subset 2^X$ is called centered iff any finite subfamily of $\mathcal{R}$ has a non-empty intersection.

Theorem 3.1 *The following condtions are equivalent.*

1. *Every open covering $\mathcal{S}$ of X contains an open subcovering;*
2. *Every centered family $\mathcal{R} \subset 2^X$ has a non-empty intersection;*
3. *Every net in X has an accumulation point.*

Proof. 1.$\Leftrightarrow$2. The condition 1. can be reformulated as follows. If $\mathcal{S}$ is a family of open sets in X such that for any finite subfamily $\mathcal{S}_0$ we have $\underset{U \in \mathcal{S}_0}{\cup} U \neq X$, then $\underset{U \in \mathcal{S}}{\cup} U \neq X$. Since the complements of open sets are closed, we immediately get 2.

2.$\Rightarrow$3. Set

$$F_i := \{x_j \ : \ i \leq j\}^{\mathrm{cl}}, \quad i \in I.$$

The family $\{F_i \ : \ i \in I\}$ is closed and centered. Thus $\underset{i \in I}{\cap} F_i$ is nonempty. But this is the set of accumullation points of $(x_i)_{i \in I}$.

2.$\Leftarrow$3. Let $\mathcal{R}$ be a centered family of closed sets. Consider the index set Σ consisting of finite subsets of $\mathcal{R}$ ordered by inclusion. For any $\sigma := \{F_1, \dots, F_n\} \in \Sigma$ we choose $x_\sigma \in \overset{n}{\underset{j=1}{\cap}} F_j$. Then $(x_\sigma)_{\sigma \in \Sigma}$ has an accumulation point $x \in X$.

Let $F \in \mathcal{R}$. Let U be a neighborhood of x. Then there exists $\sigma \in \Sigma$ such that $\sigma = \{F_1, \dots, F_n\}$, $F \leq \{F_1, \dots, F_n\}$ and $x_\sigma \in U$. One of $F_1, \dots, F_n$ equals F, hence $x_\sigma \in \overset{n}{\underset{j=1}{\cap}} F_j \subset F$. Hence $F \cap U \neq \emptyset$. But F is closed. Thus $x_\sigma \in F$. $\square$

A space satisfying the above conditions is called compact.

A subset of a topological space is called precompact if its closure is compact.

Theorem 3.2 *If $f : X \to Y$ is a continuous function and X is compact, then $f(X)$ is compact.*

Proof. If $\mathcal{S}$ is an open covering of $f(X)$, then $\{f^{-1}(A) \ : \ A \in \mathcal{S}\}$ is an open covering of X. If $\mathcal{S}_1$ is a finite open covering of $f(X)$ contained in $\mathcal{S}$, then $\{f^{-1}(A) \ : \ A \in \mathcal{S}_1\}$ is a finite open covering of X. $\square$

Theorem 3.3 *Every closed set of a compact space is compact.*

Proof. Let A be a closed set in a compact space X. If $(x_i)_{i \in I}$ is a net in A then we can choose a subnet $(x_j)j \in J$ convergent in X. Since A is closed, its limit belongs to A. $\square$

Theorem 3.4 *If $(X_i)_{i \in I}$ is a family of disjoint compact spaces, then $\underset{i \in I}{\cup} X_i$ is compact iff I is finite.*

Theorem 3.5 (Tikhonov) *If $(X_i)_{i \in I}$ is a family of topological spaces, then $\underset{i \in I}{\times} X_i$ is compact iff for every $i \in I$ X_i is compact.*

Theorem 3.6 (Dini) *If X is compact, $f_n, f \in C(X, \mathbb{R})$, $f_{n+1} \geq f_n$, $f = \sup f_n$, then $f_n \to f$ uniformly.*

3.2 Proof of Tikhonov's theorem

Let $\mathcal{W} \subset 2^X$ we say that $\mathcal{W}$ is of a finite type iff

1. $\emptyset \in \mathcal{W}$;
2. $A \in \mathcal{W} \Leftrightarrow$ for any finite $B \subset A$, $B \in \mathcal{W}$.

Lemma 3.7 (Teichmüller-Tukey lemma) *Let $\mathcal{W}$ be of a finite type and $A \in \mathcal{W}$. Then there exists $M \in \mathcal{W}$ that contains A and is maximal wrt $\subset$.*

Lemma 3.8 *Let $\mathcal{R}_0 \subset 2^X$ be a centered family. Then there exists a centered family containing $\mathcal{R}_0$ which is maximal wrt $\subset$.*

Proof. We note that the class of centered families in 2^X is of the finite type and then apply Lemma 3.7. $\square$

Lemma 3.9 *Let $\mathcal{R} \subset 2^X$ be a centered family maximal wrt $\subset$. Then*

1. $A_1, \dots, A_k \in \mathcal{R} \;\Rightarrow\; A_1 \cap \dots \cap A_k \in \mathcal{R}$;
2. $A_0 \cap A \neq \emptyset,\ A \in \mathcal{R} \Rightarrow A_0 \in \mathcal{R}$

Proof. 1. We easily see that if $\mathcal{R}$ is centered, $A_1, \dots, A_k \in \mathcal{R}$, then $\mathcal{R} \cup \{A_1 \cap \dots \cap A_k\}$ is centered.

2. If $\mathcal{R}$ is centered, satisfies 1. and $A_0 \cap A \neq \emptyset$, $A \in \mathcal{R}$, then we easily see that $\mathcal{R} \cup \{A_0\}$ is centered. $\square$

Proof of Theorem 3.5 Let $\mathcal{R}_0$ be a centered family of closed subsets of $X := \underset{i \in I}{\times} X_i$. Let $\pi_i : X \to X_i$ be the coordinate projections. Let $\mathcal{R}$ be a maximal centered family of subsets of X containing $\mathcal{R}_0$. For any $i \in I$

$$\{\pi_i(A)^{\mathrm{cl}} \;:\; A \in \mathcal{R}\}.$$

is a centered family of closed subsets of X_i. Hence there exists $x_i \in \underset{A \in \mathcal{R}}{\cap} \pi_i(A)^{\mathrm{cl}}$. We will show that

$$x := (x_i)_{i \in I} \in \underset{A \in \mathcal{R}_0}{\cap} A.$$

Let $A \in \mathcal{R}$. Let W_i be a neighborhood of x_i in X_i. We have $\pi_i(A) \cap W_i \neq \emptyset$. Hence $\pi_i^{-1}(W_i) \cap A \neq \emptyset$. By Lemma 3.9 2., $\pi_i^{-1}(W_i) \in \mathcal{R}$. By Lemma 3.9 1., , if $i_1, \dots, i_n$ is a finite subset of I, then $\overset{n}{\underset{j=1}{\cap}} \pi_{i_j}^{-1}(W_{i_j})$ belongs to $\mathcal{R}$. Thus

$$A \cap \overset{n}{\underset{j=1}{\cap}} \pi_{i_j}^{-1}(W_{i_j}) \neq \emptyset.$$

Hence $x \in A^{\mathrm{cl}}$. In particular, if $A \in \mathcal{R}_0$, then $x \in A$ $\square$

3.3 One-point compactificaton

Definition 3.10 *Let X be a topological space. Then $X^{\mathrm{comp}} := X \cup \{\infty\}$ is called the one-point compactification of X if it is equipped with the following topology: $A \subset X^{\mathrm{comp}}$ is open in X^{comp}, iff $A \cap X$ is open in X and if $\infty \in A$, then there exists a compact $K \subset X$ such that $X \backslash K \subset A$.*

Theorem 3.11 *For any space X, its one-point compactification X^{comp} is compact. X is dense in X^{comp} iff X is not compact.*

Proof. Let $\mathcal{S}$ be an open covering of X^{comp}. Then there exists $A_0 \in \mathcal{S}$ such that $\infty \in A_0$. Since A_0 is open, there exists a compact $K \subset X$ such that $A \supset X \backslash K$. Let

$$\mathcal{S}' := \{A \cap X \ : \ A \in \mathcal{S}\}.$$

Then $\mathcal{S}'$ covers X and hence also K. We can choose a finite subcovering $A_1 \cap X, \ldots, A_n \cap X$ of K. Then $A_0, A_1, \ldots, A_n$ is finite subcovering of X^{comp}. □

Definition 3.12 *Let X be a topological space. $C_0(X)$ is the set of continuous functions with a compact support. $C_\infty(X)$ is the set of of continuos functions on X such that for any $\epsilon > 0$ there exists a compact $K \subset X$ such that $|f(x)| \leq \epsilon$ on $X \backslash K$.*

$C_\infty(X)$ can be identified with

$$\{f \in C(X^{\text{comp}}) \ : \ f(\infty) = 0\}.$$

Then

$$C(X^{\text{comp}}) = C_\infty(X) \oplus \mathbb{C}.$$

3.4 Compact Hausdorff spaces

Theorem 3.13 *If X is Hausdorff and $Y \subset X$ is compact, then Y is closed.*

Proof. Let $(x_i)_{i \in I}$ be a net in Y convergent to $x \in X$. Y is compact, hence it possesses an accumulation point in Y. Since X is Hausdorff, the accumulation point is only one and equals x. Hence $x \in Y$. □

Theorem 3.14 *If $f : X \to Y$ is continuous and injective and X is compact Hausdorff, then*

$$f^{-1} : f(X) \to X$$

is continuous.

Proof. Let F be closed in X. Then it is compact. So $f(F)$ is compact. Since Y is Hausdorff, $f(F)$ is closed in Y. So $f(F)$ is closed in $f(X)$.

Since f is injective, $f(F) = (f^{-1})^{-1}(F)$. Thus the preimages of closed sets wrt f^{-1} are closed. So f^{-1} is continuous. □

Corollary 3.15 *Let $(X, \mathcal{T})$ be a compact Hausdorff space.*

1. *If $\mathcal{S}$ is a weaker topology, then it is not Hausdorff;*
2. *If $\mathcal{S}$ is a stronger topology, then it is not compact.*

Theorem 3.16 *Let X be a Tikhonov space, K be its compact subset and C its closed subset. Then there exists a continuous function $f : X \to [0, 1]$ such that $f = 0$ on K and $f = 1$ on C.*

Theorem 3.17 *If X is Hausdorff and $C, D \subset X$ are compact, then there exist disjoint open sets $U, V \subset X$ such that*

$$C \subset U, \quad D \subset V.$$

Proof. **Step 1** We will show that if C is compact and $x \in X \backslash C$, then we will find disjoint open U, V such that $x \in U$ and $C \subset V$.

For any $y \in C$ we will find open disjoint sets U_y, V_y such that $x \in U_y$, $y \in V_y$. Then $\{V_y \ : \ y \in C\}$ is an open covering of C. Let $\{V_{y_1}, \ldots, V_{y_n}\}$ be its finite subcovering. Then $U := \bigcap_{i=1}^{n} U_{y_i}$, $V := \bigcup_{i=1}^{n} V_{y_i}$ has the property we are looking for.

Step 2 Let C, D be compact. For any $x \in D$ we will find disjoint open U_x, V_x such that $x \in U_x$, $C \subset V_x$. Now $\{U_x \ : \ x \in D\}$ is an open covering of D. Let $\{U_{x_1}, \ldots, U_{x_n}\}$ be its finite subcovering $U := \bigcup_{i=1}^{n} U_{x_i}$, $V := \bigcap_{i=1}^{n} V_{a_i}$ has the property described in the theorem. □

Theorem 3.18 *Every compact Hausdorff space is normal.*

Proof. Let C, D be disjoint closed sets in X. Then they are compact. Hence we can apply Theorem. □

Theorem 3.19 (Stone-Weierstrass) *Let X be a compact Hausdorff space and $\mathfrak{A} \subset C(X)$ be a self-adjoint algebra that contains constants and separates points. Then it is uniformly dense in $C(X)$.*

Theorem 3.20 *Let X, Y be compact Hausdorff spaces and let $p : X \to Y$ be continuous. Then*

1. *$p^{\#}$ is surjective iff p is injective.*
2. *$p^{\#}$ is injective iff p is surjective.*
3. *$p^{\#}$ is bijective iff p is bijective (in this case it is homeomorphic).*

Proof. 1.$\Rightarrow$ follows by theorem 2.8.

1.$\Leftarrow$. Let p be injective. p is a bijection from X to $p(X)$. Hence it has a continuous inverse. Let $g \in C(Y)$. We can define f_0 on $p(X)$ by $f_0(y) := g(p^{-1}y)$. f_0 is continuous on. By Tietze's theorem, we can extend it to a continuous function f on Y. Then $p^{\#} f = g$. Hence $p^{\#}$ is surjective.

2.$\Rightarrow$ follows by Theorem 2.8

2.$\Leftarrow$. follows by Theorem 1.10. □

3.5 Locally compact spaces

Definition 3.21 *X is locally compact iff every point in X has a compact neighborhood.*

Clearly, every compact space is locally compact.

Theorem 3.22

1. *Every closed subset of a locally compact space is locally compact;*
2. *Every open subset of a locally compact Hausdorff space is locally compact Hasdorff.*

Proof. 1. Let F be a closed set of a locally compact space X. Let $x \in F$ and let K be a compact neighborhood of x in X. Then $K \cap F$ is a compact neighborhood of x in F.

2. Let U be an open subset of a locally compact Hausdorff space X. Let $x \in U$ and let K be a compact neighborhood of x in X. Then $K \backslash U$ is closed in K, hence compact. Applying Theorem **??** to disjoint compact sets $\{x\}$ and $K \backslash U$ we will find disjoint sets A, B open in X such that $x \in A$ and $K \backslash U \subset B$. Then $K \backslash B$ is a compact neighborhood of x in U. □

Theorem 3.23 *Let X be a topological space. Its one-point compactification X^{comp} is Hausdorff iff X is a Hausdorff locally compact space.*

Proof. $\Leftarrow$ Let $x \in X$. Let K be a compact neighborhood of x. Then there exists a set $U \subset K$ open in X such that $x \in U$. Then U and $V := X^{\mathrm{comp}} \backslash K$ are disjoint sets open in X^{comp} with $x \in U$ and $\infty \in V$.

$\Rightarrow$. Suppose that $x \in X$. Let U, V be disjoint open subsets of X^{comp} such that $x \in U$ and $\infty \in V$. Then there exists a compact $K \subset X$ such that $K \supset X \backslash V$. Thus K is a compact neighborhood of x. □

Theorem 3.24 *Every locally compact Hausdorff space is Tikhonov.*

Proof. Let X be a locally compact Hausdorff space. Then X^{comp} is compact Hausdorff, hence Tikhonov. But a subset of a Tikhonov space is Tikhonov. □

Theorem 3.25 (Stone-Weierstrass) *Let X be a locally compact Hausdorff space and $\mathfrak{A} \subset C_\infty(X)$ be a self-adjoint algebra of functions such that for every $x \in X$ there exists $f \in \mathfrak{A}$ with $f(x) \neq 0$. Then $\mathfrak{A}$ is uniformly dense in $C_\infty(X)$.*

Example 3.26 *Consider the discrete spaces X and Y, where X is infinite countable and Y is infinite countable. Then $X^{\mathrm{comp}} \times Y^{\mathrm{comp}}$ is Hausdorff compact, hence $Z := X^{\mathrm{comp}} \times Y^{\mathrm{comp}} \backslash \{(\infty, \infty)\}$ is locally compact Hausdorff. Consider the closed subsets $A := X \times \{\infty\}$ and $B := \{\infty\} \times Y$. Then there do not exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Thus Z is not normal*

3.6 σ-compact spaces

We say that X is σ-compact if it is a countable union of compact spaces.

A subset of a topological space is called σ-precompact if its cosure is σ-compact

Theorem 3.27 *If X is σ-compact, then*

1. *from every open covering of X we can choose a countable subcovering.*
2. *Every closed subset of X is σ-compact*

Theorem 3.28 *Every locally compact, σ-compact Hausdorff space is normal.*

Proof. Let E, F be closed subsets of X. **Step 1** There exists a countable open covering $\{A_1, A_2, \ldots\}$ of E such that $A_i^{\mathrm{cl}} \cap F = \emptyset$. In fact, for any $x \in E$ we will find an open A_x such that $x \in A_x$ and $A_x^{\mathrm{cl}} \cap F = \emptyset$. Since E is σ-compact, we can choose a countable subcovering.

Step 2 There exists a countable open covering $\{B_1, B_2, \ldots\}$ of F such that $B_i^{\mathrm{cl}} \cap E = \emptyset$.

Step 3 We set

$$\tilde{A}_n := A_n \backslash \bigcup_{k \le n} B_k^{\mathrm{cl}}, \qquad \tilde{B}_n := B_n \backslash \bigcup_{k \le n} A_k^{\mathrm{cl}}.$$

Then $\tilde{A}_n$ and $\tilde{B}_m$ are disjoint for any n, m.

Step 4 We set

$$A := \bigcup_{n=0}^{\infty} \tilde{A}_n, \quad B := \bigcup_{n=0}^{\infty} \tilde{B}_n.$$

Then A, B are disjoint open subsets of X satisfing $E \subset A$ and $F \subset B$. □

3.7 Stone-Cech compactification

Let X be a topological space We say that ϕ is a character on $C_{\rm b}(X)$ iff ϕ is a homomorphism of the algebra $C_{\rm b}(X)$ into $\mathbb{C}$. We denote by $X^{\rm Cech}$ the set of all characters of $C_{\rm b}(X)$ equipped with the topology of pointwise convergence. We have the continuous map

$$X \ni x \mapsto \pi_x \in X^{\rm Cech},$$

where $\pi_x(f) := f(x)$.

Theorem 3.29 *$X^{\rm Cech}$ is a compact space. Moreover, if X is Tikhonov, then $X^{\rm Cech}$ is Hausdorff compact and $x \to \pi_x$ is a homeomorphism onto its image and the image is dense in $X^{\rm Cech}$.*

4 Metrizable spaces

4.1 Spaces with a countable bases of neighborhoods

A sequence is a net with the index set equal to $\mathbb{N}$.

Theorem 4.1 *If X for any x has a countable basis of neighborhoods, then we can replace "nets" with sequences in Theorems* **??**.

4.2 Metric and metrizable spaces

Let (X, d) be a metric space. The topology whose basis equals

$$\{K(x,r) \ : \ x \in X, \ r > 0\}$$

is called the topology generated by the metric. A topology $\mathcal{T}$ is metrizable iff there exists a metric such that $\mathcal{T}$ is generated by the metric.

Theorem 4.2 *If Y is a dense subset in X, then*

$$\{K(y,1/n) \ : \ y \in Y, \ n \in \mathbb{N}\}$$

Note that $\{K(x,1/n) \ : \ n \in \mathbb{N}\}$, is a basis of neighborhoods of x. Thus in a metric space all points have a countable basis of neighborhoods.

4.3 Basic constructions in metric spaces

If Y is a subset of a metric space, (X,d) then clearly Y is a metric space.

If (X_i, d_i) is a family of disjoint metric spaces, then we can equip $\underset{i\in I}{\cup} X_i$ with the metric

$$d(x_i, y_j) := \min\{d_i(x_i, y_j), 1\}, \ i = j, \quad d(x_i, y_j) = 1, \ i \neq 1.$$

If (X_n, d_n) is a countable family of metric spaces, then $X := \underset{n\in\mathbb{N}}{\times} X_n$ is a metrizable space where the metric can be chosen as

$$d(x,y) := \sum_{n=1} 2^{-n} \frac{d_n(x_n, y_n)}{d_n(x_n, y_n) + 1}.$$

4.4 Uniform convergence

Theorem 4.3 *Let X be a topological space and Y be a metric space. Then*

1. *if $f_n \in C(X,Y)$ and $f_n \to f$ uniformly, then $f \in C(X,Y)$.*
2. *equipped with the metric $\min(\sup(d(f(x),g(x)) \ : \ x \in X),1)$, $C(X,Y)$ becomes a metric space.*
3. *If Y is complete, then $C(X,Y)$ is complete.*

4.5 Normality of metric spaces

Theorem 4.4 *Every metrizable space is normal.*

Proof. Let $C, D \subset X$ be closed and disjoint For any $x \in C$ there exists $r_x > 0$ such that $K(x,r) \cap D = \emptyset$. Set

$$U := \bigcup_{x \in C} K(x, \frac{r_x}{3}).$$

Similarly we construct V. Clearly, U i V are open and disjoint. □

Theorem 4.5 *A normal space with a countable basis can be embedded in $[0,1]^{\mathbb{N}}$, hence is metrizable.*

4.6 Compactness in metric spaces

Theorem 4.6 *If X is metrizable, then it is compact if from every sequence we can choose a convergent subsequence.*

Theorem 4.7 *If X is metric and compact then it is bounded.*

Theorem 4.8 *$Y \subset \mathbb{R}^n$ is compact iff it is closed and bounded.*

4.7 δ-open and σ-closed sets in metrizable spaces

Theorem 4.9 *Let X be a metrizable space. Then every closed set is δ-open.*

Proof. Let C be closed. Define

$$C_n := \{x \in X \ : \ \mathrm{d}(x,C) < 1/n\}.$$

Then C_n are open and

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Hence C is a δ-open set. □