Introduction to C^*-algebras

Jan Dereziński

Department of Mathematical Methods in Physics
Warsaw University
Hoża 74, 00-682, Warszawa, Poland

May 23, 2001

Lecture Notes 2001

Contents

1 Algebras

1.1 Algebras

Let $\mathfrak{A}$ be a vector space over $\mathbb{C}$. $\mathfrak{A}$ is called an associative algebra iff it is equipped with a multiplication satisfying

$$\begin{aligned}&A(BC) = (AB)C,\\&A(B+C) = AB + AC, \quad (B+C)A = BA + CA,\\&(\alpha\beta)(AB) = (\alpha A)(\beta B)\end{aligned}$$

(We will say simply an algebra instead of an associative algebra) $\mathfrak{B} \subset \mathfrak{A}$ is caled a subalgebra if it is a linear subspace and $A, B \in \mathfrak{B} \Rightarrow AB \in \mathfrak{B}$. Clearly, a subalgebra is also an algebra.

Let $\mathcal{X}$ be a vector space. Clearly, $L(\mathcal{X})$ is an algebra.

A subalgebra of $L(\mathcal{X})$ is called a a concrete algebra.

An identity of an algebra $\mathfrak{A}$ is an element $\mathbb{I} \in \mathfrak{A}$ such that

$$A = \mathbb{I}A = A\mathbb{I}, \quad A \in \mathfrak{A}.$$

Any algebra has at most one identity. In fact, if $\mathbb{I}_1, \mathbb{I}_2$ are identities, then

$$\mathbb{I}_1 = \mathbb{I}_1\mathbb{I}_2 = \mathbb{I}_2.$$

We say that $\mathfrak{A}$ is unital if it possesses an identity. In what follows, for $\lambda \in \mathbb{C}$ we will simply write λ instead of $\lambda\mathbb{I}$.

$P \in \mathfrak{A}$ is called an idempotent iff $P^2 = P$. $P\mathfrak{A}P$ is a subalgebra called the reduced algebra.

An idempotent P is called finite discrete iff $P\mathfrak{A}P$ is finite dimensional. It is called abelian iff $P\mathfrak{A}P$ is commutative.

1.2 Commutant

Fix an algebra $\mathfrak{A}$. Let $\mathfrak{B} \subset \mathfrak{A}$. $\mathrm{Alg}(\mathfrak{B})$ denotes the smallest subalgebra of $\mathfrak{A}$ containing $\mathfrak{B}$. $\mathrm{Alg}(\mathfrak{B})$ will be called the subalgebra generated by $\mathfrak{B}$.

The relative commutant of $\mathfrak{B}$, denoted $\mathfrak{B}'$ is defined as

$$\mathfrak{B}' := \{A \in \mathfrak{A} \ : \ AB = BA, \ B \in \mathfrak{B}\}$$

Clearly, $\mathfrak{B}'$ is a subalgebra of $\mathfrak{A}$ and $\mathfrak{B}' = \mathrm{Alg}(\mathfrak{B})'$. If $\mathfrak{A}$ contains 1, then so does $\mathfrak{B}'$.

$\mathfrak{B}''$ is a subalgebra of $\mathfrak{A}$ containing $\mathfrak{B}$.

If there is a risk of confusion, we will write $\mathfrak{B}' \cap \mathfrak{A}$ instead of $\mathfrak{B}'$.

The center of $\mathfrak{A}$ is defined as

$$\mathfrak{A} \cap \mathfrak{A}' = \{A \in \mathfrak{A} \ : \ AB = BA, \ B \in \mathfrak{A}\}.$$

1.3 Direct sums

If $\mathfrak{A}_1$, $\mathfrak{A}_2$ are algebras, then we can define $\mathfrak{A}_1 \oplus \mathfrak{A}_2$.

If $\mathfrak{A}$ is an algebra and $P \in \mathfrak{A} \cap \mathfrak{A}'$ is an idempotent, then clearly $P\mathfrak{A} = P\mathfrak{A}P$ is a subalgebra. $\mathfrak{A}$ is naturally isomorphic to $P\mathfrak{A} \oplus (1-P)\mathfrak{A}$.

1.4 Homomorphisms

Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras. A map $\phi : \mathfrak{A} \to \mathfrak{B}$ is called a homomorphism if it is linear and preserves the multiplication, that means it satisfies

1. $\phi(\lambda A) = \lambda \phi(A)$;
2. $\phi(A + B) = \phi(A) + \phi(B)$;
3. $\phi(AB) = \phi(A)\phi(B)$.

A homomorphism of $\mathfrak{A}$ into $L(\mathcal{X})$ is called a representation of $\mathfrak{A}$ in $\mathcal{X}$.

If $\mathfrak{A}$ is a unital algebra and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then $\phi(1)$ is an idempotent in $\mathfrak{B}$. ϕ is called unital iff

$$\phi(1) = 1.$$

1.5 Ideals

$\mathfrak{B}$ is a left ideal of an algebra $\mathfrak{A}$ iff $A \in \mathfrak{A}$, $B \in \mathfrak{B} \Rightarrow AB \in \mathfrak{B}$. Similarly we define the right ideal.

If $A \in \mathfrak{A}$ is an idempotent, then $\mathfrak{A}A$ is a left ideal.

$\mathfrak{B}$ is called a two-sided ideal if it is a left and right ideal. In what follows we will write an ideal instead of a two-sided ideal.

We say that an ideal $\mathfrak{I}$ is proper iff $\mathfrak{I} \neq \mathfrak{A}$. We say that an ideal $\mathfrak{I}$ is nontrivial iff $\mathfrak{I} \neq \mathfrak{A}$ and $\mathfrak{I} \neq \{0\}$.

Theorem 1.1 *The kernel of a homomorphism is an ideal. If $\mathfrak{I}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{I}$ has a natural structure of an algebra. The map*

$$\mathfrak{A} \ni A \mapsto A + \mathfrak{I} \in \mathfrak{A}/\mathfrak{I}$$

is a homomorphism whose kernel equals $\mathfrak{I}$.

Theorem 1.2 *Let $\pi : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism.*

1. *If $\mathfrak{K}$ is an ideal in $\mathfrak{B}$, then $\pi^{-1}(\mathfrak{K})$ is an ideal in $\mathfrak{A}$ containing* $\mathrm{Ker}\pi$.

2. *If π is surjective and $\mathfrak{I}$ is an ideal in $\mathfrak{A}$, then $\pi(\mathfrak{I})$ is an ideal in $\mathfrak{A}$.*
3. *If π is surjective, then 1. and 2. give a bijection between ideals in $\mathfrak{A}$ containing* $\mathrm{Ker}\pi$ *and ideals in $\mathfrak{B}$.*

1.6 Maximal ideals

We say that an algebra is simple if it has no nontrivial ideals.

$\mathfrak{I}$ is a maximal ideal if it is a proper ideal such that if $\mathfrak{K}$ is a proper ideal containing $\mathfrak{I}$, then $\mathfrak{I} = \mathfrak{K}$. The set of maximal ideals of the algebra $\mathfrak{A}$ is denoted $\mathrm{MI}(\mathfrak{A})$

Theorem 1.3 *Let $\mathfrak{A}$ be a unital algebra with a maximal ideal $\mathfrak{I}$. Then $\mathfrak{A}/\mathfrak{I}$ is simple.*

Proof. $\mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$ is a surjective homomorphism. We can thus apply Theorem 1.2 3. □

Theorem 1.4 *1. If $\mathfrak{A}$ is unital and $\mathfrak{I} \subset \mathfrak{A}$ is a proper ideal, then there exists a maximal ideal containing $\mathfrak{I}$.*

2. Every ideal of codimension 1 is maximal.

Proof. 1. follows by the Kuratowski-Zorn lemma. 2. is obvious. □

If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then we have a map $\pi^{\mathrm{MI}} : \mathrm{MI}(\mathfrak{B}) \to \mathrm{MI}(\mathfrak{A})$ given by

$$\pi^{\mathrm{MI}}(\mathfrak{K}) := \pi^{-1}(\mathfrak{K}), \quad \mathfrak{K} \in \mathrm{MI}(\mathfrak{B}).$$

1.7 Adjoining a unit

Let $\mathfrak{A}$ be an agebra. Introduce the algebra $\mathfrak{A}_1$ equal as a vector space $\mathfrak{A} \oplus \mathbb{C}$ with the product

$$(\lambda, A)(\mu, B) := (\lambda\mu, \lambda B + \mu A + AB).$$

Then $\mathfrak{A}_1$ is a unital algebra and $\mathfrak{A}$ is its ideal of codimension 1.

If $\mathfrak{A}$ is non-unital, $\mathfrak{B}$ is unital and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then there exists a unique extension $\pi_1 : \mathfrak{A}_1 \to \mathfrak{B}$ such that $\pi(1) = 1$.

1.8 Left regular representation

The so-called left regular representation

$$\mathfrak{A} \ni A \mapsto \lambda(A) \in L(\mathfrak{A})$$

is defined by

$$\lambda(A)B := AB, \quad A, B \in \mathfrak{A}.$$

If $\mathfrak{A}$ is unital, then λ is injective. If $\mathfrak{A}$ is not unital, then λ can be extended to a representation

$$\mathfrak{A} \ni A \mapsto \lambda_1(A) \in L(\mathfrak{A}_1)$$

in the obvious way, which is injective.

In any case, we see that every algebra is isomorphic to a concrete algebra.

1.9 Invertible elements

Let $\mathfrak{A}$ be an algebra. $A \in \mathfrak{A}$ is invertible in $\mathfrak{A}$ iff there exists an element $A^{-1} \in \mathfrak{A}_1$ such that

$$AA^{-1} = A^{-1}A = 1$$

Theorem 1.5 *If $\mathfrak{I} \subset \mathfrak{A}$ is a proper left or right ideal, then no elements of $\mathfrak{I}$ are invertible.*

Theorem 1.6 *1. If A is invertible, its inverse is unique*

2. If A, B are invertible, then

$$(AB)^{-1} = B^{-1}A^{-1}, \quad A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}.$$

1.10 Invertible elements in commutative algebras

Theorem 1.7 *Let $\mathfrak{A}$ be a commutative unital algebra. Let $A \in \mathfrak{A}$ be non-invertible. Then*

1. $\mathfrak{I} := \{AB \ : \ B \in \mathfrak{A}\}$ is a proper ideal;

2. There exists a maximal ideal containing A;

Proof. Clearly, $\mathfrak{I}$ is an ideal such that $1 \notin \mathfrak{I}$. This shows 1. 2. follows from Theorem **??**. □

1.11 Spectrum, resolvent

Let $\mathfrak{A}$ be a nital algebra and $A \in \mathfrak{A}$. The resolvent set of A, denoted $\mathrm{rs}_{\mathfrak{A}}(A)$, is defined as

$$\mathrm{rs}_{\mathfrak{A}}(A) := \{z \in \mathbb{C} \ : \ z - A \text{ is invertible in } \mathfrak{A}\}. \tag{1.1}$$

The spectrum of A, denoted $\mathrm{sp}_{\mathfrak{A}}(A)$ is defined as

$$\mathrm{sp}_{\mathfrak{A}}(A) := \mathbb{C} \setminus \mathrm{rs}_{\mathfrak{A}}(A).$$

If $\mathfrak{A}$ is non-unital, then in (1.1) we demand that $(z - A)$ is invertible n $\mathfrak{A}_1$.

Theorem 1.8 *1. $\mathrm{sp}_{\mathfrak{A}}(AB) \cup \{0\} = \mathrm{sp}_{\mathfrak{A}}(BA) \cup \{0\}$.*

2. If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and $B \in \mathfrak{B}$, then $\mathrm{sp}_{\mathfrak{B}}(B) \supset \mathrm{sp}_{\mathfrak{A}}(B)$.

3. If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a unital homomorphism, then $\mathrm{sp}_{\mathfrak{A}}(A) \supset \mathrm{sp}_{\pi(\mathfrak{A})}(\pi(A))$.

Proof. To prove 1., let $z \notin \mathrm{sp}(AB) \cup \{0\}$. Let $C := (z - AB)^{-1}$. Then $z^{-1}(1 + BCA)$ is the inverse of $z - BA$. □

A is nilpotent of degree n iff $A^n = 0$. A is quasinilpotent iff $\mathrm{sp}_{\mathfrak{A}}(A) = \{0\}$.

Theorem 1.9 *1. Every nilpotent A is quasinilpotent.*

2. If P is an idempotent not equal to 0 or 1, then $\mathrm{sp}_{\mathfrak{A}}(P) = \{0, 1\}$.

Proof. 1. Let $z \neq 0$. Then $\sum_{j=0}^{\infty} z^{-j-1}A^j$ is a finite sum and is the inverse of $(z - A)$.

2. We check that for $z \neq 0, 1$, $(z-1)^{-1}P + z^{-1}(1 - P)$ is the inverse of $z - P = (z-1)P + z(1 - P)$. □

Theorem 1.10 *Let $A, B \in \mathfrak{A}$, $z \in \mathrm{rs}(A)$ and $AB = BA$. Then*

$$B(z-A)^{-1} = (z-A)^{-1}B.$$

Proof. We have

$$(z-A)B = B(z-A).$$

Then we multiply both sides by $(z-A)^{-1}$. □

If $A \in \mathfrak{A}$, we define

$$\widetilde{\mathrm{Alg}}_{\mathfrak{A}}(A) := \mathrm{Alg}(\{A, (z-A) \ : \ z \in \mathrm{rs}_{\mathfrak{A}}(A)\}).$$

Clearly, by Theorem 1.10, $\widetilde{\mathrm{Alg}}_{\mathfrak{A}}(A)$ is commutative and contains 1.

1.12 Functional calculus

Let $K \subset \mathbb{C}$. The rational functions with poles outside K form a commutative algebra that we denote $R(K)$. If $f \in R(K)$, then $f(z) = p(z)q(z)^{-1}$, where $p(z)$, $q(z)$ are polynomials and $q(z) = (z-\lambda_1)^{m_1}\cdots(z-\lambda_n)^{m_n}$ with $z_i \notin K$.

Let $\mathfrak{A}$ be a unital algebra, $A \in \mathfrak{A}$ and $f \in R(\mathrm{sp}(A))$. We define

$$f(A) := p(A)(A-z_1)^{-m_1}\cdots(A-z_n)^{-m_n}. \tag{1.2}$$

Note that the commutativity of $\widetilde{\mathrm{Alg}}(A)$ guarantees that (1.2) does not depend on the order of $\lambda_1, \dots, \lambda_n$.

Theorem 1.11

$$R(\mathrm{sp}(A)) \ni f \mapsto f(A) \in \widetilde{\mathrm{Alg}}(A) \subset \mathfrak{A} \tag{1.3}$$

is a unital homomorphism. Moreover,

1. *if*

$$R(\mathrm{sp}(A)) \ni f \mapsto \pi(f) \in \mathfrak{A} \tag{1.4}$$

is a unital homomorphism satisfying $\pi(\mathrm{id}) = A$, where $\mathrm{id}(z) = z$, then $\pi(A) = f(A)$;

2. $\mathrm{sp}(f(A)) = f(\mathrm{sp}(A))$;

3. $g \in R(f(\mathrm{sp}(A))) \Rightarrow g \circ f(A) = g(f(A))$.

Proof. Elementary reasoning shows that (1.3) is a homomorphism.

Let us show 1. To this end it is sufficient to show tha if If $\lambda \in \mathrm{rs}(A)$, then

$$\pi(\lambda - \mathrm{id})^{-1}) = (\lambda - A)^{-1}. \tag{1.5}$$

We know that $\pi(\lambda - 1) = \lambda - A$. Moreover, $(\lambda - \mathrm{id})^{-1} \in R(\mathrm{sp}(A))$ and $(\lambda - \mathrm{id})^{-1}(\lambda - \mathrm{id}) = 1$. Hence

$$\pi((\lambda - \mathrm{id})^{-1})(\lambda - A) = (\lambda - A)\pi((\lambda - \mathrm{id})^{-1}) = 1.$$

Therefore, (1.5) is true.

Let us prove 2. First we show that if $f \in R(\mathrm{sp}(A))$, then

$$\mathrm{sp}(f(A)) \subset f(\mathrm{sp}(A)). \tag{1.6}$$

If $\mu \notin f(\mathrm{sp}(A)))$, then $z \mapsto f(z) - \mu \neq 0$ on $\mathrm{sp}(A)$. Therefore $z \mapsto (f(z) - \mu)$ belongs to $R(\mathrm{sp}(A))$ Therefore $(f(A) - \mu)^{-1}$ exists in $\mathfrak{A}$. Hence $\mu \in \mathrm{rs}(f(A))$, which means that (1.6) is true.

Let us prove

$$\mathrm{sp}(f(A)) \supset f(\mathrm{sp}(A)). \tag{1.7}$$

Let $\mu \notin \mathrm{sp}(f(A))$. Obviously, $(f(A)-\mu)^{-1}$ exists in $\mathfrak{A}$. Let $\lambda \in \mathbb{C}$, $\mu = f(\lambda)$. Then $z \mapsto g(z) := (f(z)-\mu)(z-\lambda)^{-1}$ belongs to $R(\mathrm{sp}(A)$. Hence $g(A)$ is well defined as an element of $\mathfrak{A}$.

Then we check that $g(A)(f(A)-\mu)^{-1}$ is the inverse of $\lambda - A$. Hence $\lambda \notin \mathrm{sp}(A)$. Therefore $\mu \notin f(\mathrm{sp}(A))$, which proves (1.7). □

1.13 Characters

A nonzero homomorphism of $\mathfrak{A}$ into $\mathbb{C}$ is called a character. We define $\mathrm{Char}(\mathfrak{A})$ to be the set of characters of $\mathfrak{A}$. For any $A \in \mathfrak{A}$ let $\hat{A} \in C(\mathrm{Char}(\mathfrak{A}))$ be the function

$$\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \hat{A}(\phi) := \phi(A) \in \mathbb{C}. \tag{1.8}$$

$\mathrm{Char}(\mathfrak{A})$ is endowed with the weakest topology such that (1.8) is continuous for any $A \in \mathfrak{A}$. Note that thus $\mathrm{Char}(\mathfrak{A})$ becomes a Hausdorff space and a net (ϕ_α) in $\mathrm{Char}(\mathfrak{A})$ converges to $\phi \in \mathrm{Char}(\mathfrak{A})$ iff for any $A \in \mathfrak{A}$, $\phi_\alpha \to \phi(A)$.

Theorem 1.12

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C(\mathrm{Char}(\mathfrak{A})) \tag{1.9}$$

is a homomorphism. Moreover, the range of (1.9) separates points.

Proof. Let $A, B \in \mathfrak{A}$, $\phi \in \mathrm{Char}(\mathfrak{A})$. Then

$$\hat{A}(\phi)\hat{B}(\phi) = \phi(A)\phi(B) = \phi(AB) = \widehat{AB}(\phi).$$

If $\phi, \psi \in \mathrm{Char}(\mathfrak{A})$. If $\phi \neq \psi$, then there exists $A \in \mathfrak{A}$ such that $\phi(A) \neq \psi(A)$. Hence the range of (1.9) separates points. □

Theorem 1.13 *Let $\pi : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism. Then $\pi^{\#} : \mathrm{Char}(\mathfrak{B}) \to \mathrm{Char}(\mathfrak{A})$ defined by*

$$(\pi^{\#}\psi)(A) := \psi(\pi(A)), \quad \psi \in \mathrm{Char}(\mathfrak{B}),$$

is continuous.

Proof. Let (ψ_i) be a net in $\mathrm{Char}(\mathfrak{B})$ converging to $\psi \in \mathfrak{B}$. Let $A \in \mathfrak{A}$. Then

$$\pi^{\#}(\psi_i)(A)\psi_i(\pi(A)) \to \psi(\pi(A)) = \pi^{\#}(\psi)(A).$$

Hence $\pi^{\#}(\psi_i) \to \pi^{\#}(\psi)$. □

Theorem 1.14 *Let $\mathfrak{A}$ be an algebra*

1. *If $\phi \in \mathrm{Char}(\mathfrak{A})$, then $\mathrm{Ker}\phi$ is a maximal ideal.*

2. *Let $\mathfrak{A}$ be unital and $\phi \in \mathrm{Char}(\mathfrak{A})$. Then $\phi(1) = 1$.*

3. *If $\mathfrak{A}$ is unital and $\mathfrak{I}$ is an ideal of codimension 1, then there exists a unique character ϕ such that $\mathfrak{I} = \ker\phi$.*

Proof. 1. $\ker \phi$ is an ideal, because ϕ is a homomorphism. It is of codimension 1 because ϕ is a linear functional onto $\mathbb{C}$.

2. is obvious

3. If $A \in \mathfrak{I}$ and $\lambda \in \mathbb{C}$ we set $\phi(A + \lambda) := \lambda$. $\square$

Theorem 1.15 *Let $\mathfrak{A}$ be a non-unital algebra. For any $\phi \in \mathrm{Char}(\mathfrak{A})$ there exists a unique extenion of ϕ to a character ϕ_1 on $\mathfrak{A}_1$. It is given by $\phi_1(\lambda + A) = \lambda + \phi(A)$. Moreover, on $\mathfrak{A}_1$ there exists a unique $\psi \in \mathrm{Char}(\mathfrak{A}_1)$ such that $\mathrm{Ker}\psi = \mathfrak{A}$. The map*

$$\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \phi_1 \in \mathrm{Char}(\mathfrak{A}_1)\backslash\{\psi\}$$

is a homeomorphism.

Theorem 1.16 *If $A \in \mathfrak{A}$, then*

$$\mathrm{sp}(A) \supset \{\phi(A) \ : \ \phi \in \mathrm{Char}(\mathfrak{A})\}.$$

and $\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \phi(A) \in \mathrm{sp}_{\mathfrak{A}}(A)$ is a continuous map

Proof. If $\mathfrak{A}$ is non-unital, then we adjoin the identity and extend ϕ appropriately.

Let $\phi \in \mathrm{Char}(\mathfrak{A})$ and $\phi(A) = \lambda$. Then $\phi(A - \lambda) = 0$. Hence $A - \lambda$ belongs to a maximal ideal. Hence it is not invertible. Hence $\lambda \in \mathrm{sp}(A)$. $\square$

2 Banach algebras

2.1 Banach algebras

An algebra $\mathfrak{A}$ over $\mathbb{C}$ is called a normed algebra, if it is equipped with a norm $\mathfrak{A} \ni A \mapsto \|A\| \in \mathbb{R}$ such that

$$\|AB\| \leq \|A\|\|B\|.$$

It is called a Banach algebra if it is complete in the norm $\|\cdot\|$.

If $\mathfrak{A}$ is a Banach algebra, then every norm closed subalgebra of $\mathfrak{A}$ is a Banach algebra. If $\mathfrak{B} \subset \mathfrak{A}$, then the smallest Banach algebra containing $\mathfrak{B}$ is denoted by $\mathrm{Ban}(\mathfrak{B})$.

Let $\mathcal{X}$ be a Banach space. Recall that $B(\mathcal{X})$ denotes the set of bounded operators on $\mathcal{X}$. Clearly, $B(\mathcal{X})$ is a Banach algebra. Every norm closed subalgebra of $B(\mathcal{X})$ is a Banach algebra. Such Banach algebras are called concrete Banach algebras.

2.2 Ideals and quotient algebras

Theorem 2.1 *Let $\mathfrak{I}$ be a two-sided closed ideal of $\mathfrak{A}$. Then $\mathfrak{A}/\mathfrak{I}$ is a Banach algebra and*

$$\mathfrak{A} \ni A \mapsto A + \mathfrak{I} \in \mathfrak{A}/\mathfrak{I} \tag{2.10}$$

is a norm-decreasing homorphism of Banach algebras with the kernel equal to $\mathfrak{I}$.

Proof. By Theorem 1.1 we know that (2.10) is a homomorphism of algebras with the kernel equal to $\mathfrak{I}$. We also know that $\mathfrak{A}/\mathfrak{I}$ is a Banach space and (2.10) is norm decreasing. Hence it is enough to show that

$$\|(A + \mathfrak{I})(B + \mathfrak{I})\| \leq \|A + \mathfrak{I}\|\|B + \mathfrak{I}\|.$$

$\square$

2.3 Maximal ideals

Theorem 2.2 *Let $\mathfrak{I}$ be a maximal ideal in a unital Banach algebra $\mathfrak{A}$. Then $\mathfrak{I}$ is closed.*

Proof. Invertible elements do not belong to any proper ideal. But a neighborhood of 1 consists of invertible elements. Hence the closure of any proper ideal does not contain 1.

By the continuity of operations, the closure of an ideal is an ideal. Hence if $\mathfrak{I}$ is any proper ideal, then $\mathfrak{I}^{\mathrm{cl}}$ is also a proper ideal. □

2.4 Unit

Let $\mathfrak{A}$ be a unital Banach algebra. Then $\|1\| \geq 1$. Besides, if λ is the regular representation, then

$$\|A\| \leq \|\lambda(A)\| \|1\| \leq \|1\| \|A\|.$$

Thus the norms $\|A\|$ an $\|\lambda(A)\|$ are equivalent. Note also that $\|\lambda(1)\| = 1$.

This means, that if $\mathfrak{A}$ is a unital Banach algebra, then by replacing the initial norm with the equivalent norm $\lambda(A)$ we can always assume that $\|1\| = 1$. We will make always this assumption.

Theorem 2.3 *If $\mathfrak{A}$ is a unital Banach algebra such that $\|1\| = 1$, then the regular representation $\mathfrak{A} \ni A \mapsto \lambda(A) \in L(\mathfrak{A})$ is isometric.*

Theorem 2.4 *Let $\mathfrak{A}$ be a Banach algebr. Equip $\mathfrak{A}_1$ with the norm*

$$\|\lambda + A\| := \|A\| + |\lambda|.$$

Then $\mathfrak{A}_1$ is a Banach algebra and $\mathfrak{A} \to \mathfrak{A}_1$ is an isometry.

Note, however, that there maybe other (even more natural) norms on $\mathfrak{A}_1$ extending the norm on $\mathfrak{A}$.

Theorem 2.5 *Let $\mathfrak{A}$ be a unital Banach algebra. Then 1 is an extreme point of the unit ball $(\mathfrak{A})_1$.*

Proof. $\mathfrak{A}$ can be isometrically embedded in a Banach space. Hence the theorem follows from tha fact that if $\mathcal{X}$ is a Banach space, then 1 is an extreme point in $(B(\mathcal{X}))_1$. □

2.5 Approximate units

If $\mathfrak{A}$ does not have a unit, then we can use the so-called approximate unit.

We say that a net $(E_\alpha) \subset (\mathfrak{A})_1$ is a left appproximate unit in $\mathfrak{A}$, if for any $A \in \mathfrak{A}$, $\|E_\alpha A - A\| \to 0$.

Let $\mathfrak{A}$ be a Banach algebra without a unit. In $\mathfrak{A}_1$ we define

$$\|\lambda + A\|_1 := \sup_{\|B\| \leq 1} \|\lambda B + AB\|.$$

Clearly, $\|\cdot\|_1$ is a seminorm satisfying

$$\|(\lambda + A)(\mu + B)\|_1 \leq \|(\lambda + A)\|_1 \|(\mu + B)\|_1.$$

Theorem 2.6 *If $\mathfrak{A}$ possesses an approximate unit, then $\|\cdot\|_1$ is a norm and $\mathfrak{A} \to \mathfrak{A}_1$ is an isometry. Moreover, the regular representation $\mathfrak{A} \ni A \mapsto \lambda(A) \in L(\mathfrak{A})$ is isometric.*

Recall that

$$\|A + \mathfrak{I}\| := \inf\{\|A + I\| \ : \ I \in \mathfrak{I}\}.$$

Theorem 2.7 *Suppose that $\mathfrak{J}$ is an ideal in $\mathfrak{A}$ and $\mathfrak{J}$ possesses a left approximate unit (E_α) such that*

$$\|1 - E_\alpha\| \leq 1. \tag{2.11}$$

Then the norm in $\mathfrak{A}/\mathfrak{J}$ is given by

$$\|A + \mathfrak{J}\| = \lim_\alpha \|A(1 - E_\alpha)\|.$$

Proof. Let $A \in \mathfrak{A}$ and $I \in \mathfrak{J}$. Using first $\|E_\alpha I - I\| \to 0$ and then $\|1 - E_\alpha\| \leq 1$, we get

$$\limsup_\alpha \|(1 - E_\alpha)A\| = \limsup_\alpha \|(1 - E_\alpha)(A + I)\| \leq \|A + I\|.$$

Hence

$$\limsup_\alpha \|(1 - E_\alpha)A\| \leq \inf\{\|A + I\| \ : \ I \in \mathfrak{J}\}.$$

Moreover,

$$\begin{aligned} \liminf_\alpha \|(1 - E_\alpha)A\| &\geq \inf \|(1 - E_\alpha)A\| \\ &\geq \inf\{\|A + I\| \ : \ I \in \mathfrak{J}\}. \end{aligned}$$

□

2.6 Invertible elements

Let $\mathfrak{A}$ be a unital Banach algebra.

Theorem 2.8 *Let $A \in \mathfrak{A}$ be invertible and $B \in \mathfrak{A}$ such that*

$$\|BA^{-1}\| \leq 1.$$

Then $A + B$ is invertible and

$$(A + B)^{-1} = \sum_{j=0}^{\infty} (-1)^j A^{-1} (BA^{-1})^j.$$

Moreover,

$$\begin{aligned} &\|(A + B)^{-1}\| \leq \|A^{-1}\|(1 - \|BA^{-1}\|)^{-1}, \\ &\|A^{-1} - (A + B)^{-1}\| \leq \|A^{-1}BA^{-1}\|(1 - \|BA^{-1}\|)^{-1}. \end{aligned}$$

In particular, invertible elements form an open subset of $\mathfrak{A}$ on which the inverse is a continuous function.

2.7 Spectrum

Theorem 2.9 *Let $A \in \mathfrak{A}$. Then*

1. *If $\|(\lambda - A)^{-1}\| = c$, then $\{z : \ |z - \lambda| < c^{-1}\} \subset \mathrm{rs}_{\mathfrak{A}}(A)$. Consequently, $\|(z - A)^{-1}\| \geq (\mathrm{dist}(z, \mathrm{sp}_{\mathfrak{A}}(A)))^{-1}$.*
2. *$\{|z| > \|A\|\}$ is contained in $\mathrm{rs}_{\mathfrak{A}}(A)$ and $\|(z - A)^{-1}\| \geq (\mathrm{dist}(z, \mathrm{sp}_{\mathfrak{A}}(A)))^{-1}$.*
3. *$\mathrm{sp}_{\mathfrak{A}}(A)$ is a compact subset of $\mathbb{C}$.*
4. *$(z - A)^{-1}$ is analytic on $\mathrm{rs}_{\mathfrak{A}}(A)$*

5. $\mathrm{sp}_{\mathfrak{A}}(A) \neq \emptyset$

Proof. 1. For $|z-\lambda| < c^{-1}$, we have $\|(z-\lambda)(\lambda - A)^{-1}\| = |z-\lambda|c < 1$ Hence we can apply Theorem 2.8.

2.We check that $\sum_{n=0}^{\infty} z^{-n-1}A^n$ is convergent for $|z| > \|A\|$ and equals $(z-A)^{-1}$.

3. follows from 1. and 2.

4. We check that the resolvent is differentiable in the complex sense:

$$h^{-1}\big((z+h-A)^{-1} - (z-A)^{-1}\big) = -((z+h-A)^{-1}(z-A)^{-1} \to -(z-A)^{-2}.$$

5. $(z-A)^{-1}$ is an analytic function tending to zero at infinity. Hence it cannot be analytic everywhere, unless it is zero, which is impossible. □

Theorem 2.10 *Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ and $1, A \in \mathfrak{B}$.*

1. *$\mathrm{rs}_{\mathfrak{B}}(A)$ is an open and closed subset of $\mathrm{rs}_{\mathfrak{A}}(A)$ containing a neighborhood of ∞.*
2. *The connected components of $\mathrm{rs}_{\mathfrak{A}}(A)$ and of $\mathrm{rs}_{\mathfrak{B}}(A)$ containing a neighborhood of infinity coincide.*
3. *If $\mathrm{rs}_{\mathfrak{A}}(A)$ is connected, then $\mathrm{rs}_{\mathfrak{A}}(A) = \mathrm{rs}_{\mathfrak{B}}(A)$.*

Proof. $\mathrm{rs}_{\mathfrak{B}}(A)$ is open in $\mathbb{C}$. Hence also in $\mathrm{rs}_{\mathfrak{A}}(A)$.

Let $z_0 \in \mathrm{rs}_{\mathfrak{A}}(A)$ and $z_n \in \mathrm{rs}_{\mathfrak{B}}(A)$, $z_n \to z_0$. Then $(z_n - A)^{-1} \to (z_0 - A)^{-1}$ in $\mathfrak{A}$, hence also in $\mathfrak{B}$. Therefore, $z_0 \in \mathrm{rs}_{\mathfrak{B}}(A)$. Hence $\mathrm{rs}_{\mathfrak{B}}(A)$ is closed in $\mathrm{rs}_{\mathfrak{A}}(A)$. This proves 1.

2. and 3. follow immediately from 1. □

If $A \in \mathfrak{A}$, define $\widetilde{\mathrm{Ban}}(A)$ to be the closed algebra generated by $\widetilde{\mathrm{Alg}}(A)$.

Theorem 2.11 1. *If $\mathrm{rs}_{\mathfrak{A}}(A)$ is connected, then $\widetilde{\mathrm{Ban}}(A) = \mathrm{Ban}(A, 1)$.*

2. *If $\mathrm{rs}_{\mathfrak{A}}(A)$ is disconnected, choose one point $\lambda_1, \lambda_2, \dots$ in every connected component of $\mathrm{rs}_{\mathfrak{A}}(A)$ that does not contan a neighborhood of infinity. Then $\widetilde{\mathrm{Ban}}(A) = \mathrm{Ban}(A, (\lambda_1 - A)^{-1}, (\lambda_2 - A)^{-1}, \dots)$.*

2.8 Spectral radius

Spectral radius of $A \in \mathfrak{A}$ is defined as

$$\mathrm{sr}_{\mathfrak{A}}(A) := \sup_{\lambda \in \mathrm{sp}_{\mathfrak{A}}(A)} |\lambda|.$$

Note that by Theorem 2.10, if $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$ containing A, then $\mathrm{sr}_{\mathfrak{B}}(A) = \mathrm{sr}_{\mathfrak{A}}(A)$. Hence, in what follows, we will denote the spectral radius simply by $\mathrm{sr}(A)$.

Lemma 2.12 *Let a sequence of reals (c_n) satisfy*

$$c_n + c_m \geq c_{n+m}.$$

Then

$$\lim_{n\to\infty} \frac{c_n}{n} = \inf \frac{c_n}{n}.$$

Proof. Fix $m \in \mathbb{N}$. Let $n = mq + r$, $r < m$. We have

$$c_n \leq qc_m + c_r.$$

So

$$\frac{c_n}{n} \leq \frac{qc_m}{n} + \frac{c_r}{n}.$$

Hence

$$\limsup_{n\to\infty} \frac{c_n}{n} \leq \frac{c_m}{m}.$$

Thus,

$$\limsup_{n\to\infty} \frac{c_n}{n} \leq \inf \frac{c_m}{m}.$$

□

Theorem 2.13 *Let $A \in \mathfrak{A}$. Then*

$$\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

exists and equals $\mathrm{sr}(A)$. *Besides,* $\mathrm{sr}(A) \leq \|A\|$.

Proof. Let

$$c_n := \log \|A^n\|.$$

Then

$$c_n + c_m \geq c_{n+m}$$

Hence there exists

$$\lim_{n\to\infty} \frac{c_n}{n}.$$

Consequently, there exists

$$r := \lim_{n\to\infty} \|A^n\|^{1/n}.$$

By the d'Alembert criterion, the series

$$\sum_{n=0}^{\infty} A^n z^{-1-n}. \tag{2.12}$$

is absolutely convergent for $|z| > r$, and divergent for $|z| < r$. We easily check that (2.12) equals $(z-A)^{-1}$.
□

2.9 Functional calculus

Let $K \subset \mathbb{C}$ be compact. By $A(K)$ let us denote the set of analytic functions on a neighborhood of K. It is a commutative algebra.

Definition 2.14 *Let $f \in A(\mathrm{sp}(A))$. Let γ be a contour in the domain of f that encircles* $\mathrm{sp}(A)$ *counterclockwise. We define*

$$f(A) := \frac{1}{2\pi i} \int_\gamma (z - A)^{-1} f(z) \mathrm{d}z \tag{2.13}$$

Clearly, the definition is independent of the choice of the contour

Theorem 2.15 *Let $A \in \mathfrak{A}$. Then*

$$A(\mathrm{sp}(A)) \ni f \mapsto f(A) \in \widetilde{\mathrm{Ban}}(A) \subset \mathfrak{A} \tag{2.14}$$

is a unital homomorphism satisfying

1. $\mathrm{id}(A) = A$, *for* $\mathrm{id}(z) = z$;
2. *If* $f \in R(\mathrm{sp}(A))$, *then* $f(A)$ *coincides with* $f(A)$ *defined in (1.2);*
3. *If* $f(z) := \sum_{n=0}^{\infty} f_n z^n$ *is an analytic function defined by a series absolutely convergent in a disk of radius greater than* $\mathrm{sr}(A)$, *then*
$$f(A) = \sum_{n=0}^{\infty} f_n A^n;$$
4. $\mathrm{sp}(f(A)) = f(\mathrm{sp}(A))$
5. $g \in A(f(\mathrm{sp}(A))) \Rightarrow g \circ f(A) = g(f(A))$,
6. $\|f(A)\| \leq c_{\gamma,A} \sup_{z\in\gamma} |f(z)|$.

Proof. From the formula

$$(z - A)^{-1} = \sum_{n=0}^{\infty} z^{-n-1} A^n, \quad |z| > \mathrm{sr}(A),$$

we get that $1(A) = 1$ and $\mathrm{id}(A) = A$.

It is clear that $f \to f(A)$ is linear. Let us show that it is multiplicative Let $f_1, f_2 \in A(\mathrm{sp}(A)$. Choose a contour γ_2 around the contour γ_1, both in the domains of f_1 and f_2.

$$\begin{aligned}
&\int_{\gamma_1} f_1(z_1)(z_1 - A)^{-1}\mathrm{d}z_1 \int_{\gamma_2} f_2(z_2)(z_2 - A)^{-1}\mathrm{d}z_2 \\
&= \int_{\gamma_1}\int_{\gamma_2} f_1(z_1)f_2(z_2)\left((z_1 - A)^{-1} - (z_2 - A)^{-1}\right)(z_2 - z_1)^{-1}\mathrm{d}z_1\mathrm{d}z_2 \\
&= \int_{\gamma_1} f_1(z_1)(z_1 - A)^{-1}\mathrm{d}z_1 \int_{\gamma_2} (z_2 - z_1)^{-1} f_2(z_2)\mathrm{d}z_2 \\
&+ \int_{\gamma_2} f_2(z_2)(z_2 - A)^{-1}\mathrm{d}z_2 \int_{\gamma_1} (z_1 - z_2)^{-1} f_1(z_1)\mathrm{d}z_1.
\end{aligned}$$

But

$$\begin{aligned}
&\int_{\gamma_1} (z_1 - z_2)^{-1} f_1(z_1)\mathrm{d}z_1 = 0, \\
&\int_{\gamma_2} (z_2 - z_1)^{-1} f_2(z_2)\mathrm{d}z_2 = 2\pi\mathrm{i} f_2(z_1).
\end{aligned}$$

Thus

$$f_1(A)f_2(A) = f_1 f_2(A). \tag{2.15}$$

Let us prove 4. First we will show

$$\mathrm{sp}(f(A)) \subset f(\mathrm{sp}(A)). \tag{2.16}$$

If $\mu \notin f(\mathrm{sp}(A))$, then the function $z \mapsto f(z) - \mu \neq 0$ on $\mathrm{sp}(A)$. Therefore, $z \mapsto (f(z) - \mu)^{-1}$ belongs to $A(\mathrm{sp}(A))$. Thus there exists $(f(A) - \mu)^{-1}$ in $\mathfrak{A}$ and therefore, $\mu \notin \mathrm{sp}(f(A))$. This implies (2.16).

Let us now show

$$\mathrm{sp}(f(A)) \supset f(\mathrm{sp}(A)). \tag{2.17}$$

Let $\mu \notin \mathrm{sp}(f(A))$. This clearly implies that $(f(A) - \mu)^{-1}$ exists as an element of $\mathfrak{A}$.

If μ does not belong to the image of f, then of course it does not belong to $f(\mathrm{sp}(A))$. Let us assume that $\mu = f(\lambda)$. Then the function

$$z \mapsto g(z) := (f(z) - \mu)(z - \lambda)^{-1}$$

belongs to $A(\mathrm{sp}(A))$. Hence $g(A)$ is well defined as an element of $\mathfrak{A}$. We check that $g(A)(f(A) - f(\lambda))^{-1} = (\lambda - A)^{-1}$. Hence $\lambda \not\in \mathrm{sp}(A)$. Thus $\mu \not\in f(\mathrm{sp}(A))$. Consequently, (2.17) holds.

Let us show now 5. Notice that if $w \not\in f(\mathrm{sp}(A))$, then the function $z \mapsto (w - f(z))^{-1}$ is analytic on a neighborhood of

$$(w - f(A))^{-1} = \frac{1}{2\pi i} \int_\gamma (w - f(z))^{-1}(z - A)^{-1} \mathrm{d}z.$$

We compute

$$\begin{aligned}
& g(f(A)) \\
&= \tfrac{1}{2\pi i} \textstyle\int_{\tilde\gamma} g(w)(w - f(A))^{-1} \mathrm{d}w \\
&= \tfrac{1}{(2\pi i)^2} \textstyle\int_{\tilde\gamma} \int_\gamma g(w)(w - f(z))^{-1}(z - A)^{-1} \mathrm{d}w \mathrm{d}z \\
&= \tfrac{1}{2\pi i} \textstyle\int_\gamma (z - A)^{-1} \mathrm{d}z \int_{\tilde\gamma} g(w)(w - f(z))^{-1} \mathrm{d}w \\
&= \tfrac{1}{2\pi i} \textstyle\int_\gamma g(f(z))(z - A)^{-1} \mathrm{d}z.
\end{aligned}$$

□

2.10 Spectral idempotents

Let Ω be a subset of $B \subset \mathbb{C}$. Ω_1 will be called an isolated subset of B, if $\Omega \cap (\mathrm{sp}(A) \backslash \Omega)^{\mathrm{cl}}$ and $\Omega^{\mathrm{cl}} \cap (\mathrm{sp}(A) \backslash \Omega) = \emptyset$ (or Ω is closed and open in the relative topology of B).

Let $\Omega \subset \mathrm{sp}(A)$ be an isolated subset of $\mathrm{sp}(A)$. We define the function

$$1_\Omega(z) := \begin{cases} 1 & z \text{ belongs to a neighborhood of } \Omega, \\ 0 & z \text{ belongs to a neighborhood of } \mathrm{sp}(A) \backslash \Omega. \end{cases}$$

Clearly, $1_\Omega \in A(\mathrm{sp}(A))$ and $1_\Omega^2 = 1_\Omega$ Hence $1_\Omega(A)$ is an idempotent.

If γ is a contour around Ω outside of $\mathrm{sp}(A) \backslash \Omega$, then

$$1_\Omega(A) := \frac{1}{2\pi \mathrm{i}} \int_\gamma (z - A)^{-1} \mathrm{d}z$$

This operator will be called the spectral idempotent of the operator A onto Ω. It commutes with A. If $\mathfrak{B} := 1_\Omega(A) \mathfrak{A} 1_\Omega(A)$ is the reduced algebra, then

$$\mathrm{sp}_{\mathfrak{B}}(A 1_\Omega(A)) = \mathrm{sp}(A) \cap \Omega.$$

If Ω_1 i Ω_2 are two isolated subsets of $\mathrm{sp}(A)$, then

$$1_{\Omega_1}(A) 1_{\Omega_2}(A) = 1_{\Omega_1 \cap \Omega_2}(A)$$

2.11 Isolated eigenvalues

Assume now that λ is an isolated point of $\mathrm{sp}(A)$. Set

$$P := 1_\lambda(A), \qquad N := (A - \lambda) P.$$

Definition 2.16 *We say that λ is a semisimple eigenvalue if $N = 0$ If $N^n = 0$, then we say that λ is nilpotent of degree n. It is easy to see that if $A \in L(\mathcal{X})$, then the degree of nilpotence of λ is less than or equal to* $\dim P$.

Proposition 2.17 *The operator N is quasinilpotent, satisfies $PN = NP = N$ and can be written as*

$$N = f(A), \quad f(z) := (z - \lambda)1_\lambda(z). \tag{2.18}$$

Besides,

$$(z - A)^{-1}P = (z - \lambda)^{-1}P + \sum_{j=1}^{\infty} N^j (z - \lambda)^{-j+1}.$$

and $(z - A)^{-1}(1 - P)$ is analytic in the neighborhood of λ. If N is nilpotent of degree n, then there exist $\delta > 0$ and C such that

$$\|(z - A)^{-1}\| \le C|z - \lambda|^{-n}, \quad z \in B(\lambda, \delta). \tag{2.19}$$

Proof. Clearly, $AP = A1_\lambda(A)$ and $\lambda P = \lambda 1_\lambda(A)$. This shows (2.18). Then note that $f(z) = 0$ for $z \in \mathrm{sp}(A)$. Hence $\mathrm{sp}(N) = \{0\}$.

Using the Laurent series expansion we get

$$(z - A)^{-1} = \sum_{n=-\infty}^{\infty} C_n (z - \lambda)^n,$$

where

$$C_n = \frac{1}{2\pi \mathrm{i}} \int_\gamma (z - A)^{-1} (z - \lambda)^{-n-1} \mathrm{d}z.$$

Clearly, $C_{-1} = P$ and $C_{-2} = N$. Besides, by Theorem 2.15 we obtain

$$C_{-1-n}C_{-1-m} = C_{-1-n-m}.$$

□

2.12 Characters

Theorem 2.18 *Let $\mathfrak{A}$ be a unital algebra.*

1. *Let ϕ be a character on $\mathfrak{A}$. Then it is continuous and $\|\phi\| = 1$.*
2. $\mathrm{Char}(\mathfrak{A})$ *is a compact space.*
3. *The Gelfand transform*
$$\mathfrak{A} \ni A \mapsto \hat{A} \in C(\mathrm{Char}(A))$$
is a norm decreasing homomorphism of Banach algebras.

Proof. 1. $\ker \phi$ is a maximal ideal. Hence it is closed. Hence ϕ is continuous.

Suppose that $\|\phi\| > 1$. Then for some $A \in \mathfrak{A}$, $\|A\| < 1$ we have $|\phi(A)| > 1$. Now $A^n \to 0$ and $|\phi(A^n)| = |\phi(A)|^n \to \infty$, which means that ϕ is not continuous.

2. and 3. follows asily from 1. □

2.13 Gelfand-Mazur theorem

Theorem 2.19 *Let $\mathfrak{A}$ be a unital algebra such that all non-zero elements are invertible. Then $\mathfrak{A} = \mathbb{C}$.*

Proof. Let $A \in \mathfrak{A}$. By Theorem 2.9 there exists $\lambda \in \mathrm{sp}(A)$. Thus $\lambda - A$ is not invertible. Hence $\lambda - A = 0$. Hence $A = \lambda$. □

2.14 Commutative Banach algebras

Let $\mathfrak{A}$ be a commutative unital Banach algebra.

Theorem 2.20 *Every maximal ideal has codimension 1. Hence the map*

$$\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \mathrm{Ker}\phi \in \mathrm{MI}(\mathfrak{A})$$

is a bijection.

Proof. Let ϕ be a character. Then we know that $\ker \phi$ has codimension 1 and hence is a maximal ideal by Theorem 1.4.

Conversely, let $\mathfrak{I}$ be a maximal ideal. If it has a codimension 1, then it is the kernel of a character by Theorem **??**. Thus it is sufficient to show that every maximal ideal has the codimension 1.

Let $\mathfrak{I}$ be a ideal of $\mathfrak{A}$. Then $\mathfrak{A}/\mathfrak{I}$ is a commutative Banach algebra and $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$ is a homomorphism. Assume that the codimension of $\mathfrak{I}$ is not 1. This means that $\mathfrak{A}/\mathfrak{I}$ is not $\mathbb{C}$. By the Gelfand-Mazur theorem, $\mathfrak{A}/\mathfrak{I}$ contains non-invertible elements. Every such an element is contained in a proper ideal $\mathfrak{K}$. By theorem 1.2, $\pi^{-1}(\mathfrak{K})$ is a proper ideal of $\mathfrak{A}$ containing $\mathfrak{I}$. Hence $\mathfrak{I}$ is not maximal. □

-

Theorem 2.21 *For each $A \in \mathfrak{A}$,*

$$\mathrm{sp}(A) = \{\phi(A) \;\; \phi \in \mathrm{Char}(\mathfrak{A})\}.$$

Hence

$$\mathrm{sr}(A) = \sup\{|\hat{A}(\phi)| \;:\; \phi \in \mathrm{Char}(\mathfrak{A})\} = \|\hat{A}\|.$$

Proof. The inclusion $\supset$ was proven in Theorem 1.16.

Let $z \in \mathrm{sp}(A)$. Then $(z - A)$ is not invertible. Hence there exists a maximal ideal containing $z - A$. Therefore, exists $\phi \in \mathrm{Char}(\mathfrak{A})$ such that $\phi(z - A) = 0$. Hence $z = \phi(A) = \hat{A}(\phi)$. □

Theorem 2.22 *Let $A \in \mathfrak{A}$. The following conditions are equivalent:*

1. *A belongs to the intersection of all maximal ideals;*
2. *For all $\phi \in \mathrm{Char}(\mathfrak{A})$ we have $\phi(A) = 0$*
3. *$\hat{A} = 0$;*
4. *$\mathrm{sr}(A) = 0$;*
5. *$\limsup \|A^n\|^{1/n} = 0$.*

The set of $A \in \mathfrak{A}$ satisfying the conditions of Theorem 2.22 is called the radical of $\mathfrak{A}$. It is a closed ideal of $\mathfrak{A}$.

Theorem 2.23 *Let $A \in \mathfrak{A}$ and $\mathfrak{A}$ be unital. Set $\mathfrak{C} := \widetilde{\mathrm{Ban}_{\mathfrak{A}}}(A)$. Then*

$$\mathrm{Char}(\mathfrak{C}) \ni \phi \mapsto \phi(A) \in \mathrm{sp}_{\mathfrak{A}}(A) = \mathrm{sp}_{\mathfrak{C}}(A)$$

is a homeomorphism. Thus $\mathrm{sp}_{\mathfrak{A}}(A)$ can be identified with $\mathrm{Char}(\mathfrak{C})$. If $f \in A(\mathrm{sp}_{\mathfrak{A}}(A))$ and $\phi \in \mathrm{sp}_{\mathfrak{A}}(A)$, then

$$\phi(f(A)) = f(\phi).$$

Thus

$$\mathfrak{C} \ni B \mapsto \hat{B} \in R(\mathrm{sp}(A))$$

is an extension of the inverse of (2.14).

3 C^*-algebras

3.1 $*$-algebras

Let $\mathfrak{A}$ be an algebra. A mapping $\mathfrak{A} \ni A \mapsto A^* \in \mathfrak{A}$ is an involution iff

$$A^{**} = A, \quad (AB)^* = B^*A^*, \quad (\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*.$$

An algebra with an involution is called a $*$-algebra.

Let $\mathfrak{A}$ be a $*$-algebra. $A \in \mathfrak{A}$ is invertible iff A^* is, and $(A^{-1})^* = (A^*)^{-1}$.

A subset $\mathfrak{B}$ of $\mathfrak{A}$ is called self-adjoint iff $B \in \mathfrak{B} \Rightarrow B^* \in \mathfrak{B}$.

Let $\mathfrak{A}$, $\mathfrak{B}$ be $*$-algebras. A homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is called a $*$-homomorphism iff $\pi(A^*) = \pi(A)^*$.

3.2 Special elements of a $*$-algebra

$A \in \mathfrak{A}$ is called normal if $AA^* = A^*A$. It is called self-adjoint if $A^* = A$. $\mathfrak{A}_{\mathrm{sa}}$ denotes the set of self-adjoint elements of $\mathfrak{A}$

$P \in \mathfrak{A}$ is called a projection if it is a self-adjoint idempotent. $P(\mathfrak{A})$ denotes the set of projections of $\mathfrak{A}$.

$U \in \mathfrak{A}$ is called a partial isometry iff U^*U is a projection. If this is the case, then UU^* is also a projection. U^*U is called the right support of U and UU^* is called the left support of U.

U is called an isometry if $U^*U = 1$.

U is called a unitary if $U^*U = UU^* = 1$. $U(\mathfrak{A})$ denotes the set of unitary elements of $\mathfrak{A}$.

3.3 C^*-algebras

$\mathfrak{A}$ is a C^*-algebra if it is a Banach algebra equipped with an involution $*$ satisfying

$$\|A^*A\| = \|A\|^2. \tag{3.20}$$

$$\|A^*\| = \|A\|. \tag{3.21}$$

We can weaken the conditions (3.20) and (3.21) in the definition of a C^* -algebra as follows:

Theorem 3.1 *If $\mathfrak{A}$ is a Banach algebra with an involution $*$ satisfying*

$$\|A\|^2 \leq \|A^*A\|, \tag{3.22}$$

then it is a C^*-algebra.

Proof. Clearly,

$$\|A^*A\| \leq \|A^*\|\|A\|. \tag{3.23}$$

Hence, by (3.22), $\|A\| \leq \|A^*\|$. Using $A^{**} = A$, this gives $\|A^*\| \geq \|A\|$. Hence (3.21) is true.

(3.21) and (3.23) give $\|A^*A\| \leq \|A\|^2$. This and (3.22) imply (3.20). □

Let $\mathfrak{A}$ be a fixed C^*-algebra. A subset of $\mathfrak{A}$ is a C^*-algebra iff it is a self-adjoint closed algebra. If $\mathfrak{B} \subset \mathfrak{A}$, then $C^*(\mathfrak{B})$ will denote the smallest C^*-subalgebra in $\mathfrak{A}$ containing $\mathfrak{B}$.

Let $\mathcal{X}$ be a Hilbert space. Then $B(\mathcal{X})$ is a C^*-algebra. A C^*-algebra inside $B(\mathcal{X})$ is called a concrete C^*-algebra.

Theorem 3.2 *If $1 \in \mathfrak{A}$, then $\|1\| = 1$.*

Proof. By the uniqueness of the identity, we have $1 = 1^*$. Hence $\|1\|^2 = \|1^*1\| = \|1\|$. □

3.4 Algebras without a unit

Theorem 3.3 *$\mathfrak{A}$ be an C^*-algebra without a unit. Then the algebra $\mathfrak{A}_1$ with the norm given by*

$$\|\lambda + A\| := \sup_{B\neq 0} \frac{\|\lambda B + AB\|}{\|B\|},$$

and the involution $(\lambda + A)^* := (\overline{\lambda} + A^*)$ is a C^*-algebra.

Proof. Step 1. Recall from the theory of Banach algebras that $\|\cdot\|_1$ is a seminorm on $\mathfrak{A}_1$ that satisfies

$$\|(\lambda + A)(\mu + B)\|_1 \leq \|\lambda + A\|_1\|\mu + B\|_1.$$

Step 2. We show that $\|\cdot\|_1$ coincides on $\mathfrak{A}$ with $\|\cdot\|$. In fact, $\|A\|_1 \leq \|A\|$ is obvious for any Banach algebra. The converse inequality follows by

$$\|A\|_1 \geq \frac{\|AA^*\|}{\|A^*\|} = \|A\|.$$

Step 3. For any $\mu < 1$ there exists B such that $\|B\| = 1$ and $\mu\|\lambda + A\| \leq \|\lambda B + AB\|$. Then

$$\mu^2\|\lambda + A\|^2 \leq \|\lambda B + AB\|^2 = \|B^*(\lambda + A)^*(\lambda + A)B\| \leq \|(\lambda + A)^*(\lambda + A)\|.$$

This proves

$$\|\lambda + A\|^2 \leq \|(\lambda + A)^*(\lambda + A)\|.$$

3.5 Spectral radius of a normal element

Theorem 3.4 *Let $A \in \mathfrak{A}$ be normal. Then*

$$\mathrm{sr}(A) = \|A\|.$$

Proof.

$$\|A^2\|^2 = A^{2*}A^2\| = \|(A^*A)^2\| = \|A^*A\|^2 = \|A\|^4.$$

Thus $\|A^{2^n}\| = \|A\|^{2^n}$. Hence $\|A^{2^n}\|^{2^{-n}} = \|A\|$. □

3.6 Spectrum of elements of C^*-algebras

Theorem 3.5 *Let $A \in \mathfrak{A}$.*

1. *Let V be isometric. Then* $\mathrm{sp}(V) \subset \{|z| \leq 1\}$.
2. *U is unitary iff U is normal and* $\mathrm{sp}(U) \subset \{z \ : \ |z| = 1\}$.
3. *A is self-adjoint iff A is normal and* $\mathrm{sp}(A) \subset \mathbb{R}$.

Proof. 1. We have $\|V\|^2 = \|V^*V\| = \|1\| = 1$. Hence, $\mathrm{sp}(V) \subset \{|z| \leq 1\}$.

2.$\Rightarrow$. Clearly, U is normal.

U is an isometry, hence $\mathrm{sp}(U) \subset \{|z| \leq 1\}$.

U^{-1} is also an isometry, hence $\mathrm{sp}(U^{-1}) \subset \{|z| \leq 1\}$. This implies $\mathrm{sp}(U) \subset \{|z| \geq 1\}$.

3.$\Rightarrow$. For $|\lambda^{-1}| \geq \|A\|$, $1 + \mathrm{i}\lambda A$ is invertible. We check that $U := (1 - \mathrm{i}\lambda A)(1 + \mathrm{i}\lambda A)^{-1}$ is unitary. Hence, by 2.$\Rightarrow$, $\mathrm{sp}(U) \subset \{|z| = 1\}$. By the spetral mapping theorem, $\mathrm{sp}(A) \subset \mathbb{R}$.

The proofs of 2.$\Leftarrow$ and 3.$\Leftarrow$ are postponed. □

3.7 Invariance of spectrum

Lemma 3.6 *Let A be invertible in $\mathfrak{A}$. Then A^{-1} belongs to $C^*(1, A)$.*

Proof. First assume that A is self-adjoint. Then $\mathrm{sp}_{\mathfrak{A}}(A) \subset \mathbb{R}$. Hence $\mathrm{rs}_{\mathfrak{A}}(A)$ is connected. But $\mathfrak{C} := C^*(1, A) = \mathrm{Ban}(1, A)$. Hence, by Theorem 2.10,

$$\mathrm{rs}_{\mathfrak{C}}(A) = \mathrm{rs}_{\mathfrak{A}}(A) \tag{3.24}$$

A is invertible iff $0 \in \mathrm{rs}_{\mathfrak{A}}(A)$. By (3.24), this means that $0 \in \mathrm{rs}_{\mathfrak{C}}(A)$ and hence $A^{-1} \in \mathfrak{C}$.

Next assume that A be an arbitrary invertible element of $\mathfrak{A}$. Clearly, A^* is invertible in $\mathfrak{A}$ and $(A^*)^{-1} = (A^{-1})^*$. Likewise, A^*A is invertible in $\mathfrak{A}$ and $(A^*A)^{-1} = (A^*)^{-1}A^{-1}$. But A^*A is self-adjoint and hence $(A^*A)^{-1} \in C^*(1, A^*A) \subset C^*(1, A)$. Next we check that $A^{-1} = (A^*A)^{-1}A^*$. □

Theorem 3.7 *Let $\mathfrak{B} \subset \mathfrak{A}$ be C^*-algebras and $A, 1 \in \mathfrak{B}$. Then* $\mathrm{sp}_{\mathfrak{B}}(A) = \mathrm{sp}_{\mathfrak{A}}(A)$.

Proof. By Lemma 3.6, $\mathrm{sp}_{\mathfrak{A}}(A) = \mathrm{sp}_{\mathfrak{C}}(A)$, where $\mathfrak{C} := C^*(1, A)$. But $\mathfrak{C} \subset \mathfrak{B} \subset \mathfrak{A}$. □

Motivated by the above theorem, when speaking about C^*-algebras, we will write $\mathrm{sp}(A)$ instead of $\mathrm{sp}_{\mathfrak{A}}(A)$.

3.8 Holomorphic functional calculus for normal operators

If K is a compact subset of $\mathbb{C}$ let $A^\infty(K)$ be the completion of $A(K)$ in $C(K)$. It is a Banach algebra and we can identify $\mathrm{Char}(A^\infty(K))$ with K.

The following theorem will be improved later on when we fully develop the Gelfand theory.

Theorem 3.8 *Let $\mathfrak{A}$ be unital and $A \in \mathfrak{A}$ be normal. Then there exists a unique continuous isomorphism*

$$A^\infty(\mathrm{sp}(A)) \ni f \mapsto f(A) \in \mathrm{Ban}(1, A) \subset \mathfrak{A},$$

such that

1. $\mathrm{id}(A) = A$ *if* $\mathrm{id}(z) = z$.
   Moreover, we have

2. *If $f \in A(\mathrm{sp}(A))$, then $f(A)$ coincides with $f(A)$ defined in (2.13).*

3. $\mathrm{sp}(f(A)) = f(\mathrm{sp}(A))$.

4. $g \in A^\infty(f(\mathrm{sp}(A))) \Rightarrow g \circ f(A) = g(f(A)$.

5. $\|f(A)\| = \sup |f|$.

3.9 Commutative C^*-algebras

Theorem 3.9 *Let $\mathfrak{A}$ be a unital commutative C^*-algebra. Then the Gelfand transform*

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C(\mathrm{Char}(\mathfrak{A}))$$

is a $*$-isomorphism.

Proof. Step 1 We already know that it is a norm-decrasing homomorphism by Theorem 2.18.

Step 2 Let us show that if $\phi \in \mathrm{Char}(\mathfrak{A})$ and $A \in \mathfrak{A}$, then $\phi(A^*) = \overline{\phi(A)}$. Let $A \in \mathfrak{A}$. Then $\mathrm{Re}A$ and $\mathrm{Im}A$ are self-adjoint. Hence, $\phi(\mathrm{Re}A)$, $\phi(\mathrm{Im}A) \in \mathbb{R}$. By linearity, this implies $\phi(A^*) = \overline{\phi(A)}$. Thus the Gelfand transform is a $*$-homomorphism.

Step 3 Next we note that every $A \in \mathfrak{A}$ is normal. Hence $\|A\| = \mathrm{sr}\|A\|$ by Theorem 3.4. But we know by Theorem 2.18 that $\|\hat{A}\| = \mathrm{sr}(A)$. This show that the Gelfand transform is isometric.

Step 4 We know that the image of the Gelfand transform is dense and $\mathfrak{A}$ is complete. We proved also that it is isometric. Hence it is bijective. □

Completion of proof of Theorem 3.5 2$\Leftarrow$. Consider the algebra $\mathfrak{C} := C^*(1, U)$. By the normality of U, it is commutative. Let $\phi \in \mathrm{Char}(\mathfrak{C})$, Then $\phi(U^*U) = \phi(U) \in \mathrm{sp}(U) \subset \{|z| = 1\}$. Hence $\phi(U^*)\phi(U) = \overline{\phi(U)}\phi(U) = 1$. Hence $U^*U = 1$.

3$\Leftarrow$. Consider the algebra $\mathfrak{C} := C^*(1, A)$. By the normality of A, it is commutative. Let $\phi \in \mathrm{Char}(\mathfrak{C})$, Then $\phi(A) \in \mathrm{sp}(A) \subset \mathbb{R}$. Hence $\phi(A^*) = \phi(A)$. Hence $A^* = A$. □

Theorem 3.10 *Let $\mathfrak{A}$ be a commutative C^*-algebra. Then the Gelfand transform*

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C_\infty(\mathrm{Char}(\mathfrak{A}))$$

is a $*$-isomorphism and $\mathrm{Char}(\mathfrak{A}_1)$ is the one-point compactification of $\mathrm{Char}(\mathfrak{A})$.

3.10 Functional calculus for normal operators

Theorem 3.11 *Let $\mathfrak{A}$ be unital and $A \in \mathfrak{A}$ be normal. Then there exists a unique continuous unital $*$-homomorphism*

$$C(\mathrm{sp}(A)) \ni f \mapsto f(A) \in C^*(1, A) \subset \mathfrak{A},$$

such that

1. $\mathrm{id}(A) = A$ *if* $\mathrm{id}(z) = z$.

   Moreover, we have

2. *If $f \in A(\mathrm{sp}(A))$, then $f(A)$ coincides with $f(A)$ defined in Theorem 3.8.*

3. $\mathrm{sp}(f(A)) = f(\mathrm{sp}(A))$.

4. $g \in C(f(\mathrm{sp}(A))) \Rightarrow g \circ f(A) = g(f(A)$.

5. $\|f(A)\| = \sup |f|$.

Proof. If f is a polynomial, that is $f(z) = \sum a_{nm} z^n \overline{z}^m$, we set

$$f(A) := \sum a_{nm} A^n A^{*m}.$$

$C^*(1, A)$ is a commutative algebra. Let ϕ be a character on $C^*(1, A)$. Then we easily check that $\phi(f(A)) = f(\phi(A))$. Hence $\mathrm{sp}(f(A)) = f(\mathrm{sp}(A))$.

Clearly, $f(A)$ is normal. Hence

$$\|f(A)\| = \mathrm{sr}(f(A)) = \sup |f|.$$

Therefore, on polynomials the map $f \to f(A)$ is isometric. Since polynomials are dense in a complete metric space $C(\mathrm{sp}(A))$ and polynomials in A, A^* are dense in a complete metric space $C^*(1, A)$, there is exactly one continuous extension of this map to the whole $C(\mathrm{sp}(A))$, which is an isometric bijection of $C(\mathrm{sp}(A))$ to $C^*(1, A)$.

Clearly, on polynomials, the map $f \mapsto f(A)$ is a $*$-homomorphism. Since the multiplication, and involution are continuous both in $C(\mathrm{sp}(A))$ and $C^*(1, A)$, this map is a homomorphism on $C(\mathrm{sp}(A))$. □

If $\mathfrak{A}$ is not unital, either we can adjoin the identity and consider the algebra $\mathfrak{A}_1$, or we can use the following version of the above theorem:

Theorem 3.12 *Let $A \in \mathfrak{A}$ be normal. Then there exists a unique continuous $*$-homomorphism*

$$C_\infty(\mathrm{sp}(A) \backslash \{0\}) \ni f \mapsto f(A) \in C^*(A) \subset \mathfrak{A},$$

such that $\mathrm{id}(A) = A$ *if* $\mathrm{id}(z) = z$.

3.11 Fuglede's theorem

Theorem 3.13 *Let $A, B \in \mathfrak{A}$ and let B be normal. Then $AB = BA$ implies $AB^* = B^*A$.*

Proof. For $\lambda \in \mathbb{C}$, the operator $U(\lambda) := \mathrm{e}^{\lambda B^* - \overline{\lambda} B} \in C^*(1, B)$ is unitary and

$$U(\lambda) \mathrm{e}^{\lambda B^*} = \mathrm{e}^{\lambda B}.$$

Hence

$$\mathrm{e}^{-\lambda B^*} A \mathrm{e}^{\lambda B^*} = U(-\lambda) A U(\lambda)^* \tag{3.25}$$

is a uniform,ly bounded analytic function. Hence is constant. Differentiating it wrt λ we get $[A, B^*] = 0$. □

3.12 Functional calculus for a family of normal operators

Let $\{A_i \ : \ i \in I\}$ be a family of normal commuting elements of $\mathfrak{A}$. By Theorem 3.13, the algebra $\mathfrak{C} := C^*(1, A_i \ : \ i \in I\}$ is commutative. Define

$$\mathrm{sp}(A_i \ : \ i \in I\} := \{(\phi(A_i)) \in \mathop{\times}_{i \in I} \mathrm{sp}(A_i) \ : \ \phi \in \mathrm{Char}(\mathfrak{C})\}.$$

Theorem 3.14

$$\mathrm{Char}(\mathfrak{C}) \ni \phi \mapsto (\phi(A_i)) \in \mathrm{sp}(A_i \ : \ i \in I)$$

is a homeomorphism.

Proof. The continuity is obvious.

Polynomials are dense in $\mathfrak{C}$, hence if $\phi(A_i) = \psi(A_i)$, $i \in I$, then $\phi = \psi$. Therefore, it is a bijection. But a continuous bijection on a compact set is a homeomorphism. □

Theorem 3.15 *Let $\mathfrak{A}$ be unital and (A_i), $i \in I$ be a family of commuting normal elements of $\mathfrak{A}$ be normal. Then there exists a unique continuous unital $*$-homomorphism*

$$C(\mathrm{sp}(A_i \ : \ i \in I)) \ni f \mapsto f(A_i \ : \ i \in I) \in C^*(1, A_i \ : \ i \in I) \subset \mathfrak{A},$$

such that

1. $\mathrm{id}_j(A_i \ : \ i \in I) = A_j$ *if* $\mathrm{id}_j(z_i \ : \ i \in I) = z_j$.

   Moreover, we have

2. *If $f(z_i \ : i \in I) = f_j(z_j)$ for $f_j \in C(\mathrm{sp}(A_j))$, then $f(A_i \ : \ i \in I)$ coincides with $f_j(A_j)$ defined in Theorem 3.11.*

3. $\mathrm{sp}(f(A_i \ : \ i \in I)) = f(\mathrm{sp}(A_i \ : \ i \in I))$.

4. $\|f(A)\| = \sup |f|$.

3.13 Positive elements

Let $A \in \mathfrak{A}$. We say that A is positive iff A is self-adjoint and $\mathrm{sp}(A) \subset [0, \infty[$. $\mathfrak{A}_+$ will denote the set of positive elements in $\mathfrak{A}$. We will write $A \geq B$ iff $A - B \in \mathfrak{A}_+$. We will write $A > B$ iff $A \geq B$ and $A \neq B$.

Lemma 3.16 1. *Let A be self-adjoint. Then $\|\lambda - A\| \leq \lambda$ implies $A \geq 0$ and $\|A\| \leq 2\lambda$.*

2. $A \geq 0$ *iff* $\|1 - A/\|A\|\| \leq 1$.

Theorem 3.17 1. $A \in \mathfrak{A}_+$ *and* $\lambda > 0$ *implies* $\lambda A \in \mathfrak{A}_+$.

2. $A, B \in \mathfrak{A}_+$ *implies* $A + B \in \mathfrak{A}_+$.

3. $A, -A \in \mathfrak{A}_+$ *implies* $A = 0$.

4. $\mathfrak{A}_+$ *is closed.*
   In other words, $\mathfrak{A}_+$ is a closed pointed cone.

Proof. 2. We have $\|1 - A/\|A\|\| \leq 1$ and $\|1 - B/\|B\|\| \leq 1$. Hence

$$\big\|\|A\| + \|B\| - A - B\big\| = \big\|\|A\|(1 - A/\|A\|) + \|B\|(1 - A/\|A\|)\big\| \leq \|A\| + \|B\|.$$

Hence $A + B \geq 0$.

3. $\mathrm{sp}(A), \mathrm{sp}(-A) \subset [0, \infty[$ implies $\mathrm{sp}(A) = \{0\}$. But A is self-adjoint. Hence $A = 0$.

4. Let $A_n \to A$. Then $\|A_n\| \to \|A\|$. $A_n \in \mathfrak{A}_+$ iff $\|A_n - \|A_n\|\| \leq \|A_n\|$. By taking the limit, $\|A - \|A\|\| \leq \|A\|$. Hence $A \in \mathfrak{A}_+$. □

Theorem 3.18 *Let $A \in \mathfrak{A}_+$ and $n \in \mathbb{N}$. Then there exists a unique $B \in \mathfrak{A}_+$ such that $B^n = A$.*

Proof. $[0,\infty[\ni\lambda\mapsto\lambda^{1/n}$ is a continuous function. Hence $B:=A^n$ is well defined. Clearly, B satisfies the requirements of the theorem.

Let $B\in\mathfrak{A}_+$, $B^n=A$. Clearly,

$$BA=B^{n+1}=AB. \tag{3.26}$$

Let $\mathfrak{C}:=C^*(1,B,A)$. By (3.26), $\mathfrak{C}$ is commutative. If $\phi\in\mathrm{Char}(\mathfrak{C})$, then $\phi(A)=\phi(B^n)=\phi(B)^n$. Moreover, $\phi(B)>0$. Hence $\phi(B)=\phi(A)^{1/n}$. Hence B is uniquely determined, and equals $A^{1/n}$. □

Theorem 3.19 (Jordan decomposition of a self-adjoint operator.) *Let $A\in\mathfrak{A}$ be self-adjoint. Then there exist unique $A_+,A_-\in\mathfrak{A}_+$ such that $A_+A_-=A_-A_+=0$ and $A=A_+-A_-$.*

Proof. The functions $|x|_+:=\max(x,0)$ and $|x|_-:=\max(-x,0)$ are continuous. Hence A_+ and A_- can be defined as $|A|_+$ and $|A|_-$ by the functional calculus.

Assume that A_- and A_+ satisfy the conditions of the theorem. Then

$$A^2=A_-^2+A_+^2=(A_++A_-)^2.$$

By the uniqueness of the positive square root, $|A|=A_++A_-$. Hence $A_+=\frac{1}{2}(|A|+A)$ and $A_-=\frac{1}{2}(|A|-A)$. □

Theorem 3.20 *Let $A\in\mathfrak{A}$. The following conditions are equivalent*

1. $A\geq 0$.
2. *There exists $B\in\mathfrak{A}$ such that $A=B^*B$.*

Proof. 1. $\Rightarrow$ 2. is contained in Theorem 3.18.

Let us prove 1. $\Leftarrow$ 2. Clearly, B^*B is self-adjoint. Let $B^*B=A_+-A_-$ be its Jordan decomposition. Clearly

$$(BA_-)^*(BA_-)=A_-(A_+-A_-)A_-=-A_-^3\in-\mathfrak{A}_+.$$

Let $BA_-=S+\mathrm{i}T$. Then

$$(BA_-)(BA_-)^*=-(BA_-)^*(BA_-)+2(S^2+T^2)\in\mathfrak{A}_+,$$

using the fact that $\mathfrak{A}_+$ is a convex cone.

But by Theorem 1.8

$$\mathrm{sp}\big((BA_-)^*(BA_-)\big)\cup\{0\}=\mathrm{sp}\big((BA_-)(BA_-)^*\big)\cup\{0\}.$$

Hence $\mathrm{sp}\big((BA_-)^*(BA_-)\big)=\{0\}$. Consequently, $(BA_-)^*(BA_-)=0$. Consequently, $A_-^3=0$. By the uniqueness of the positive third root, $A_-=0$. □

Theorem 3.21 *1. Let A is self-adjoint, then $-\|A\|\leq A\leq\|A\|$.*
In what follows, let $0\leq B\leq A$. Then

2. $\|B\|\leq\|A\|$,
3. *If $D^*D\leq 1$, then $DD^*\leq 1$.*
4. $0\leq C^*BC\leq C^*AC$.
5. $0\leq(\lambda+A)^{-1}\leq(\lambda+B)^{-1},\quad 0<\lambda$.

6. $B(\lambda+B)^{-1} \leq A(\lambda+A)^{-1}$.

7. $0 \leq B^\theta \leq A^\theta, \quad 0 \leq \theta \leq 1,$

Proof. 1. $\mathrm{sp}(A) \subset [-\|A\|, \|A\|]$. Hence $\|A\| - A \geq 0$.

2. By 1., $A \leq \|A\|$. Hence, $B \leq \|A\|$. Hence $\mathrm{sp}(B) \subset [0, \|A\|]$. Therefore, $\|B\| \leq \|A\|$.

3. Clearly, $\|D^*D\| \leq 1$. Hence $\|DD^*\| \leq 1$. Hence, by 1., $DD^* \leq 1$.

4. $C^*(A-B)C = \left((A-B)^{\frac{1}{2}}C\right)^*(A-B)^{\frac{1}{2}}C$.

5. Clearly, $\lambda + A \geq \lambda + B \geq \lambda$. Hence $\lambda + A$ and $\lambda + B$ are positive invertible. By 3., applied with $C = (\lambda+A)^{-\frac{1}{2}}$, for $D := (\lambda+B)^{\frac{1}{2}}(\lambda+A)^{-\frac{1}{2}}$ we have $1 \geq D^*D$. Hence $1 \geq DD^*$.

6. follows immediately from 5.

7. We use 6. and

$$A^\theta = c_\theta \int_0^\infty \lambda^{\theta-1} A(\lambda+A)^{-1} \mathrm{d}\lambda.$$

□

Theorem 3.22 *Let $B_i \in \mathfrak{A}_+$, $\|B_i\| < 1$, $i = 1, 2$. Then there exists $B \in \mathfrak{A}_+$, such that $\|B\| < 1$ and $B_i \leq B$, $i = 1, 2$.*

Proof. Set $A_i := (1 - B_i)^{-1}B_i$. Note that $A_i \in \mathfrak{A}_+$ and

$$B_i = (1 + A_i)^{-1}A_i, \tag{3.27}$$

We set

$$B := (1 + A_1 + A_2)^{-1}(A_1 + A_2). \tag{3.28}$$

Clearly $B \in \mathfrak{A}_+$, $\|B\| < 1$ Clearly, $A_i \leq A_1 + A_2$, $i = 1, 2$. Hence, by (3.27), (3.28) and Theorem 3.21, we get $B_i \leq B$. □

3.14 Positive approximate units

Let $\mathfrak{J}$ be a closed right ideal of $\mathfrak{A}$. Then a positive left approximate unit of $\mathfrak{J}$ is defined to be a net $\{E_\alpha\}$ of elements of $\mathfrak{J}$ such that

1. $0 \leq E_\alpha \leq 1$,
2. $\alpha \leq \beta$ implies $E_\alpha \leq E_\beta$,
3. $\lim_\alpha \|E_\alpha A - A\| = 0$ for all $A \in \mathfrak{A}$.

The following theorem implies that every ideal possesses a positive approximate unit.

Theorem 3.23 *Let $\mathfrak{J}$ be a right ideal of $\mathfrak{A}$. Then $\mathfrak{E} := \{A \in \mathfrak{J}_+ \ : \ \|A\| < 1\}$ ordered by $\leq$ is a positive approximate unit in $\mathfrak{J}$.*

Proof. By Theorem 3.22, $\mathfrak{E}$ is a directed set.

Let $A \in \mathfrak{J}$. Then, for any $\lambda > 0$, then clearly, $|A| := (A^*A)^{\frac{1}{2}} \in \mathfrak{J}$. Therefore, $E_\lambda := |A|(\lambda^{-1} + |A|)^{-1} \in \mathfrak{E}$. Clearly, $(1 - E_\lambda)A = (1 + \lambda|A|)A$ has norm less than λ^{-1}. □

3.15 Quotient algebras

Theorem 3.24 *Let $\mathfrak{I}$ be a closed ideal of a C^*-algebra $\mathfrak{A}$. Then $\mathfrak{A}/\mathfrak{I}$ is a C^*-algebra and we have a short exact sequence of $*$-homomorphisms:*

$$0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0.$$

If (E_α) is a positive approximate unit, thenThe norm in $\mathfrak{A}/\mathfrak{I}$ is given by

$$\|A + \mathfrak{I}\| = \lim_\alpha \|A(1 - E_\alpha)\|. \tag{3.29}$$

Proof. The approximate unite $\mathfrak{E}$ satisfies the condition (2.11), so (3.29) holds. Now let $A \in \mathfrak{A}$ and $I \in \mathfrak{I}$.

$$\begin{aligned} \|A + \mathfrak{I}\|^2 &= \lim_\alpha \|A(1 - E_\alpha)\|^2 \\ &= \lim_\alpha \|(1 - E_\alpha)A^*A(1 - E_\alpha)\|^2 \\ &= \lim_\alpha \|(1 - E_\alpha)(A^*A + I)(1 - E_\alpha)\|^2 \\ &\leq \|A^*A + I\|. \end{aligned}$$

Hence

$$\|A + \mathfrak{I}\|^2 \leq \|(A + \mathfrak{I})(A + \mathfrak{I})^*\|.$$

3.16 Homomorphisms of C^*-algebras

Theorem 3.25 *I. Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a $*$-homomorphism. Then*

1. $\|\pi(A)\| \leq \|A\|$;
2. $\pi(\mathfrak{A})$ *is a C^*-algebra.*
3. *The following conditions are equivalent*
   (a) $\mathrm{Ker}\pi = \{0\}$,
   (b) $\|\pi(A)\| = \|A\|$.

Proof. First we would like to argue that it is sufficient to assume that π preserves the identity. If $\mathfrak{A}$ has a unit, then $\pi(1) = P$ is a projection in $\mathfrak{B}$. We can replace $\mathfrak{B}$ with $P\mathfrak{B}P$, and then consider $\pi : \mathfrak{A} \to P\mathfrak{B}P$.

If $\mathfrak{A}$ does not have a unit, we simply adjoin the unit to $\mathfrak{A}$, if needed also to $\mathfrak{B}$, and consider the extended $*$-homomorphism $\pi_1 : \mathfrak{A}_1 \to \mathfrak{B}_1$ such that $\pi_1(1) = 1$.

Proof of 1. Clearly, if $A \in \mathfrak{A}$, then $\mathrm{sp}(\pi(A)) \subset \mathrm{sp}(A)$. If A is self-adjoint, then

$$\|\pi(A)\| = \mathrm{sr}(\pi(A)) \leq \mathrm{sr}(A) = \|A\|.$$

For an arbitrary $B \in \mathfrak{A}$,

$$\|\pi(B)\|^2 = \|\pi(B)^*\pi(B)\| = \|\pi(B^*B)\| \leq \|B^*B\| = \|B\|^2.$$

Proof of 3. 3.1$\Rightarrow$3.2. π is injective. Hence there exists $\pi^{-1} : \pi(\mathfrak{A}) \to \mathfrak{A}$. By 1., $\|\pi^{-1}(\pi(A))\| \leq \|\pi(A)\|$.

3.1$\Leftarrow$3.2. is obvious.

Proof of 2. Clearly, $\tilde{\pi} : \mathfrak{A}/\mathrm{Ker}\pi \to \pi(\mathfrak{A})$ is a $*$-isomorphism. By 3.2. it is also isometric. Since $\mathfrak{A}/\ker\mathfrak{A}$ is a C^*-algebra by Theorem 3.24, so is $\pi(\mathfrak{A})$. $\square$

3.17 Linear functionals

Let ω be a linear functional on $\mathfrak{A}$. The adjoint functional ω^* is defined by

$$\omega^*(A) := \overline{\omega(A^*)}.$$

We say that ω is self-adjoint iff $\omega^* = \omega$, or equivalently, if $\omega(A) \in \mathbb{R}$ for A self-adjoint. Note that if ω is self-adjoint, then its norm as the functional over $\mathfrak{A}$ is the same as the norm of its restriction to $\mathfrak{A}_{\mathrm{sa}}$:

$$\begin{aligned} \|\omega\| &= \sup\{|\omega(A)| \ : \ A \in (\mathfrak{A})_1\} \\ &= \sup\{|\omega(A)| \ : \ A \in (\mathfrak{A}_{\mathrm{sa}})_1\}. \end{aligned}$$

We say that ω is positive iff

$$\omega(A) \geq 0, \qquad A \in \mathfrak{A}_+.$$

The set of continuous functionals over $\mathfrak{A}$ will be denoted $\mathfrak{A}^\#$. The set of positive functionals over $\mathfrak{A}$ will be denoted $\mathfrak{A}_+^\#$.

Theorem 3.26 *If ω is a positive functionals, then it is self-adjoint and*

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B). \tag{3.30}$$

Proof. If A is self-adjoint, then we can decompose A as $A = -A_- + A_+$ with A_-, A_+ positive. Hence $\omega(A) \in \mathbb{R}$.

To prove (3.30), we note that for any $\lambda \in \mathbb{C}$,

$$\omega\big((A + \lambda B)^*(A + \lambda B)\big) \geq 0.$$

□

Theorem 3.27 *Let ω be a linear functional on a unital C^*-algebra $\mathfrak{A}$. The following conditions are equivalent:*

1. *ω is positive*
2. *ω is continuous and $\|\omega\| = \omega(1)$*

Proof. 1.⇒2. **Step 1** Let $A \in \mathfrak{A}_+$. Then $A \leq \|A\|$. Hence $|\omega(A)| = \omega(A) \leq \|A\|\omega(1)$.

Step 2 Let $B \in \mathfrak{A}$. Then, by (3.30), using the positivity of B^*B and Step 1, we get

$$|\omega(B)|^2 \leq \omega(1)\omega(B^*B) \leq \omega(1)^2\|\|B^*B\| = \omega(1)^2\|B\|^2.$$

Hence $\|\omega\|^2 \leq \omega(1)^2$.

1.⇐ 2. It is enough to assume that $\|\omega\| = 1$.

Step 1 Let A be self-adjoint. Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\omega(A) = \alpha + \mathrm{i}\beta$. It is enough to assume that $\omega(1) = \|\omega\| = 1$. Clearly,

$$\|\gamma - \mathrm{i}A\| = \sqrt{\gamma^2 + \|A\|^2}, \quad \omega(\gamma - \mathrm{i}A) = \gamma + \beta - \mathrm{i}\alpha.$$

But

$$|\omega(\gamma - \mathrm{i}A)|^2 \leq \|\gamma - \mathrm{i}A\|^2.$$

Hence

$$(\gamma + \beta)^2 + \alpha^2 \leq (\gamma^2 + \|A\|^2).$$

For large $|\gamma|$, this is possible only if $\beta = 0$. Hence ω is self-adjoint.
Step 2 Let $A \in \mathfrak{A}_+$. Then $\|1 - \frac{A}{\|A\|}\| \leq 1$. Therefore,

$$\left|\omega(1) - \omega(A)/\|A\|\right| \leq 1.$$

But $\omega(1) = 1$, and $\omega(A)$ is real. Hence $\omega(A) \geq 1$. □

Theorem 3.28 *Let ω be a linear functional on a non-unital C^*-algebra. The following conditions are equivalent:*

1. *ω is positive*
2. *ω is continuous and for some positive approximate identity $\{E_\alpha\}$ of $\mathfrak{A}$*
$$\|\omega\| = \lim_\alpha \omega(E_\alpha^2).$$
3. *ω is continuous and if the functional $\omega_1 : \mathfrak{A}_1 \to \mathbb{C}$ is given by $\omega_1(\lambda + A) := \lambda\|omega\| + \omega(A)$, then ω_1 is a positive functional on $\mathfrak{A}_1$*

Moreover, ω_1 is the unique functional on $\mathfrak{A}_1$ that extends ω and satisfies $\|\omega\| = \|\omega_1\|$.

Proof. 1⇒ 2. **Step 1.** We want to show that

$$c := \sup\{\omega(A) \ : \ 0 \leq A \leq 1\}$$

is finite. Suppose that it is not true, $0 \leq A_n \leq 1$ and $\omega(A_n) \to \infty$. Then we will find $\lambda_n \geq 0$ such that $\sum \lambda_n < \infty$ and $\sum \lambda_n \omega(A_n) = \infty$. But $A := \sum \lambda_n A_n$ is convergent and

$$\sum \lambda_n \omega(A_n) \leq \omega(A) < \infty,$$

which is a contradiction.

Step 2. If $A \in \mathfrak{A}$, then $A = \sum_{j=0}^3 \mathrm{i}^j A_j$ with $A_j \in \mathfrak{A}_+$ and $\|A_j\| \leq \|A\|$. Hence

$$|\omega(A)| \leq \sum_{j=0}^{3} \omega(A_j) \leq 4c\|A\|.$$

Hence ω is continuous.

Step 3. Let E_α be a positive approximate unit. $\omega(E_\alpha)$ is an increasing bounded net, so $c := \lim_\alpha \omega(E_\alpha)$ exists Since $\|E_\alpha\| \leq 1$, we have $c \leq \|\omega\|$.

Step 4 Let $A \in \mathfrak{A}$. Then

$$|\omega(E_\alpha A)|^2 \leq \omega E_\alpha^2)\omega(A^*A) \leq \omega(E_\alpha^2)\|\omega\|\|A^*A\| \leq c\|\omega\|\|A\|^2.$$

Moreover, $E_\alpha A \to A$ and ω is continuous, hence the right hand side goes to $|\omega(A)|^2$. Hence $|\omega(A)|^2 \leq c\|\omega\|$ Therefore, $\|\omega\| \leq c$.

2⇒3 It is obvious that $\|\omega_1\| \geq \|\omega\|$. Let us prove the converse inequality.

Let E_α be a positive approximative unit. We have

$$\omega_1(\lambda + A) = \lim_\alpha \omega(\lambda E_\alpha + E_\alpha A).$$

Hence

$$\begin{aligned} \||\omega_1(\lambda + A)| &= \lim_\alpha |\omega(\lambda E_\alpha + E_\alpha A)| \leq \lim_\alpha \|\omega\|\|\lambda E_\alpha + E_\alpha A\| \\ &\leq \|\omega\| \limsup_\alpha \|E_\alpha\|\|\lambda + A\| = \|\omega\|\|\lambda + A\|. \end{aligned}$$

Hence, $\|\omega_1\| \leq \|\omega\|$.

Thus we proved that $\|\omega\| = \|\omega_1\|$. Therefoore, $\omega_1(1) = \|\omega_1\|$. Therefore, ω is positive by the previous theorem.

3⇒1 is obvious.

3.18 The GNS representation

Let $(\mathcal{H}, \pi)$ be a $*$-representation of $\mathfrak{A}$, $\Omega \in \mathcal{H}$ and $\omega \in \mathfrak{A}_+^\#$. We say that Ω is a vector representative of ω iff

$$\omega(A) = (\Omega|A\Omega).$$

We say that Ω is cyclic iff $\pi(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$. $(\mathcal{H}, \pi, \Omega)$ is called a cyclic $*$-representation iff $(\pi, \mathcal{H})$ is a $*$-representation and Ω is a cyclic vector.

Theorem 3.29 *Let ω be a state on $\mathfrak{A}$. Then there exists a cyclic $*$-representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ such that Ω_ω is a vector representative of ω. Such a representation is unique up to a unitary equivalence.*

Proof. For $A, B \in \mathfrak{A}$, $\omega(A^*B)$ is a pre-Hilbert scalar product on $\mathfrak{A}$. Define $\mathfrak{N}_\omega := \{A \in \mathfrak{A} \ : \ \omega(A^*A) = 0\}$. Then $\mathfrak{N}$ is a closed left ideal. The scalar product on $\mathfrak{A}/\mathfrak{N}_\omega$ is well defined. Let $\mathcal{H}_\omega$ be the completion of $\mathfrak{A}/\mathfrak{N}_\omega$.

The left regular representation

$$\mathfrak{A} \ni A \mapsto \lambda(A) \in L(\mathfrak{A}), \quad \lambda(A)B := AB,$$

preserves $\mathfrak{N}$. Hence we can define he represetation π_ω on $\mathfrak{A}/\mathfrak{N}_\omega$ by

$$\pi_\omega(A)(B + \mathfrak{N}_\omega) := AB + \mathfrak{N}_\omega.$$

We have

$$\begin{aligned} \|\pi_\omega(A)(B + \mathfrak{N}_\omega)\|^2 &= \inf\{\omega(B_1^* A^* A B_1) \ : \ B_1 \in B + \mathfrak{N}_\omega\} \\ &\leq \|A^*A\| \inf\{\omega(B_1^* B_1) \ : \ B_1 \in B + \mathfrak{N}_\omega\} = \|A\|^2 \|B + \mathfrak{N}_\omega\|^2. \end{aligned}$$

Hence $\pi_\omega(A)\| \leq \|A\|$ and π_ω extends to a linear map on $\mathcal{H}_\omega$.

We set $\Omega_\omega := 1 + \mathfrak{N}_\omega$. Clearly, $\pi_\omega(A)\Omega_\omega = A + \mathfrak{N}_\omega$, hence Ω_ω is cyclic. □

3.19 Existence of a faithful representation

Theorem 3.30 *Let $\mathfrak{A}$ be a unital C^*-algebra and $A \in \mathfrak{A}_{\rm sa}$ be strictly positive. Then there exists a state ω on $\mathfrak{A}$ such that $\omega(A) = \|A\|$.*

Proof. Let $A = A_+ - A_-$ be the Jordan decomposition of A. Then $\|A\| = \max\{\|A_+\|, \|A_-\|\}$. It is enough to assume that $\|A\| = \|A_+\|$.

By the Hahn-Banach theorem applied to the subspace $\mathbb{C}A_+ \subset \mathfrak{A}$, there exists $\omega \in \mathfrak{A}^\#$ with $\omega\| = 1$ and $\omega(A_+) = \|A_+\|$. By Theorem 3.27, ω is positive. Now

$$\omega(A_+) + \omega(A_-) = \omega(A_+ + A_-) \leq \|A_+ + A_-\| = \|A_+\| = \omega(A_+).$$

Hence $\omega(A_-) = 0$. Therefore,

$$\omega(A) = \omega(A_+) - \omega(A_-) = \omega(A_+) = \|A_+\| = \|A\|.$$

□

Theorem 3.31 *There exists an injective representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$.*

Proof. . For any $A \in \mathfrak{A}_{\rm sa}$ there exists a state ω_A such that $\omega_A(A) \neq 0$. Let $(\pi_A, \mathcal{H}_A, \Omega_A)$ be the corresponding GNS representation. Then $(\Omega_A|\pi_A(A)\Omega_A) = \omega(A)$. Hence $\pi_A(A) \neq 0$. Set

$$\mathcal{H} := \underset{A \in \mathfrak{A}_{\rm sa}}{\oplus} \mathcal{H}_A, \quad \pi := \underset{A \in \mathfrak{A}_{\rm sa}}{\oplus} \pi_A.$$

Then π is a representation of $\mathfrak{A}$ in $\mathcal{H}$ and for any $A \in \mathfrak{A}_{\rm sa}$, $\pi(A) \neq 0$. Since self-adjoint elements span $\mathfrak{A}$, π is injective. □