It is important to recall that the operation of contraction explicitly depends on the state (as it involves $\hat{A}$.)

If all contractions are numbers, the average value becomes:

$$
\langle \Phi | \hat{A}_1 \hat{A}_2 \hat{A}_3 \ldots \hat{A}_n | \Phi \rangle = \hat{A}_1 \langle \Phi | \hat{A}_2 \hat{A}_3 \ldots \hat{A}_n | \Phi \rangle + \hat{A}_1 \hat{A}_2 \langle \Phi | \hat{A}_3 \ldots \hat{A}_n | \Phi \rangle \\
+ c \hat{A}_1 \hat{A}_3 \langle \Phi | \hat{A}_2 \ldots \hat{A}_n | \Phi \rangle + \ldots + c^{n-2} \hat{A}_1 \hat{A}_n \langle \Phi | \hat{A}_2 \hat{A}_3 \ldots | \Phi \rangle,
$$

i.e., the average of a product of operators is expressed in terms of products of averages involving less operators. The condition that all contractions are numbers is only fulfilled for certain classes of states $|\Phi\rangle$.

Such states are called **product states**.

Many-body product states are such states in which all contractions of creation and annihilation operators are numbers.

By applying the above recurrence relation, one can formulate the **Wick's theorem**:

If all the contractions of operators appearing in the product are numbers, then the average of the product becomes a linear combination of products of all possible contractions and self-contractions.

The coefficients appearing in this linear combination are various powers of $c$. 
Let us introduce

\[ \hat{A} \hat{D}_1 \hat{D}_2 \ldots \hat{D}_k \hat{B} \equiv c^k \hat{A} \hat{B} \hat{D}_1 \hat{D}_2 \ldots \hat{D}_k \]

Using this definition, the expression for the average value of a product, becomes

\[
\langle \Phi | \hat{A}_1 \hat{A}_2 \hat{A}_3 \ldots \hat{A}_n | \Phi \rangle = \hat{A}_1 \langle \Phi | \hat{A}_2 \hat{A}_3 \ldots \hat{A}_n | \Phi \rangle + \langle \Phi | \hat{A}_1 \hat{A}_2 \hat{A}_3 \ldots \hat{A}_n | \Phi \rangle \\
+ \langle \Phi | \hat{A}_1 \hat{A}_2 \hat{A}_3 \ldots \hat{A}_n | \Phi \rangle + \ldots + \langle \Phi | \hat{A}_1 \hat{A}_2 \hat{A}_3 \ldots \hat{A}_n | \Phi \rangle.
\]

The coefficient \( c^k \) is given by a number of permutations needed to bring the two operators next to each other. In order to calculate the average value of a product, we need to calculate all possible pairwise contractions. Note that for the product states;

\[ \hat{A} \hat{B} = \hat{A}_- \hat{B} - c \hat{B} \hat{A}_- = \langle \Phi | \hat{A}_- \hat{B} - c \hat{B} \hat{A}_- | \Phi \rangle = \langle \Phi | \hat{A}_- \hat{B} | \Phi \rangle \]

That is

\[ \hat{A} \hat{B} = \langle \Phi | \hat{A} \hat{B} | \Phi \rangle - \langle \Phi | \hat{A} | \Phi \rangle \langle \Phi | \hat{B} | \Phi \rangle \]

Deviation of a product average from the product of averages

In practice, we do not need the annihilating part \( A_\).
Example: product of four operators:

\[
\langle \Phi | \hat{A} \hat{B} \hat{C} \hat{D} | \Phi \rangle = \hat{A} \hat{B} \hat{C} \hat{D} + \hat{A} \hat{B} \hat{C} \hat{D} c + \hat{A} \hat{B} \hat{C} \hat{D} c^2 + \hat{A} \hat{B} \hat{C} \hat{D} + \hat{A} \hat{B} \hat{C} \hat{D} c + \hat{A} \hat{B} \hat{C} \hat{D} c^2 + \hat{A} \hat{B} \hat{C} \hat{D} .
\]

\[
\langle \Phi | \hat{A} \hat{B} \hat{C} \hat{D} | \Phi \rangle = \langle \Phi | \hat{A} \hat{B} | \Phi \rangle \langle \Phi | \hat{C} \hat{D} | \Phi \rangle + c \langle \Phi | \hat{A} \hat{C} | \Phi \rangle \langle \Phi | \hat{B} \hat{D} | \Phi \rangle + \langle \Phi | \hat{A} \hat{D} | \Phi \rangle \langle \Phi | \hat{B} \hat{C} | \Phi \rangle - (1 + c) \langle \Phi | \hat{A} | \Phi \rangle \langle \Phi | \hat{B} | \Phi \rangle \langle \Phi | \hat{C} | \Phi \rangle \langle \Phi | \hat{D} | \Phi \rangle .
\]

this term vanishes for Fermions!
Wick’s theorem for Slater determinants

\[ |\Phi\rangle = a^+_{\mu_1} \cdots a^+_{\mu_A} |0\rangle \]

\[
a_{\mu_0} = 0 \quad (\langle \Phi | a_{\mu} | \Phi \rangle = 0) \]

\[
a_{\mu_-} = \begin{cases} 
0 & \text{for} \quad \mu \in \{\mu_i\}; \\
 a_{\mu} & \text{for} \quad \mu \not\in \{\mu_i\}
\end{cases} 
\]

\[
a_{\mu_+} = \begin{cases} 
 a_{\mu} & \text{for} \quad \mu \in \{\mu_i\}; \\
0 & \text{for} \quad \mu \not\in \{\mu_i\}
\end{cases} \]

\[
a^+_{\mu_0} = 0 \]

\[
a^+_{\mu_-} = \begin{cases} 
 a^+_{\mu} & \text{for} \quad \mu \in \{\mu_i\}; \\
 0 & \text{for} \quad \mu \not\in \{\mu_i\}
\end{cases} \]

\[
a^+_{\mu_+} = \begin{cases} 
 0 & \text{for} \quad \mu \in \{\mu_i\}; \\
 a^+_{\mu} & \text{for} \quad \mu \not\in \{\mu_i\}
\end{cases} \]

Of course, in this case \( c=-1 \).
Let us now calculate simple contractions:

\[ a^\mu_\mu a^\nu_\nu = \sum_{i=1}^{A} \delta_{\mu\mu_i} \delta_{\nu\mu_i}, \]

\[ a^\mu_\mu a^\nu_\nu^+ = \sum_{i=A+1}^{M} \delta_{\mu\mu_i} \delta_{\nu\mu_i}, \]

\[ a^\mu_\mu a^\nu_\nu = \bar{a}^\mu_\mu a^\nu_\nu = 0, \]

\[ a^\mu_\mu = \bar{a}^\mu_\mu = 0. \]
What if Slated determinant is expressed in another basis?

\[ |\Phi'\rangle = a'^+_{\mu_1} \ldots a'^+_{\mu_A} |0\rangle \]

\[ a'^+_{\mu} = \sum_{\nu} U_{\mu\nu} a^+_\nu \]

\[ a_{\mu_0} = 0 \]

\[ a_{\mu_-} = \sum_{\nu \not\in \{\mu_i\}} U^T_{\mu\nu} a'_\nu = \sum_{i=A+1}^{M} \sum_{\nu} U^T_{\mu\nu_i} U^{*}_{\nu_i\nu} a_{\nu}, \]

\[ a_{\mu_+} = \sum_{\nu \in \{\mu_i\}} U^T_{\mu\nu} a'_\nu = \sum_{i=1}^{A} \sum_{\nu} U^T_{\mu\nu_i} U^{*}_{\nu_i\nu} a_{\nu} \]

\[ a^+_{\mu_0} = 0 \]

\[ a^+_{\mu_-} = \sum_{\nu \in \{\mu_i\}} U^+_{\mu\nu} a'^+_{\nu} = \sum_{i=1}^{A} \sum_{\nu} U^+_{\mu\nu_i} U_{\nu_i\nu} a^+_{\nu}, \]

\[ a^+_{\mu_+} = \sum_{\nu \not\in \{\mu_i\}} U^+_{\mu\nu} a'^+_{\nu} = \sum_{i=A+1}^{M} \sum_{\nu} U^+_{\mu\nu_i} U_{\nu_i\nu} a^+_{\nu}, \]
\[
\begin{align*}
\overline{a_\mu^+ a_\nu} &= \sum_{i=1}^{A} U_{\mu \nu_i}^+ U_{\nu_i \nu}, \\
\overline{a_\mu a_\nu^+} &= \sum_{i=A+1}^{M} U_{\mu \nu_i}^T U_{\nu_i \nu}^*, \\
\overline{a_\mu^+ a_\nu^+} &= a_\mu a_\nu = 0,
\end{align*}
\]

... and all self-contractions vanish!