Doubling the single-particle space

It is convenient to double the dimension of the one-particle space (such doubling is useful in the context of pairing theory)

\[ c_k = \begin{cases} \alpha_{\mu} & \text{for } k = \mu, \\ \alpha_{\mu}^+ & \text{for } k = M + \mu. \end{cases} \]

Here, \( k = 1, 2, \ldots, 2M \)

This new space is referred to as \( \text{the quasiparticle space} \)

By introducing the matrix (that exchanges lower and upper components)

\[ \mathcal{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

we can write

\[ \tilde{c} = \mathcal{F}c \]
\[ \tilde{c}^+ = c^+ \mathcal{F} \]
Consequently,

\[
c_{k-} = \sum_{m=1}^{2M} (1 - \mathcal{R})_{km} c_m
\]

\[
c_{k+} = \sum_{m=1}^{2M} \mathcal{R}_{km} c_m - \xi_k
\]

where:

\[
\mathcal{R} = \begin{pmatrix}
\rho & \kappa \\
-\kappa^* & 1 - \rho^T
\end{pmatrix}
\]

\[
\xi = \begin{pmatrix}
x \\
y^*
\end{pmatrix}
\]

Using this notation, we can write the contractions of the original creation and annihilation operators in the compact form:

\[
\sqrt{c_m c_k^+} = (1 - \mathcal{R})_{mk}
\]

\[
\sqrt{c_k} = \xi_k
\]

Also, the usual commutation relations can be written as:

\[
c_m c_k^+ + \tilde{c}_k \tilde{c}_m^+ = \delta_{mk}
\]
Consider the matrix elements

\[
\frac{\langle \Phi_2 | c_m c_k^+ | \Phi_1 \rangle}{\langle \Phi_2 | \Phi_1 \rangle} = c_m c_k^+ + c_m c_k^+ \\
= (1 - \mathcal{R} + \xi \eta^+)_{mk}
\]

\[
\frac{\langle \Phi_2 | \tilde{c}_k \tilde{c}_m^+ | \Phi_1 \rangle}{\langle \Phi_2 | \Phi_1 \rangle} = \tilde{c}_k \tilde{c}_m^+ + \tilde{c}_k \tilde{c}_m^+ \\
= [\mathcal{F}(1 - \mathcal{R} + \xi \eta^+)\mathcal{F}]_{km}
\]

where

\[
\eta_k^+ = c_k^+
\]

\[
\eta^+ = \left( \begin{array}{c} a^+ \\ a \end{array} \right) = \left( \begin{array}{c} y^+ \\ x^T \end{array} \right)
\]

or

\[
\eta = \left( \begin{array}{c} y \\ x^* \end{array} \right) = \mathcal{F} \left( \begin{array}{c} x^* \\ y \end{array} \right) = \mathcal{F} \xi^*
\]

By noting that

\[
[\mathcal{F} \xi \eta^+ \mathcal{F}]^T = \xi \eta^+
\]

we obtain
\( R = \mathcal{F}(1 - R^T)\mathcal{F} + 2\xi \eta^+ \)

or

\[
\begin{align*}
\rho' & = \rho - 2xy^+ \\
\kappa^T & = -\kappa + 2xx^T \\
\kappa'T & = -\kappa' - 2yy^T
\end{align*}
\]

Let us now consider the condition:

\[ c_{k-} |\Phi_1\rangle = 0 \quad \frac{\langle \Phi_2 | c_{k-} |\Phi_1\rangle}{\langle \Phi_2 | \Phi_1\rangle} = 0 \]

\[
\begin{align*}
R\xi &= \xi \\
R^2 &= R
\end{align*}
\]

Similarly:

\[
\frac{\langle \Phi_2 | c_{m}^+ c_{k-} |\Phi_1\rangle}{\langle \Phi_2 | \Phi_1\rangle} = 0
\]

\[
\frac{\langle \Phi_2 | c_{k-} c_{m}^+ |\Phi_1\rangle}{\langle \Phi_2 | \Phi_1\rangle} = 0
\]

\[
1 - R - (1 - R)(1 - R + \xi \eta^+) = 0
\]
Let us now show that the self-contraction vanish

\[
\begin{align*}
\rho^2 - \kappa \kappa'^* &= \rho \\
\rho'^2 - \kappa'^T \kappa'^+ &= \rho' \\
\rho \kappa - \kappa \rho'^T &= 0 \\
\kappa'^* \rho - \rho'^T \kappa'^* &= 0
\end{align*}
\]

Therefore:

\[
\begin{align*}
(\hat{A})^* &= \hat{A}^+ \\
(\hat{A}\hat{B})^* &= \hat{B}^+ \hat{A}^+
\end{align*}
\]

\[
\begin{align*}
y &= x \\
\rho^+ &= \rho \\
\rho'^+ &= \rho' \\
\kappa' &= -\kappa^T
\end{align*}
\]

In the doubled space:

\[
\begin{align*}
\eta &= \xi \\
\mathcal{R}^+ &= \mathcal{R}
\end{align*}
\]
Hence, the condition cannot be met, and

\[ 2\xi^+\xi = 1 \]

\[ \xi = 0 \]

All self-contractions vanish!

\[ \xi_k = 0 \quad \text{i.e.,} \quad \begin{cases} x_\mu = a_\mu = \langle \Phi | a_\mu | \Phi \rangle = 0 \\ x_\mu^* = a_\mu^+ = \langle \Phi | a_\mu^+ | \Phi \rangle = 0 \end{cases} \]

\[ \rho' = \rho'^+ = \rho = \rho^+ \]

\[ \kappa' = -\kappa'^T = \kappa = -\kappa^T \]

Generalized density matrix:

\[ \mathcal{R} = \mathcal{R}^+ = \mathcal{R}^2 = \begin{pmatrix} \rho & \kappa \\ -\kappa^* & 1 - \rho^* \end{pmatrix} \]
\[
(1 - \rho)_{\mu\nu} = \langle \Phi | a_\mu a_\nu^+ | \Phi \rangle = a_{\mu} a_{\nu}^+
\]
\[
\kappa_{\mu\nu} = \langle \Phi | a_\nu a_\mu | \Phi \rangle = a_{\nu} a_{\mu}
\]
\[
\kappa^*_{\mu\nu} = \langle \Phi | a_\mu^+ a_\nu^+ | \Phi \rangle = a_{\mu}^+ a_{\nu}^+
\]
\[
\rho_{\mu\nu} = \langle \Phi | a_\nu^+ a_\mu | \Phi \rangle = a_{\nu}^+ a_{\mu}
\]

\[
\rho^2 + \kappa \kappa^+ = \rho
\]
\[
\rho \kappa - \kappa \rho^* = 0
\]

\(\rho\) density matrix of the product state

\(\kappa\) pair tensor of the product state