Doubling the single-particle space

It is convenient to double the dimension of the one-particle space (such doubling is useful in the context of pairing theory)

$$c_k = \begin{cases} a_\mu & \text{for} \quad k = \mu, \\ a_\mu^+ & \text{for} \quad k = M + \mu. \end{cases}$$

$$\begin{aligned} c &\equiv \begin{pmatrix} a \\ a^+ \end{pmatrix} \\ \tilde{c} &\equiv \begin{pmatrix} a^+ \\ a \end{pmatrix} \\ c^+ &\equiv \begin{pmatrix} a^+, a \end{pmatrix} \\ \tilde{c}^+ &\equiv \begin{pmatrix} a, a^+ \end{pmatrix} \end{aligned}$$

Here, *k=1,2,…, 2M*

This new space is referred to as …

the quasiparticle space

By introducing the matrix (that exchanges lower and upper components)

$$\mathcal{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we can write

$$\begin{aligned} \tilde{c} &= \mathcal{F} c \\ \tilde{c}^+ &= c^+ \mathcal{F} \end{aligned}$$

Consequently,

$$
\begin{aligned}
c_{k-} &= \sum_{m=1}^{2M} (1-\mathcal{R})_{km}\, c_m \\
c_{k+} &= \sum_{m=1}^{2M} \mathcal{R}_{km} c_m \ - \xi_k
\end{aligned}
$$

where:

$$
\mathcal{R} = \left(\begin{array}{cc} \rho & \kappa \\ -\kappa'^{*} & 1-\rho'^{T} \end{array} \right)
$$

$$
\xi = \left(\begin{array}{c} x \\ y^{*} \end{array} \right)
$$

Using this notation, we can write the contractions of the original creation and annihilation operators in the compact form:

$$
\overbracket{c_m c_k^{+}} = (1-\mathcal{R})_{mk}
$$

$$
\overbracket{c_k} = \xi_k
$$

Also, the usual commutation relations can be written as:

$$
c_m c_k^{+} + \tilde{c}_k \tilde{c}_m^{+} = \delta_{mk}
$$

Consider the matrix elements

$$
\begin{aligned}
\frac{\langle \Phi_2 | c_m c_k^+ | \Phi_1 \rangle}{\langle \Phi_2 | \Phi_1 \rangle} &= \overline{c_m c_k^+} + \overline{c_m}\;\overline{c_k^+} \\
&= (1 - \mathcal{R} + \xi \eta^+)_{mk}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\langle \Phi_2 | \tilde{c}_k \tilde{c}_m^+ | \Phi_1 \rangle}{\langle \Phi_2 | \Phi_1 \rangle} &= \overline{\tilde{c}_k \tilde{c}_m^+} + \overline{\tilde{c}_k}\;\overline{\tilde{c}_m^+} \\
&= \left[\mathcal{F} (1 - \mathcal{R} + \xi \eta^+) \mathcal{F} \right]_{km}
\end{aligned}
$$

where

$$\eta_k^+ = \overline{c_k^+}$$

$$\eta^+ = \left(\ \overline{a^+}\, , \overline{a} \ \right) = \left(\ y^+ , x^T \ \right)$$

or

$$\eta = \left(\begin{array}{c} y \\ x^* \end{array} \right) = \mathcal{F} \left(\begin{array}{c} x^* \\ y \end{array} \right) = \mathcal{F} \xi^*$$

By noting that

$$[\mathcal{F} \xi \eta^+ \mathcal{F}]^T = \xi \eta^+$$

we obtain

$$\mathcal{R} = \mathcal{F}(1 - \mathcal{R}^T)\mathcal{F} + 2\xi\eta^+$$

or

$$\begin{aligned}
\rho' &= \rho - 2xy^+ \\
\kappa^T &= -\kappa + 2xx^T \\
\kappa'^T &= -\kappa' - 2yy^T
\end{aligned}$$

Let us now consider the condition:

$$c_{k-}|\Phi_1\rangle = 0 \qquad \frac{\langle\Phi_2|c_{k-}|\Phi_1\rangle}{\langle\Phi_2|\Phi_1\rangle} = 0$$

$$\Downarrow$$

$$\begin{aligned}
\mathcal{R}\xi &= \xi \\
\mathcal{R}^2 &= \mathcal{R}
\end{aligned}$$

Similarly:

$$\frac{\langle\Phi_2|c_m^+ c_{k-}|\Phi_1\rangle}{\langle\Phi_2|\Phi_1\rangle} = 0$$

$$\overbracket{c_k c_m^+} - \frac{\langle\Phi_2|c_{k-} c_m^+|\Phi_1\rangle}{\langle\Phi_2|\Phi_1\rangle} = 0$$

$$1 - \mathcal{R} - (1 - \mathcal{R})(1 - \mathcal{R} + \xi\eta^+) = 0$$

$$\boxed{\mathcal{R} = \mathcal{R}^2}$$

Projector !

$$\begin{aligned}
\rho^2 - \kappa\kappa'^* &= \rho \\
\rho'^2 - \kappa^T\kappa'^+ &= \rho' \\
\rho\kappa - \kappa\rho'^T &= 0 \\
\kappa'^*\rho - \rho'^T\kappa'^* &= 0
\end{aligned}$$

Let us now show that the self-contraction vanish

$$\begin{aligned}
(\overbracket{\hat{A}})^* &= \overbracket{\hat{A}^+} \\
(\overbracket{\hat{A}\hat{B}})^* &= \overbracket{\hat{B}^+\hat{A}^+}
\end{aligned}$$

Therefore:

$$\begin{aligned}
y &= x \quad \Leftarrow \quad (\overbracket{a_\mu})^* = \overbracket{a_\mu^+} \\
\rho^+ &= \rho \\
\rho'^+ &= \rho' \\
\kappa' &= -\kappa^T
\end{aligned}$$

In the doubled space:

$$\begin{aligned}
\eta &= \xi \\
\mathcal{R}^+ &= \mathcal{R}
\end{aligned}$$

$$Tr(\mathcal{R}) = M + \xi^{+}\xi \qquad\qquad \xi = \xi(2\xi^{+}\xi)$$

Must be an integer number

Hence, the conditipn $2\xi^{+}\xi = 1$ cannot be met, and

$$\xi = 0$$

All self-contractions vanish!

$$\xi_k = 0 \quad \text{i.e.,} \quad \left\{ \begin{array}{lclcl} x_\mu & = & a_\mu & = & \langle\Phi|a_\mu|\Phi\rangle = 0 \\ x^*_\mu & = & a^+_\mu & = & \langle\Phi|a^+_\mu|\Phi\rangle = 0 \end{array} \right.$$

$$\begin{array}{rcl} \rho' = \rho'^{+} & = & \rho = \rho^{+} \\ \kappa' = -\kappa'^{T} & = & \kappa = -\kappa^{T} \end{array}$$

Generalized density matrix:

$$\mathcal{R} = \mathcal{R}^{+} = \mathcal{R}^{2} = \begin{pmatrix} \rho & \kappa \\ -\kappa^* & 1-\rho^* \end{pmatrix}$$

$$
\begin{aligned}
(1-\rho)_{\mu\nu} &= \langle\Phi|a_\mu a_\nu^+|\Phi\rangle = \overbracket{a_\mu a_\nu^+} \\
\kappa_{\mu\nu} &= \langle\Phi|a_\nu a_\mu|\Phi\rangle = \overbracket{a_\nu a_\mu} \\
\kappa^*_{\mu\nu} &= \langle\Phi|a_\mu^+ a_\nu^+|\Phi\rangle = \overbracket{a_\mu^+ a_\nu^+} \\
\rho_{\mu\nu} &= \langle\Phi|a_\nu^+ a_\mu|\Phi\rangle = \overbracket{a_\nu^+ a_\mu}
\end{aligned}
$$

$$
\begin{aligned}
\rho^2 + \kappa\kappa^+ &= \rho \\
\rho\kappa - \kappa\rho^* &= 0
\end{aligned}
$$

ρ density matrix of the product state

κ pair tensor of the product state