Bogoliubov Transformation, Group Structure

Any *F*-even unitary matrix in *2M* will correspond to some Bogoliubov transformation. The set of such matrices forms a group (a subgroup of the unitary group of degree *2M*)

$$\mathcal{F}\mathcal{A}_1\mathcal{A}_2\mathcal{F} = (\mathcal{F}\mathcal{A}_1\mathcal{F})(\mathcal{F}\mathcal{A}_2\mathcal{F}) = \mathcal{A}_1^*\mathcal{A}_2^* = (\mathcal{A}_1\mathcal{A}_2)^*$$

Let us now make a unitary rotation of all matrices *A* and define:

$$\mathcal{B} \equiv \mathcal{U}\mathcal{A}\mathcal{U}^+$$

with

$$\mathcal{U} = \sqrt{\frac{i}{2}}\begin{pmatrix} 1 & , & -i \\ i & , & -1 \end{pmatrix}$$

$$\begin{aligned} \mathcal{U}^+\mathcal{U} &= 1, \\ \mathcal{U}^T\mathcal{U} = -\mathcal{U}\mathcal{U}^T &= \mathcal{F} \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \mathcal{B}^* &= \mathcal{B} \\ \mathcal{B}^T\mathcal{B} &= 1 \end{aligned}$$

Consequently, matrices *B* are real and orthogonal! They form the *O(2M)* group - orthogonal group in *2M*

O(2M) can be decomposed into *SO(2M)* (subgroup of orthogonal matrices with Det=+1; proper orthogonal matrices) and a subset of orthogonal matrices with Det=-1 (improper orthogonal matrices). The corresponding Bogoliubov transformations will be called proper and improper, respectively

Any element of *SO(2M)* can be represented as

$$\mathcal{B} = \exp(\mathcal{G})$$

where *G* is real and antisymmetric:

$$\begin{aligned} \mathcal{G}^T &= -\mathcal{G} \\ \mathcal{G}^* &= \mathcal{G} \end{aligned}$$

Of course, $Det(\mathcal{B}) = \exp\left[Tr(\mathcal{G})\right] = 1$

Any improper matrix can be expressed as a product of a proper matrix and one (selected) improper matrix.

Proper Bogoliubov Transformations

Any proper Bogoliubov matrix transformation can be represented by

$$\mathcal{A} = \exp(i\mathcal{Y})$$

where

$$\begin{aligned} \mathcal{Y}^+ &= \mathcal{Y} \\ \mathcal{F}\mathcal{Y}\mathcal{F} &= -\mathcal{Y}^* \end{aligned}$$

Note that *Y* is *F*-odd

$\mathcal{A} = \mathcal{U}^+\mathcal{B}\mathcal{U}$ $\qquad$ $\mathcal{Y} = -i\mathcal{U}^+\mathcal{G}\mathcal{U}$ $\qquad$ $\mathcal{F}\mathcal{Y}\mathcal{F} = -\mathcal{Y}^*$

Bogoliubov Transformations in the Fock space

A hermitian and *F*-odd matrix in *2M* can be represented by an operator acting in the Fock space:

$$\hat{Y} = \frac{1}{2}\sum_{kl} \mathcal{Y}_{kl} c_k^+ c_l$$

Hermitian operator in the Fock space

Looks like one-body but is not. Why?

Expressing Y explicitly in terms of a^+ and a, one can write

$$\mathcal{Y} = \begin{pmatrix} Y & ; & -X^* \\ X & ; & -Y^* \end{pmatrix}$$

where

$$\begin{aligned} Y^+ &= Y \\ X^T &= -X \end{aligned}$$

$$\hat{Y} = \sum_{\mu\nu}\left(\frac{1}{2}X^+_{\mu\nu}a^+_\mu a^+_\nu + Y_{\mu\nu}a^+_\mu a_\nu + \frac{1}{2}X_{\mu\nu}a_\mu a_\nu\right) - \frac{1}{2}\mathrm{Tr}(Y)$$

The most general hermitian operator involving all pairs of creation and annihilation operators

For any proper matrix Bogoliubov transformation

$$\mathcal{A} = \exp(i\mathcal{Y})$$

one can find a Bogoliubov transformation in the Fock space

$$\hat{A} = \exp(i\hat{Y})$$

such that

$$\hat{A}c\hat{A}^{+} = \mathcal{A}^{+}c = \gamma$$

Proof:

$$\exp(D)E\exp(-D) = E + \sum_{n=1}^{\infty} \tfrac{1}{n!}[D,[D,\ldots[D,E]\ldots]]_n,$$

$$\hat{A}c\hat{A}^{+} = c + \sum_{n=1}^{\infty} \tfrac{i^n}{n!}[\hat{Y},[\hat{Y},\ldots[\hat{Y},c]\ldots]]_n.$$

$$\begin{aligned}
[\hat{Y},c_m] &= \frac{1}{2}\sum_{kl}\mathcal{Y}_{kl}[c_k^{+}c_l,c_m] = \frac{1}{2}\sum_{kl}\mathcal{Y}_{kl}\left(\mathcal{F}_{lm}c_k^{+} - \delta_{km}c_l\right) \\
&= \frac{1}{2}\sum_{k}\left(\mathcal{F}\mathcal{Y}^{T}\mathcal{F} - \mathcal{Y}\right)_{mk}c_k = -\sum_{k}\mathcal{Y}_{mk}c_k
\end{aligned}$$

$$[\hat{Y},[\hat{Y},\ldots[\hat{Y},c_m]\ldots]]_n = \sum_{k}(-\mathcal{Y})_{mk}^{n}\,c_k$$

$$\boxed{\hat{A} c_m \hat{A}^+ = \sum_k \left(\exp(-i\mathcal{Y})\right)_{mk} c_k = \sum_k \mathcal{A}^+_{mk} c_k}$$

The Bogoliubov transformation in the Fock space takas us from the old fermion operators to quasiparticles

$$\hat{A} a^+_{\mu_1} \ldots a^+_{\mu_A} |0\rangle = \alpha^+_{\mu_1} \ldots \alpha^+_{\mu_A} |\Phi\rangle.$$

$$|\Phi\rangle \equiv \hat{A}|0\rangle$$

Note that the Bogoliubov transformation is linear in a^+ and a. This guarantees that the Wick's theorem holds for all the vacua $|\Phi\rangle$ obtained from the real particle vacuum $|0\rangle$

Improper Bogoliubov Transformations

Consider the hermitian and unitary operator

$$\hat{A}^{(\mu)} \equiv a_\mu^+ + a_\mu$$

$$\hat{A}^{(\mu)} a_\nu^+ \hat{A}^{(\mu)+} = \begin{cases} -a_\nu^+ & \text{for } \nu \neq \mu \\ a_\nu & \text{for } \nu = \mu. \end{cases}$$

We see that A is the Bogoliubov transformation in the Fock space.
The corresponding matrix transformation can be written as:

$$A_{\nu\nu'}^{(\mu)} = \begin{cases} -1 & \text{for } \nu = \nu' \neq \mu \\ 0 & \text{other cases,} \end{cases}$$

$$B_{\nu\nu'}^{(\mu)} = \begin{cases} 1 & \text{for } \nu = \nu' = \mu \\ 0 & \text{other cases.} \end{cases}$$

$$Det(\mathcal{A}^{(\mu)}) = -1 \quad \Longrightarrow$$ improper Bogoliubov transformation

All improper Bogoliubov transformations can be written as

$$\begin{aligned} \hat{A} &= \exp(i\hat{Y})\hat{A}^{(1)} \\ \mathcal{A} &= \exp(i\mathcal{Y})\mathcal{A}^{(1)} \end{aligned}$$

Let us introduce the particle-number-parity operator:

$$\hat{\pi}_N = \exp(i\pi\hat{N})$$

The operator $e^{i\hat{Y}}$ does not change parity π_N and $\hat{A}^{(\mu)}$ changes it. Consequently.

$$\det \mathcal{A} = \pm 1 \qquad \Leftrightarrow \qquad \hat{\pi}_N \hat{A} = \pm \hat{A} \hat{\pi}_N$$

- Proper Bpgoliubov transformations conserve the particle-number-parity
- Improper Bogoliubov transformations change the particle-number-parity

Product States -summary

- The simplest states in the Fock space that obey Wick's theorem. (It is easy to calculate expectation values of operators in product states!)

- Correspond to vacua of quasi-particle operators α_k

- Form a complete set in the Fock space

- They are eigenstates of one-body operators. (Describe non-interacting fermions)

- Can be used in perturbative calculations

- All product states can be written in a form

$$\left.\begin{array}{l} |\Phi\rangle_{\text{even}} \\ |\Phi\rangle_{\text{odd}} \end{array}\right\} = \exp(i\hat{Y}) \left\{ \begin{array}{c} 1 \\ a_1^+ \end{array} \right\} |0\rangle.$$

$$\begin{aligned} |\Phi\rangle_{\text{even}} &= \exp\left(\frac{i}{2}\sum_{kl}\mathcal{Y}_{kl}c_k^+c_l\right)|0\rangle \\ &= \exp\left(i\sum_{\mu\nu}\left\{\frac{1}{2}X_{\mu\nu}^+a_\mu^+a_\nu^+ + Y_{\mu\nu}a_\mu^+a_\nu + \frac{1}{2}X_{\mu\nu}a_\mu a_\nu\right\} - \frac{i}{2}\mathrm{Tr}Y\right)|0\rangle \end{aligned}$$

A parameterization of a product state… What is the number of parameters?

$$\begin{array}{lll} X^T = -X & \Rightarrow & M(M-1) \ \text{real parameters} \\ Y^+ = Y & \Rightarrow & M^2 \ \text{real parameters} \end{array}$$

In reality, it is enough to use only *M(M-1)* real parameters!