BCS model

$$\hat{H}_{BCS}\alpha^+_{k_1}\alpha^+_{k_2}\cdots\alpha^+_{k_n}|0\rangle = (E_0 + E_{k_1} + E_{k_2} + \cdots + E_{k_n})\alpha^+_{k_1}\alpha^+_{k_2}\cdots\alpha^+_{k_n}|0\rangle$$

$$E_k = \sqrt{\tilde{\varepsilon}_k^2 + \Delta_k^2}$$

Quasiparticle energy

State-independent monopole pairing force:

$$\hat{V}_{pair} = -G\hat{P}^+\hat{P} = -G\sum_{i,j>0} c_i^+ c_{\bar{i}}^+ c_{\bar{j}} c_j$$

$$\hat{P} \equiv \sum_{i>0} c_i^+ c_{\bar{i}}^+ \qquad \left(\bar{v}_{ijkl} = -G\delta_{j\bar{i}}\delta_{l\bar{k}}\,\mathrm{sgn}(j)\,\mathrm{sgn}(k)\right)$$

$$\Delta = G\sum_{i>0} u_i v_i \Rightarrow \Delta\left(\frac{G}{2}\sum_{i>0}\frac{1}{E_i} - 1\right) = 0$$

- **Fermi level can be extracted from experimental binding energies:** $\lambda = \frac{dE}{dN}$
- **Energy gap in even-even nuclei:** $E_{k_1} + E_{k_2} \geq 2\Delta$
- **High level density in odd systems:** $E_k - E_{k_0} \approx \sqrt{\tilde{\varepsilon}_k^2 + \Delta^2} - \Delta$
- **Pairing gap can be extraxted from experimental odd-even mass differences:**

$$\Delta \cong -\tfrac{1}{2}\left[E_{g.s.}^{(N+2)} + E_{g.s.}^{(N)} - 2E_{g.s.}^{(N+1)}\right]$$

The gap equation

- Schematic solution for the case of uniform density of single-particle states

$$\frac{2}{G} = \sum_{k>0} \frac{1}{E_k} \Rightarrow 1 = \frac{G}{2} \int_a^b \frac{1}{\sqrt{\varepsilon^2 + \Delta^2}} g(\varepsilon) d\varepsilon$$

$$\text{If } g(\varepsilon) \approx \bar{g} \text{ and } G\bar{g} << 1 \text{ then } \Delta \propto e^{-(1/\bar{g}G)}$$

- Anticorrelation between pairing and shell effects

$$E_{pair} = E_0 - E_0^{G=0} \approx -\bar{g}\Delta^2 \approx -2 \text{ MeV}$$

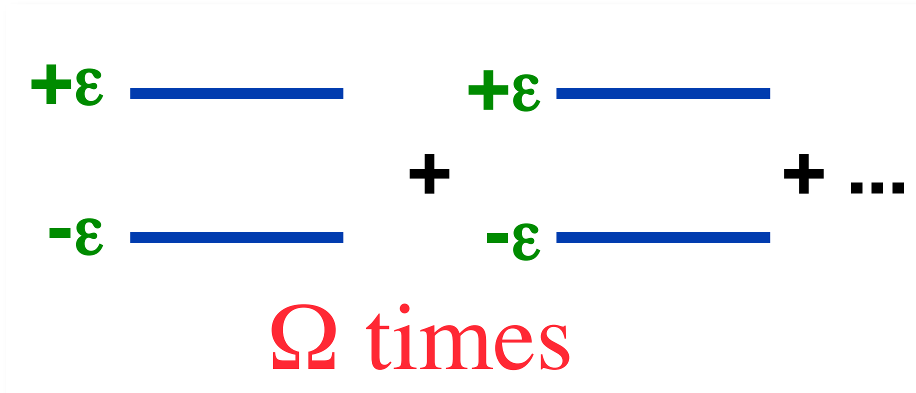
For well deformed nuclei

- Blocking effect

$$\alpha_k^+ |0\rangle = c_k^+ \prod_{i \neq k} \left(u_i + v_i c_i^+ c_{\bar{i}}^+ \right) |-\rangle$$

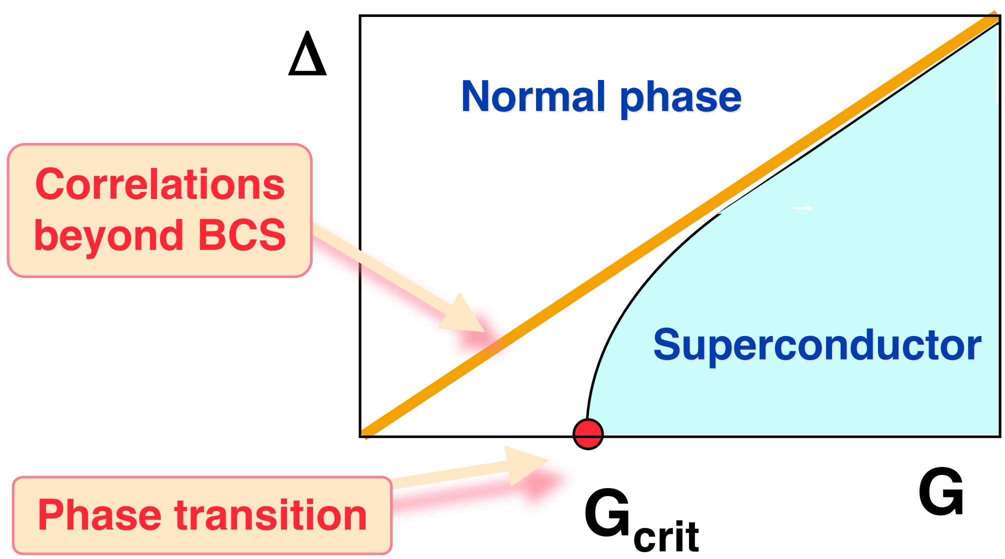
$$\Delta_k = G \sum_{i \neq k} u_i v_i, \quad N = 1 + \sum_{i \neq k} 2 v_i^2$$

Simple pairing model



$$N = 2\Omega \text{ particles } \Rightarrow \lambda = 0 \quad (\text{assuming } Gv^2 < \varepsilon)$$

$$\frac{1}{G} = \frac{\Omega}{E} \Rightarrow \Delta = \varepsilon\sqrt{\left(\frac{G}{G_{crit}}\right)^2 - 1}, \quad G_{crit} = \frac{\varepsilon}{\Omega}$$



- ❑ Correlations are important, especially for small values of G
- ❑ Static description breaks down when the single-particle level density is small

Recall that v_k^2 is the occupation probability for state k. Thus, the pairing interaction destabilizes the Fermi surface!

Inserting these results into Eq. (1) yields the *Gap Equation*

$$\Delta_k = -\frac{1}{2}\sum_{l>0} \overline{v}_{k\bar{k}l\bar{l}} \frac{\Delta_k}{\sqrt{\tilde{\varepsilon}_k^2 + \Delta_k^2}}$$

This equation, together with the Equation for the energies $\tilde{\varepsilon}_k$ has to be solved iteratively.

Simple examples and illustration:

(1) Pure pairing force: $\overline{v}_{ijkl} = -G, \quad t_{k,l} = \delta_{kl}\varepsilon_l$
Gap equation

$$\Delta = \frac{G}{2}\sum_{l>0} \frac{\Delta}{\sqrt{(\varepsilon_k - \mu)^2 + \Delta_k^2}}$$

For vanishing single-particle energies $\varepsilon_k = 0$ and a single-j shell one recovers the results of the seniority model. The gap becomes

$$\Delta = G\sqrt{\frac{n}{2}\left(\Omega - \frac{N}{2}\right)} = \frac{G\Omega}{2}$$

at half filling. Comparison with the seniority model shows that 2Δ is the energy to break a pair and equals the excitation energy!

(2) Analytical solution for simple model.

Assume constant interaction G, constant gap Δ, and constant density of states ρ, and restrict summation to the vicinity δ of the Fermi surface. Gap equation:

$$1 = \frac{G}{2} \int_{\mu-\delta}^{\mu+\delta} d\varepsilon \frac{\rho(\varepsilon)}{\sqrt{(\varepsilon-\mu)^2 + \Delta^2}} = G\rho \ \mathrm{arsinh} \frac{\delta}{\Delta}$$

For weak pairing $G\rho \ll 1$ one finds

$$\frac{\Delta}{\delta} \propto \exp\left(\frac{-1}{|G|\rho}\right)$$

Gap is nonperturbative in interaction G!

The particle number variance is

$$(\Delta N)^2 \approx 2\rho\Delta \ \mathrm{atan} \frac{\delta}{\Delta} = \begin{cases} \pi\rho\Delta & \text{for weak pairing } \Delta \ll \delta, \\ 2\rho\delta & \text{for strong pairing.} \end{cases}$$

BCS essentials

1. The Fermi surface of a Fermi gas is unstable with respect to *attractive* interactions. The gap equation has a nontrivial solution $\Delta \neq 0$ whenever the interaction or the density of states is sufficiently large.

2. The finite excitation gap causes superfluidity: The system cannot absorb arbitrarily small perturbations and therefore remains inert (See, e.g., Landau & Lifshitz, Statistical Mechanics II).