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How weakly bound are the strongly interacting systems?

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Quantum field theory in four minutes

Minute 1

- **Classical mechanics:**
  - Lagrangian:

$$\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, t) = T - U$$

  - Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

  - Hamiltonian:

$$\mathcal{H}(q_i, p_i, t) = \sum_i p_i \dot{q}_i - \mathcal{L} \quad , \quad \text{for} \quad p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Minute 2

- **Quantum mechanics:**
  - Quantization:

$$\hat{q}_i = q_i \quad , \quad \hat{p}_i = \frac{\partial}{\partial q_i}$$

$$\mathcal{H}(q_i, p_i, t) \longrightarrow \hat{\mathcal{H}}(\hat{q}_i, \hat{p}_i, t)$$

  - Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}\Psi(q_i, t) = \hat{\mathcal{H}}(\hat{q}_i, \hat{p}_i, t)\Psi(q_i, t)$$

Quantum field theory in four minutes

Minute 3

- Classical field theory:
  - Generalized variables:

$$i \longrightarrow \vec{x}$$

$$q_i \longrightarrow \psi_{\vec{x}} \equiv \psi(\vec{x})$$

  - Local Lagrangian density $\mathcal{L}[\psi(\vec{x}), \partial_\mu \psi(\vec{x})]$ defines the Lagrangian:

$$L = \int d^3\vec{x} \; \mathcal{L}[\psi(\vec{x}), \partial_\mu \psi(\vec{x})]$$

  - Action is relativisticaly invariant, $x \equiv (\vec{x}, t)$:

$$I = \int dt \; L[\psi(\vec{x}), \dot{\psi}(\vec{x})] = \int d^4x \; \mathcal{L}[\psi(x), \partial_\mu \psi(x)]$$

Minute 4

- Quantum field theory:
  - Quantization:

$$\hat{\psi}(\vec{x}) = \psi(\vec{x}) \quad , \quad \widehat{\text{momentum}} = \frac{\delta}{\delta \psi(\vec{x})}$$

  - Some details:
    1. Spin
    2. Statistics
    3. Perturbation theory
    4. ...

Quantum electrodynamics (QED)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}_e\gamma^\mu[\partial_\mu + ieA_\mu]\psi_e - m_e\bar{\psi}_e\psi_e$$

- Space-time index μ=0,1,2,3
- Units: $\hbar = c = 1$, $e = \sqrt{4\pi\alpha}$, $1/\alpha \simeq 137$
- Tensor of the electromagnetic field: $F^{\mu\nu}=\partial^\mu A^\nu - \partial^\nu A^\mu$ (contains electric and magnetic fields)
- Vector potential of the photon field: A^μ
- Dirac four-spinor of the electron field: ψ_e
- Elementary charge: e (electron charge is $q=-e$)
- Electron mass: m_e
- Dirac 4×4 matrices γ^μ:

$$\gamma^0=-i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \vec{\gamma}=-i\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix},\ \gamma_5=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

- Pauli 2×2 matrices $\vec{\sigma}$:

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Quantum chromodynamics (QCD)

$$\mathcal{L} = -\frac{1}{4} F^{\alpha}_{\mu\nu} F^{\mu\nu}_{\alpha} - \sum_{n} \bar{\psi}_n \gamma^{\mu} [\partial_{\mu} - i g A^{\alpha}_{\mu} t_{\alpha}] \psi_n - \sum_{n} m_n \bar{\psi}_n \psi_n$$

- Color index α=1,. . . ,8
- Flavor index n=1,. . . ,6 (or n=1,2 for nuclei)
- Tensors of the color fields: $F^{\mu\nu}_{\alpha} = \partial^{\mu} A^{\nu}_{\alpha} - \partial^{\nu} A^{\mu}_{\alpha}$ (contains gluon fields)
- Vector potentials of the gluon fields: A^{μ}_{α}
- Dirac four-spinors of the quark fields: ψ_n
- Color charge (strong coupling constant): g
- Quark masses: m_n
- Generators of the SU(3) color group: 3×3 matrices t_{α}

Chiral symmetry and the isospin

$$\mathcal{L}_\chi = -\bar{q}\gamma^\mu D_\mu q = -\bar{u}\gamma^\mu D_\mu u - \bar{d}\gamma^\mu D_\mu d$$

- Covariant derivative: $D_\mu = \partial_\mu - igA_\mu^\alpha t_\alpha$
- Quark iso-spinor: $q = \begin{pmatrix} u \\ d \end{pmatrix}$
- Quark masses neglected for a moment.
- $\mathcal{L}_\chi$ is invariant with respect to the SU(2)$\times$SU(2) group generated by the mixing of the u and d quarks.
- SU(2)$\times$SU(2) generators:
  1° the isospin matrices: $\vec{t} = \frac{1}{2}\vec{\tau}$

  $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  2° the $\gamma_5\times$isospin matrices: $\vec{x} = \gamma_5\vec{t}$
  (remember that $(\gamma_5)^2 = 1$)
- SU(2)$\times$SU(2) commutation relations:
  $\vec{t}_L = \frac{1}{2}(1+\gamma_5)\vec{t} = \frac{1}{2}(\vec{t}+\vec{x}) \quad \longleftarrow \quad$ left-handed
  $\vec{t}_R = \frac{1}{2}(1-\gamma_5)\vec{t} = \frac{1}{2}(\vec{t}-\vec{x}) \quad \longleftarrow \quad$ right-handed
  $[t_{Li}, t_{Lj}] = i\epsilon_{ijk}t_{Lk}, \qquad [t_{Ri}, t_{Rj}] = i\epsilon_{ijk}t_{Rk}, \qquad [t_{Li}, t_{Rj}] = 0$
- Group isomorphism: SU(2)$\times$SU(2) $\equiv$ O(4)

Chiral symmetry breaking

$$\mathcal{L}_\chi = -g\bar{u}\gamma^\mu D_\mu u - \bar{d}\gamma^\mu D_\mu d - m_u\bar{u}u - m_d\bar{d}d$$

- Quark masses are small as compared to the QCD scale: $m_u \approx 3\,\mathrm{MeV}$, $m_d \approx 6\,\mathrm{MeV}$, $\Lambda_{QCD} \approx 1000\,\mathrm{MeV}$
- Quark masses weakly break the chiral symmetry
- Chiral symmetry is strongly broken in the real world: pairs of particles having similar masses and opposite parities are not observed.
- Strong chiral symmetry breaking: SU(2)×SU(2) broken while the isospin SU(2) conserved.
- Effective theories are needed to describe complicated composite objects like mesons and nucleons.
- Fields of composite objects can be treated as elementary fields.
- Lagrangians of effective fields can be built based on the symmetry requirements.
- Pion mass is given by the quark masses: $m_\pi = -4(m_u + m_d)\langle\Phi_4^+\rangle_{\mathrm{vac}}/F_\pi^2$

Nonlinear σ model

Let ϕ_n, n=1,2,3,4, denote the pseudoscalar (real) fields described by the Lagrangian

$$\mathcal{L}_\sigma = -\tfrac{1}{2}\partial_\mu\phi_n\partial^\mu\phi_n - \tfrac{1}{2}\mathcal{M}^2\phi_n\phi_n - \tfrac{1}{4}g(\phi_n\phi_n)^2$$

$\mathcal{L}_\sigma$ is explicitly invariant with respect to rotations in 4 dimensions: O(4) $\equiv$ SU(2)$\times$SU(2) symmetry. However, the potential energy depends only on the radial variable $\sigma^2=\phi_n\phi_n$

$$V(\phi) = V(\sigma) = \tfrac{1}{2}\mathcal{M}^2\sigma^2 + \tfrac{1}{4}g\sigma^4$$

For $g>0$ and $\mathcal{M}^2<0$ it has the minimum at

$$\sigma = |\mathcal{M}|/\sqrt{g}$$

and does not depend on the orientation of ϕ in the 4-dim space.

By picking one solution $\bar{\phi}$ out of the infinitely-many existing ones, we break the $O(4)$ symmetry.

The mass-matrix (the stiffness of the potential) calculated at $\bar{\phi}$ reads:

$$\mathcal{M}_{nm} = \frac{\partial^2 V}{\partial\phi_n\partial\phi_m} = 2g\bar{\phi}_n\bar{\phi}_m$$

It has one eigenvalue equal to $m^2 = 2g\sigma^2 = -2\mathcal{M}^2$ (eigenvector $\bar{\phi}$) and three eigenvalues m^2=0 (three vectors orthogonal to $\bar{\phi}$) — the Goldstone bosons — the three pions: $\vec{\pi} = (\pi_+, \pi_0, \pi_-)$.

Transformation to angle-radial variables

In order to separate out the radial variable σ in the 4-dimensional space we set:

$$\vec{\phi} = \frac{2\vec{z}}{1+\vec{z}^2}\sigma \qquad (n = 1, 2, 3)$$

$$\phi_4 = \frac{1-\vec{z}^2}{1+\vec{z}^2}\sigma$$

The Lagrangian expressed in the new fields $\vec{z}$ and σ reads

$$\mathcal{L}_\sigma = -2\sigma^2\frac{\partial_\mu\vec{z}\partial^\mu\vec{z}}{(1+\vec{z}^2)^2} - \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \tfrac{1}{2}\mathcal{M}^2\sigma^2 - \tfrac{1}{4}g\sigma^4$$

- Pion field $\vec{\pi}$ is equal to $\vec{z}$ up to a normalization constant:

$$\vec{\pi} = F\vec{z}$$

- Details of the part depending on σ are irrelevant (high-energy part).
- Pions fields must couple to other fields (e.g. to nucleons) through the covariant derivatives $\vec{D}_\mu = \frac{\partial_\mu\vec{z}}{1+\vec{z}^2}$

Nucleon-pion Lagrangian

$$\mathcal{L}_{N\pi} = -\bar{N}\left(\gamma^\mu \partial_\mu + g_\phi[\phi_4 + 2i\gamma_5 \vec{t}\cdot\vec{\phi}]\right) N$$

In the angle-radial variables this reads:

$$\mathcal{L}_{N\pi} = -\bar{\widetilde{N}}\left(\gamma^\mu \partial_\mu + g_\phi \sigma + 2i\vec{t}\cdot(\vec{z}\times\gamma^\mu \vec{D}_\mu) + 2ig_A\gamma_5\vec{t}\cdot\gamma^\mu\vec{D}_\mu\right)\widetilde{N}$$

- Nucleon iso-spinor: $N = \begin{pmatrix} p \\ n \end{pmatrix}$
- Transformed nucleon field: $\widetilde{N} = (1 + 2i\gamma_5 \vec{t}\cdot\vec{z})N/\sqrt{1+\vec{z}^2}$
- Nucleon mass results from the chiral symmetry breaking: $m_N = g_\phi \sigma$
- Axial-vector coupling constant g_A introduced “by hand”, because the last term is separately chiral invariant.
- Nucleons must couple through the covariant derivatives $\vec{\mathcal{D}}_\mu = \partial_\mu + 2i\vec{t}\cdot(\vec{z}\times\vec{D}_\mu)$
- In the lowest order the nucleons interact by a one-pion-exchange potential (OPEP).

Argonne v18 interaction

$$V(NN) = V^{\mathrm{EM}}(NN) + V^{\pi}(NN) + V^{\mathrm{R}}(NN)$$

- R.B. Wiringa, V.G.J. Stoks, and R. Schiavilla, Phys. Rev. C51, 38 (1995)
- Electromagnetic part: EM
- One-pion-exchange part: π
- Intermediate and short-range phenomenological part: R

$$V^{\pi}(pp) = f_{pp}^2 \tfrac{1}{3} m_\pi \left[Y(r) \sigma_i \cdot \sigma_j + T(r) S_{ij} \right]$$

$$Y(r) = \frac{e^{-m_\pi r}}{r} \left(1 - e^{-cr^2} \right)$$

$$T(r) = \left(1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2} \right) \frac{e^{-m_\pi r}}{r} \left(1 - e^{-cr^2} \right)^2$$

$$V^{\mathrm{R}} = V^{\mathrm{c}} + V^{\mathrm{l2}} L^2 + V^{\mathrm{t}} S_{12} + V^{\mathrm{ls}} L \cdot S + V^{\mathrm{ls2}} (L \cdot S)^2$$

$$V^{\mathrm{i}}(r) = I^{\mathrm{i}} T^2(r) + \left[P^{\mathrm{i}} + m_\pi r Q^{\mathrm{i}} + (m_\pi r)^2 R^{\mathrm{i}} \right] \left(1 + e^{(r-r_0)/a} \right)^{-1}$$

Effective Lagrangians

Instead of solving simple Lagrangians to all orders one may solve corrected Lagrangians to low orders:

$$\mathcal{L}^{\text{eff}}_{\pi\pi} = -\tfrac{1}{2}F^2\vec{D}_\mu\vec{D}^\mu - \tfrac{1}{4}c_4\left(\vec{D}_\mu\vec{D}^\mu\right)^2 - \tfrac{1}{4}c'_4\left(\vec{D}_\mu\vec{D}_\nu\right)\left(\vec{D}^\mu\vec{D}^\nu\right) + \dots$$

- Pion coupling constant: $F = 2\sigma$
- $\mathcal{L}^{\text{eff}}_{\pi\pi}$ gives a good description of the $\pi\pi$ scattering lengths.

$$\mathcal{L}^{\text{eff}}_{N\pi} = -\bar{\widetilde{N}}\left(\gamma^\mu\vec{\mathcal{D}} + m_N + 2ig_A\gamma_5\vec{t}\cdot\gamma^\mu\vec{D}_\mu\right)\widetilde{N}$$
$$-c_{2\alpha\beta}\left(\bar{\widetilde{N}}\Gamma_\alpha\widetilde{N}\right)\left(\bar{\widetilde{N}}\Gamma_\beta\widetilde{N}\right) + \dots$$

- Projectors on different spin-isospin channels: Γ_α
- $\mathcal{L}^{\text{eff}}_{N\pi}$ gives a good description of the NN scattering lengths.
- Supplemented by the next order gives a good description of the NN phase shifts: D.R. Entem and R. Machleidt, Phys. Lett. B524, 93 (2002).

Weak nuclear binding

Energy scales in nuclear physics

- QCD coupling constant: $\Lambda_{QCD} \approx 1000\,\mathrm{MeV}$
- Nucleon masses: $m_p = 938\,\mathrm{MeV}$, $m_n = 940\,\mathrm{MeV}$
- Pion masses: $m_{\pi^0} = 135\,\mathrm{MeV}$, $m_{\pi^\pm} = 140\,\mathrm{MeV}$
- OPEP range: $R_\pi \approx 1/m_\pi \approx 2\,\mathrm{fm}$
- Deuteron binding energy $B_D = 2.225\,\mathrm{MeV}$
- Pole of the $^3S_1$ (deuteron) scattering amplitude:
  $k = \sqrt{m_N B_D} = 45\,\mathrm{MeV}$
- Pole of the $^3S_0$ (nn) scattering amplitude:
  $k = 8i\,\mathrm{MeV}$
- $^3S_1$ scattering length: $a = 5.417\,\mathrm{fm}$
- $^3S_0$ scattering length: $a = -23.714\,\mathrm{fm}$
- Binding energy of nuclei: $B/A \approx 8\,\mathrm{MeV}$

General many-body problem

We assume a one- plus two-body Hamiltonian

$$\hat{H} = T_{xy} a_x^+ a_y + \frac{1}{4} V_{xyx'y'} a_x^+ a_y^+ a_{y'} a_{x'}$$

where $x \equiv (\vec{r}, \sigma, \tau)$, $x' \equiv (\vec{r}', \sigma', \tau')$, etc., and the summation-integration $\int d^3\vec{r}' \sum_{\sigma'\tau'}$ is implied for every repeating index. The energy of an arbitrary state $|\Psi\rangle$ is then given by

$$E = \langle\Psi|\hat{H}|\Psi\rangle = T_{xy}\rho_{yx} + \frac{1}{4} V_{xyx'y'} \rho_{x'y'xy}$$

where the one- and two-body density matrices read

$$\rho_{yx} = \langle\Psi|a_x^+ a_y|\Psi\rangle \quad , \quad \rho_{x'y'xy} = \langle\Psi|a_x^+ a_y^+ a_{y'} a_{x'}|\Psi\rangle$$

Let us assume that

- $M \simeq 10^4$ is the number of space-spin-isospin points
- $A \simeq 200$ is the number of particles

Important dimensions are

- $\begin{pmatrix} M \\ A \end{pmatrix} \simeq 10^{425}$ is the dimension of the A-particle Hilbert space
- $M^2 + (M(M-1)/2)^2 \simeq 10^{16}$ is the number of real parameters defining the energy

Hartree-Fock approximation

The Hartree-Fock approach relies on assuming that the ground-state energy can by calculated by:

- approximating the two-body density matrix by a product of one-body density matrices:

$$\boxed{\rho_{x'y'xy} = \rho_{x'x}\rho_{y'y} - \rho_{x'y}\rho_{y'x}}$$

- minimizing the energy over the one-body density matrices.

Then we have the Hartree-Fock approximation to the energy:

$$\begin{aligned} E &= T_{xy}\rho_{yx} + \frac{1}{4}V_{xyx'y'}\rho_{x'y'xy} \\ &\simeq T_{xy}\rho_{yx} + \frac{1}{2}\Gamma_{xx'}\rho_{x'x} = \frac{1}{2}\left(T_{xy} + h_{xy}\right)\rho_{yx} \end{aligned}$$

for

$$\begin{aligned} \Gamma_{xx'} &= V_{xyx'y'}\rho_{y'y} && \Longleftarrow \quad \text{HF potential} \\ h_{xy} &= T_{xy} + \Gamma_{xy} && \Longleftarrow \quad \text{HF Hamiltonian} \\ 0 &= h_{xz}\rho_{zy} - \rho_{xz}h_{zy} && \Longleftarrow \quad \text{HF equation} \end{aligned}$$

One-body Symmetry

Consider a one-body symmetry operator $\hat{P}$ such that

$$\hat{P}^{+} a_x \hat{P} = P_{xy} a_y \quad , \quad \hat{P}^{+} a_x^{+} \hat{P} = P_{xy}^{*} a_y^{+}$$

and

$$P^{+}_{xx'} T_{x'y'} P_{y'y} = T_{xy} \quad , \quad P^{+}_{zz'} P^{+}_{tt'} V_{z't'x'y'} P_{x'x} P_{y'y} = V_{ztxy}$$

$\hat{P}$ can be an arbitrary symmetry. However, for simplicity we specify it to the simplest case of the parity symmetry: $P_{xx'} \equiv \delta(\vec{r} + \vec{r}')$.

The exact energy of the arbitrary state $|\Psi\rangle$ depends only on the scalar (parity invariant) part of the one- and two-body density matrices:

$$\begin{aligned} E = \langle \Psi | \hat{H} | \Psi \rangle \; &= \; T_{xy} \rho_{yx} + \frac{1}{4} V_{xyzt} \rho_{ztxy} \\ &= \; T_{xy} \rho^{(+)}_{yx} + \frac{1}{4} V_{xyzt} \rho^{(+)}_{ztxy} \end{aligned}$$

for

$$\begin{aligned} \rho^{(\pm)}_{yx} \; &= \; \frac{1}{2} \left(P_{yy'} \rho_{y'x'} P^{+}_{x'x} \pm \rho_{yx} \right) \\ \rho^{(\pm)}_{ztxy} \; &= \; \frac{1}{2} \left(P_{zz'} P_{tt'} \rho_{z't'x'y'} P^{+}_{x'x} P^{+}_{y'y} \pm \rho_{ztxy} \right) \end{aligned}$$

Spontaneous Symmetry Breaking

We have a choice of two different approximations:

- **Symmetry-conserving Hartree-Fock approximation:**

$$\boxed{\boldsymbol{\rho^{(+)}_{x'y'xy} = \rho^{(+)}_{x'x}\rho^{(+)}_{y'y} - \rho^{(+)}_{x'y}\rho^{(+)}_{y'x}}}$$

- **Symmetry-breaking Hartree-Fock approximation:**

$$\boxed{\boldsymbol{\rho_{x'y'xy} = \rho_{x'x}\rho_{y'y} - \rho_{x'y}\rho_{y'x}}}$$

$$\boldsymbol{\rho^{(+)}_{x'y'xy} = \rho^{(+)}_{x'x}\rho^{(+)}_{y'y} - \rho^{(+)}_{x'y}\rho^{(+)}_{y'x} + \rho^{(-)}_{x'x}\rho^{(-)}_{y'y} - \rho^{(-)}_{x'y}\rho^{(-)}_{y'x}}$$

$\boxed{\text{SYMMETRY BREAKING}} \Longleftrightarrow \boxed{\text{CORRELATIONS}}$

Selfconsistent point symmetries

Symmetry	Definition	Square	A	$\hat{\boldsymbol{J}}_y$
Parity	$\hat{P}$	$\hat{P}^2 = 1$		Yes
Time reversal	$\hat{T} = -i\sigma_y \hat{K}$	$\hat{T}^2 = -1$	Yes	
Signature x	$\hat{R}_x = \exp\left(-i\pi \hat{J}_x\right)$	$\hat{R}_x^2 = -1$		
Signature y	$\hat{R}_y = \exp\left(-i\pi \hat{J}_y\right)$	$\hat{R}_y^2 = -1$		Yes
Signature z	$\hat{R}_z = \exp\left(-i\pi \hat{J}_z\right)$	$\hat{R}_z^2 = -1$		
Simplex x	$\hat{S}_x = \hat{P}\hat{R}_x$	$\hat{S}_x^2 = -1$		
Simplex y	$\hat{S}_y = \hat{P}\hat{R}_y$	$\hat{S}_y^2 = -1$		Yes
Simplex z	$\hat{S}_z = \hat{P}\hat{R}_z$	$\hat{S}_z^2 = -1$		
T-Signature x	$\hat{R}_x^T = \hat{T}\hat{R}_x$	$(\hat{R}_x^T)^2 = 1$	Yes	Yes
T-Signature y	$\hat{R}_y^T = \hat{T}\hat{R}_y$	$(\hat{R}_y^T)^2 = 1$	Yes	
T-Signature z	$\hat{R}_z^T = \hat{T}\hat{R}_z$	$(\hat{R}_z^T)^2 = 1$	Yes	Yes
T-Simplex x	$\hat{S}_x^T = \hat{T}\hat{P}\hat{R}_x$	$(\hat{S}_x^T)^2 = 1$	Yes	Yes
T-Simplex y	$\hat{S}_y^T = \hat{T}\hat{P}\hat{R}_y$	$(\hat{S}_y^T)^2 = 1$	Yes	
T-Simplex z	$\hat{S}_z^T = \hat{T}\hat{P}\hat{R}_z$	$(\hat{S}_z^T)^2 = 1$	Yes	Yes

Nuclear shapes - multipole moments

Symmetry planes:	one x-z	two x-z & y-z	two x-z & x-y	three
Conserved:	$\hat{S}_y$	$\hat{S}_y$ & $\hat{S}_x^T$	$\hat{S}_y$ & $\hat{S}_z^T$	$\hat{S}_y$ & $\hat{S}_x^T$ & $\hat{S}_z^T$
Q_{10}	X	X	0	0
Q_{11}	X	0	X	0
Q_{20}	X	X	X	X
Q_{21}	X	0	0	0
Q_{22}	X	X	X	X
Q_{30}	X	X	0	0
Q_{31}	X	0	X	0
Q_{32}	X	X	0	0
Q_{33}	X	0	X	0
Q_{40}	X	X	X	X
Q_{41}	X	0	0	0
Q_{42}	X	X	X	X
Q_{43}	X	0	0	0
Q_{44}	X	X	X	X

X $\equiv$ non-zero value allowed

Symmetries of local densities

Conserved:	$\hat{S}_x^T$	$\hat{S}_y$	$\hat{S}_z^T$	Conserved:	$\hat{S}_x^T$	$\hat{S}_y$	$\hat{S}_z^T$
Plain:	y-z	x-z	x-y	Plain:	y-z	x-z	x-y
$\boldsymbol{\rho}$	$+$	$+$	$+$	$\boldsymbol{\Delta\rho}$	$+$	$+$	$+$
$\boldsymbol{\tau}$	$+$	$+$	$+$	$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}}$	$+$	$+$	$+$
$\boldsymbol{s}_1$	$-$	$-$	$+$	$\boldsymbol{\nabla}_1\boldsymbol{\rho}$	$-$	$+$	$+$
$\boldsymbol{s}_2$	$+$	$+$	$+$	$\boldsymbol{\nabla}_2\boldsymbol{\rho}$	$+$	$-$	$+$
$\boldsymbol{s}_2$	$+$	$-$	$-$	$\boldsymbol{\nabla}_3\boldsymbol{\rho}$	$+$	$+$	$-$
$\boldsymbol{T}_1$	$-$	$-$	$+$	$\boldsymbol{\Delta s}_1$	$-$	$-$	$+$
$\boldsymbol{T}_2$	$+$	$+$	$+$	$\boldsymbol{\Delta s}_2$	$+$	$+$	$+$
$\boldsymbol{T}_3$	$+$	$-$	$-$	$\boldsymbol{\Delta s}_3$	$+$	$-$	$-$
$\boldsymbol{j}_1$	$+$	$+$	$-$	$(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{s}})_1$	$+$	$+$	$-$
$\boldsymbol{j}_2$	$-$	$-$	$-$	$(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{s}})_2$	$-$	$-$	$-$
$\boldsymbol{j}_3$	$-$	$+$	$+$	$(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{s}})_3$	$-$	$+$	$+$
$\boldsymbol{J}_{11}$	$-$	$-$	$-$	$(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{j}})_1$	$-$	$-$	$+$
$\boldsymbol{J}_{21}$	$+$	$+$	$-$	$(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{j}})_2$	$+$	$+$	$+$
$\boldsymbol{J}_{31}$	$+$	$-$	$+$	$(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{j}})_3$	$+$	$-$	$-$
$\boldsymbol{J}_{12}$	$+$	$+$	$-$				
$\boldsymbol{J}_{22}$	$-$	$-$	$-$				
$\boldsymbol{J}_{32}$	$-$	$+$	$+$				
$\boldsymbol{J}_{13}$	$+$	$-$	$+$				
$\boldsymbol{J}_{23}$	$-$	$+$	$+$				
$\boldsymbol{J}_{33}$	$-$	$-$	$-$				

Local densities

Density matrix:

$$\rho(r\sigma, r'\sigma') = \langle\Phi|a^+(r'\sigma')a(r\sigma)|\Phi\rangle$$

Scalar and vector part:

$$\begin{aligned} \rho(r, r') &= \sum_\sigma \rho(r\sigma, r'\sigma) \\ \vec{s}(r, r') &= \sum_{\sigma\sigma'} \rho(r\sigma, r'\sigma')\langle\sigma'|\vec{\sigma}|\sigma\rangle \end{aligned}$$

Symmetries:

$$\begin{aligned} \rho^T(r, r') &= \rho^*(r, r') &= \rho(r', r) \\ \vec{s}^T(r, r') &= -\vec{s}^*(r, r') &= -\vec{s}(r', r) \end{aligned}$$

Local densities:

Matter:	$\rho(r) = \rho(r, r)$	1
Momentum:	$\vec{j}(r) = (1/2i)[(\vec{\nabla}-\vec{\nabla}')\rho(r, r')]_{r=r'}$	3
Kinetic:	$\tau(r) = [\vec{\nabla} \cdot \vec{\nabla}'\rho(r, r')]_{r=r'}$	1
Spin:	$\vec{s}(r) = \vec{s}(r, r)$	3
Spin momentum:	$J_{\mu\nu}(r) = (1/2i)[(\nabla_\mu - \nabla'_\mu)s_\nu(r, r')]_{r=r'}$	9
Spin kinetic:	$\vec{T}(r) = [\vec{\nabla} \cdot \vec{\nabla}'\vec{s}(r, r')]_{r=r'}$	3

Energy density

Density	Derivative	Symmetry			Energy density
		T	P	space	
$\rho(r)$		$+$	$+$	scalar	ρ^2
	$\vec{\nabla}\rho(r)$	$+$	$-$	vector	$\vec{\nabla}\rho\cdot\vec{J}$
	$\Delta\rho(r)$	$+$	$+$	scalar	$\rho\Delta\rho$
$\tau(r)$		$+$	$+$	scalar	$\rho\tau$
$J^{(0)}(r)$		$+$	$-$	scalar	$J^{(0)}J^{(0)}$
	$\vec{\nabla}J^{(0)}(r)$	$+$	$+$	vector	
$\vec{J}(r)$		$+$	$-$	vector	$\vec{J}^2$
	$\vec{\nabla}\cdot\vec{J}(r)$	$+$	$+$	scalar	$\rho\vec{\nabla}\cdot\vec{J}$
	$\vec{\nabla}\times\vec{J}(r)$	$+$	$+$	vector	
$J^{(2)}_{\mu\nu}(r)$		$+$	$-$	tensor	$\sum_{\mu\nu}J^{(2)}_{\mu\nu}J^{(2)}_{\mu\nu}$
$\vec{s}(r)$		$-$	$+$	vector	$\vec{s}^2$
	$\vec{\nabla}\cdot\vec{s}(r)$	$-$	$-$	scalar	
	$\vec{\nabla}\times\vec{s}(r)$	$-$	$-$	vector	$\vec{j}\cdot\vec{\nabla}\times\vec{s}$
	$\Delta\vec{s}(r)$	$-$	$+$	vector	$\vec{s}\cdot\Delta\vec{s}$
$\vec{j}(r)$		$-$	$-$	vector	$\vec{j}^2$
	$\vec{\nabla}\cdot\vec{j}(r)$	$-$	$+$	scalar	
	$\vec{\nabla}\times\vec{j}(r)$	$-$	$+$	vector	$\vec{s}\cdot\vec{\nabla}\times\vec{j}$
$\vec{T}(r)$		$-$	$+$	vector	$\vec{s}\cdot\vec{T}$

Local density approximation

We assume that the total energy of a nuclear system has a form of a spatial integral of the local energy density:

$$\mathcal{E} = \int d^3r \mathcal{H}(r)$$

for

$$\mathcal{H}(r) = \frac{\hbar^2}{2m}\tau_0 + \mathcal{H}_0(r) + \mathcal{H}_1(r),$$

where

$$\mathcal{H}_t(r) = \mathcal{H}_t^{\text{even}}(r) + \mathcal{H}_t^{\text{odd}}(r),$$

and

$$\begin{array}{rclllll}
\mathcal{H}_t^{\text{even}}(r) & = & C_t^\rho \rho_t^2 & + \; C_t^{\Delta\rho}\rho_t\Delta\rho_t & + \; C_t^\tau \rho_t\tau_t & & + \; \ldots \\
\mathcal{H}_t^{\text{odd}}(r) & = & C_t^s \vec{s}_t^{\,2} & + \; C_t^{\Delta s}\vec{s}_t\cdot\Delta\vec{s}_t & + \; C_t^T \vec{s}_t\cdot\vec{T}_t & & + \; \ldots \\
 & & \ldots & + \; C_t^J \overset{\leftrightarrow}{J}_t^{2} & + \; C_t^{\nabla J}\rho_t\vec{\nabla}\cdot\vec{J}_t, & & \\
 & & \ldots & + \; C_t^j \vec{j}_t^{\,2} & + \; C_t^{\nabla j}\vec{s}_t\cdot(\vec{\nabla}\times\vec{j}_t), & &
\end{array}$$

Here, t=0 and 1 denotes the isoscalar and isovector terms, e.g.:

$$\rho_0 = \rho_n + \rho_p \quad , \quad \rho_1 = \rho_n - \rho_p,$$

Mean-field Hamiltonian

Mean-field potentials:

$$
\begin{aligned}
\Gamma_t^{\text{even}} &= -\vec{\nabla}\cdot M_t(r)\vec{\nabla} &&+ U_t(r) &&+ \ldots\\
\Gamma_t^{\text{odd}} &= -\vec{\nabla}\cdot(\vec{\sigma}\cdot\vec{C}_t(r))\vec{\nabla} &&+ \vec{\sigma}\cdot\vec{\Sigma}_t(r) &&+ \ldots
\end{aligned}
$$

$$
\begin{aligned}
&\ldots + \tfrac{1}{2i}\big(\overset{\leftrightarrow}{\nabla}\sigma\cdot\overset{\leftrightarrow}{B}_t(r) + \overset{\leftrightarrow}{B}_t(r)\cdot\overset{\leftrightarrow}{\nabla}\sigma\big),\\
&\ldots + \tfrac{1}{2i}(\vec{\nabla}\cdot\vec{I}_t(r) + \vec{I}_t(r)\cdot\vec{\nabla}).
\end{aligned}
$$

where

$$
\begin{aligned}
U_t &= 2C_t^{\rho}\rho_t + 2C_t^{\Delta\rho}\Delta\rho_t + C_t^{\tau}\tau_t + C_t^{\nabla J}\vec{\nabla}\cdot\vec{J}_t,\\
\vec{\Sigma}_t &= 2C_t^{s}\vec{s}_t + 2C_t^{\Delta s}\Delta\vec{s}_t + C_t^{T}\vec{T}_t + C_t^{\nabla j}\vec{\nabla}\times\vec{j}_t,\\
M_t &= C_t^{\tau}\rho_t,\\
\vec{C}_t &= C_t^{T}\vec{s}_t,\\
\overset{\leftrightarrow}{B}_t &= 2C_t^{J}\overset{\leftrightarrow}{J}_t - C_t^{\nabla J}\overset{\leftrightarrow}{\nabla}\rho_t,\\
\vec{I}_t &= 2C_t^{j}\vec{j}_t + C_t^{\nabla j}\vec{\nabla}\times\vec{s}_t,
\end{aligned}
$$

Neutron and proton mean-field Hamiltonians:

$$
\begin{aligned}
h_n &= -\frac{\hbar^2}{2m}\Delta + \Gamma_0^{\text{even}} + \Gamma_0^{\text{odd}} + \Gamma_1^{\text{even}} + \Gamma_1^{\text{odd}},\\
h_p &= -\frac{\hbar^2}{2m}\Delta + \Gamma_0^{\text{even}} + \Gamma_0^{\text{odd}} - \Gamma_1^{\text{even}} - \Gamma_1^{\text{odd}}.
\end{aligned}
$$

HF equation for single-particle wave functions:

$$
h_\alpha\psi_{i,\alpha}(\vec{r}\sigma) = \epsilon_{i,\alpha}\psi_{i,\alpha}(\vec{r}\sigma),
$$

where i numbers the neutron ($\alpha=n$) and proton ($\alpha=p$) eigen-states.

Local gauge transformation

For an arbitrary non-local interaction

$$V(r_1', r_2'; r_1, r_2)$$

the Hartree-Fock interaction energy has the form:

$$\begin{aligned}\mathcal{E}^{\rm int} &= \int dr_1' dr_2' dr_1 dr_2 V(r_1', r_2'; r_1, r_2) \times \\ &\quad (\rho(r_1, r_1')\rho(r_2, r_2') - \rho(r_2, r_1')\rho(r_1, r_2')),\end{aligned}$$

and for a local interaction

$$V(r_1', r_2'; r_1, r_2) = \delta(r_1' - r_1)\delta(r_2' - r_2)V(r_1, r_2),$$

reduces to:

$$\begin{aligned}\mathcal{E}^{\rm int} &= \int dr_1 dr_2 V(r_1, r_2) \times \\ &\quad (\rho(r_1)\rho(r_2) - \rho(r_2, r_1)\rho(r_1, r_2))\end{aligned}$$

This interaction energy is invariant with respect to the local gauge transformation:

$$|\Psi'\rangle = \exp\left\{ i \sum_{j=1}^{A} \phi(r_j) \right\} |\Psi\rangle,$$

which transforms the density matrix as:

$$\rho'(r, r') = \exp\left\{ i(\phi(r) - \phi(r')) \right\} \rho(r, r').$$

Gauge-invariant energy density

Gauge transformation of the density matrix:

$$\rho'(r,r') = \exp\{i(\phi(r) - \phi(r'))\}\, \rho(r,r').$$

gives the following relations for the local densities:

$$\begin{aligned}
\rho' &= \rho \\
\tau' &= \tau + 2\vec{j}\cdot\vec{\nabla}\phi + \rho(\vec{\nabla}\phi)^2, \\
s'_k &= s_k, \\
j'_k &= j_k + \rho\nabla_k\phi, \\
T'_k &= T_k + 2\textstyle\sum_l J_{kl}\nabla_l\phi, + s_k(\vec{\nabla}\phi)^2, \\
J'_{kl} &= J_{kl} + s_l\nabla_k\phi,
\end{aligned}$$

and the gauge-invariant energy density reads:

$$\begin{aligned}
\mathcal{H}_t(r) &= C_t^\rho\rho_t^2 + C_t^s\vec{s}_t^{\,2} + C_t^{\Delta\rho}\rho_t\Delta\rho_t + C_t^{\Delta s}\vec{s}_t\cdot\Delta\vec{s}_t \\
&+ C_t^\tau\left(\rho_t\tau_t - \vec{j}_t^{\,2}\right) + C_t^T(\vec{s}_t\cdot\vec{T}_t - \overleftrightarrow{J}_t^{\,2}) \\
&+ C_t^{\nabla J}(\rho_t\vec{\nabla}\cdot\vec{J}_t + \vec{s}_t\cdot(\vec{\nabla}\times\vec{j}_t)).
\end{aligned}$$

Coupling constants for the Skyrme interactions

For the Skyrme interaction, the time-odd coupling constants C_t^s, $C_t^{\Delta s}$, and C_t^T are unique functions of the time-even coupling constants C_t^ρ, $C_t^{\Delta\rho}$, and C_t^τ.

Table I

Coupling constants in the gauge-invariant energy density obtained for the SIII, SkP, and SkM* Skyrme interactions. Zero-order coupling constants C_t^ρ and C_t^s are given in MeV·fm^3 and the remaining (second-order) ones are given in MeV·fm^5.

	t=0			t=1		
	SIII	SkM*	SkP	SIII	SkM*	SkP
$C_t^\rho(\rho=0)$	−423.3	−991.9	−1099.4	268.8	390.1	580.6
$C_t^\rho(\rho=\rho_{\rm nm})$	−296.4	−237.7	−335.6	141.2	150.8	188.4
$C_t^{\Delta\rho}$	−63.0	−68.2	−60.1	17.0	17.1	35.1
C_t^τ	44.4	34.7	0.0	−30.6	−34.1	−44.6
$C_t^{\nabla J}$	−90.0	−97.5	−75.0	−30.0	−32.5	−25.0
$C_t^s(\rho=0)$	14.1	271.1	152.3	141.1	330.6	366.5
$C_t^s(\rho=\rho_{\rm nm})$	56.4	31.7	−31.4	98.8	91.2	78.5
$C_t^{\Delta s}$	17.0	17.1	−4.2	17.0	17.1	9.8
C_t^T	−30.6	−34.1	7.7	−30.6	−34.1	−41.1

Zero-order coupling constants for the Skyrme interactions

Table III

Zero-order coupling constants as functions of parameters of the Skyrme interactions, expressed by the formula: $C = \frac{1}{8}(at_0 + bt_0x_0) + \frac{1}{48}\rho^\alpha(at_3 + bt_3x_3)$.

	a	b
C_0^ρ	3	0
C_1^ρ	−1	−2
C_0^s	−1	2
C_1^s	−1	0

Zero-order time-odd coupling constants as functions of the time-even coupling constants, expressed by the formula: $C = \frac{1}{3}(aC_0^\rho + bC_1^\rho)$.

	a	b
C_0^s	−2	−3
C_1^s	−1	0

Second-order coupling constants for the Skyrme interactions

Table IV

Second-order coupling constants as functions of parameters of the Skyrme interactions, expressed by the formula: $C=\frac{1}{64}(at_1+bt_1x_1+ct_2+dt_2x_2)$.

	a	b	c	d
$C_0^{\Delta\rho}$	-9	0	5	4
$C_1^{\Delta\rho}$	3	6	1	2
C_0^{τ}	12	0	20	16
C_1^{τ}	-4	-8	4	8
$C_0^{\Delta s}$	3	-6	1	2
$C_1^{\Delta s}$	3	0	1	0
C_0^{T}	-4	8	4	8
C_1^{T}	-4	0	4	0

Second-order time-odd coupling constants as functions of the time-even coupling constants, expressed by the formula: $C=\frac{1}{24}(aC_0^{\Delta\rho}+bC_1^{\Delta\rho}+cC_0^{\tau}+dC_1^{\tau})$.

	a	b	c	d
$C_0^{\Delta s}$	-12	-12	3	9
$C_1^{\Delta s}$	-4	-4	3	-3
C_0^{T}	16	48	-4	12
C_1^{T}	16	-16	4	-12

Problem, Solution, and the Bottom Line

- Problem: The Skyrme force time-odd mean-field components uniquely depend on the time-even mean-field components. What does it mean?
- Solution: Read the Classic !!

> "The Skyrme force must be understood as a very special average of the effective interaction over the occupied states of the Fermi sea, and the values and density dependence of the parameters contain a combination of information about both the true effective interaction and the Fermi sea over it was averaged."
> "Since the Skyrme force parameters are essentially determined by fitting observable properties of spin-saturated nuclei, [...] it is evident that the apparent spin dependence [in its energy density $\mathcal{H}(\vec{r})$] is purely illusory."
> " To derive the actual spin dependence from the effective interaction, it is easy to simply recalculate $\mathcal{H}(\vec{r})$ for the SE, TE, SO, and TO components of the two-body [finite-range] effective interaction separately..."
> J.W. Negele, *Lecture Notes in Physics 40* (1975) pp. 285, 288.

- Bottom Line: The time-odd components in the mean field should be derived from the finite range interaction by using the Local-Density-Approximation.

  They may have the form of the Skyrme energy density, but their parameters *are not* equal to those given by the Skyrme force.

Example: no spin, no isospin

For an arbitrary non-local interaction

$$V(\mathbf{r}_1', \mathbf{r}_2'; \mathbf{r}_1, \mathbf{r}_2)$$

the Hartree-Fock interaction energy has the form:

$$\begin{aligned}\mathcal{E}^{\rm int} = & \tfrac{1}{2}\int d^3r_1' d^3r_2' d^3r_1 d^3r_2 V(r_1', r_2'; r_1, r_2) \times \\ & (\rho(r_1, r_1')\rho(r_2, r_2') - \rho(r_2, r_1')\rho(r_1, r_2')),\end{aligned}$$

while for a local interaction

$$V(r_1', r_2'; r_1, r_2) = \delta(r_1' - r_1)\delta(r_2' - r_2)V(r_1, r_2),$$

the interaction energy reduces to:

$$\mathcal{E}^{\rm int} = \tfrac{1}{2}\int d^3r_1 d^3r_2 V(r_1, r_2) \times (\rho(r_1)\rho(r_2) - \rho(r_2, r_1)\rho(r_1, r_2))$$

where $\rho(r_1) \equiv \rho(r_1, r_1)$ and $\rho(r_2) \equiv \rho(r_2, r_2)$.

Short range approximation (direct term)

Defining the "center-of-mass" and relative coordinates by

$$\begin{aligned} \boldsymbol{R} &= \tfrac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2) \\ \boldsymbol{r} &= \boldsymbol{r}_1 - \boldsymbol{r}_2 \end{aligned}$$

we have the short range approximation:

$$\begin{aligned} \rho(\boldsymbol{r}_1) = \rho(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{r}) &= \rho(\boldsymbol{R}) + \tfrac{1}{2}r^i\nabla_i\rho(\boldsymbol{R}) + \tfrac{1}{8}r^ir^j\nabla_i\nabla_j\rho(\boldsymbol{R}) + \ldots \\ \rho(\boldsymbol{r}_2) = \rho(\boldsymbol{R} - \tfrac{1}{2}\boldsymbol{r}) &= \rho(\boldsymbol{R}) - \tfrac{1}{2}r^i\nabla_i\rho(\boldsymbol{R}) + \tfrac{1}{8}r^ir^j\nabla_i\nabla_j\rho(\boldsymbol{R}) + \ldots \end{aligned}$$

and

$$\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2) = \rho^2(\boldsymbol{R}) + \tfrac{1}{4}r^ir^j(\rho(\boldsymbol{R})\nabla_i\nabla_j\rho(\boldsymbol{R}) - [\nabla_i\rho(\boldsymbol{R})][\nabla_j\rho(\boldsymbol{R})]) + \ldots$$

where $\nabla_i = \partial/\partial R^i$.

For the scalar interaction $V(\boldsymbol{r}_1, \boldsymbol{r}_2) = V(|\boldsymbol{r}_1 - \boldsymbol{r}_2)| = V(|\boldsymbol{r}|)$ the direct interaction energy is given by a local energy density:

$$\mathcal{E}^{\text{int}}_{\text{dir}} = \tfrac{1}{2}\int d^3R[V_0\rho^2 + \tfrac{1}{4}V_2(\rho\Delta\rho - (\vec{\nabla}\rho)^2)] + \ldots$$

where the coupling constants V_0 and V_2 are given by the integrals of the interaction:

$$\begin{aligned} V_0 &= \int d^3r V(|\boldsymbol{r}|) &&= 4\pi\int dr r^2 V(r) \\ V_2 &= \tfrac{1}{3}\int d^3r r^2 V(|\boldsymbol{r}|) &&= \tfrac{4}{3}\pi\int dr r^4 V(r) \end{aligned}$$

Local density approximation (exchange term)

The "parabolic" local density approximation (LDA) is defined by

$$\rho(r_1,r_2) \;=\rho(R,r) \;=\rho(R)+r^i\partial_i\rho(R,r)+\tfrac{1}{2}r^ir^j\partial_i\partial_j\rho(R,r)+\ldots$$

where the derivatives $\partial_i{=}\partial/\partial r^i$ are always calculated at $r^i{=}0$.

The "parabolic" LDA does not ensure that $\rho(r_1,r_2)\longrightarrow 0$ for large $|r|{=}|r_1{-}r_2|$. We improve it by introducing three functions of $|r|$, $\pi_0(r)$, $\pi_1(r)$, and $\pi_2(r)$, which vanish at large $|r|$, i.e., we define the LDA by:

$$\begin{aligned}
\rho(r_1,r_2) \;&=\rho(R,r) &=\;& \pi_0(r)\rho(R)+\pi_1(r)r^i\partial_i\rho(R,r) \\
&& & +\tfrac{1}{2}\pi_2(r)r^ir^j\partial_i\partial_j\rho(R,r)+\ldots \\
\rho(r_2,r_1) \;&=\rho(R,-r) &=\;& \pi_0(r)\rho(R)-\pi_1(r)r^i\partial_i\rho(R,r) \\
&& & +\tfrac{1}{2}\pi_2(r)r^ir^j\partial_i\partial_j\rho(R,r)+\ldots
\end{aligned}$$

with supplementary conditions:
$\pi_0(0){=}\pi_1(0){=}\pi_2(0){=}1$, $\pi_0'(0){=}\pi_1'(0){=}0$, and $\pi_0''(0){=}0$.
Local gauge invariance requires that
$\pi_1^2(r){=}\pi_0(r)\pi_2(r)$.

The product of nonlocal densities in the exchange integral now reads

$$\begin{aligned}
\rho(r_1,r_2)\rho(r_2,r_1) \;=\;& \pi_0^2(r)\rho^2(R) \\
& +\pi_0(r)\pi_2(r)r^ir^j\rho(R)\partial_i\partial_j\rho(R,r) \\
& -\pi_1^2(r)r^ir^j[\partial_i\rho(R,r)][\partial_j\rho(R,r)]+\ldots
\end{aligned}$$

and gives the exchange interaction energy:

$$\mathcal{E}^{\text{int}}_{\text{exch}}=\tfrac{1}{2}\int d^3R[V_0'\rho^2+\tfrac{1}{4}V_2'(\rho\Delta\rho-4(\rho\tau-\vec{j}^2))]+\ldots$$

where the coupling constants V_0' and V_2' are given by the integrals of the interaction:

$$\begin{aligned}
V_0' &= \;\int d^3r\pi_0^2(r)V(|r|) &&= 4\pi\int drr^2\pi_0^2(r)V(r) \\
V_2' &= \tfrac{1}{3}\int d^3rr^2\pi_0(r)\pi_2(r)V(|r|) &&= \tfrac{4}{3}\pi\int drr^4\pi_0(r)\pi_2(r)V(r)
\end{aligned}$$

Example: central interaction

For an arbitrary central finite-range local interaction

$$\hat{V}(r_1,r_2) = W(r_1,r_2)+B(r_1,r_2)P_\sigma - H(r_1,r_2)P_\tau - M(r_1,r_2)P_\sigma P_\tau$$

we have the interaction energy

$$\begin{aligned}\mathcal{E}^{\text{int}} = \int d^3r \textstyle\sum_{t=0,1} \quad [\quad & C_t^\rho \rho_t^2 + C_t^{\Delta\rho}\rho_t\Delta\rho_t \ + C_t^\tau(\rho_t\tau_t - \vec{j}_t^{\,2}) \\ + \ & C_t^s \vec{s}_t^{\,2} + C_t^{\Delta s}\vec{s}_t\cdot\Delta\vec{s}_t + C_t^T(\vec{s}_t\cdot\vec{T}_t - \overset{\leftrightarrow}{J}{}_t^2)]\end{aligned}$$

for

$$8\begin{pmatrix} C_0^\rho \\ C_1^\rho \\ C_0^s \\ C_1^s \end{pmatrix} = \begin{pmatrix} 4 & -1 & 2 & -2 \\ 0 & -1 & 0 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} W_0 + M_0' \\ M_0 + W_0' \\ B_0 + H_0' \\ H_0 + B_0' \end{pmatrix}$$

$$32\begin{pmatrix} C_0^{\Delta\rho} \\ C_1^{\Delta\rho} \\ C_0^\tau \\ C_1^\tau \\ C_0^{\Delta s} \\ C_1^{\Delta s} \\ C_0^T \\ C_1^T \end{pmatrix} = \begin{pmatrix} 8 & -1 & 4 & -2 & -4 & 2 & -2 & 4 \\ 0 & -1 & 0 & -2 & -4 & 0 & -2 & 0 \\ 0 & 4 & 0 & 8 & 0 & -8 & 0 & -16 \\ 0 & 4 & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 4 & 0 & 2 & -2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 & 0 & -8 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} W_2 \\ W_2' \\ B_2 \\ B_2' \\ H_2 \\ H_2' \\ M_2 \\ M_2' \end{pmatrix}$$

where (for $X \equiv W$, B, H, or M)

$$\begin{aligned} X_0 &= \int d^3r X(|r|) \quad &, \quad X_0' &= \int d^3r \pi_0^2(r) X(|r|) \\ X_2 &= \tfrac{1}{3}\int d^3r r^2 X(|r|) \quad &, \quad X_2' &= \tfrac{1}{3}\int d^3r r^2 \pi_0(r)\pi_2(r) X(|r|) \end{aligned}$$