PART I  —  QUANTUM METROLOGY WITH UNCORRELATED NOISE


PART II  —  BEATING THE SHOT NOISE LIMIT DESPITE THE UNCORRELATED NOISE

\textbf{(Classical) Quantum Metrology}

\textbf{Atomic Spectroscopy: "Phase" Estimation}

\(N\) two-level atoms (qubits) in a \textbf{separable} state

\[|\psi_0\rangle^{\otimes N} = +^{\otimes N} = \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]^{\otimes N}\]

unitary rotation

\[U_\varphi = \text{e}^{i\frac{\varphi}{2}\hat{\sigma}_z}\]

\[\hat{H}_N = \frac{\omega}{2} \sum_{i=1}^{N} \hat{\sigma}_z^{(i)}\]

output state \textbf{(separable)}

\[|\psi_\varphi\rangle^{\otimes N} = (U_\varphi|\psi_0\rangle)^{\otimes N} = \left[\frac{1}{\sqrt{2}}\left(\text{e}^{-i\frac{\varphi}{2}}|0\rangle + \text{e}^{i\frac{\varphi}{2}}|1\rangle\right)\right]^{\otimes N}\]

uncorrelated measurement – POVM:

\[\hat{M}_{x_k} = \bigotimes_{k=1}^{N} \hat{M}_{x_k}\]

\[X_{\varphi}^N \sim p(X^N|\varphi) = p(X|\varphi)^N\]

\[p(x_i|\varphi) = \langle \psi_\varphi | \hat{M}_{x_i} | \psi_\varphi \rangle\]

\[\varDelta \varphi \geq \frac{1}{\sqrt{F_{cl}[p(X^N|\varphi)]}} = \frac{1}{\sqrt{F_{cl}[p(X|\varphi)]}}\frac{1}{\sqrt{N}}\]

\[F_{cl}[p(X^N|\varphi)] = \sum_x \left[\frac{\partial \varphi}{p(x|\varphi)}\right]^2 = NF_{cl}[p(X|\varphi)] \leq F_Q[|\psi_\varphi\rangle^{\otimes N}] = NF_Q[|\psi_\varphi\rangle]\]

\textbf{Classical Fisher Information}

\textbf{Quantum Fisher Information} (measurement independent)

\textbf{Shot Noise}

always saturable in the limit \(N \to \infty\)
**QUANTUM FISHER INFORMATION**

- **QFI of a pure state** $\rho_\varphi = |\psi_\varphi\rangle \langle \psi_\varphi|$: 
  $$F_Q[|\psi_\varphi\rangle] = 4 \left( \langle \dot{\psi}_\varphi | \dot{\psi}_\varphi \rangle - |\langle \psi_\varphi | \dot{\psi}_\varphi \rangle|^2 \right)$$

- **QFI of a mixed state** $\rho_\varphi = \sum_i p_i(\varphi) |e_i(\varphi)\rangle \langle e_i(\varphi)|$:
  $$\partial_\varphi \rho_\varphi = \frac{1}{2} (\rho_\varphi L[\rho_\varphi] + L[\rho_\varphi] \rho_\varphi)$$
  $$F_Q[\rho_\varphi] = \text{Tr} \left\{ \rho_\varphi L[\rho_\varphi] \right\} = \sum_k \frac{(p'_k)^2}{p_k} + 4 \sum_{j<k \atop p_j + p_k \neq 0} \frac{(p_j - p_k)^2}{p_j + p_k} |\langle e'_j | e_k \rangle|^2$$

$L[\rho_\varphi]$ – Symmetric Logarithmic Derivative

Evaluation requires eigen-decomposition of the density matrix, which size grows exponentially, e.g. $d=2^N$ for $N$ qubits.

- **Geometric interpretation – QFI is a local quantity**

Two PDFs $p(x), q(x)$:
  $$F(p, q) = \sum_x \sqrt{p(x) q(x)}, \quad D^2(p, q) = 2(1-F(p, q)),$$
  $$D(p_\varphi, p_\varphi + \delta_\varphi) = \frac{1}{2} \sqrt{F_{cl}[p_\varphi]} \delta_\varphi + O(\delta_\varphi^2)$$

Two q. states $\rho, \sigma$:
  $$F(\rho, \sigma) = \text{Tr} \left\{ \sqrt{\rho \sigma} \sqrt{\rho \sigma} \right\}, \quad D^2_E(\rho, \sigma) = 2(1-F(\rho, \sigma)),$$
  $$D_B(\rho_\varphi, \rho_\varphi + \delta_\varphi) = \frac{1}{2} \sqrt{F_Q[\rho_\varphi]} \delta_\varphi + O(\delta_\varphi^2)$$

The necessity of the asymptotic limit of repetitions $N \to \infty$ is a consequence of locality.

- **Purification-based definition of the QFI** $\rho_\varphi = \text{Tr}_E\{ |\Psi(\varphi)\rangle \langle \Psi(\varphi)| \}$

[Escher et al, Nat. Phys. 7(5), 406 (2011)] – 
  $$F_Q[\rho_\varphi] = \min_{\Psi(\varphi)} F_Q[|\Psi(\varphi)\rangle] = 4 \min_{\Psi(\varphi)} \left\{ \langle \dot{\Psi}(\varphi) | \dot{\Psi}(\varphi) \rangle - |\langle \Psi(\varphi) | \dot{\Psi}(\varphi) \rangle|^2 \right\}$$

  $$F_Q[\rho_\varphi] = 4 \min_{\Psi(\varphi)} \langle \dot{\Psi}(\varphi) | \dot{\Psi}(\varphi) \rangle$$

Due to locality of the QFI need to consider only purifications that differ in: $\{ |\Psi(\varphi)\rangle, |\dot{\Psi}(\varphi)\rangle \}$
CLAIM: Any infinitesimal source of uncorrelated decoherence acting independently on each atom will "decorrelate" the atoms, so that we may attain the ultimate precision in the $N \to \infty$ limit with $k = 1$, but at the price of scaling ...

Heisenberg Limit
**Observations**

- Infinitesimal uncorrelated disturbance forces asymptotic (classical) shot noise scaling.
- The bound then "makes sense" for a single shot $(k = 1)$.
- Does this behaviour occur for decoherence of a generic type?

**Realistic Quantum Metrology**

Atomic Spectroscopy: "Phase" Estimation

With dephasing noise added:

\[
\frac{d\rho^N}{dt} = i\frac{\omega}{2} [\rho^N, \hat{H}_N] - \frac{\gamma}{2} \sum_{i=1}^{N} [\sigma_z^{(i)} \rho^N \sigma_z^{(i)} - \rho^N]
\]

\[|\psi_0^N\rangle \rightarrow \text{estimator} \quad \bar{\phi}(x) \]

\[\Delta\bar{\phi}_{N \to \infty} \geq \sqrt{\frac{c_Q(\eta)}{N}} = \sqrt{1 - \eta^2} \frac{1}{\sqrt{N}}
\]

Constant factor improvement over Shot Noise

Achievable with $k = 1$ and spin-squeezed states

The properties of the single use of a channel – $\Lambda_\varphi$ – dictate the asymptotic ultimate scaling of precision.
In order of their power and range of applicability:

- **Classical Simulation (CS) method**
  - Stems from the possibility to **locally simulate** quantum channels via **classical probabilistic mixtures**:
    \[
    \Lambda_\varphi \leftrightarrow \Phi + O(\delta \varphi^2) \quad \text{where} \quad p_\varphi = \sum_i p_i(\varphi) |e_i\rangle\langle e_i|, \quad \Lambda_\varphi[\varrho] = \Phi[\varrho \otimes p_\varphi] + O(\delta \varphi^2) = \sum_i p_i(\varphi) \Pi_i[\varrho] + O(\delta \varphi^2)
    \]
  - Optimal simulation corresponds to a **simple, intuitive, geometric representation**.
  - Proves that **almost all** (including full rank) channels asymptotically scale classically.
  - Allows to **straightforwardly derive bounds** (e.g. dephasing channel considered).

- **Quantum Simulation (QS) method**
  - Generalizes the concept of local classical simulation, so that the parameter-dependent state does not need to be diagonal:
    \[
    \Lambda_\varphi \leftrightarrow \Phi + O(\delta \varphi^2) \quad \text{where} \quad \Lambda_\varphi[\varrho] = \Phi[\varrho \otimes \sigma_\varphi] + O(\delta \varphi^2)
    \]
  - Proves **asymptotic shot noise** also for a **wider class of channels** (e.g. optical interferometry with loss).

- **Channel Extension (CE) method**
  - Applies to **even wider class** of channels, and provides the **tightest** lower bounds on \( c_Q(\eta) \).
    (e.g. amplitude damping channel)
    \[
    \max_{|\psi_0^N\rangle} F_Q[\Lambda_\varphi^N[|\psi_0^N\rangle]] \leq \max_{|\psi_{\text{ext}}^N\rangle} F_Q[(\Lambda_\varphi \otimes \mathbb{I})^\otimes N[|\psi_{\text{ext}}^N\rangle]] = \min_K \left\{ N \|\alpha_K\| + N (N - 1) \|\beta_K\|^2 \right\}
    \]
  - Efficiently calculable numerically by means of **Semi-Definite Programming** even for **finite \( N \)** !!!.
as a Markov chain:

\[ \varphi \rightarrow \Lambda_\varphi^N [\lvert \psi_0^N \rangle] \rightarrow \tilde{\varphi} \]

\[ \Delta \tilde{\varphi} \geq \frac{1}{\sqrt{F_Q[\rho_\varphi^N]}} \]
CLASSICAL/QUANTUM SIMULATION OF A CHANNEL

as a Markov chain:

\[ \varphi \xrightarrow{p_\varphi} \Phi[p_\varphi \otimes \bullet]^N(\lvert \psi_0^N \rangle) \rightarrow \tilde{\varphi} \quad \Rightarrow \quad \Delta \tilde{\varphi} \geq \frac{1}{\sqrt{F_Q[\varphi^N]}} \]
**CLASSICAL/QUANTUM SIMULATION OF A CHANNEL**

as a Markov chain:

\[
\varphi \rightarrow \sigma_\varphi \rightarrow \Phi[\sigma_\varphi \otimes \bullet] \otimes^N (|\psi_0^N\rangle) \rightarrow \tilde{\varphi} \quad \Rightarrow \quad \Delta \tilde{\varphi} \geq \frac{1}{\sqrt{F_Q[\varrho_\varphi^N]}}
\]
But how to verify if this construction is possible and what is the optimal ("worse") classical/quantum simulation giving the tightest lower bound on the ultimate precision?

\[ \frac{1}{\sqrt{F_{Q}[\sigma_{\varphi}]}} \cdot \frac{1}{\sqrt{N}} \]
The "worst" classical simulation

The set of quantum channels (CPTP maps) is \textit{convex}

**Locality:**
Quantum Fisher Information at a given \( \varphi : F_Q [ \Lambda^N [ |\psi_0^N \rangle] ] \)
depends only on:

\[ \Lambda_\varphi \quad \partial_\varphi \Lambda_\varphi \]

We want to construct the "local classical simulation" of the form:

\[ \Lambda_\varphi [\rho] = \Phi[\rho \otimes p_\varphi] + O(\delta \varphi^2) = \sum_i p_i(\varphi) \Lambda_i[\rho] + O(\delta \varphi^2) \]

The "worst" local classical simulation:

\[ \Lambda_\varphi = p_+(\varphi) \Lambda_+ + p_-(\varphi) \Lambda_- + O(d \varphi^2) \]

\[ \Lambda_\pm = \Lambda_\varphi \pm \frac{d \Lambda_\varphi}{d \varphi} \epsilon_\pm \]

\[ F_Q \leq F_Q^{CS} = N F_{cl}[p_{\pm}(\varphi)] = \frac{N}{\epsilon_+ \epsilon_-} \]

\[ c_Q \geq c_Q^{CS} = \epsilon_+ \epsilon_- , \Delta \tilde{\varphi} \geq \sqrt{\frac{\epsilon_+ \epsilon_-}{N}} \]

Does \textbf{not} work for \( \varphi \)-extremal channels, e.g \( \text{unitaries} \ U_\varphi \).
## Gallery of decoherence models

<table>
<thead>
<tr>
<th>Depolarization</th>
<th>Dephasing</th>
<th>Lossy interferometer</th>
<th>Spontaneous emission</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Depolarization" /></td>
<td><img src="image2" alt="Dephasing" /></td>
<td><img src="image3" alt="Lossy interferometer" /></td>
<td><img src="image4" alt="Spontaneous emission" /></td>
</tr>
<tr>
<td>Inside the set of quantum channels full rank</td>
<td>on the boundary, non-extremal, not ( \varphi )-extremal</td>
<td>on the boundary, non-extremal, but ( \varphi )-extremal</td>
<td>on the boundary, extremal</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>( \mathbf{CS} )</th>
<th>( \mathbf{QS} )</th>
<th>( \mathbf{CE} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1 + 3\eta}{4\eta} \cdot \frac{1 - \eta}{\eta} \cdot \frac{1}{\sqrt{N}} )</td>
<td>( \frac{1}{\eta} \cdot \frac{1}{\sqrt{N}} )</td>
<td>( \frac{1 + 2\eta}{2\eta} \cdot \frac{1 - \eta}{\eta} \cdot \frac{1}{\sqrt{N}} )</td>
</tr>
</tbody>
</table>

\( c_{Q}^{CE} \geq c_{Q}^{QS} \geq c_{Q}^{CS} \)  \( \Delta \tilde{\varphi} \geq \Delta \tilde{\varphi}_{CE} \geq \Delta \tilde{\varphi}_{QS} \geq \Delta \tilde{\varphi}_{CS} \)
CONSEQUENCES ON REALISTIC SCENARIOS

“PHASE ESTIMATION” IN ATOMIC SPECTROSCOPY WITH DEPHASING ($\eta = 0.9$)

\[ \Delta \phi \leq \frac{1}{\eta} \cdot \frac{1}{N} \]
better than Heisenberg Limit region

\[ \Delta \phi \geq \frac{1}{\eta} \cdot \frac{1}{\sqrt{N}} \]
worse than classical region

\[ N_{th} = \frac{1}{1 - \eta^2} \approx 5.26 \]

input correlations dominated region
worth investing in GHZ states
e.g. $N=3$
[D. Leibfried et al, Science, 304 (2004)]

uncorrelated decoherence dominated region
worth investing in S-S states
e.g. $N=10^5$ !!!!!!!

GHZ strategy
S-S strategy
finite-N bound
asymptotic bound

\[ \Delta \phi \geq \frac{1}{\eta} \cdot \frac{1}{\sqrt{N}} \]

\[ \frac{\sqrt{1 - \eta^2}}{\eta} \cdot \frac{1}{\sqrt{N}} \]

N
CONSEQUENCES ON REALISTIC SCENARIOS

PERFORMANCE OF GEO600 GRAVITATIONAL-WAVE INTERFEROMETER

A gravitational wave observatory operating beyond the quantum shot-noise limit

The LIGO Scientific Collaboration

\[ \text{N-number of photons} \approx 10^{12}/\text{ns} \]

ATOMIC MAGNETOMETRY WITH DEPHASING BEYOND SQL

inspired by the apparatus in [Wasilewski et al, Phys. Rev. Lett. 104, 133601 (2010)]

\[
B = \begin{pmatrix}
B_x \\
B_y \\
B_z
\end{pmatrix} = B \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}
\]

Sensed magnetic field

\[
b = \begin{pmatrix}
0 \\
0 \\
b
\end{pmatrix} = be_3
\]

Evolution of \(N\) atoms described by the density matrix \(\rho^N(t)\):

\[
\frac{d\rho^N}{dt} = i\frac{\omega_b}{2} \left[ \rho^N, \sum_{n=1}^{N} \sigma_3^{(n)} \right] - \frac{\gamma B}{2} \sum_{n=1}^{N} \left[ \sum_{i=1}^{3} \alpha_i \sigma_i^{(n)} \rho^N \sigma_i^{(n)} - \rho^N \right]
\]

Estimation of \(\varphi = \omega b t\), \(\omega_b \propto b \approx 0\)

- evolution time of the atomic cloud
- extra free parameter
**Atomic Magnetometry with Dephasing Beyond SQL**

*Inspired by the apparatus in* [Wasilewski et al, Phys. Rev. Lett. 104, 133601 (2010)]

Evolution of \( N \) atoms described by the density matrix \( \rho^N(t) \):

\[
\frac{d\rho^N}{dt} = \frac{i}{2} \omega_b \left[ \rho^N, \sum_{n=1}^{N} \sigma_3^{(n)} \right] - \frac{\gamma B}{2} \sum_{n=1}^{N} \left[ \sum_{i=1}^{3} \alpha_i \rho^N \sigma_i^{(n)} \sigma_i^{(n)} - \rho^N \right]
\]

Estimation of \( \omega_b \propto b \approx 0 \quad \rightarrow \quad t \) - evolution time of the atomic cloud – *extra free parameter*

**Atomic Spectroscopy Experiment - Resources:** Total time of the experiment \( T \), number of particles involved \( N \):

\[
\Delta \tilde{\omega}_b \geq \frac{1}{\sqrt{T} \cdot F_Q [\rho^N_{\omega_b}(t)]}
\]

*Optimize over \( t \)*

\[
f[\Lambda_{\omega_b}^N] = \max_{t} \left\{ F_Q [\rho^N_{\omega_b}(t)] / t \right\}
\]

\[
t_{\text{opt}}(N) = \argmax_{t} \left\{ F_Q [\rho^N_{\omega_b}(t)] / t \right\}
\]

\[
\Delta \tilde{\omega}_b \sqrt{T} \geq \frac{1}{\sqrt{f[\Lambda_{\omega_b}^N]}}
\]

\( t_{\text{opt}}(N) \to 0 \) as \( N \to \infty \)
Atomic Magnetometry with Dephasing Beyond SQL

inspired by the apparatus in [Wasilewski et al, Phys. Rev. Lett. 104, 133601 (2010)]

Evolution of $N$ atoms described by the density matrix $\rho^N(t)$:

$$\frac{d\rho^N}{dt} = \frac{i}{2} \omega_b \left[ \rho^N, \sum_{n=1}^{N} \sigma_3^{(n)} \right] - \frac{\gamma_B}{2} \sum_{n=1}^{N} \left[ \sum_{i=1}^{3} \alpha_i \sigma_i^{(n)} \rho^N \sigma_i^{(n)} - \rho^N \right]$$

Estimation of $\omega_b \propto b \approx 0 \quad \rightarrow \quad t$ - evolution time of the atomic cloud – extra free parameter

Parallel dephasing: $\alpha_3 = 1, \alpha_1 = \alpha_2 = 0$ two Kraus operators – non-full rank channel – SN-bounding Methods apply for any $t$

Parallelogram

$\Delta \omega_b \sqrt{T} \geq \min_t \frac{1}{\sqrt{F_Q[\rho_b^N(t)]}} \geq \min_t \sqrt{c_{\parallel}(t)} \geq \sqrt{\frac{2\gamma_B}{N}}$

Ellipsoid-like dephasing: $\exists_{i \neq j}, \alpha_i \neq 0, \alpha_j \neq 0$ four Kraus operators – full rank channel – SN-bounding Methods apply for any $t$

$\Delta \omega_b \sqrt{T} \geq \min_t \frac{1}{\sqrt{F_Q[\rho_b^N(t)]}} \geq \min_t \sqrt{c_{\parallel}(t)} \geq \sqrt{\frac{\text{const}}{N}}$

$c_{\parallel}(t) = 2\gamma_B + O(t)$

t_{\text{opt}}(N) \rightarrow 1/\sqrt{N} \quad \text{as} \quad N \rightarrow \infty$

$c_{\parallel}(t) = \text{const} + O(t)$
**Atomic Magnetometry with Transversal Dephasing**

- **Transversal dephasing:**
  \[ \alpha_1 = 1, \quad \alpha_2 = \alpha_3 = 0 \]

- **four Kraus operators** – full rank channel – **SN-bounding Methods apply**, but...
  \[ t_{\text{opt}}(N) \to \left( \frac{3}{\gamma_B \omega_b^2 N} \right)^{1/3} \]

- **Beyond the Shot Noise !!!**
  \[ c_\perp(t) = \frac{\gamma_B^2 \omega_b}{12} t^3 + O(t^5) \]

\[ \Delta \tilde{\omega}_b \sqrt{T} \geq \min_t \frac{1}{\sqrt{F_{Q|\rho_{\omega_b}^N(t)}}} \geq \min_t \sqrt{\frac{c_\perp(t)}{N}} \geq \left( \frac{3 \omega_b \sqrt{\gamma_B}}{8} \right)^{1/3} \cdot \frac{1}{N^{5/6}} \]

- **(a) Saturability with the GHZ states**
  - **(dotted)** – parallel dephasing *with* \( t \)-optimisation
  - **(dashed)** – transversal dephasing *without* \( t \)-optimisation
  - **(solid)** – transversal dephasing *with* \( t \)-optimisation

- **(b) Impact of parallel component**

CONCLUSIONS

- **Classically**, for *separable* input states, the ultimate precision is bound to **Shot Noise scaling** $1/\sqrt{N}$, which can be attained in a single experimental shot ($k = 1$).

- For **noiseless** unitary evolution highly *entangled* input states ($GHZ, N00N$) allow for ultimate precision that follows the **Heisenberg scaling** $1/N$, but attaining this limit may in principle require infinitely many repetitions of the experiment ($k \rightarrow \infty$).

- The consequences of the *dehorence* acting independently on each particle:
  - The **Heisenberg scaling** is lost and only a **constant factor quantum enhancement** over classical estimation strategies is possible.
  - In the $N \rightarrow \infty$ limit, the **optimal input states** have a **simpler form** and are easier to produce. (Conjecture) They achieve the ultimate precision in a single shot ($k=1$):
    - **Atomic**: (noisy regime) *spin-squeezed states* v.s. (noiseless regime) *GHZ states*
    - **Optical**: (noisy regime) *squeezing of light* v.s. (noiseless regime) *NOON states*
  - The asymptotically **optimal measurement schemes** become far less complex:
    - **Atomic**: (noisy regime) *Ramsey measurement* v.s. (noiseless regime) *parity measurement*
    - **Optical**: (noisy regime) *Photon counting* v.s. (noiseless regime) *parity measurement*
  - However, finding the **optimal form of input states** is still an issue.
    - **Classical scaling suggests local correlations**:
CONCLUSIONS

○ We have formulated three methods: Classical Simulation, Quantum Simulation and Channel Extension; that may efficiently lower-bound the constant factor of the quantum asymptotic enhancement for a generic channel by the properties of its single use (Kraus representation).

○ The CS method has a geometric picture and proves the $c_Q \sqrt{N}$ for all full-rank channels and more (e.g. dephasing) due to the convexity of the space of the CPTP maps.

○ The CE method may also be applied numerically for finite $N$ as a semi-definite program.

○ After allowing the form of channel to depend on $N$, what is achieved by the (experimentally-motivated) single experimental-shot period, $(t)$, optimisation, we establish a channel that, despite being full-rank for any finite $t$, achieves the ultimate super-classical $1/N^{5/6}$ asymptotic — the transversal dephasing.


THANK YOU FOR YOUR ATTENTION