Uniqueness of scalar field energy and gravitational energy in the radiating regime

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The usual approaches to the definition of energy give an ambiguous result for the energy of fields in the radiating regime. We show that for a massless scalar field in Minkowski space–time the definition may be rendered unambiguous by adding the requirement that the energy cannot increase in retarded time. We present a similar theorem for the gravitational field, proved elsewhere, which establishes that the Trautman–Bondi energy is the unique (up to a multiplicative factor) functional, within a natural class, which is monotonic in time for all solutions of the vacuum Einstein equations admitting a smooth "piece" of conformal null infinity $\mathscr{I}$.

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In Lagrangian field theories one can usually define the energy of an isolated system by a volume integral over a spacelike hypersurface S stretching to spatial infinity — say, $S = \{x^0 = \tau\}$ in Minkowskian or quasi–Minkowskian coordinates. The integrand may be ambiguous, because there is a freedom in the choice of the Lagrangian, but, as discussed in detail below, for a massless scalar field and for hypersurfaces stretching to spatial infinity the total energy so defined does not depend upon this choice. Moreover the boundary conditions guarantee that the result will not depend upon τ, a property which is known as the conservation of energy. The same is true for the Maxwell or Yang–Mills–Higgs fields with the usual asymptotic conditions. The corresponding conserved *energy at spatial infinity* in general relativity is the ADM mass (though one needs to further require that the energy be a Hamiltonian in an appropriate function space to get rid of the ambiguities that arise).

When there is radiation, one would like to measure the amount of energy radiated away. This cannot be achieved by integrals over hypersurfaces S extending to spatial infinity: for massless fields, the radiation propagates along null directions, and the energy transport is defined where t and r both approach infinity but with a finite difference between them. This region is avoided by hypersurfaces S. In this case to measure the energy left in the system at a retarded time $t - r = u_0$ one can integrate an appropriate integrand (*cf.* Eqs. (2)–(3) below)

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on a null hypersurface $t - r = u_0$, or on any hypersurface Σ which asymptotes to that hypersurface; the result will not depend upon that choice when an appropriate rate of approach is imposed. For definiteness we will use hyperboloids $t = u + \sqrt{1 + x^2 + y^2 + z^2}$ in Minkowski space.

Now the choice of the Lagrangian becomes important, because the weaker asymptotic decay of the fields in the radiation regime, as compared to the asymptotic behaviour on hypersurfaces of constant Minkowskian time, will give different values for the energy integral when a different Lagrangian function is used. So one needs a prescription for the choice of this "energy". Here we prove that for massless scalar fields the requirement that the energy of the sources be non-increasing, which has the obvious physical motivation that outgoing radiation always carries energy away from rather than towards the source, gives a unique prescription, which is in fact the formula normally used. We also state a similar result, proved by a similar method with additional technical complications which will be given elsewhere, for gravitational radiation in general relativity, where the resulting energy is the Trautman–Bondi (TB) mass (given by Freud [1], Trautman [2], Bondi *et al.* [3], and Sachs [4]).

To be more specific, consider a Lagrangian theory of fields ϕ^A defined on a manifold M with a Lagrange function density

$$\mathcal{L} = \mathcal{L}[\phi^A, \partial_\mu \phi^A, \ldots, \partial_{\mu_1} \ldots \partial_{\mu_k} \phi^A] \,, \tag{1}$$

for some $k \in \mathbb{N}$, where ∂_μ denotes partial differentiation with respect to x^μ. Suppose further that there exists a function t on M such that M can be decomposed as $\mathbb{R} \times \Sigma$, where $\Sigma \equiv \{t = 0\}$ is a hypersurface in M and the vector $\partial/\partial t$ is tangent to the $\mathbb{R}$ factor. The proof of the Noether theorem, as presented *e.g.* in [Section 10.1] of Ref. [5], shows that the vector density

$$\begin{aligned} E^\lambda = {} & -\mathcal{L} X^\lambda + X^\mu \textstyle\sum_{\ell=0}^{k-1} \phi^A{}_{,\alpha_1 \ldots \alpha_\ell \mu} \\ & \times \textstyle\sum_{j=0}^{k-\ell-1} (-1)^j \partial_{\gamma_1} \ldots \partial_{\gamma_j} \left(\frac{\partial \mathcal{L}}{\partial \phi^A{}_{,\lambda \alpha_1 \ldots \alpha_\ell \gamma_1 \ldots \gamma_j}} \right) \end{aligned} \tag{2}$$

has vanishing divergence, $E^\lambda{}_{,\lambda} = 0$, when the fields ϕ_A are sufficiently smooth and satisfy the variational equations associated with a sufficiently smooth $\mathcal{L}$ (*cf.* also Ref. [6]). Here $\phi^A{}_{,\alpha_1 \ldots \alpha_\ell} = \partial_{\alpha_1} \ldots \partial_{\alpha_\ell} \phi^A$, and $X^\mu \partial_\mu = \partial_t$. The total energy associated with the hypersurface Σ can be defined by the formula

$$E(\Sigma) = \int_\Sigma E^\lambda \mathrm{d}S_\lambda, \tag{3}$$

with $\mathrm{d}S_\lambda = \partial_\lambda \lrcorner \mathrm{d}x^0 \wedge \ldots \mathrm{d}x^3$, where $\lrcorner$ denotes contraction. The addition to $\mathcal{L}$ of a functional of the form

$$\partial_\lambda (Y^\lambda [\phi^A, \partial_\alpha \phi^A, \ldots, \partial_{\alpha_1} \ldots \partial_{\alpha_{k-1}} \phi^A]) \,, \tag{4}$$

where k is as in (1), which does not affect the field equations will change $E(\Sigma)$ by a boundary integral (see, *e.g.*, Ref. [7] for an explicit formula for $\Delta E^{\mu\lambda}$):

$$E(\Sigma) \longrightarrow \hat{E}(\Sigma) = E(\Sigma) + \int_{\partial\Sigma} \Delta E^{\mu\lambda} dS_{\mu\lambda} \ , \qquad (5)$$

where $S_{\alpha\beta} = \partial_\alpha \lrcorner \partial_\beta \lrcorner \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^3$. If $\partial\Sigma$ is a "sphere at infinity" the integral over $\partial\Sigma$ has to be understood by a limiting process. Unless the boundary conditions at $\partial\Sigma$ force all such boundary integrals to give a zero contribution, if one wants to define energy for radiating systems using this framework one has to have a criterion for choosing a "best" functional, within the class of all functionals obtainable in this way. In several cases of interest, including a massless scalar field and general relativity, such boundary integrals do not automatically vanish.

As an example, consider a scalar field ϕ in Minkowski space–time, taking $S = \{t = 0\}$ as Σ. Assuming that ϕ satisfies the rather strong fall–off conditions on S

$$\partial_{\alpha_1} \ldots \partial_{\alpha_j} \phi = o(r^{-2}), \quad 0 \leq j \leq 2(k-1) \ , \qquad (6)$$

where k is the integer appearing in (1), the boundary integral in (5) will vanish for all smooth Y^μ's in eq. (4). This shows that eq. (3) leads to a well–defined notion of energy on this space of fields on S (whatever the Lagrange function $\mathcal{L}$), as long as the volume integral there converges.

Now take a massless scalar field in Minkowski space–time, so that $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, and the field equations are

$$\Box\phi = 0 \ . \qquad (7)$$

Let Σ be the hyperboloid $t = \sqrt{1 + x^2 + y^2 + z^2}$. In that case solutions of eq. (7) which are obtained by evolving compactly supported data on $\{t = 0\}$ (see eq. (9) below) do not satisfy (6) on Σ, and the boundary term in (5) does not vanish in general. Thus, even for scalar fields in Minkowski space–time, a supplementary condition singling out a preferred E^λ is needed in the radiation regime.

Now for various field theories on the Minkowski background, including the scalar field, and for the gravitational field, one can impose some further conditions on E^λ such as Lorentz covariance, and dependence upon the first derivatives of the field only, which render it unique [8–10]. However, the condition in [9] that the energy–momentum pseudo–tensor of the gravitational field should be quadratic in the first derivatives of the metric tensor appears to us too restrictive. Thus it seems worthwhile to find a more natural criterion, which would encompass both general relativity and field theories on a Minkowski background, and which would single out a preferred expression for energy in the radiation regime.

In this letter we point out that the requirement of *monotonicity of energy in retarded time* allows one to single out an energy expression in a unique way, within a natural class of "energies".

Let us start with the case of a massless scalar field in Minkowski space–time. The variational formalism described above leads one to consider functionals of the form

$$H[\phi,t] = E(\Sigma_t) + \int_{\partial\Sigma_t} H^{\alpha\beta} \mathrm{d}S_{\alpha\beta} \ , \qquad (8)$$
$$E(\Sigma_t) = \int_{\Sigma_t} T^{\mu}{}_{\nu} X^{\nu}\, dS_{\mu}, \quad X^{\nu}\partial_{\nu} = \partial_t \ ,$$

where $H^{\alpha\beta}$ is a twice continuously differentiable function of $\phi(x)$, $\partial_{\alpha_1}\phi(x)$, …, $\partial_{\alpha_1}\ldots\partial_{\alpha_n}\phi(x)$, for some n. The indices α refer to Minkowski coordinates, in which the $H^{\alpha\beta}$'s *depend upon the coordinates through the fields only.* [1] Here as before the Σ_t's are unit hyperboloids in Minkowski space–time, $T^{\mu}{}_{\nu}$ is the standard energy–momentum tensor for the scalar field (with the normalization determined by eq. (2)), and the integral in (8) is understood as a limit as R tends to infinity of integrals on coordinate balls of radius R included in Σ_t.

Before analyzing convergence of functionals (8) we need to specify the class of fields ϕ of interest. Consider solutions of (7) which have smooth compactly supported initial data on the hyperplane $\{x^0 = 0\}$, where x^0 is a standard Minkowski coordinate. Using conformal covariance of eq. (7) it can be shown that there will exist smooth functions $c(u,\theta,\phi)$ and $d(u,\theta,\phi)$ defined on $(-\infty,\infty)\times S^2$ such that

$$\phi(u,r,\theta,\phi) - \frac{c(u,\theta,\phi)}{r} - \frac{d(u,\theta,\phi)}{r^2} = O(r^{-3}), \qquad (9)$$

with $u = x^0 - r$. Moreover (9) is preserved under differentiation in the obvious way. (The hypothesis that the initial data are compactly supported is not necessary, and is made only to avoid unnecessary technical discussions.) In what follows we will only consider solutions of (7) satisfying (9). Again standard results on the wave equation together with conformal covariance show that *given arbitrary functions c on, say $[u_0-1,u_0+1]\times S^2$ and d_0 on S^2 there exists a solution of the wave equation defined on Minkowski space–time such that (9) holds, with*

$$\frac{\partial d}{\partial u} = -\frac{1}{2}\Delta_2 c \ , \qquad d(u_0,\theta,\phi) = d_0(\theta,\phi) \ . \qquad (10)$$

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[1] Explicit coordinate dependence in $H^{\alpha\beta}$ could arise if there were explicit coordinate dependence in $\mathcal{L}$; this could then lead to E^{λ}'s which do not have vanishing divergence. While the hypothesis of coordinate independence of $\mathcal{L}$ seems to us to be a natural hypothesis for the problem at hand, it would certainly be of interest to find all monotonic functionals which arise when some coordinate dependence is allowed.

Here Δ_2 denotes the Laplace operator on S^2 with the standard round metric. (Eq. (10) is obtained by inserting the expansion (9) in (7)). We claim the following:

Theorem 1 *Let H be as described above and suppose that (8) converges for all solutions of the wave equation satisfying (9). If $\frac{\mathrm{d}H}{\mathrm{d}t} \leq 0$, then for all such ϕ's*

$$\int_{\partial\Sigma_t} H^{\alpha\beta}\mathrm{d}S_{\alpha\beta} = 0 \ ,$$

so that the numerical value of H equals the standard canonical energy.

Proof: We can Taylor expand $H^{\alpha\beta}$ at $\phi = 0$ up to second order to obtain

$$H^{\alpha\beta} = H^{\alpha\beta}\Big|_{\phi=0} + \sum_{0\leq|I|\leq k} \frac{\partial H^{\alpha\beta}}{\partial\phi_I}\Big|_{\phi=0}\phi_I + \sum_{0\leq|I|,|J|\leq k} \frac{\partial^2 H^{\alpha\beta}}{\partial\phi_I\partial\phi_J}\Big|_{\phi=0}\phi_I\phi_J + r^{\alpha\beta} \ , \qquad (11)$$

where we use the symbol ϕ_I to denote objects of the form $\partial_{\alpha_1}\ldots\partial_{\alpha_\ell}\phi$, with $|I| = |(\alpha_1,\ldots,\alpha_\ell)| = \ell$. By hypothesis $H^{\alpha\beta}\Big|_{\phi=0}$ depends only upon the metric and its derivatives, so in Minkowski coordinates the coefficients $H^{\alpha\beta}\Big|_{\phi=0}$, $\frac{\partial H^{\alpha\beta}}{\partial\phi_I}\Big|_{\phi=0}$ *etc.* in (11) are constants. By well known properties of Taylor expansions and by (9) we have $r^{\alpha\beta} = o(r^{-2})$, so that $r^{\alpha\beta}$ will not contribute to H in the limit $r\to\infty$. In Minkowski coordinate systems (9) can be rewritten as

$$\frac{\partial}{\partial x^{\alpha_1}}\cdots\frac{\partial}{\partial x^{\alpha_j}}\phi = \frac{c^{(j)}}{r}n_{\alpha_1}\ldots n_{\alpha_j} + \frac{L_{\alpha_1\cdots\alpha_j}}{r^2} + O(r^{-3}), \quad (12)$$

$$\frac{\partial}{\partial x^{\alpha_1}}\cdots\frac{\partial}{\partial x^{\alpha_j}}\frac{d}{r^2} = \frac{d^{(j)}}{r^2}n_{\alpha_1}\ldots n_{\alpha_j} + O(r^{-3}) \ , \qquad (13)$$

where $c^{(m)}(t,r,\theta,\varphi) = \frac{\partial^m}{\partial u^m}\left(c(u,\theta,\varphi)\right)\big|_{u=t-r}$, similarly for $d^{(m)}$, and $n_\mu = (1,-\frac{x^i}{r})$. Here $L_{\alpha_1\cdots\alpha_j}$ is a linear function of c, its u and angular derivatives up to order $j-1$. Inserting (9) in (11) and making use of (12)–(13) might produce several terms which do not obviously converge, but those have to cancel out or integrate out to zero by our hypothesis of convergence of (8). It then follows that (8) can be rewritten as

$$H = E(\Sigma_t) + \int_{S^2} \hat{h}[c,c^{(1)},\ldots,c^{(k)},d^{(1)},\ldots,d^{(k)},\theta,\phi]\,d^2\mu \ , \quad (14)$$

for some functional $\hat{h}$, smooth in all its arguments, with $d^2\mu = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi$. Moreover $\hat{h}$ is linear in d and its u–derivatives. Eq. (10) allows one to eliminate the u–derivatives of d in terms of derivatives of c, so that (14) can be rewritten as

$$H = E(\Sigma_t) + \int_{S^2} \Big(h[c, c^{(1)}, \dots, c^{(k)}, \theta, \phi] + \alpha[d, \theta, \phi] \Big)\, d^2\mu \ , \quad (15)$$

for some functionals h and α, with α linear in d. The u–derivative of (15) gives

$$\frac{dH}{dt} = \int_{S^2} \Big(-(c^{(1)})^2 + \frac{\delta h}{\delta c} c^{(1)} + \dots + \frac{\delta h}{\delta c^{(k)}} c^{(k+1)} + \frac{\delta \alpha}{\delta d} d^{(1)} \Big)\, d^2\mu \ ,$$

Since $c^{(k+1)}$ is an arbitrary function on S^2 at fixed $c, \dots, c^{(k)}$ and d_0, we can choose it so that $\frac{dH}{dt} \le 0$ unless $\delta h / \delta c^{(k)} = 0$, $k \ge 1$. A suitable redefinition of h leads to

$$\begin{aligned} H(t) &= E(\Sigma_t) + \textstyle\int_{S^2} \left(h[c, \theta, \phi] + \alpha[d, \theta, \phi] \right) d^2\mu \ , \\ \tfrac{dH}{dt} &= \textstyle\int_{S^2} \left(\left(-c^{(1)} + \frac{\delta h}{\delta c} \right) c^{(1)} - \frac{1}{2} \frac{\delta \alpha}{\delta d} \Delta_2 c \right) d^2\mu \ , \end{aligned} \quad (16)$$

Consider now solutions of the wave equation with $c^{(1)}(u = u_0) = 0$. In this case (16) and arbitrariness of $c(u = u_0)$ imply that dH/dt will be non–positive if and only if $\Delta_2(\delta\alpha/\delta d) = 0$, which forces $\delta\alpha/\delta d$ to be a constant. We note that for any constant a the integral

$$\int_{S^2} a d \, d^2\mu \ , \quad (17)$$

is a constant of motion (see also [11, Sect. 8.2]). However, integrals of the form (17) cannot arise in the class of functionals considered here. Indeed, the identity (13) shows that all the terms which would give a non–vanishing contribution to H and which contain derivatives of d contain at least one u derivative of d. Then the only possible term which would contain d would come from the term

$$\int_{S^2} \frac{\partial H^{\alpha\beta}}{\partial \phi}\Big|_{\phi=0} \phi \, dS_{\alpha\beta} = \int_{S^2} \frac{\partial H^{rt}}{\partial \phi}\Big|_{\phi=0} (cr + d)\, d^2\mu \ ,$$

which for generic c diverges when r goes to infinity, unless identically vanishing. We thus obtain $\delta\alpha/\delta d = 0$. The right hand side of eq. (16) can be made positive by choosing $c^{(1)}(u = u_0) = \frac{1}{2}\delta h/\delta c$, unless $\delta h/\delta c = 0$, and our claim follows. □

For general relativity, the appropriate spacelike surfaces analogous to hyperboloids are space–like hypersurfaces which intersect the future null infinity $\mathcal{I}^+$ in a compact cross–section K. In [7] we show that *the TB energy is, up to a multiplicative constant $\alpha \in \mathbb{R}$, the only functional of the gravitational field, in a certain natural class of functionals, which is monotonic in time for all vacuum field configurations which admit (a piece of) a smooth null infinity $\mathcal{I}^+$*. More precisely, in [7] we show the following:

Theorem 2 *Let H be a functional of the form*

$$H[g,\,u]=\int_{S^2(u)}H^{\alpha\beta}(g_{\mu\nu},\,g_{\mu\nu,\sigma},\,\ldots,\,g_{\mu\nu,\sigma_1\ldots\sigma_k})\,\mathrm{d}S_{\alpha\beta}\,, \tag{18}$$

where the $H^{\alpha\beta}$ are twice differentiable functions of their arguments, and the integral over $S^2(u)$ is understood as a limit as ρ goes to infinity of integrals over the spheres $t=u+\rho, r=\rho$. Suppose that H is finite and monotonic in u for all vacuum metrics $g_{\mu\nu}$ satisfying

$$g_{\mu\nu}=\eta_{\mu\nu}+\frac{h^1_{\mu\nu}(u,\,\theta,\,\phi)}{r}+\frac{h^2_{\mu\nu}(u,\,\theta,\,\phi)}{r^2}+o(r^{-2})\ ,$$
$$\partial_{\sigma_1}\ldots\partial_{\sigma_i}(g_{\mu\nu}-\frac{h^1_{\mu\nu}(u,\,\theta,\,\phi)}{r}-\frac{h^2_{\mu\nu}(u,\,\theta,\,\phi)}{r^2})=o(r^{-2}), \tag{19}$$

with $1\le i\le k$, for some C^k functions $h^a_{\mu\nu}(u,\,\theta,\,\phi)$, $a=1,2$. If H is invariant under passive BMS super-translations, then the numerical value of H equals (up to a proportionality constant) the Trautman–Bondi mass.

Some comments are in order. First, the volume integral $E(\Sigma_t)$ which was present in (8) does not occur in (18), because the Trautman–Bondi mass is itself a boundary integral. Next, Theorem 2 imposes the further requirement of *passive BMS invariance*, which did not occur in the scalar field case. This requirement arises as follows: recall that the coordinate systems in which the metric satisfies (19) are, roughly speaking, defined only modulo BMS transformations. Then the requirement of *passive BMS invariance* is the rather reasonable requirement that the concept of energy be independent of the coordinate system chosen to measure this energy. We note that we believe that the requirement of monotonicity forces the energy to be invariant under (passive) super–translations, but we have not succeeded in proving this so far.

It is natural to ask why the Newman–Penrose constants of motion [12], or the logarithmic constants of motion of [13], do not occur in our results of [7]. These quantities are excluded by the hypothesis that the boundary integrand $H^{\alpha\beta}$ which appears in the integrals we consider depends on the coordinates only through the fields.

We note that a key ingredient of the proof of Theorem 2 is the Friedrich–Kannar [14,15] construction of space–times "having a piece of $\mathscr{I}$".

Let us finally mention that one can set up a Hamiltonian framework in a phase space which consists of Cauchy data on hyperboloids together with values of the fields on appropriate parts of Scri to describe the dynamics in the radiation regime [16]. Unsurprisingly, the Hamiltonians one obtains in such a formalism are again not unique, but the non–uniqueness can be controlled in a very precise way. The Trautman–Bondi mass turns out to be a Hamiltonian, and an appropriate version of the uniqueness Theorem 2 can be used to single out the TB mass amongst the family of all possible Hamiltonians. In the

Hamiltonian framework the freedom of multiplying the functional by a constant disappears.

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