

TRAUTMAN-BONDI MASS FOR SCALAR FIELD AND GRAVITY

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Abstract

The energy at null infinity is presented with the help of a simple example of a massless scalar field in Minkowski spacetime. It is also discussed for Einstein gravity. In particular, various aspects of the loss of the energy in the radiating regime are shown.

1. INTRODUCTION

It turns out that the case of the massless scalar field in Minkowski space-time already exhibits all the essential features of the problem at hand, while avoiding various technicalities which arise when one wishes to describe Einstein gravity. In General Relativity we present a new variational formulation on hypersurfaces which are space-like inside and light-like near future null infinity \mathcal{I}^+ . The formulae, we obtain, correspond to the mass loss formula.

2. SCALAR FIELD ON A HYPERBOLOID

Let us consider a scalar field ϕ in a flat Minkowski space M with the metric

$$\eta_{\mu\nu}dx^\mu dx^\nu = \rho^{-2} \left(-\rho^2 ds^2 + \frac{2dsd\rho}{\sqrt{1+\rho^2}} + \frac{d\rho^2}{1+\rho^2} + d\theta^2 + \sin^2\theta d\varphi^2 \right) \quad (2.1)$$

Let us fix a coordinate chart (x^μ) on M such that $x^1 = \theta$, $x^2 = \varphi$ (spherical angles), $x^3 = \rho$ and $x^0 = s$, and let us denote by $\overset{\circ}{\gamma}_{AB}$ a metric on a unit sphere ($\overset{\circ}{\gamma}_{AB}dx^A dx^B := d\theta^2 + \sin^2\theta d\varphi^2$).

We shall consider an initial value problem on a hyperboloid Σ

$$\Sigma_\tau := \{x \in M \mid x^0 = \tau = \text{const.}\}$$

for our scalar field ϕ with a density of the Lagrangian (corresponding to the wave equation)

$$L := -\frac{1}{2}\sqrt{-\det \eta_{\mu\nu}}\eta^{\mu\nu}\phi_\mu\phi_\nu = -\frac{1}{2}\rho^{-2}\sin\theta\left[\rho^2(\phi_3)^2 - \frac{(\phi_0)^2}{1+\rho^2} + \frac{2\phi_3\phi_0}{\sqrt{1+\rho^2}} + \overset{\circ}{\gamma}{}^{AB}\phi_A\phi_B\right]$$

We use the following convention for indices: Greek indices μ, ν, \dots run from 0 to 3; k, l, \dots are coordinates on a hyperboloid Σ_τ and run from 1 to 3; A, B, \dots are coordinates on $S(\tau, \rho)$ and run from 1 to 2, where $S(\tau, \rho) := \{x \in \Sigma_\tau \mid x^3 = \rho = \text{const.}\}$.

The Euler-Lagrange equations for scalar field ϕ can be described by the following generating formula (see e.g. [7])

$$\delta L = (p^\mu \delta \phi)_{,\mu} = p^\mu_{,\mu} \delta \phi + p^\mu \delta \phi_\mu \quad (2.2)$$

where $L = L(\phi, \phi_\mu)$ is the Lagrangian density of the theory. When we integrate (2.2) over any finite three-volume $V \subset \Sigma$, the following variational equation

$$\delta \int_V L = \int_V (p^0 \delta \phi)_{,0} + \int_{\partial V} p^3 \delta \phi$$

can be written¹. In particular, the definition $p^\mu = \frac{\partial L}{\partial \phi_\mu}$ of the canonical momenta p^μ enables one to calculate the time and radial components of p^μ

$$p^0 = \frac{\partial L}{\partial \phi_0} = \rho^{-2} \sin \theta \left(\frac{\phi_0}{1+\rho^2} - \frac{\phi_3}{\sqrt{1+\rho^2}} \right)$$

$$p^3 = \frac{\partial L}{\partial \phi_3} = -\rho^{-2} \sin \theta \left(\frac{1}{\sqrt{1+\rho^2}} \phi_0 + \rho^2 \phi_3 \right)$$

Let us observe that in general the integral $\int_V L$ is not convergent for $V = \Sigma$, if we assume the usual asymptotics of a typical solution ϕ of the wave equation, namely $\phi = O(\rho)$ and $\phi_3 = O(1)$. The same problem with “infinities” we meet in p^0 and p^3 .

Let \overline{M} denote the standard conformal completion at future null infinity \mathcal{I}^+ of M , let $\overline{\Sigma}_\tau$ be the closure of Σ_τ in \overline{M} , set $S_\tau := S(\tau, \rho = 0) = \partial \overline{\Sigma}_\tau = \overline{\Sigma}_\tau \cap \mathcal{I}^+$. The reader not familiar with the notion of Scri can simply think of the S_τ 's as “spheres at infinity” on the hypersurfaces Σ_τ . ($\mathcal{I}^+ = \{x \in \overline{M} \mid \rho = 0\}$.)

The conformal rescaling of the metric $\eta_{\mu\nu}$

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \quad (2.3)$$

¹For the simplicity of notation we assumed that ∂V is a sphere $S(\tau, \rho)$. In general case instead of p^3 should be normal component p^\perp of the canonical momentum p^μ (see [7]).

enables one to “renormalize” L by adding a full divergence

$$\begin{aligned}\bar{L} &:= -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\psi_\mu\psi_\nu + \frac{1}{12}\sqrt{-g}R(g)\psi^2 = \\ &= -\frac{1}{2}\Omega^2\sqrt{-\eta}\eta^{\mu\nu}\psi_\mu\psi_\nu + \frac{1}{2}\Omega\sqrt{-\eta}\square\Omega\psi^2 = L + \frac{1}{2}\partial_\nu(\sqrt{-\eta}\eta^{\mu\nu}\Omega_{,\mu}\Omega\psi^2)\end{aligned}\quad (2.4)$$

where we used a new field variable $\psi := \Omega^{-1}\phi$ which is a natural one close to the null infinity. The generating formula takes the following form with respect to the new variable ψ

$$\delta \int_V \bar{L} = \int_V (\pi^0 \delta\psi)_{,0} + \int_{\partial V} \pi^3 \delta\psi$$

and the Euler-Lagrange equations we write explicitly

$$\begin{aligned}\pi^0 &= \frac{\partial \bar{L}}{\partial \psi_0} = \rho^{-2}\Omega^2 \sin \theta \left(\frac{\psi_0}{1 + \rho^2} - \frac{\psi_3}{\sqrt{1 + \rho^2}} \right) \\ \pi^3 &= \frac{\partial \bar{L}}{\partial \psi_3} = -\rho^{-2}\Omega^2 \sin \theta \left(\frac{1}{\sqrt{1 + \rho^2}}\psi_0 + \rho^2\psi_3 \right) \\ \pi^A &= \frac{\partial \bar{L}}{\partial \psi_A} = -\rho^{-2}\Omega^2 \sin \theta \overset{\circ}{\gamma}{}^{AB}\psi_B \\ \partial_\mu \pi^\mu &= \frac{\partial \bar{L}}{\partial \psi} = \frac{1}{6}\sqrt{-g}R(g)\psi\end{aligned}$$

It is easy to check that all terms are finite at null infinity, provided $\psi = O(1)$, $\psi_3 = O(1)$ and $\rho^{-1}\Omega = O(1)$. For the special case $\Omega = \rho$ see [12]. It is convenient to “normalize” Ω by assuming that

$$\lim_{\rho \rightarrow 0^+} \rho^{-1}\Omega = 1 \quad (2.5)$$

From the above Euler-Lagrange equations one can easily obtain the conformal wave equation

$$\square_g \psi + \frac{1}{6}R(g)\psi = 0 \quad (2.6)$$

Following the usual practice ([6], [7]), we perform the Legendre transformation between π^0 and ψ_0 . It gives us the following Hamiltonian density

$$H := \pi^0\psi_0 - \bar{L} = \rho^{-2}\Omega^2 \frac{1}{2} \sin \theta \left[(\rho\psi_{,3})^2 + \frac{1}{1 + \rho^2}(\psi_0)^2 + \overset{\circ}{\gamma}{}^{AB}\psi_A\psi_B - \frac{1}{6}\rho^{-2}\Omega^2 R(g)\psi^2 \right] \quad (2.7)$$

and the following variational relation

$$-\delta \int_V H = \int_V (\dot{\pi}\delta\psi - \dot{\psi}\delta\pi) + \int_{\partial V} \pi^3 \delta\psi \quad (2.8)$$

where here $\pi := \pi^0$.

Although the density H given by (2.7) depends explicitly on the choice of the conformal factor Ω , it can be easily checked that when V becomes Σ (extending to the future null infinity) the numerical value of the Hamiltonian

$$\mathcal{H} := \int_{\Sigma} H$$

is independent on the choice of the conformal factor provided $\partial_0\Omega = 0$. This can be easily seen from the following relations

$$\begin{aligned} \dot{\pi}\delta\psi - \dot{\psi}\delta\pi &= \dot{p}^0\delta\phi - \dot{\phi}\delta p^0 \\ \pi^3\delta\psi - p^3\delta\phi &= \frac{1}{2}\delta\sin\theta\Omega\Omega_3\psi^2 \end{aligned}$$

More precisely, the conformal factor Ω with the asymptotics on \mathcal{I}^+

$$\Omega|_{\mathcal{I}^+} = 0, \quad \partial_3\Omega|_{\mathcal{I}^+} = 1$$

and $\hat{\Omega} \equiv 1$ (corresponding to the density of the lagrangian L) give the same numerical value of the Hamiltonian \mathcal{H} because $\Omega\Omega_3\psi^2$ vanishes on \mathcal{I}^+ . Moreover, each term in the variational formula

$$-\delta\mathcal{H} = \int_{\Sigma} (\dot{\pi}\delta\psi - \dot{\psi}\delta\pi) + \int_{\partial\Sigma} \pi^3\delta\psi \quad (2.9)$$

possesses a universal character not depending on the particular choice of the conformal factor Ω .

However, when we pass from (2.8) to (2.9), integrating the relation (2.8) over hyperboloid Σ , we quickly realize that the boundary term

$$\int_{\partial\Sigma_\tau} \pi^3\delta\psi = \int_{S_\tau} \sin\theta\dot{\psi}\delta\psi \left(\lim_{\rho \rightarrow 0^+} \rho^{-2}\Omega^2 \right) = - \int_{\partial\Sigma_\tau} \pi\delta\psi$$

does not vanish for the usual asymptotics² of the field ψ . One can try to get a closed Hamiltonian system by assuming that $\dot{\psi}|_{\partial\Sigma} = 0$ and then the energy will be conserved in time. But this is not the case we would like to describe. We want to consider the situation with any data $\psi|_{\mathcal{I}^+}$. In this case we can define Trautman-Bondi energy³, but it would be no longer conserved, formally we can treat it as a Hamiltonian of the opened Hamiltonian system. The variational formula (2.8) enables one to define the TB energy

²The difference in the asymptotics of the field at null and spatial infinity is the main obstruction which does not allow to consider the boundary data at the spatial and at the null infinity in the same way.

³The explicit explanation, why we are convinced that the name of Trautman [18] should be associated with the notion of the mass in the radiating regime, is given in [14] but for gravity. We extend here this notion for the energy of a massless scalar field in the radiating regime.

together with its changes in time. The boundary data depends explicitly on time and we are able to compare the initial data with different boundary conditions. Moreover, if we extend our phase space by adding a piece of \mathcal{S}^+ , we are able to describe a usual closed Hamiltonian system, where Trautman-Bondi energy is the true Hamiltonian of the dynamics [15].

Let us express the density of the Hamiltonian in terms of conformally rescaled phase variables (π, ψ)

$$H(\pi, \psi) := \frac{1}{2}\rho^{-2}\Omega^2 \sin\theta \left[(\rho\psi_3)^2 + \left(\frac{\pi\rho^2\sqrt{1+\rho^2}}{\Omega^2 \sin\theta} + \psi_3 \right)^2 + \overset{\circ}{\gamma}{}^{AB}\psi_A\psi_B - \frac{1}{6}\rho^{-2}\Omega^2 R(g)\psi^2 \right] \quad (2.10)$$

The Hamilton equations are the following

$$\dot{\psi} = \frac{\rho^2\pi}{\Omega^2 \sin\theta} (1 + \rho^2) + \psi_3 \sqrt{1 + \rho^2} \quad (2.11)$$

$$\dot{\pi} = (\pi\sqrt{1+\rho^2})_{,3} + [\rho^{-2}\Omega^2(1+\rho^2)\sin\theta\psi_3]_{,3} + (\rho^{-2}\Omega^2 \sin\theta \overset{\circ}{\gamma}{}^{AB}\psi_B)_{,A} + \frac{1}{6}\rho^{-2}\Omega^2 \sin\theta R(g)\psi \quad (2.12)$$

and they obviously correspond to the wave equation (2.6).

The variational formula (2.9) describes an opened Hamiltonian system because we are not allowed to kill the boundary term. Our Hamiltonian is not conserved in time⁴

$$-\partial_0 \mathcal{H} = \int_{\partial\Sigma} \pi^3 \dot{\psi} = \int_{S_\tau} \sin\theta (\dot{\psi})^2 \left(\lim_{\rho \rightarrow 0^+} \rho^{-2}\Omega^2 \right) = \int_{S_\tau} \sin\theta (\dot{\psi})^2 \quad (2.13)$$

and the last equality holds for Ω obeying the boundary condition (2.5). Formally, the result (2.13) can be obtained from (2.9) if we replace variation δ with ∂_0 , but it can be also checked by a direct computation, using equations (2.11) and (2.12).

2.1. Trautman-Bondi Mass as a Hamiltonian

For $\tau > \tau_0$ let $N_{[\tau_0, \tau]} := \cup_{u \in [\tau_0, \tau]} S_u$, so $N_{[\tau_0, \tau]}$ is a null hypersurface contained in \mathcal{S}^+ with boundary $S_\tau \cup S_{\tau_0}$.

An attempt to treat separately the hyperboloid and Scri leads to the various difficulties in the Hamiltonian approach. In particular, we have learned from [12] that there is no possibility to get a nice Hamiltonian system for the hyperboloidal foliation. However, the ADM energy assigned to the hyperboloid Σ_τ plus $N_{]-\infty, \tau]}$ – a piece of Scri between hyperboloid and spatial infinity (cf. [12]), enables one to remove an infinite tail $N_{]-\infty, \tau_0]}$ and apply the remaining Trautman-Bondi energy as a Hamiltonian. More precisely, let us consider a Hamiltonian system on a surface $\Sigma_\tau \cup N_{[\tau, \tau_0]}$. We propose the following variational formula, which is a direct consequence of the considerations⁵ given in subsection 2.4 of [12]

$$-\delta \left(\int_{\Sigma_\tau} H + \int_{N_{[\tau, \tau_0]}} H \right) = \int_{\Sigma_\tau} (\dot{\pi}\delta\psi - \dot{\psi}\delta\pi) + \int_{N_{[\tau, \tau_0]}} (\dot{\pi}\delta\psi - \dot{\psi}\delta\pi) + \int_{\partial\Sigma_\tau} \pi^3 \delta\psi + \int_{\partial N_{[\tau, \tau_0]}} \pi^{\bar{u}} \delta\psi$$

⁴The conservation law is an obvious feature of the ADM energy defined on the spacelike hypersurface Σ which is stretching to the spatial infinity.

⁵In the paper [12] they were applied for the ADM energy.

where the density of the Hamiltonian on N is defined by $H := \pi\dot{\psi}_{,u}$. The motivation for this choice of H on N is given in [12]. Roughly speaking, it is a consequence of the same universal formula (2.2) but integrated over a null surface.

The following relations (see [12])

$$\pi^3|_{\partial\Sigma} = -\sin\theta\dot{\psi} = \pi^{\bar{u}}, \quad \partial\Sigma_\tau = S_\tau, \quad \partial N_{[\tau,\tau_0]} = S_\tau \cup S_{\tau_0}, \quad \partial(\Sigma_\tau \cup N_{[\tau,\tau_0]}) = S_{\tau_0}$$

enable one to obtain the following variational formula

$$-\delta m_{TB} = \int_{\Sigma_\tau \cup N_{[\tau,\tau_0]}} (\dot{\pi}\delta\psi - \dot{\psi}\delta\pi) + \int_{S_{\tau_0}} \pi\delta\psi \quad (2.14)$$

where $m_{TB} := \int_{\Sigma_\tau \cup N_{[\tau,\tau_0]}} H = \int_{\Sigma_{\tau_0}} H$ is the TB energy at retarded time τ_0 . Killing the term at S_{τ_0} in (2.14) by an appropriate choice of the boundary conditions, our system becomes Hamiltonian as a usual infinite dimensional dynamical system. This can be achieved, assuming for example that

$$\delta\psi|_{S_{\tau_0}} = 0$$

which simply means that ψ is fixed at the time τ_0 . The precise meaning of those heuristic considerations will be given in [15]. Let us only stress that the quantity m_{TB} may be rendered unambiguous by adding the requirement that it cannot increase in retarded time. This particular result has been shown in [14].

2.2. Energy-Momentum Tensor and Non-conservation Laws

Let us consider the energy-momentum tensor for the scalar field ϕ

$$T^\mu{}_\nu = \frac{1}{\sqrt{-\eta}} (p^\mu\phi_{,\nu} - \delta^\mu{}_\nu L)$$

where $\eta := \det \eta_{\mu\nu}$ and by $\delta^\mu{}_\nu$ we have denoted the Kronecker's delta. For the Lagrangian L describing scalar field ϕ the canonical energy momentum is symmetric. From Nöther theorem we have

$$\partial_\mu (\sqrt{-\eta} T^\mu{}_\nu X^\nu) = 0$$

for a Killing vector field X^μ , and integrating the above formula we obtain

$$\partial_0 \int_\Sigma \sqrt{-\eta} T^0{}_\nu X^\nu = - \int_{\partial\Sigma} \sqrt{-\eta} T^3{}_\nu X^\nu \quad (2.15)$$

Usually, when Σ is a space-like surface with the end at spatial infinity, the boundary term on the right-hand side vanishes and the equation (2.15) expresses conservation law for the appropriate generator related to the vector field X . On the contrary, for the hyperboloid the right-hand side does not vanish and (2.15) expresses *non-conservation*

law. It can be easily verified that for the energy and angular momentum we have respectively

$$\int_{\Sigma} \sqrt{-\eta} T^0_0 = \int_{\Sigma} H = \mathcal{H} \quad (2.16)$$

$$J_z := \int_{\Sigma} \sqrt{-\eta} T^0_{\varphi} = \int_{\Sigma} \pi \dot{\psi}_{,\varphi} \quad (2.17)$$

Moreover, for the energy ($X^\mu = \delta_0^\mu$) the boundary term on the right-hand side of (2.15) can be expressed in terms of $\dot{\psi}$

$$- \int_{\partial\Sigma} T^3_0 \rho^{-4} \sin \theta d\theta d\varphi = \int_{\partial\Sigma \cap \mathcal{I}^+} \dot{\psi}^2 \sin \theta d\theta d\varphi \quad (2.18)$$

Equalities (2.16) and (2.18) show obviously the equivalence between (2.15) and (2.13). This way we obtain the following result:

Theorem 2.1 *For massless scalar field the energy loss formulae in Hamiltonian form (2.13) and in Nöther form (2.15) are equivalent.*

In [12] we have shown that the same is true in electrodynamics. In gravity there is no energy-momentum tensor but the formula analogous to (2.13) is also true. The equation (2.13) is a special case of the more general equality, which can be formulated for any Killing vector field $X = X^\mu \partial_\mu$ in the following form

$$\partial_0 \left(\int_{\Sigma} \sqrt{-\eta} T^0_{\nu} X^\nu \right) = - \int_{\Sigma} \pi^3 X^A \psi_{,A} + \pi^3 X^0 \dot{\psi} \quad (2.19)$$

One can check by a direct computation that $X^3|_{\mathcal{I}^+} = 0$, which simply means that the Killing field X (related to Poincaré group) is tangent to the future null infinity \mathcal{I}^+ .

Previous equation (2.13) (in the form of (2.19)) corresponds to the vector field $X_H := \partial_0$.

The vector field corresponding to the linear momentum in z direction

$$X_P := - \frac{\cos \theta}{\sqrt{1 + \rho^2}} \partial_0 - \rho^2 \cos \theta \partial_3 - \rho \sin \theta \partial_\theta, \quad X_P|_{\mathcal{I}^+} = - \cos \theta \partial_0$$

gives also the loss formula

$$- \partial_0 P_z = \int_{\partial\Sigma} \pi^3 X_P^0 \dot{\psi} = - \int_{S(s,0)} \sin \theta \cos \theta (\dot{\psi})^2 \quad (2.20)$$

where $P_z := \int_{\Sigma} \sqrt{-\eta} T^0_{\mu} X_P^\mu$.

The angular momentum along z -axis is related to the vector field $X_J = \partial_\varphi$, and (2.19) takes the form

$$- \partial_0 J_z = \int_{\partial\Sigma} \pi^3 X_J^A \psi_{,A} = \int_{S(s,0)} \sin \theta \dot{\psi} \psi_{,\varphi} \quad (2.21)$$

Similarly, we can take a boost generator along z -axis

$$X_K := -\rho\sqrt{1+\rho^2}\cos\theta\partial_3 - \sqrt{1+\rho^2}\sin\theta\partial_\theta + sX_P, \quad X_K|_{\mathcal{I}^+} = sX_P|_{\mathcal{I}^+} + \hat{\partial}_\varphi$$

where $\hat{\partial}_A := \varepsilon_A{}^B\partial_B$, and the corresponding particular form of the formula (2.19) is the following

$$-\partial_0 K_z = \int_{\partial\Sigma} \pi^3 X_K^0 \dot{\psi} + \pi^3 X_K^A \psi_A = -s\partial_0 P_z - \int_{S(s,0)} \sin^2\theta \dot{\psi} \psi_\theta \quad (2.22)$$

for $K_z := \int_{\Sigma} T_\mu^0 X_K^\mu$ or

$$-\partial_0 K_z + s\partial_0 P_z = \int_{S(s,0)} \sin\theta \dot{\psi} \hat{\partial}_\varphi \psi$$

The equations (2.13) and (2.20)–(2.22) express the *non-conservation* laws for the Poincaré group generators defined at null infinity.

3. GENERAL RELATIVITY

3.1. Generating Formula for Einstein Equations

The variation of the Hilbert Lagrangian (see [11])

$$L = \frac{1}{16\pi} \sqrt{|g|} R \quad (3.1)$$

may be calculated as follows

$$\delta L = \delta \left(\frac{1}{16\pi} \sqrt{|g|} g^{\mu\nu} R_{\mu\nu} \right) = -\frac{1}{16\pi} \mathcal{G}^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{16\pi} \sqrt{|g|} g^{\mu\nu} \delta R_{\mu\nu} \quad (3.2)$$

where

$$\mathcal{G}^{\mu\nu} := \sqrt{|g|} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \quad (3.3)$$

It was proved in [11] that the last term in (3.2) is a boundary term (a complete divergence). For this purpose we denote

$$\pi^{\mu\nu} := \frac{1}{16\pi} \sqrt{|g|} g^{\mu\nu} \quad \text{and} \quad A_{\mu\nu}^\lambda := \Gamma_{\mu\nu}^\lambda - \delta_{(\mu}^\lambda \Gamma_{\nu)}^\kappa \quad (3.4)$$

We have

$$\partial_\lambda A_{\mu\nu}^\lambda = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_{(\mu} \Gamma_{\nu)}^\lambda{}_\lambda = R_{\mu\nu} + A_{\mu\sigma}^\lambda A_{\nu\lambda}^\sigma - \frac{1}{3} A_{\mu\lambda}^\lambda A_{\nu\sigma}^\sigma \quad (3.5)$$

Hence, we obtain an identity

$$\partial_\lambda (\pi^{\mu\nu} \delta A_{\mu\nu}^\lambda) = \pi^{\mu\nu} \delta R_{\mu\nu} + (\nabla_\lambda \pi^{\mu\nu}) \delta A_{\mu\nu}^\lambda \quad (3.6)$$

Due to the metricity of Γ we have $\nabla_\lambda \pi^{\mu\nu} = 0$. This way we obtain

$$\pi^{\mu\nu} \delta R_{\mu\nu} = \partial_\lambda (\pi^{\mu\nu} \delta A_{\mu\nu}^\lambda) = \partial_\kappa (\pi_\lambda^{\mu\nu\kappa} \delta \Gamma_{\mu\nu}^\lambda) \quad (3.7)$$

where we denote

$$\pi_\lambda^{\mu\nu\kappa} := \pi^{\mu\nu} \delta_\lambda^\kappa - \pi^{\kappa(\nu} \delta_\lambda^{\mu)} \quad (3.8)$$

Inserting (3.7) into (3.2) we have

$$\delta L = -\frac{1}{16\pi} \mathcal{G}^{\mu\nu} \delta g_{\mu\nu} + \partial_\lambda (\pi^{\mu\nu} \delta A_{\mu\nu}^\lambda) \quad (3.9)$$

We conclude that Euler-Lagrange equations $\mathcal{G}^{\mu\nu} = 0$ are equivalent to the following generating formula, analogous to the (2.2) in field theory

$$\delta L = \partial_\lambda (\pi^{\mu\nu} \delta A_{\mu\nu}^\lambda) \quad (3.10)$$

or, equivalently,

$$\delta L = \partial_\kappa (\pi_\lambda^{\mu\nu\kappa} \delta \Gamma_{\mu\nu}^\lambda) \quad (3.11)$$

This formula is a starting point for the derivation of canonical gravity. Let us observe, that it is valid not only in the present, purely metric, context but also in any variational formulation of General Relativity. For this purpose let us rewrite (3.9) without using *a priori* the metricity condition $\nabla_\lambda \pi^{\mu\nu} = 0$. This way we obtain the following, universal formula

$$\delta L = -\frac{1}{16\pi} \mathcal{G}^{\mu\nu} \delta g_{\mu\nu} - (\nabla_\kappa \pi_\lambda^{\mu\nu\kappa}) \delta \Gamma_{\mu\nu}^\lambda + \partial_\kappa (\pi_\lambda^{\mu\nu\kappa} \delta \Gamma_{\mu\nu}^\lambda) \quad (3.12)$$

It may be proved that, in this form, the formula remains valid also in the metric-affine approach and in the purely-affine one ([9]). In metric-affine formulation, the vanishing of $\nabla_\lambda \pi^{\mu\nu}$ is not automatic: it is a part of field equations. We see that, again, the entire field dynamics is equivalent to (3.11). Finally, in the purely-affine formulation of General Relativity the Einstein equations are satisfied “from the very beginning” whereas the metricity condition for the connection becomes the dynamical equation. We conclude that also in this case the entire information about the field dynamics is contained in the generating formula (3.11). The universality of (3.11) enables one to apply the same tool for different regimes, namely spatial and null infinity.

Let (P^{kl}, g_{kl}) be a standard Cauchy data in ADM form [4], [5] on a space-like surface. In [10] and [11] the ADM aspect of the formula (3.11) has been developed. In particular, the following theorem has been proposed:

Theorem 3.1 *Dynamical Einstein equations are equivalent to the following formula:*

$$\int_V \dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} = \int_{\partial V} \check{Q}^{AB} \delta g_{AB} - 2 \int_{\partial V} \left[n^A \delta P^3_A + \frac{N^3}{\sqrt{\hat{g}^{33}}} \delta \frac{P^{33}}{\sqrt{\hat{g}^{33}}} + N \delta(\lambda k) \right] \quad (3.13)$$

where $\check{Q}_{AB} = N^3 P_{AB} - N \lambda k_{AB} + (Nk - \frac{\hat{g}^{3l}}{\sqrt{\hat{g}^{33}}} N_{,l}) \lambda g_{AB}$, $N = (-g^{00})^{-\frac{1}{2}}$ is the lapse function, $N^k = \hat{g}^{kl} g_{0l}$ is the shift, $\lambda := \sqrt{\det g_{AB}}$, \check{g}^{AB} is the 2-dimensional inverse to g_{AB} , $n^A = \check{g}^{AB} g_{0B}$ and by k_{AB} we have denoted extrinsic curvature of the 2-surface ∂V embedded in V (k is its 2-dimensional trace).

This is an example of a homogeneous generating formula on a space-like hypersurface. It enables one to derive quasi-local Hamiltonians in General Relativity (cf. [11]). The boundary term in (3.13) can be also written in an equivalent form described by eq. (80) in [11]. We will show in the sequel that the same formula (3.11) possesses an equally important status at null infinity and leads to the similar result.

3.2. Metrics of Bondi-Sachs Type

It has been proved in [12] that the generating formula (3.11) is applicable to the situation considered in [1], [2], [3], [13]. The curved space-time M equipped with a pseudoriemannian metric of the form

$$g_{\mu\nu} dx^\mu dx^\nu = -\frac{V}{r} e^{2\beta} du^2 - 2e^{2\beta} du dr + r^2 \gamma_{AB} (dx^A - U^A du)(dx^B - U^B du) \quad (3.14)$$

enables one to consider the initial value problem on a light-like hypersurface

$$C := \{x \in M \mid x^0 = u = \text{const.}, r \geq r_0\}$$

with the boundary $\partial C = S(u, r = r_0) \cup S(u, r = \infty)$, where

$$S(u, r = r_0) := \{x \in M \mid x^0 = u = \text{const.}, r = r_0\}$$

and $S(u, r = \infty) = S_u \subset \mathcal{I}^+$ is a sphere at the future null infinity. We also assume that

$$\sqrt{\det \gamma_{AB}} = \sin \theta$$

The following boundary integral at null infinity, proposed by Trautman and Bondi, defines the mass in the radiating regime

$$m_{TB} := \frac{1}{8\pi} \int_{S_u} r - V = \frac{1}{4\pi} \int_{S_u} M \sin \theta \quad (3.15)$$

where $V = r - 2M + O(r^{-1})^6$. We would like to stress that in general the definition (3.15) is correct only on a $\{u = \text{const.}\}$ cross-section of \mathcal{I}^+ . If we consider any (cross-) section $s : S^2 \rightarrow \mathcal{I}^+$ of the null infinity \mathcal{I}^+ , we can extend (3.15) to the following:

Definition

$$16\pi m_{TB} := \int_{S^2} \left(4M - \overset{\circ}{\chi}{}^{AB}{}_{\parallel AB} \right) (s(\theta, \phi)) \sin \theta d\theta d\phi \quad (3.16)$$

⁶The asymptotic behaviour of the full metric $g_{\mu\nu}$ in the form (3.14) is given in [2] and summarized in [12].

$$16\pi p^k := \int_{S^2} \left(4M - \overset{\circ}{\chi}{}^{AB}{}_{||AB}\right) (s(\theta, \phi)) \frac{z^k}{r} \sin\theta d\theta d\phi \quad (3.17)$$

which enables one to prove the following theorem (cf. Section 8.1 in [12]):

Theorem 3.2 *The energy-momentum four-vector at null infinity is invariant with respect to the passive supertranslations.*

Let us choose a (3+1)-foliation of space-time and integrate (3.10) over a three-dimensional null-volume $V \subset C$ ($\partial V = S_1 \cup S_2$)

$$\delta \int_V L = \int_V (\pi^{\mu\nu} \delta A_{\mu\nu}^0) \cdot + \int_{\partial V} \pi^{\mu\nu} \delta A_{\mu\nu}^3 \quad (3.18)$$

We use here adapted coordinates; this means that the coordinate x^3 is constant on the boundary ∂V . Adapted coordinates simplify considerably derivation of the final formula. We stress, however, that all our results have an independent, geometric meaning. To rewrite them in a coordinate-independent form it is sufficient to replace “dots” by Lie derivatives \mathcal{L}_X , where X is the vector field generating our one-parameter group of transformations, which we are describing. In adapted coordinates $X := \frac{\partial}{\partial x^0}$. Moreover, the upper index “3” has to be replaced everywhere by the sign “ \perp ”, denoting the transversal component with respect to the world tube. This way our results have a coordinate-independent meaning as relations between well defined geometric objects and not just their specific components.

The following homogeneous generating formula can be directly obtained (cf. [12]) from (3.18)

$$\begin{aligned} 0 &= \int_V \dot{\Pi}_{AB} \delta \psi^{AB} - \dot{\psi}^{AB} \delta \Pi_{AB} - \delta \int_{\partial V} 2V \sin\theta + \\ &+ \frac{1}{2} \int_{\partial V} \sin\theta (rV \gamma_{AB,3} - r^2 \dot{\gamma}_{AB} - 2U_{A||B} + r^2 e^{-2\beta} U_{,3}^C \gamma_{CA} U_B) \delta \gamma^{AB} + \\ &+ \int_{\partial V} 2r^2 \sin\theta \left(\frac{2V}{r^2} - U^B{}_{||B} + U_A U_{,3}^A \right) \delta \beta - r^2 \sin\theta e^{-2\beta} U_A \delta U_{,3}^A \end{aligned} \quad (3.19)$$

where we have introduced the following asymptotic variables (Π_{AB}, ψ^{AB}) analogous to the (π, ψ) describing massless scalar field

$$\begin{aligned} \psi^{AB} &:= r\gamma^{AB} - r \overset{\circ}{\gamma}{}^{AB}, \quad \psi_{AB} := r\gamma_{AB} - r \overset{\circ}{\gamma}{}_{AB} \\ \Pi_{AB} &:= -\frac{1}{2} \sin\theta (r\gamma_{AB})_{,3} + \frac{1}{2} \sin\theta \left(r \overset{\circ}{\gamma}{}_{AB} \right)_{,3} \end{aligned}$$

The equivalent form of the homogeneous formula (3.13) expressed in terms of the natural geometric objects assigned to the world tube $\{x^3 = \text{const.}\}$ ((80) in [11]) can be rewritten as follows

$$0 = \int_V \dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} + 2 \int_{\partial V} \dot{\lambda} \delta \alpha - \dot{\alpha} \delta \lambda + \int_{\partial V} \delta (n^2 Q^{00}) + n^2 \delta Q^{00} - 2n^A \delta Q_A^0 + Q^{AB} \delta g_{AB} \quad (3.20)$$

where Q^{ab} is the tensor density built up from extrinsic curvature of the (one-time-like-two-space-like) world tube in a similar way as ADM momentum for space-like hypersurface, $\lambda := \sqrt{\det g_{AB}}$, $\alpha := \operatorname{arcsinh} \frac{g^{30}}{\sqrt{-g^{00}g^{33}}}$, $n^A := \check{g}^{AB}g_{0B}$ and $n := \sqrt{n_A n^A - g_{00}}$.

The equation (3.20) enables one to compose generating formulae on a space-like and null hypersurfaces along two-dimensional boundary, where they meet. More precisely, let O be a space-like hypersurface with $\partial O = S(u, r_0)$, so $O \cup C$ gives a typical example of such composition. One can directly check in Bondi coordinates that the boundary terms on a sphere $S(u, r_0)$ in (3.20) and (3.19) corresponding respectively to O and C are exactly the same. This way we can write the variational formula on a truncated cone $O \cup C$, which is space-like inside and light-like near Scri. One can also take a space-like hyperboloidal hypersurface Σ_u , which approaches \mathcal{I}^+ in an appropriate way by moving the “sticking together” sphere $S(u, r_0)$ to the null infinity along cone C . The above observations confirm the fact that TB mass is not sensitive on the particular choice of the internal shape of the hypersurface but depends only on its boundary, which is a section of \mathcal{I}^+ (cf. Definition of m_{TB} before Theorem 3.2).

This way, passing to the limit⁷, the composition of the formula (3.19) together with (3.20) takes the form

$$-16\pi\delta m_{TB} = \int_O \dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl} + \int_C \dot{\Pi}_{AB}\delta\psi^{AB} - \dot{\psi}^{AB}\delta\Pi_{AB} - \frac{1}{2} \int_{S_u} \sin\theta\dot{\psi}_{AB}\delta\psi^{AB} \quad (3.21)$$

Similarly to (2.13), one can denote the non-conservation law for the TB mass as follows

$$-16\pi\partial_0 m_{TB} = -\frac{1}{2} \int_{S_u} \sin\theta\dot{\psi}_{AB}\dot{\psi}^{AB} \left(= \frac{1}{2} \int_{S_u} \sin\theta \overset{\circ}{\chi}_{AB,u} \overset{\circ}{\chi}{}^{AB}{}_{,u} \right) \quad (3.22)$$

where the last form in the brackets becomes clear when we apply the asymptotics presented in [12]. In particular, $\psi_{AB}|_{\mathcal{I}^+} = \overset{\circ}{\chi}_{AB}$ and $\psi^{AB}|_{\mathcal{I}^+} = -\overset{\circ}{\chi}{}^{AB}$.

The formula (3.22) is an example of the non-conservation law similar to (2.13). It expresses the central result of the classical paper [1] and is valid in the form (3.22) for much wider asymptotics than considered in the original papers [1], [2], [3]. The property described by this law, namely monotonicity in time for all vacuum field configurations, leads to the uniqueness property of the TB energy, which we summarize in the last section.

Remark. Quasi-local Hamiltonian $\frac{1}{16\pi} \int_{\partial V} n^2 Q^{00}$ renormalized by an additive constant gives simultaneously TB mass and ADM mass, depending on the limit we take.

4. UNIQUENESS OF THE TRAUTMAN-BONDI MASS

In [14] it is shown that the TB energy is, up to a multiplicative constant $\alpha \in \mathbb{R}$, the only functional of the gravitational field, in a certain natural class of functionals, which is

⁷This makes sense even for a polyhomogeneous asymptotics considered in [13].

monotonic in time for all vacuum field configurations which admit (a piece of) a smooth null infinity \mathcal{I}^+ . More precisely, it is shown the following:

Theorem 4.1 *Let H be a functional of the form*

$$H[g, u] = \int_{S^2(u)} H^{\alpha\beta}(g_{\mu\nu}, g_{\mu\nu,\sigma}, \dots, g_{\mu\nu,\sigma_1\dots\sigma_k}) dS_{\alpha\beta}, \quad (4.1)$$

where the $H^{\alpha\beta}$ are twice differentiable functions of their arguments, and the integral over $S^2(u)$ is understood as a limit as r_0 goes to infinity of integrals over the spheres $t = u + r_0, r = r_0$. Suppose that H is monotonic in u for all vacuum metrics $g_{\mu\nu}$ for which H is finite, provided that $g_{\mu\nu}$ satisfies

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \frac{h_{\mu\nu}^1(u, \theta, \varphi)}{r} + \frac{h_{\mu\nu}^2(u, \theta, \varphi)}{r^2} + o(r^{-2}), \\ \partial_{\sigma_1} \dots \partial_{\sigma_i} (g_{\mu\nu} - \frac{h_{\mu\nu}^1(u, \theta, \varphi)}{r} - \frac{h_{\mu\nu}^2(u, \theta, \varphi)}{r^2}) &= o(r^{-2}), \end{aligned} \quad (4.2)$$

with $1 \leq i \leq k$, for some C^k functions $h_{\mu\nu}^a(u, \theta, \varphi)$, $a = 1, 2$. If H is invariant under passive BMS super-translations, then the numerical value of H equals (up to a proportionality constant) the Trautman–Bondi mass.

Theorem 4.1 imposes the further requirement of *passive BMS invariance*, which did not occur in the scalar field case. We note that we believe that the requirement of monotonicity forces the energy to be invariant under (passive) super-translations, but we have not succeeded in proving this so far. The proof of Theorem 4.1 is similar to the proof for the scalar field, but technically rather more involved. A key ingredient of the proof is the Friedrich-Kannar [16], [17] construction of space-times “having a piece of \mathcal{I} ”. Let us finally mention that one can set up a Hamiltonian framework in a phase space, which consists of Cauchy data on hyperboloids together with values of the fields on appropriate parts of Scri, to describe the dynamics in the radiation regime [15]. Unsurprisingly, the Hamiltonians one obtains in such a formalism are again not unique, but the non-uniqueness can be controlled in a very precise way. The Trautman-Bondi mass turns out to be a Hamiltonian, and an appropriate version of the uniqueness Theorem 4.1 can be used to single out the TB mass amongst the family of all possible Hamiltonians. In the Hamiltonian framework the freedom of multiplying the functional by a constant disappears.

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