

# Unconstrained degrees of freedom for gravitational waves, $\beta$ -foliations and spherically symmetric initial data

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## Abstract

A new parameterization of unconstrained degrees of freedom for gravitational field, used in [6], has been generalized to one-parameter family of such parameterizations, depending on a real parameter  $\beta \in [0, 2]$ . The description introduced in [6] corresponds to the special choice  $\beta = 0$ . The method is closely related to the proof of the positivity of the energy presented in [3] where  $\beta$ -foliations have been introduced (see also applications to black holes dynamics in [4], [6] and [7]).

Spherically symmetric initial data corresponding to trivial degrees of freedom is analyzed along these lines. In particular, the quasi-local energy content of the Schwarzschild initial data is analyzed for different choices of the  $\beta$ -gauge.

## 1 Introduction

We consider a compact, smooth, 3-dimensional manifold  $V$ , diffeomorphic to  $K(0, r_0, r_1)$ :

$$K(0, r_0, r_1) := \{ \vec{x} \in \mathbb{R}^3 \mid r_0 \leq \sqrt{\sum_{i=1}^3 (x^i)^2} \leq r_1 \} .$$

Denote by  $\partial V$  its boundary. Limiting cases  $r_0 \rightarrow 0$  and/or  $r_1 \rightarrow \infty$  may be also considered.

Consider the spacetime “tube”  $M = V \times \mathbb{R}^1$  and its boundary  $T = \partial V \times \mathbb{R}^1$  which is a 1-time-like and 2-space-like hyper-surface in our spacetime. Choose coordinates  $(x^\mu)$  on  $M$  in such a way that  $(x^1, x^2)$  are coordinates on  $\partial V$  (e.g. spherical angles  $\theta$

and  $\varphi$ ),  $x^3 = r$  is a „radial” coordinate which is constant on  $\partial V$ . Moreover, denote by  $x^0$  the time coordinate. So we have

$$V_t := \{x \in M : x^0 = t\} = \bigcup_{r \in [r_0, r_1]} S(r) \quad \text{where } S(r) := \{x \in V : x^3 = r\}$$

$$T = \{x \in M : x^3 = r_0\} \cup \{x \in M : x^3 = r_1\}$$

We use the following convention: Greek indices  $\mu, \nu, \dots$  label spacetime coordinates and run from 0 to 3 ; Latin indices  $k, l, \dots$  label space coordinates on  $V$  and run from 1 to 3 ; Capital indices  $A, B, \dots$  label coordinates on  $\partial V$  („spherical angels”) and run from 1 to 2.

Consider Cauchy data  $(g_{kl}, P^{kl})$  for Einstein equations in the 3–dimensional bounded volume  $V_t$  with boundary  $\partial V_t$ . This means that  $g_{kl}$  is a Riemannian metric on  $V_t$  and  $P^{kl}$  is a symmetric tensor density which we identify with the A.D.M. momentum (see [1])

$$P^{kl} = \sqrt{\det g_{mn}} (g^{kl} T_{\mu\nu} n^\mu n^\nu - K^{kl}) .$$

Here,  $K$  is the second fundamental form (external curvature) of the imbedding of  $V_t$  into the space–time  $M$  which we are going to construct.

The 12 functions  $(g_{kl}, P^{kl})$  must fulfill 4 Gauss–Codazzi constraints:

$$(\det g_{mn}) \mathcal{R} - P^{kl} P_{kl} + \frac{1}{2} (P^{kl} g_{kl})^2 = 16\pi (\det g_{mn}) T_{\mu\nu} n^\mu n^\nu , \quad (1)$$

$$P_i{}^l{}_{|l} = 8\pi \sqrt{\det g_{mn}} T_{i\mu} n^\mu . \quad (2)$$

where  $T_{\mu\nu}$  is the energy momentum tensor of the matter. By  $\mathcal{R}$  we denote the (3–dimensional) scalar curvature of  $g_{kl}$ , whereas  $n^\mu$  is a future timelike 4–vector normal to the hypersurface  $V_t$ . The geometric structure used in (1) and (2) (the covariant derivative „|”, rising and lowering of the indices etc.) is the one defined by the 3–metric  $g_{kl}$ .

Einstein equations and the definition of the metric connection imply the first order (in time) differential equations for  $g_{kl}$  and  $P^{kl}$  (see [1] or [2] p. 525) and contain the lapse function  $N$  and the shift vector  $N^k$  as free parameters, canonically conjugate to the four constraints (1) and (2):

$$\dot{g}_{kl} = \frac{2N}{\sqrt{g}} \left( P_{kl} - \frac{1}{2} g_{kl} P \right) + N_{k|l} + N_{l|k} , \quad (3)$$

where  $g := \det g_{mn}$  and  $P := P^{kl} g_{kl}$ ,

$$\begin{aligned} \dot{P}^{kl} = & -N\sqrt{g} \left( \mathcal{R}^{kl} - \frac{1}{2} g^{kl} \mathcal{R} \right) - \frac{2N}{\sqrt{g}} \left( P^{km} P_m{}^l - \frac{1}{2} P P^{kl} \right) + (P^{kl} N^m)_{|m} + \\ & + \frac{N}{2\sqrt{g}} g^{kl} \left( P^{kl} P_{kl} - \frac{1}{2} P^2 \right) - N^k{}_{|m} P^{ml} - N^l{}_{|m} P^{mk} + \sqrt{g} (N^{kl} - g^{kl} N^{|m}{}_{|m}) + \\ & + 8\pi N \sqrt{g} T_{mn} g^{km} g^{ln} . \end{aligned} \quad (4)$$

## 2 Reduced phase space of Cauchy data

We want to analyze Cauchy data in terms of the 2+1 decomposition of the initial surface. To describe the independent degrees of freedom of the gravitational field we use the following objects  $(w^k, s_l)$ :

$$w^k := \lambda g^{3k} (g^{33})^{\beta-1} \quad (5)$$

where  $\lambda := \sqrt{\det g_{AB}}$  is the 2-dimensional volume form and

$$s_A := (w^3)^{-1} P^3{}_A \quad s_3 := - (w^3)^{-1} \left( \frac{1}{2} S + P^3{}_A \frac{g^{3A}}{g^{33}} \right), \quad (6)$$

where  $\tilde{g}^{AB}$  is the 2-dimensional inverse of two-metric  $g_{AB}$ , whereas by  $S$  we denote  $\tilde{g}^{AB} P_{AB}$ . As will be seen later, the data  $(w^k, s_l)$  describe partially reduced phase space of gravitational field.

Denote by  $\sigma_{AB}$  the standard metric on a unit sphere  $S^2$ , ( $\sigma_{AB} dx^A \otimes dx^B = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi$ ) and by  $\sigma = \sqrt{\det \sigma_{AB}} (= \sin \theta)$  the corresponding volume element on the unit sphere. On each sphere  $S(r)$  we introduce the following two-metric

$$\mu_{AB} := \sigma \lambda^{-1} g_{AB}, \quad (7)$$

conformally equivalent to  $g_{AB}$ . Its inverse metric is given by  $\mu^{AB} = \sigma^{-1} \lambda \tilde{g}^{AB}$ .

It is easy to check that the left-hand sides of the vector constraints (2) may be rewritten as follows:

$$P_A{}^k|_k = \left( \lambda \tilde{g}^{AC} S_{AB} \right)_{||C} + (s_A w^k)_{,k} - w^k s_{k,A} + \frac{1}{2} \left( \frac{P^{33}}{g^{33}} + \beta S \right) g^{33},{}_A \quad (8)$$

$$\begin{aligned} \frac{1}{g^{33}} P^{3k}|_k &= \left( \frac{P^{33}}{g^{33}} \frac{w^k}{w^3} \right)_{,k} + \left( \frac{w^3 \tilde{g}^{AB} s_B}{g^{33}} \right)_{,A} - \lambda \tilde{g}^{AC} S_{BC} \left( \frac{w^B}{w^3} \right)_{||A} + \\ &+ \frac{1}{2} S_{AB} \left( \lambda \tilde{g}^{AB} \right)_{,3} + \frac{w^l}{w^3} s_l w^k{}_{,k} + \frac{1}{2} \left( \frac{P^{33}}{g^{33}} + \beta S \right) \frac{w^k}{w^3} g^{33},{}_k \end{aligned} \quad (9)$$

where  $S_{AB} := \lambda^{-1} (P_{AB} - \frac{1}{2} S)$  and “||” denotes 2-dimensional covariant derivative with respect to the 2-metric  $\mu_{AB}$ .

To describe effectively<sup>1</sup> the reduced phase space, i.e. the space of classes of gauge equivalent pairs  $(g_{kl}, P^{kl})$ , one is free to impose four gauge conditions which enable us to pick up a single representative within each gauge-equivalence class.

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<sup>1</sup>The procedure proposed here does not cover the entire phase space but only an open neighbourhood of the flat initial data.

We propose the following conditions:

$$\frac{P^{33}}{g^{33}} + \beta S = 0, \quad (10)$$

$$\left(\frac{w^k}{r^2}\right)_{,k} = 0, \quad (11)$$

$$\mu_{AB} = \sigma_{AB}. \quad (12)$$

It is easy to check that the gauge condition (11) may be rewritten in the following form:

$$\frac{k}{\sqrt{g^{33}}} = \beta \frac{g^{3k}}{g^{33}} (\ln g^{33})_{,k} - \frac{2}{x^3} \quad (13)$$

where  $k$  is the two-dimensional trace of the extrinsic curvature  $k_{AB}$  of  $S(r)$  with respect to the three-metric  $g_{kl}$  on  $V_t$ .

Conditions (11) and (10) describe a specific “2+2” decomposition of spacetime. The two-parameter family of surfaces  $t = x^0 = \text{const.}$ ,  $r = x^3 = \text{const.}$  (topological spheres) is defined in terms of a nonlinear system of partial differential equations imposed on coordinates  $r$  and  $t$ , equivalent to (11) and (10)<sup>2</sup>. However, for  $\beta = 1$  the construction splits into separate equations, the first one gives a maximal three-surface  $t = \text{const.}$  and the second one corresponds to a certain spherical foliation of this maximal surface (conformally harmonic gauge in [3]).

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Condition (12) corresponds to the choice of appropriate conformal coordinates on each two-dimensional sphere  $S(r)$ . This is possible because every two-dimensional topological sphere is conformally equivalent to a unit sphere and (12) describes precisely this equivalence. However, coordinates  $(x^A)$  are not fixed uniquely by the above condition but only up to the six-parameter family of conformal transformations of the unit sphere. We have, therefore, the residual, six-dimensional gauge freedom on each  $S(r)$ . Using this freedom we may annihilate the six-dimensional dipole component of the vector  $w^k$ .

Due to gauge conditions (11) and (12) we can simplify vector constraints as follows:

$$\sigma S_A^B \parallel_B + (s_A w^k)_{,k} - w^k s_{k,A} = 8\pi j_A \quad (14)$$

where  $j_A := \sqrt{\det g_{mn}} T_{A\mu} n^\mu$  and

$$2\beta \left(\frac{s_m w^m w^k}{w^3}\right)_{,k} + \left[\sigma s^A (g^{33})^{\beta-1}\right]_{,A} - \sigma S^A_B \left(\frac{w^B}{w^3}\right) \parallel_A + \frac{2w^l}{r} s_l = 8\pi j_3 \quad (15)$$

where  $j_3 := \sqrt{\det g_{mn}} T_{k\mu} n^\mu \frac{g^{3k}}{g^{33}}$ . Observe that for  $\beta = 1$  the “conformal factor”  $g^{33}$  does not enter into the equation.

Equation (14) is a two-dimensional internal equation on each slice  $S(r)$  separately and it enables one to reconstruct  $S_{AB}$  from the reduced data  $(s_k, w^l)$ . There are,

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<sup>2</sup>They are quite close to the gauge conditions discussed in [10].

however, additional constraints which must be fulfilled by this data. These constraints are visible if we contract (14) with an arbitrary generator  $\xi^A$  of the six-dimensional conformal group on  $S(r)$ . Such a generator fulfills the conformal Killing equation

$$\xi_{A||B} + \xi_{B||A} = \sigma_{AB} \xi^C{}_{||C},$$

and, whence,

$$\sigma S_A{}^B{}_{||B} \xi^A = \left( \sigma S_A{}^B \xi^A \right)_{||B} = \partial_B \left( \sigma S_A{}^B \xi^A \right).$$

The integral of this expression over  $S(r)$  vanishes identically. Consequently, we have six residual constraints imposed on the reduced data  $(s_k, w^l)$  on each sphere  $S(r)$ :

$$\int_{S(r)} \xi^A \left[ (s_A w^k)_{,k} - w^k s_{k,A} \right] = 8\pi \int_{S(r)} \xi^A j_A. \quad (16)$$

They enable us to calculate the (six-dimensional) dipole part of  $s_A$ , canonically conjugate to the dipole part of  $w^A$ , which was annihilated by the residual gauge condition.

Equation (15) provides a relation between variables  $s_k$ , which are no longer independent parameters. If we know  $s_A$  and  $w^3 \neq 0$  than (15) can be viewed as an equation for the unknown function  $s_3$  with given  $s_A$ . Similarly, the gauge condition (11) enables us to calculate  $w^3$  once we know  $w^A$ .

We conclude that, similarly as in [6], the (partially) reduced canonical data  $(w^k, s_l)$ , fulfilling the gauge condition (11) and the residual gauge condition (the dipole part of  $w^A$  vanishes) contain all the information about the complete Cauchy data  $(P^{kl}, g_{kl})$  satisfying the constraint equations (2), (1) and gauge conditions (11), (12). In particular, to calculate the ‘‘conformal factor’’  $g^{33}$  we must solve the scalar constraint (1), which can be rewritten as follows (see also [6]):

$$\begin{aligned} & 2 \left( \frac{\lambda w^k}{w^3} k \right)_{,k} - 2 \partial_A \left[ \lambda \tilde{g}^{AB} \left( \frac{1}{\sqrt{g^{33}}} \right)_{,B} \right] + \frac{\lambda}{\sqrt{g^{33}}} \left( R + \frac{1}{2} k^2 \right) = \\ & = \frac{\sqrt{g^{33}}}{\lambda} \left( P_{kl} P^{kl} - \frac{1}{2} P^2 \right) + \frac{\lambda}{\sqrt{g^{33}}} \left( k_{AB} k^{AB} - \frac{1}{2} k^2 + 16\pi\rho \right) \end{aligned} \quad (17)$$

where  $\rho := T_{\mu\nu} n^\mu n^\nu$  is a matter density,  $k_{AB}$  is an extrinsic curvature of two-surface  $S(r)$  and  $R$  is a scalar curvature of induced two-metric  $g_{AB}$  (see [3], [4] and [6]).

Let us notice that the following identity holds:

$$\begin{aligned} P_{kl} P^{kl} - \frac{1}{2} P^2 &= \frac{1}{2} \left( \frac{P^{33}}{g^{33}} \right)^2 + \frac{2}{g^{33}} \tilde{g}^{AB} P^3{}_A P^3{}_B + \lambda \tilde{g}^{AC} \lambda \tilde{g}^{DB} S_{AB} S_{CD} - \frac{P^{33}}{g^{33}} S = \\ &= \frac{2}{g^{33}} (w^3)^2 \tilde{g}^{AB} s_A s_B + \lambda \tilde{g}^{AC} \lambda \tilde{g}^{DB} S_{AB} S_{CD} + 2\beta(\beta + 2) (w^k s_k)^2 \end{aligned} \quad (18)$$

where the last equality is implied by the gauge condition  $\frac{P^{33}}{g^{33}} + \beta S = 0$ . Let us observe that (18) is nonnegative for  $\beta \geq 0$ . Moreover, due to gauge condition (11) and identity (18), the scalar constraint (17) takes the following form:

$$\begin{aligned}
& 2 \left[ \left( \sqrt{g^{33}} k + \frac{g^{33}}{r} \right) \frac{\lambda w^k}{w^3} \right]_{,k} + \lambda R + \left[ \lambda \tilde{g}^{AB} (\log g^{33})_{,B} \right]_{,A} = \frac{2}{\lambda} (w^3)^2 \tilde{g}^{AB} S_A S_B + \\
& + \frac{g^{33}}{\lambda} \left( \lambda \tilde{g}^{AC} \lambda \tilde{g}^{DB} S_{AB} S_{CD} + 2\beta(\beta + 2)(w^k s_k)^2 \right) + \frac{\lambda}{2g^{33}} \beta(2 - \beta) \left( \frac{w^k}{w^3} g^{33} \right)_{,k}^2 + \\
& + \lambda \left( k_{AB} k^{AB} - \frac{1}{2} k^2 + \frac{1}{2} \tilde{g}^{AB} (\log g^{33})_{,A} (\log g^{33})_{,B} + 16\pi\rho \right). \tag{19}
\end{aligned}$$

It is easy to check that for  $\beta \in [0, 2]$  the right-hand side is nonnegative. This observation enabled us to show the positivity of ADM mass (see [3]). For this purpose we observed that the integral of the left hand side over the whole  $K(0, r_0, r_1)$  for  $r_0 \rightarrow 0$  and  $r_1 \rightarrow \infty$  gives the surface term at infinity, proportional to the ADM mass.

### 3 Symplectic structure and the complete reduction

Let us observe the following identity

$$-P^{kl} dg_{kl} = 2s_k dw^k + \sigma S_{AB} d\mu_{AB} + \left( \frac{P^{33}}{g^{33}} + \beta S \right) d \ln g^{33} \tag{20}$$

which may be easily checked by direct computation, using definitions (5), (6) and (7). It leads to the partial reduction of the phase space if we impose gauge conditions (10) and (12). More precisely, the last two terms drop out and we are left with

$$-P^{kl} dg_{kl} = 2s_k dw^k.$$

Defining

$$\frac{w^k}{r^2} = D^k,$$

and

$$2r^2 \cdot s_k = A_k,$$

we get the following symplectic structure

$$-P^{kl} dg_{kl} = A_k dD^k, \tag{21}$$

together with the constraint (11), which now reads:

$$\partial_k D^k = 0.$$

This structure is formally equivalent to the structure of classical electrodynamics, where  $D^k$  is the electric displacement vector-density and  $A_k$  is the vector potential for the magnetic field. Constraint (15) plays a role of the non-linear ‘‘Coulomb gauge condition’’ in electrodynamics. The analogy is not complete, because here we have also the residual gauge condition (dipole part of  $D^k$  vanishes), dual to the residual constraints (16).

Further reduction of the phase space  $(D^k, A_l)$  may be performed if we observe that only two variables among  $D^k$  and the two variables among  $A_l$  are independent. Equations (11) and (15), together with the scalar constraint (1), define a subspace in the space of variables  $(D^k, A_l)$ , corresponding to the two degrees of freedom of the gravitational field. To parameterize this space by independent variables, we may represent

$$A_l = \tilde{A}_l + \partial_l \phi ,$$

and impose a further gauge condition:

$$\tilde{A}_B = \varepsilon_{BC} \mu^{CD} \partial_D \varphi .$$

Inserting this *Ansatz* into (21) and integrating over the entire  $V$  we obtain the following symplectic form:

$$\Omega = \int_V \left( d D^3 \wedge d \tilde{A}_3 + d (\varepsilon^{BC} \partial_B D_C) \wedge d \varphi \right) .$$

It is obvious that the above four functions contain the entire information about  $(D^k, \tilde{A}_l)$ . To reconstruct the physical data  $(P^{kl}, g_{kl})$  we must solve constraint equations. In particular, equation (15) is a three-dimensional, elliptic equation for the function  $\phi$ .

## 4 Spherically symmetric initial data

Let us assume that our initial data are spherically symmetric and the gauge condition is compatible with spherical foliation. This simply means that  $s_A = w_A = 0$  and moreover  $w^3 = r^2 \sigma$  which is consistent with (11). The energy–momentum tensor is no longer free, we have that  $j_A = 0$ ,  $T_{3A} = 0$  and moreover that the traceless part of  $T_{AB}$  vanishes:

$$T_{AB} - \frac{1}{2} g_{AB} \tilde{g}^{CD} T_{CD} = 0$$

The vector constraints (14), (15) simplify drastically:

$$\sigma S_A^B{}_{||B} - w^3 s_{3,A} = 0 \tag{22}$$

$$2\beta \left( s_3 w^3 \right)_{,3} + \frac{2w^3}{r} s_3 = j_3 \tag{23}$$

Assuming that  $\partial_A j_3 = 0$  (spherical symmetry of  $j_3$ ) we deduce from (23) that  $\partial_A s_3 = 0$  and from (22) we get  $S_{AB} = 0$ .

The scalar constraint is also very simple:

$$\begin{aligned} & 2 \left[ \left( \sqrt{g^{33}}k + \frac{g^{33}}{r} \right) \lambda + r\sigma \right]_{,3} = \\ & = 2\beta(\beta + 2) \frac{g^{33}}{\lambda} (w^3 s_3)^2 + \frac{\lambda}{2g^{33}} \beta(2 - \beta) \left( g^{33}{}_{,3} \right)^2 + 16\pi\lambda\rho \end{aligned} \quad (24)$$

#### 4.1 Special cases $\beta = 0, \frac{1}{2}, 1$ and the Schwarzschild initial data

We would like to compare different expressions for the Schwarzschild metric, which we obtain for different  $\beta$ -foliations.

##### 4.1.1 conformal gauge $\beta = 1$

Let us start with conformally flat representation of the Schwarzschild initial data three-metric:

$$ds^2 = \left( 1 + \frac{m}{2r} \right)^4 \left( dr^2 + r^2 \sigma_{AB} dx^A dx^B \right), \quad \sigma = \sqrt{\det \sigma_{AB}}. \quad (25)$$

Obviously  $r = 0$  corresponds to the second spatial end. One can easily check that

$$w^3 = r^2 \sigma, \quad \sqrt{g^{33}} = \left( 1 + \frac{m}{2r} \right)^{-2}, \quad -\frac{k}{\sqrt{g^{33}}} = \frac{2(r - \frac{m}{2})}{r(r + \frac{m}{2})}$$

and

$$\mathcal{S} := 2 \left[ \left( \sqrt{g^{33}}k + \frac{g^{33}}{r} \right) \lambda + r\sigma \right] = 4m\sigma \frac{r}{r + \frac{m}{2}} = \begin{cases} 2m\sigma & \text{for } r = \frac{m}{2} \\ 4m\sigma & \text{for } r = \infty \end{cases}.$$

Let us observe that the surface integral

$$\frac{1}{16\pi} \int_{S(r)} \mathcal{S}, \quad (26)$$

which we obtain when integrating the scalar constraint (24) over  $V$ , gives half of the ADM mass when calculated on the minimal surface  $S(r = \frac{m}{2})$  and the entire ADM for  $S(r = \infty)$ . This means that the “energy density” – the right-hand side of the scalar constraint (24) – splits into a half which is contained outside the horizon and another half, hidden inside the horizon. The ADM mass seen at both space ends is equal to each other.

##### 4.1.2 harmonic gauge $\beta = 1/2$

Let us denote by  $R = x^3$  the solution of equation (11) for  $\beta = 1/2$ , to avoid confusion with conformal coordinate  $r$  introduced in the previous subsection. It is easy to check that  $R = r + \frac{m}{2}$ . This implies the following:

$$2 \left[ \left( \sqrt{g^{33}}k + \frac{g^{33}}{R} \right) \lambda + R\sigma \right] = 4m\sigma \frac{r + \frac{m}{4}}{r + \frac{m}{2}} = \begin{cases} 3m\sigma & \text{for } r = \frac{m}{2} \\ 4m\sigma & \text{for } r = \infty \end{cases}$$

Hence, (26) gives 3/4 of the ADM mass when calculated on the minimal surface (and, again, the entire ADM for  $S(r = \infty)$ ).

### 4.1.3 inverse mean curvature flow $\beta = 0$

This particular case has been already analyzed in [6] and [7]. The gauge condition (11) or, equivalently, equation (13), corresponds to the so-called inverse mean curvature flow (see [9]) and its solution enables one to prove the Penrose inequality (see [11] and references therein). In this case equation (11) implies that the coordinate  $x^3 = r \left(1 + \frac{m}{2r}\right)^2$  is the usual Schwarzschild coordinate. Moreover, for the Schwarzschild initial data and  $\beta = 0$  the surface integral (26) is constant on each sphere  $S(r)$  and plays a role of a *quasi-local mass*.

## 5 Conclusions

We hope that the description of unconstrained initial data for gravity presented in this short article may be useful for such applications like:

- Construction of initial data for numerical analysis (see e.g. [13])
- Description of the so-called dynamical horizons (see e.g. [12])
- Description of initial data which are sufficiently close to spherical symmetry (one can use analysis based on different  $\beta$ -foliations, analogous to the one presented in [6] for  $\beta = 0$  foliation)
- Construction of trapped surfaces (see e.g. [8])

and in other cases which are not yet discovered or not known to the authors.

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