Boundary data in canonical gravity and thermodynamics of black holes

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Abstract

Characteristic boundary value problem for the scalar wave equation and for the Einstein's theory of gravity is discussed in terms of the Hamiltonian formalism. The symplectic description of the field data on a null-like hypersurface (a wave front) is used for the quasi-local derivation of the first law of black holes thermodynamics. The zeroth law and Penrose inequalities are discussed from this point of view.

1 Introduction

Contrary to its name, the so called "first law of thermodynamics of black holes" is a purely statical formula. There is no dynamics there. The "infinitesimal change of mass" $\delta\mathscr{M}$ is usually understood as the effect of some adiabatical process, where the system moves infinitesimally slowly and its state is always an equilibrium state.

In the present paper we derive the dynamical Hamiltonian formula for the evolution of the gravitational field within a spacetime region with a light-like boundary and show that the first law of thermodynamics of black holes is a special case of this formula. We believe that this result is a step

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towards a consistent description of the dynamics (and not only statics) of black holes.

Correct approach to the Hamiltonian field theory needs always an appropriate control of boundary data. For the reader's convenience, we illustrate our approach on the simplest case of a linear scalar field theory. We make a further simplification assuming that the "spacetime" is two-dimensional. This way we obtain the theory of an elastic string. Field configuration of the string is described by its displacement function: $\mathbb{R} \times [a, b] \ni (t, x) \to \varphi(t, x) \in \mathbb{R}$, fulfilling the dynamical wave equation:

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi = \frac{\partial^2}{\partial x^2} \varphi \ , \tag{1}$$

where velocity "c" is a combination of the string's proper density (per unit length) and its elasticity coefficient. Passing to appropriate time and length units, we may always put $c = 1$. The above dynamics may be derived from the Lagrangian density

$$L = -\frac{1}{2} \sqrt{|\det g|}\ g^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) = \frac{1}{2} \left((\dot{\varphi})^2 - (\varphi')^2 \right) \ , \tag{2}$$

where $\mu, \nu = 0, 1$ and $(x^0, x^1) = (t, x)$, $g_{\mu\nu} = \mathrm{diag}(-1, +1)$, "dot" denotes the time derivative and "prime" denotes the space derivative. Euler-Lagrange equation implies that the "volume part" of the variation δL of the Lagrangian vanishes and, therefore, δL reduces to the "boundary part" or a "complete divergence":

$$\delta L(\varphi, \partial_\nu \varphi) = \partial_\mu \left(p^\mu \delta \varphi \right) = \left(\partial_\mu p^\mu \right) \delta \varphi + p^\mu \delta \left(\partial_\mu \varphi \right) \ . \tag{3}$$

Entire information about the dynamics of the string is encoded in the above *Lagrangian generating formula* (see [15] or [5] for its correct symplectic interpretation). It contains the definition of the canonical momenta:
1) the kinetic momentum

$$\pi := p^0 = \frac{\partial L}{\partial(\partial_0 \varphi)} = \partial_0 \varphi = \dot{\varphi} \ , \tag{4}$$

and 2) the stress density

$$\pi^\perp := p^1 = \frac{\partial L}{\partial(\partial_1 \varphi)} = -\partial_1 \varphi = -\varphi' \ ,$$

together with the Euler-Lagrange equation, obviously equivalent to (1):

$$\partial_\mu p^\mu = \frac{\partial L}{\partial \varphi} = 0 \ .$$

Integrating infinitesimal generating formula (3) over the entire string $[a, b]$ we obtain the finite generating formula:

$$\delta \int_a^b L = \int_a^b \left(\dot{\pi} \delta \varphi + \pi \delta \dot{\varphi} \right) + \left[\pi^\perp \delta \varphi \right]_a^b \ . \tag{5}$$

Hamiltonian description of the same dynamics is obtained *via* Legendre transformation between π and $\dot{\varphi}$, using $\pi\delta\dot{\varphi} = \delta(\pi\dot{\varphi}) - \dot{\varphi}\delta\pi$:

$$-\delta\mathscr{H} = \int_a^b (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi) + \left[\pi^{\perp}\delta\varphi\right]_a^b \ , \tag{6}$$

with

$$\mathscr{H} := \int_a^b (\pi\dot{\varphi} - L) = \frac{1}{2}\int_a^b (\pi^2 + (\varphi')^2) \ . \tag{7}$$

This formal expression acquires a precise, infinitely-dimensional, Hamiltonian meaning:

$$\dot{\pi} = -\frac{\delta\mathscr{H}}{\delta\varphi} \quad , \quad \dot{\varphi} = \frac{\delta\mathscr{H}}{\delta\pi} \quad , \tag{8}$$

as soon as the boundary terms in (6) are killed by imposing some boundary conditions. As an example consider the Dirichlet boundary conditions, i.e. restrict ourselves to an infinitely dimensional phase space of Cauchy data (φ, π), defined on $[a, b]$ and fulfilling conditions: $\varphi(a) \equiv A$, $\varphi(b) \equiv B$. Within this phase space we have $\delta\varphi(a) = \delta\varphi(b) = 0$ and equations (8) hold.

Consider now the subspace of static solutions: $\dot{\pi} = 0 = \dot{\varphi}$. Due to (8), these are points where the derivative of the functional $\mathscr{H}$ vanishes and the Hamiltonian formula (6) reduces to the formula for virtual work

$$\delta\mathscr{H} = -\left[\pi^{\perp}\delta\varphi\right]_a^b \ , \tag{9}$$

But, due to (7), $\mathscr{H}$ is manifestly convex. This implies that every static solution gives the minimal value of the Hamiltonian in the corresponding phase space. Due to equation (1) and to boundary conditions, such a solution is given by: $\pi \equiv 0$ and $\varphi(x) = A + (x - a)\frac{B-A}{b-a}$. Inserting this value into (7) we obtain the following "Penrose-like inequality":

$$\frac{(B - A)^2}{b - a} \leq \mathscr{H} \ , \tag{10}$$

analogous to the gravitational Penrose inequality relating the energy carried by Cauchy data outside of a horizon S and the energy of a black hole corresponding to the same value of appropriate boundary data on S.

Instead of controlling the string configuration at the boundary, we may control e.g. its stress by applying an appropriate force F. This leads to the Neumann control mode $\pi^{\perp}(a) = F_{\text{left}}$, $\pi^{\perp}(b) = F_{\text{right}}$, which is again a legitime Hamiltonian system

$$-\delta\widetilde{\mathscr{H}} = \int_a^b (\dot{\pi}\ \delta\varphi - \dot{\varphi}\ \delta\pi) - \left[\varphi\ \delta\pi^{\perp}\right]_a^b \ , \tag{11}$$

where $\widetilde{\mathscr{H}}$, obtained *via* Legendre transformation between φ and $\pi^\perp$ at the boundary, plays role of a free energy:

$$\widetilde{\mathscr{H}} := \mathscr{H} + \left[\varphi\ \pi^\perp\right]_a^b = \mathscr{H} - \left[\varphi\ \varphi'\right]_a^b = \frac{1}{2}\int_a^b (\pi^2 - (\varphi')^2 - 2\varphi\varphi'') \ . \quad (12)$$

Again, the boundary term in (11) vanishes due to Neumann conditions and the field dynamics reduces to (8). However, the new Hamiltonian (12) is obviously non-convex. There is, therefore, no "Penrose-like" inequality in this mode: static solutions are stationary points of the Hamiltonian $\widetilde{\mathscr{H}}$ but *are not* minimal points of such a "free energy".

The above example shows that the Hamiltonian formulation of the field theory needs an appropriate control of the boundary data, which are usually neglected in standard formulations of the Hamiltonian formalism (cf. [3]). To be able to derive first law of thermodynamics of black holes as an analog of the static formula (9), we need a formalism which incorporates boundary integrals into the Hamiltonian formulation. Such a formalism was proposed in [15] and [5]. Using this approach, evolution of gravitational field within a finite tube with a *time-like boundary* was described in [13] and then reformulated in [14] in a way very much analogous to the above formulation of the string theory. For this purpose the so called "affine variational principle" was used, where the Lagrangian function depends on the Ricci tensor of a spacetime connection Γ. In this picture, the metric tensor g arises only in the Hamiltonian formulation as the momentum canonically conjugate to Γ. Later, it was proved in [14] that the Hamiltonian dynamics obtained this way is universal and does not depend upon a specific variational formulation we start with (actually, it can be derived from field equations only, without any use of variational principles, the existence of them being a consequence of the "reciprocity" of Einstein equations – see [15] and [5]). On the contrary, the Hamiltonian picture is very sensitive to the method of controlling the boundary data. A list of natural control modes, leading to different "quasilocal Hamiltonians", is given in [14]. A conjecture about the "true mass", based on an analysis of the linearized theory [11], was also formulated there.

Unfortunately, the above results are not sufficient for purposes of thermodynamics of black holes, because the boundary in question is the *black-hole horizon*, whose metric is degenerate. We need, therefore, a generalization of these results to the case of a *wave front* (a three-dimensional submanifold whose internal metric is degenerate). Restricting our result to the special case of wave fronts, namely to non-expanding horizons, we obtain the first law of thermodynamics of black holes as a simple consequence.

2 Hamiltonian description of a mixed Cauchy-characteristic initial value problem

Controlling data on a wave front (which is a characteristic surface of the dynamical equation) is a delicate subject which, up to our knowledge, has never been described in a Hamiltonian formalism. To illustrate our approach, we first continue discussion of the linear scalar field theory, whose structure is much simpler, but the essential problems are similar to those of General Relativity. In the next Section we show how to apply these ideas to the case of gravity theory and to derive thermodynamics of black holes (a more detailed presentation of these results will be given in [6]).

Consider dynamics $(\partial^2/\partial t^2 - \Delta)\phi = 0$ of the scalar field ϕ, within a past-oriented light cone in a four-dimensional space-time

$$\mathscr{C}^- = \{(t,x) : x \in \mathbb{R}^3, \quad \|x\| < -t\} \ ,$$

where by $\| \ \|$ we denote Euclidean norm in $\mathbb{R}^3$. To describe Cauchy data on the surface $\{t = \text{const}\}$ within the interior of the cone $\mathscr{C}^-$ we must be able to identify these surfaces for different times. Let us introduce for this purpose new coordinates $(\xi^\mu) = (\tau, \xi)$ (where $\mu = 0, \dots 3$), related to the Minkowskian coordinates $(x^\mu) = (t, x)$ in the following way:

$$t = -e^{-\tau} \ , \tag{13}$$

$$x^k = \xi^k e^{-\tau} \ , \tag{14}$$

where $\tau \in \mathbb{R}^1$. For $\|\xi\| \leq 1$, the new coordinates parameterize the entire cone $\mathscr{C}^-$. To derive the Hamiltonian description of the wave equation in these coordinates, we use the textbook procedure, based on the standard, relativistic-invariant Lagrangian (the four-dimensional version of (2))

$$\mathbf{L} = L \, \mathrm{d}^4 x \ , \tag{15}$$

where

$$L = -\frac{1}{2}\sqrt{|\det g|} g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) = \frac{1}{2}\left\{(\partial_t \phi)^2 - (\nabla\phi)^2\right\} \ . \tag{16}$$

We rewrite this Lagrangian in the coordinates (τ, ξ), using the following formulae which may be easily derived from (13) and (14):

$$\partial_t = e^\tau \left(\partial_\tau + \xi^k \partial_{\xi^k}\right) \ ,$$

$$\partial_{x^k} = e^\tau \partial_{\xi^k} \ .$$

Moreover, we have

$$\mathrm{d}^4 x = \mathrm{d}t \mathrm{d}^3 x = e^{-4\tau} \mathrm{d}\tau \mathrm{d}^3 \xi = e^{-4\tau} \mathrm{d}^4 \xi \ .$$

Expressing the Lagrangian (15) in terms of new coordinates we thus obtain:

$$\mathbf{L} = \mathscr{L} \, \mathrm{d}^4 \xi \ , \tag{17}$$

where

$$\mathscr{L} = \frac{1}{2} e^{-2\tau} \left\{ \left(\frac{\partial \phi}{\partial \tau} + \xi^k \frac{\partial \phi}{\partial \xi^k} \right)^2 - \delta^{kl} \frac{\partial \phi}{\partial \xi^k} \frac{\partial \phi}{\partial \xi^l} \right\} . \tag{18}$$

The Lagrangian density $\mathscr{L}$ depends explicitly on the time variable τ, but if we re-scale the field according to:

$$\varphi := e^{-\tau} \phi ,$$

then this dependence disappears:

$$\mathscr{L} = \frac{1}{2} \left\{ \left(\frac{\partial \varphi}{\partial \tau} + \varphi + \xi^k \frac{\partial \varphi}{\partial \xi^k} \right)^2 - \delta^{kl} \frac{\partial \varphi}{\partial \xi^k} \frac{\partial \varphi}{\partial \xi^l} \right\} , \tag{19}$$

leading to an *autonomous* Hamiltonian system. The standard derivation of this system, valid for an arbitrary Lagrangian density $\mathscr{L} = \mathscr{L}(\varphi, \varphi_\mu, \xi^\mu)$, where $\varphi_\mu := \partial_\mu \varphi$, proceeds as follows: We introduce generalized momenta:

$$\pi^\mu := \frac{\partial \mathscr{L}}{\partial \varphi_\mu} , \tag{20}$$

and calculate the variation of the Lagrangian:

$$\delta \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \varphi} \delta \varphi + \pi^\mu \delta \varphi_\mu = \partial_\mu \left(\pi^\mu \delta \varphi \right) + \left(\frac{\partial \mathscr{L}}{\partial \varphi} - \partial_\mu \pi^\mu \right) \delta \varphi . \tag{21}$$

The field equation $\Box \phi = 0$ is equivalent to the vanishing of the Euler–Lagrange term in (21):

$$\frac{\partial \mathscr{L}}{\partial \varphi} - \partial_\mu \pi^\mu = 0 , \tag{22}$$

and, therefore, is equivalent to the following equation, analogous to (3) in the string theory:

$$\delta \mathscr{L} = \partial_\mu \left(\pi^\mu \delta \varphi \right) = \left(\pi \delta \varphi \right)^{\cdot} + \partial_{\xi^k} \left(\pi^k \delta \varphi \right) , \tag{23}$$

where "dot" denotes derivative with respect to the new time variable τ. We have introduced the momentum $\pi := \pi^0$ (similar to (4)), which provides the remaining piece of Cauchy data on the surface. Using formula (18) for the value of the Lagrangian we obtain:

$$\pi = \pi^0 = \frac{\partial \mathscr{L}}{\partial f_0} = \partial_\tau f + f + \xi^k \frac{\partial f}{\partial \xi^k} = \dot{\varphi} + \varphi + \xi^k \frac{\partial \varphi}{\partial \xi^k} , \tag{24}$$

$$\pi^k = \frac{\partial \mathscr{L}}{\partial f_k} = \xi^k \left(\partial_\tau f + f + \xi^l \frac{\partial f}{\partial \xi^l} \right) - \delta^{kl} \frac{\partial f}{\partial \xi^l} = \xi^k (\dot{\varphi} + \varphi) - \left(\delta^{kl} - \xi^k \xi^l \right) \frac{\partial \varphi}{\partial \xi^l} , \tag{25}$$

$$\frac{\partial \mathscr{L}}{\partial f} = \partial_\tau f + f + \xi^k \frac{\partial f}{\partial \xi^k} = \dot{\varphi} + \varphi + \xi^k \frac{\partial \varphi}{\partial \xi^k} . \tag{26}$$

To derive the Hamiltonian description of the above dynamics, we first integrate (23) over the volume $V_{\rm int} := \{\xi : \|\xi\| \leq 1\}$ in the Cauchy surface $\Sigma = \{\tau = \text{const.}\}$. This way we obtain an identity valid for fields satisfying wave equation:

$$\begin{aligned} \delta \int_{V_{\rm int}} \mathscr{L}\, d^3\xi &= \int_{V_{\rm int}} (\pi\delta\varphi)^{\cdot}\ d^3\xi + \int_{\partial V_{\rm int}} \left(\pi^k \delta\varphi\right)\ \mathrm{d}^2\sigma_k \\ &= \int_{V_{\rm int}} (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi + \delta(\pi\dot{\varphi}))\ \mathrm{d}^3\xi + \int_{\partial V_{\rm int}} \pi^{\perp}\delta\varphi\ . \quad (27) \end{aligned}$$

Similarly as in string theory (cf. equations (5) and (6)) we may perform the Legendre transformation between $\dot{\varphi}$ and π. This leads to the following formula (see [5] for the correct definition of the underlying mathematical structure):

$$-\delta H_{\rm int}(\varphi, \pi) = \int_{V_{\rm int}} (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi)\ d^3\xi + \int_{\partial V_{\rm int}} \pi^{\perp}\delta\varphi\ . \qquad (28)$$

where

$$H_{\rm int}(\varphi, \pi) := \int_{V_{\rm int}} (\pi\dot{\varphi} - \mathscr{L})\ . \qquad (29)$$

Using (24), the Hamiltonian (29) may be written explicitely in terms of Cauchy data on $V_{\rm int}$:

$$\begin{aligned} H_{\rm int}(\varphi, \pi) &:= \int_{V_{\rm int}} \left\{ \pi \left(\pi - \varphi - \xi^k \frac{\partial\varphi}{\partial\xi^k} \right) - \mathscr{L} \right\} \mathrm{d}^3\xi \\ &= \frac{1}{2}\int_{V_{\rm int}} \left\{ \pi^2 - 2\pi\xi^k \frac{\partial\varphi}{\partial\xi^k} - 2\pi\varphi + (\nabla_\xi\varphi)^2 \right\} \mathrm{d}^3\xi\ . \quad (30) \end{aligned}$$

Euler-Lagrange equation (22) implies the following Hamiltonian equations:

$$\dot{\varphi} = \pi - \varphi - \xi^k \frac{\partial\varphi}{\partial\xi^k}\ , \qquad (31)$$

$$\begin{aligned} \dot{\pi} &= \frac{\partial\mathscr{L}}{\partial f} - \partial_k\pi^k = -2\dot{\varphi} - 2\varphi - \xi^k\partial_k\dot{\varphi} + \partial_k \left[\left(\delta^{kl} - \xi^k\xi^l \right) \frac{\partial\varphi}{\partial\xi^l} \right] \\ &= -2\pi - \xi^k\partial_k\pi + \triangle_\xi\,\varphi\ . \quad (32) \end{aligned}$$

It is easy to see that they coincide with Hamilton equations derived directly from (28) and (30).

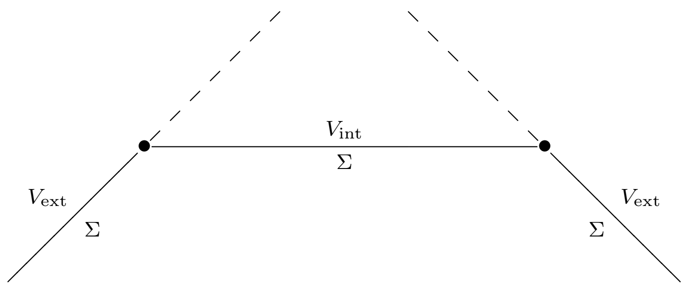
Similarly as in Section 1, the field evolution described by the above equations is not well defined (i. e. the initial value problem not well posed) unless we control appropriately boundary conditions. But, because the boundary $\partial\mathscr{C}^-$ is a characteristic surface of the wave equation, the situation differs considerably from the one described previously. Indeed, having chosen Cauchy data (φ, π) on $V_{\rm int}$ at a given instant of time, say τ_0, we still have the freedom to chose boundary data on $\partial V_{\rm int}$, *i.e.*, the values $\varphi(\tau, \xi)$ on a sphere $\{\|\xi\| = 1\}$, but *only for* $\tau \leq \tau_0$. Indeed, for $\tau \geq \tau_0$, the Cauchy data alone determine uniquely the field φ inside the light cone, whereas for $\tau \leq \tau_0$ the field may be uniquely obtained by solving the (well posed) mixed "characteristic + Cauchy" problem backward in time. This anti-symmetry contradicts apparently the Hamiltonian analysis, where initial value problem must be well posed both towards the future and the past.

The remedy for these difficulties consists in treating the data on the boundary of the light cone not as *boundary* data, but as a further piece of Cauchy data. For this purpose, we extend the parameterization (13)-(14) to $V_{\text{ext}} := \{\xi : \|\xi\| \geq 1\}$ setting:

$$t := -e^{-\tau+\|\xi\|-1} \quad \text{for } \|\xi\| > 1 \, , \tag{33}$$

$$x^k := \frac{\xi^k}{\|\xi\|} e^{-\tau+\|\xi\|-1} \quad \text{for } \|\xi\| > 1 \, , \tag{34}$$

and we consider the data (φ, π) on the entire surface $\Sigma = V_{\text{int}} \cup V_{\text{ext}}$.



Within V_{ext} we have $X = Y$, where

$$X := \partial_\tau \, , \qquad Y := -\frac{\xi^k}{\|\xi\|}\partial_{\xi^k} \, ,$$

and, whence, the dynamics consists in transporting the field data over the surface Σ along Y. Field equations reduce, therefore, to the Lie derivatives:

$$\dot{\varphi} = \pounds_X \varphi = \pounds_Y \varphi = -\frac{\xi^k}{\|\xi\|}\partial_{\xi^k}\varphi \tag{35}$$

$$\dot{\pi} = \pounds_X \pi = \pounds_Y \pi = -\partial_{\xi^k}\left(\frac{\xi^k}{\|\xi\|}\pi\right) \tag{36}$$

where (36) follows from the fact that the momentum is not a scalar field (like φ) but a scalar density. The above equations can be also derived from the standard Hamiltonian formula:

$$H_{\text{ext}}(\varphi, \pi) = \int_{\|\xi\|\geq 1} (\pi\dot{\varphi} - \mathscr{L}_{\text{ext}})\, \mathrm{d}^3\xi = \int_{\|\xi\|\geq 1} (\pi \pounds_Y \varphi)\, \mathrm{d}^3\xi \, . \tag{37}$$

where $\mathscr{L}_{\text{ext}}$ vanishes identically as a pull-back of the scalar density L via the degenerate coordinate transformation (33)-(34). Indeed, variation of the above Hamiltonian gives:

$$-\delta H_{\text{ext}}(\varphi, \pi) = \int_{V_{\text{ext}}} (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi)\ \mathrm{d}^3\xi + \int_{\partial V_{\text{ext}}} \pi^{\perp}\delta\varphi \, , \tag{38}$$

where $(\dot{\varphi}, \dot{\pi})$ are given by (35) and (36), whereas the boundary term comes from the integration by parts.

According to the standard procedure, described in the chapter 3 of [5], the momentum π on the Cauchy surface Σ is equal to the pull-back of the differential (odd) form $\pi^\mu \partial_\mu \lfloor \mathrm{d}\xi^0 \wedge \mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3$, with π^μ is given by formula (20) and coordinates ξ^μ are given by (13)-(14). Applying this formula to V_{ext} we obtain:

$$\pi = -\frac{1}{\|\xi\|^2}\left(\frac{\xi^k}{\|\xi\|}\frac{\partial\varphi}{\partial\xi^k} - \varphi\right) , \tag{39}$$

which provides the Hamiltonian constraint between the configuration φ and the momentum π. Moreover, $\pi^\perp = \pi$, as the pull-back of the same form to the surface $\{\|\xi\| = \text{const.}\} = \{\tau = \text{const.}\} = \Sigma$.

The phase space of Cauchy data on the entire Σ is, therefore, described by pairs (φ, π) defined on the whole $\mathbb{R}^3$ and fulfilling constraint (39) for $\|\xi\| \geq 1$. Moreover, φ must be continuous, whereas π may have a jump discontinuity on the surface $\|\xi\| = 1$. Assuming a sufficiently strong fall-off conditions at infinity (*e.g.*, assuming that the data are compactly supported on Σ) we also obtain cancelation of the boundary integral at infinity and the surface term in (38) reduces to the integral over the sphere $\|\xi\| = 1$.

Summing up formula (28) for H_{int} and formula (38) for H_{ext}, we finally obtain the global Hamiltonian formula

$$-\delta H(\varphi, \pi) = \int_\Sigma (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi)\ \mathrm{d}^3\xi . \tag{40}$$

with the total Hamiltonian being the sum of the internal and the external contributions:

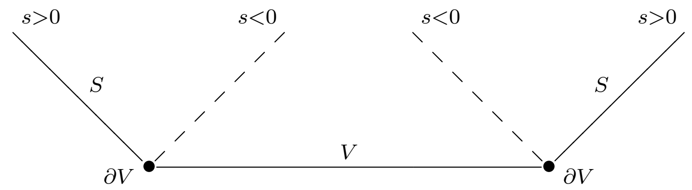
$$H := H_{\mathrm{int}} + H_{\mathrm{ext}} . \tag{41}$$

This is due to the cancellation of the boundary terms on the two-sphere $\|\xi\| = 1$. Indeed, the orientation of the sphere coming from $\partial V_{\mathrm{ext}}$ is opposite to the orientation coming from $\partial V_{\mathrm{int}}$. Moreover, the field φ is continuous across the surface. What remains is to prove also the continuity of $\pi^\perp$. This is due to the fact that $\pi^\perp$ is a pull-back of the vector density π^μ onto the family of three-hypersurfaces $\{\tau \in \mathbb{R}^1; \|\xi\| = \text{ const}\}$. But this surface depends in a continuous way upon the value of $\|\xi\|$ and, whence, there is no jump of $\pi^\perp$ across the surface $\|\xi\| = 1$.

Let us mention for the sake of completeness, that the functional analytic framework of this theory may be organized in such a way, that the phase space of Cauchy data (φ, π) acquires a *strong* symplectic structure, see [4] (a possible choice consists in choosing Sobolev spaces: $\varphi \in H^{\frac{1}{2}}$ and $\pi \in H^{-\frac{1}{2}}$).

3 Dynamics of the gravitational field inside a null hypersurface

In this Section we show how to describe the gravitational Cauchy data on a wave front. Consider, therefore, gravitational field dynamics inside a null hypersurface S:



Parameter $s = \pm 1$ labels two possible situations: an expanding or a shrinking wave front (if S is a horizon, these correspond to a black hole or a white hole case). To simplify notation we use coordinates x^μ, $\mu = 0, 1, 2, 3$, adapted to the above situation: $x^0 = t$ is constant on a chosen family of Cauchy surfaces whereas x^3 is constant on the boundary S (this does not mean that x^3 is null-like everywhere, but only on S). Coordinates x^A, $A = 1, 2$, are "angular" coordinates on the two-surface $\partial V = V \cap S$ whose topology is assumed to be that of a two-sphere. Finally, x^k, $k = 1, 2, 3$, are spatial coordinates on the Cauchy surfaces $\{x^0 = \text{const.}\}$ and x^a, $a = 0, 1, 2$, are coordinates on the boundary S.

In paper [6] we derive the following formula:

$$-\delta \mathscr{H} = \frac{1}{16\pi} \int_V \left(\dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} \right) + \frac{s}{8\pi} \int_{\partial V} (\dot{\lambda} \delta a - \dot{a} \delta \lambda)$$
$$+ \frac{s}{16\pi} \int_{\partial V} \left(\lambda l^{AB} \delta g_{AB} - 2 \left(w_0 \delta \Lambda^0 - \Lambda^A \delta w_A \right) \right) , \qquad (42)$$

where

$$\mathscr{H} = \frac{1}{8\pi} \int_V \mathscr{G}^0{}_0 + \frac{s}{8\pi} \int_{\partial V} \lambda l \equiv \frac{s}{8\pi} \int_{\partial V} \lambda l \, , \qquad (43)$$

and P^{kl} denotes external curvature of the Cauchy surface, written in ADM form. Moreover, $\lambda = \sqrt{\det g_{AB}}$ is a two-dimensional volume form and $a = -\frac{1}{2} \log |g^{00}|$. The remaining objects are constructed from a null field K tangent to S. It is not unique, since fK is also a null field for any function f on S. For purposes of the Hamiltonian formula (42) we always use the normalization compatible with the (3+1)-decomposition used here: $< K, dx^0 > = 1$. Hence, $K = \partial_0 - n^A \partial_A$. The vector-density $\Lambda^a = \lambda K^a = (\lambda, -\lambda n^A)$ is uniquely defined on S. Now we define

$$l_{ab} := -g(\partial_b, \nabla_a K) = -\frac{1}{2} \pounds_K g_{ab} \, , \qquad (44)$$

$$w_a := - < \nabla_a K, dx^0 > \, , \qquad (45)$$

where g_{ab} is the induced (degenerate) metric on S. Denoting by $\tilde{\tilde{g}}^{AB}$ the inverse two-metric, we define the null mean curvature: $l = \tilde{\tilde{g}}^{AB} l_{AB}$ (often denoted by θ – see [12]).

The volume term in (43) vanishes due to constraint equations $\mathscr{G}^0{}_\nu = 0$[1]. $\mathscr{G}^0{}_0$ is often denoted by $NH + N^k H_k$ (see e.g. [16]), where H is the scalar ("Hamiltonian") constraint and H_k are the vector ("momentum") constraints, N and N^k are the lapse and the shift functions. Constraint equations $H = 0$ i $H_k = 0$ imply vanishing of $\mathscr{G}^0{}_0$.

In [6] we give two independent proofs of the formula (42). The first one is analogous to the transition from formula (3) to formula (6). For this purpose we use Einstein equations written analogously to (3) (cf. [13]):

$$\delta L = \partial_\kappa \left(\pi^{\mu\nu} \delta A^\kappa_{\mu\nu} \right) \ , \tag{46}$$

where $\pi^{\mu\nu} := \frac{1}{16\pi} \sqrt{|g|}\ g^{\mu\nu}$, and $A^\lambda_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu} \Gamma^\kappa_{\ \nu)\kappa}$.

Integrating (46) over a volume V and using metric constraints for the connection Γ, we directly prove (42). However, an indirect proof is also provided, based on a limiting procedure, when a family S_ϵ of time-like surfaces tends to a light-like surface S. It is shown that the non-degenerate formula derived in [13] and [14] gives (42) as a limiting case for $\epsilon \to 0$.

The last term in (42) may be written in the following way

$$-\Lambda^A \delta w_A = \lambda n^A \delta w_A = n^A \delta \mathscr{W}_A - n^A w_A \delta\lambda \ ,$$

where $\mathscr{W}_A := \lambda w_A$ and $n^A := \tilde{\tilde{g}}^{AB} g_{0B}$. Denoting $\kappa := n^A w_A - w_0 = -K^a w_a$ we finally obtain the following generating formula:

$$-\delta\mathscr{H} = \frac{1}{16\pi} \int_V \left(\dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} \right) + \frac{s}{8\pi} \int_{\partial V} (\dot{\lambda} \delta a - \dot{a} \delta\lambda) \tag{47}$$

$$+ \frac{s}{16\pi} \int_{\partial V} \left(\lambda l^{AB} \delta g_{AB} + 2 \left(\kappa \delta\lambda - n^A \delta \mathscr{W}_A \right) \right) \ . \tag{48}$$

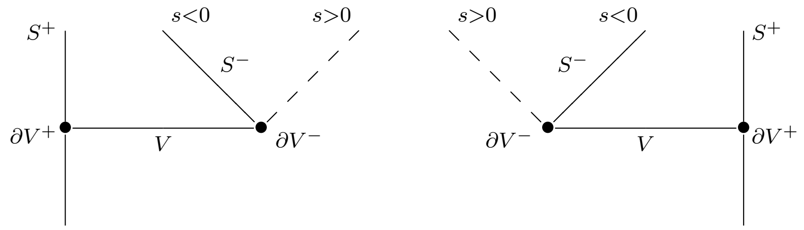
Quantity κ fulfilling: $K^a \nabla_a K = \kappa K$, is traditionally called a "surface gravity" on S. Its value is not an intrinsic property of the surface itself, but depends upon a choice of the null field K on S (i.e. the (3+1)-decomposition of spacetime). In black hole thermodynamics there is a privileged time, compatible with the Killing field of stationary solution and normalized to unity at infinity. In this case the above formula provides, as will be seen, the so called first law of black hole thermodynamics.

We stress that the symplectic structure of Cauchy data, given by two integrals in (47), is invariant with respect to spacetime diffeomorphisms (see [14]). Neglecting the last, surface integral and defining symplectic form only by the volume integral destroys this gauge invariance.

[1]In the presence of matter the volume term equals $\mathscr{G}^0{}_0 - 8\pi T^0{}_0$ and also vanishes due to constraint equations.

4 Dynamics of gravitational field outside the null surface

Consider now dynamics of the gravitational field outside a wave front S^-. We first add an external, timelike (non-degenerate) boundary S^+ and the situation is illustrated by the following figure:



where $\partial V^+ = V \cap S^+$, and $\partial V^- = V \cap S^-$. Because ∂V^- enters with negative orientation, we have: $\int_{\partial V} = \int_{\partial V^+} - \int_{\partial V^-}$. Integrating again Einstein equations written in the form (46) over V, using techniques derived in [13] and [14] to handle surface integrals over ∂V^+ and formula (42) to handle the surface integrals over ∂V^-, we obtain:

$$
\begin{aligned}
-\delta\mathscr{H} = -\delta\mathscr{H}^+ - \delta\mathscr{H}^- &= \frac{1}{16\pi}\int_V \left(\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}\right) \\
&+ \frac{1}{8\pi}\int_{\partial V^+}\left(\dot{\lambda}\delta\alpha - \dot{\alpha}\delta\lambda\right) + \frac{s}{8\pi}\int_{\partial V^-}(\dot{\lambda}\delta a - \dot{a}\delta\lambda) - \frac{1}{16\pi}\int_{\partial V^+}\mathcal{Q}^{ab}\delta g_{ab} \\
&+ \frac{s}{16\pi}\int_{\partial V^-}\left(\lambda l^{AB}\delta g_{AB} - 2\left(w_0\delta\Lambda^0 - \Lambda^A\delta w_A\right)\right) \ , \qquad (49)
\end{aligned}
$$

where α is the "hyperbolic angle" between V and S^+, whereas $\mathcal{Q}^{ab}$ is the external curvature of S^+ written in the ADM form (cf. [14]). The contribution $\mathscr{H}^+$ to the total Hamiltonian from the external boundary is written here in the form of a "free energy" proposed in [14]:

$$
\mathscr{H}^+ = -\frac{1}{8\pi}\int_{\partial V^+}\mathcal{Q}^0{}_0 - E_0 \ , \qquad (50)
$$

where the additive gauge E_0 is chosen in such a way that the entire quantity vanishes if ∂V^+ is a round sphere in a flat space. The internal contribution to the energy is given by formula (43) with ∂V replaced by ∂V^-. Shifting the external boundary to space infinity: $\partial V^+ \to \infty$, the external energy $\mathscr{H}^+$ gives the ADM mass, which we denote by $\mathscr{M}$, whereas the remaining surface integrals over ∂V^+ vanish. This way we obtain the following generating formula for the field dynamics outside of an arbitrary wave front S^- in an asymptotically flat spacetime:

$$
\begin{aligned}
-\delta\mathscr{M} - \delta\mathscr{H}^- &= \frac{1}{16\pi}\int_V\left(\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}\right) + \frac{s}{8\pi}\int_{\partial V^-}(\dot{\lambda}\delta a - \dot{a}\delta\lambda) \\
&+ \frac{s}{16\pi}\int_{\partial V^-}\left(\lambda l^{AB}\delta g_{AB} + 2\left(\kappa\delta\lambda - n^A\delta\mathscr{W}_A\right)\right) \ . \qquad (51)
\end{aligned}
$$

5 Black hole thermodynamics

In this Section we apply the above result to the situation, when the wave front S^- is a non-expanding horizon, i.e. $l = 0$ (see [12]). In this case the "internal energy" $\mathscr{H}^-$ vanishes. Moreover, Einstein equations imply $l^{AB} = 0$ (see [9]) and the definition of w_a reduces to: $\nabla_a K = -w_a K$. We obtain the following generating formula for the black hole dynamics

$$-\delta \mathscr{M} = \frac{1}{16\pi} \int_V \left(\dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} \right) + \frac{s}{8\pi} \int_{\partial V^-} (\dot{\lambda} \delta a - \dot{a} \delta \lambda) + \frac{s}{8\pi} \int_{\partial V^-} \left(\kappa \delta \lambda - n^A \delta \mathscr{W}_A \right) , \tag{52}$$

where $s = 1$ for a white hole, and $s = -1$ for a black hole.

The so called "black hole thermodynamics" consists in analysing possible stationary situations. By stationarity we understand the existence of a timelike symmetry (Killing) vector field. If such a field exists, we may always choose a coordinate system such that the Killing field becomes $\frac{\partial}{\partial x^0}$ and all the time derivatives (dots) vanish. Hence, formula (52) reduces to:

$$\delta \mathscr{M} = -\frac{s}{8\pi} \int_{\partial V^-} \left(\kappa \delta \lambda - n^A \delta \mathscr{W}_A \right) . \tag{53}$$

We assumed here that $\frac{\partial}{\partial x^0}$ is tangent to S. If this is not the case, we would have a one-parameter family of horizons. Such phenomenon corresponds to the Kundt's class of metrics (see e.g. [18]). The known metrics of this class are not asymptotically flat. We do not know whether or not this is a universal property and we exclude such a pathology by the above assumption.

We have shown in [9] that there is a canonical affine fibration $\pi : S \to B$ over a base manifold B, whose topology is assumed to be that of a sphere S^2. The affine structure of the fibers is implied by the fact that they are null-geodesic lines in M. Identity $-2l_{ab} = \pounds_K g_{ab} = 0$ implies that the metrics g on S may be projected onto the base manifold B, which acquires a Riemannian two-metric tensor h_{AB}. The degenerate metric g_{ab} on a manifold S is simply the pull back of h_{AB} from B to S: $g = \pi^* h$.

The quantity w_a is not an intrinsic property of the surface itself, but depends upon a choice of the null field K on S. Indeed, if $\tilde{K} = \exp(-\gamma) K$ then $\tilde{w}_a = w_a + \partial_a \gamma$. In particular, there are on S vector fields K such that $K^a \nabla_a K = 0$ and, consequently, $\kappa = 0$. These are null geodesic fields tangent to fibers of $\pi : S \to B$.

In case of a black hole, there is a privileged field K, compatible with the time-like symmetry of the solution, which is normalized to unity at infinity. This way the quantities κ and w_A in formula (53) become uniquely defined.

We have, therefore, two symmetry fields of the metric g_{ab} on S: ∂_0 and K. Due to normalization chosen above, we have $< \partial_0 - K, dx^0 >= 0$. Hence, the field $\vec{n} := \partial_0 - K = n^A \partial_A$ is purely space-like and projects on B. Moreover, it is a symmetry field of the Riemannian two-metric h_{AB}.

Because the conformal structure of h_{AB} is always isomorphic to the conformal structure of the unit sphere S^2, we are free to choose a coordinate system in which $h_{AB} = f\breve{h}_{AB}$ (and $\breve{h}_{AB}$ denotes the standard unit two-sphere metrics). The field $\vec{n}$ is, therefore, the symmetry field of this conformal structure. Consequently, $\vec{n}$ belongs to the six-dimensional space of conformal fields on the two-sphere. Using remaining gauge freedom, we may choose angular coordinates $(x^A) = (\theta, \phi)$ in such a way that $\vec{n}$ becomes a rotation field on the two-sphere. This means (cf. [2] or [6]) that there exists a coordinate system in which the following holds:

$$\vec{n} = -\Omega^k \epsilon_{klm} y^l \partial^m \ . \tag{54}$$

Here, Ω^k are components of a three-dimensional vector called angular velocity of the black hole, and y^k are functions on S^2 created by restricting Cartesian coordinates on $\mathbb{R}^3$ to a unit two-sphere. We can also set z-coordinate axis parallelly to angular velocity vector field. After a suitable rotation we have: $(\Omega^k) = (0, 0, \Omega)$, $z = y^3 = \cos\theta$, and:

$$\vec{n} = -\Omega \frac{\partial}{\partial \varphi} \ . \tag{55}$$

Inserting this into (53) we obtain

$$-\frac{1}{8\pi} \int_{\partial V^-} n^A \delta \mathscr{W}_A = \Omega \delta J \ , \tag{56}$$

where

$$J \equiv J_z := \frac{1}{8\pi} \int_{\partial V^-} \mathscr{W}_\varphi \ , \tag{57}$$

is the z-component of the black hole angular momentum.

Up to now we have used only the symmetry of conformal structure carried by h_{AB}. The symmetry of the metric itself implies that the conformal factor f is constant along the field $\vec{n}$. This follows from the observation that the trace of the Killing equation implies vanishing of divergence of the field $\vec{n}$:

$$0 \ = \ \partial_A(\sqrt{\det h_{CD}}\ n^A) = n^A \sqrt{\det \breve{h}_{CD}}\ \partial_A f \ , \tag{58}$$

where the fact that $\vec{n}$ is the symmetry field of the metric $\breve{h}$ has been used. Formula (55) implies that $\partial_\varphi f = 0$ and the conformal factor f must be a function of the variable θ only.

It turns out that also its canonical conjugate κ may be gauged in such a way that it is constant along the field $\vec{n}$ (see [6] for a proof).[2]

This result was obtained locally, or rather *quasi*-locally – i.e. from the analysis of the field on the horizon itself. However, the *global* theorems

[2]In case $\Omega = 0$, quantities κ and f are arbitrary functions on S^2.

on the existence of stationary solutions possessing a horizon, imply the so called 0-th law of thermodynamics of black holes (see [8]), according to which *the surface gravity κ must be constant along the horizon.* But $A := \int_{S^2} \lambda$ is the area of the horizon S. Taking this into account and using (56), we derive from (53) the "first law of black holes thermodynamics":

$$-s\delta\mathscr{M} = \frac{1}{8\pi}\kappa\delta A + \Omega\delta J \ . \tag{59}$$

Contrary to the theory proposed by Wald and Iyer in [17], the first law (59) is, in our approach, a simple consequence of the complete Hamiltonian formula (52), restricted to the stationary case. As illustrated by an example of the string dynamics, where formula (9) for virtual work was a consequence of the Hamiltonian formula (6), a similar "thermodynamics of boundary data" may be expected in any Hamiltonian field theory (see e.g. [14] for the corresponding analysis of the Maxwell electrodynamics). Also a "Penrose-like" inequality (analogous to (10) in the string theory) is satisfied as soon as the Hamiltonian is convex. We very much hope that the gravitational Penrose inequality can be proved along these lines. Preliminary results in this direction, based on the analysis of the field Hamiltonian in linearized gravity (see [11]), are promising.

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