Coherent States in LQG

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Outline:

- Motivation
- Complexifier coherent states $\psi^t_{g_1,\ldots,g_E}$
- Gauge-invariant coherent states $\Psi^t_{[g_1,\ldots,g_E]}$
- Semiclassical analysis
- Summary

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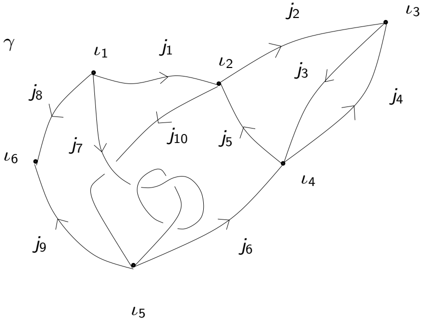
- Motivation — see talk by Rovelli
- Complexifier coherent states $\psi^t_{g_1,\ldots,g_E}$ — see talk by Thiemann, Perini
- Gauge-invariant coherent states $\Psi^t_{[g_1,\ldots,g_E]}$ — see talk by Thiemann, Perini
- Semiclassical analysis — see talk by Giesel, Thiemann
- Summary

Classical interpretation of states in LQG

In LQG, basis of $\mathcal{H}_{\mathrm{kin}}$ given by spin networks $T_{\gamma,\vec{j},\vec{\iota}}$

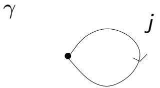


$$T_{\gamma,\vec{j},\vec{\iota}}(A) \; = \; \left(\prod_v (\iota_v)^{n_1,\ldots,n_v}_{m_1,\ldots,m_v}\right) \; \left(\prod_e \sqrt{2j_e+1}\,\pi_{j_e}\left(h_e(A)\right)^{m_e}_{n_e}\right)$$

Classical interpretation of states in LQG

The $T_{\gamma,\vec{j},\vec{\iota}}$ have geometric interpretation (eigenstates of area- and volume operator).

But: not close to 'classical geometry' (e.g. half-integer holonomy operators have zero expectation values).



$$\langle T_{\gamma,j} \,|\, \widehat{\mathrm{tr}_{\frac{1}{2}}(h)} \,|\, T_{\gamma,j} \rangle \;=\; 0$$

One needs to construct states which contain information about both canonical variables (fluxes *and* holonomies) $\Rightarrow$ semiclassical states, in order to:

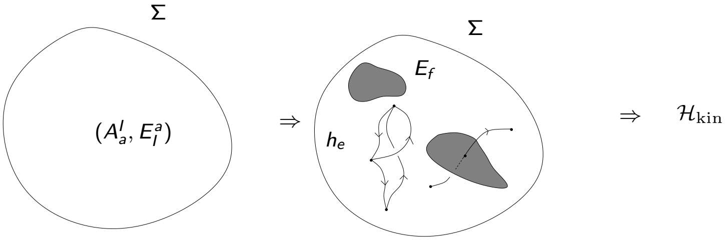
- Interpret states as "close to classical geometry" (centered around phase space points, small fluctuations)
- Check semiclassical limit of operators

Strategy:

Construct coherent states on $\mathcal{H}_\gamma$ by taking Hall's complexifier coherent states on the gauge-variant Hilbert space, and project them to the gauge-invariant subspace:

$$\begin{array}{ccccc}
\psi^t_{g_1,\ldots,g_E} & \in & L^2\left(SU(2)^E, d\mu_H\right) & \subset & L^2\left(\overline{\mathcal{A}}, d\mu_{\mathrm{AL}}\right) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\Psi^t_{[g_1,\ldots,g_E]} & \in & L^2\left(SU(2)^E/SU(2)^V, d\mu\right) \;=\; \mathcal{H}_\gamma & \subset & L^2\left(\overline{\mathcal{A}/\mathcal{G}}, d\mu_{\mathrm{AL}}\right)
\end{array}$$

Phase-space of GR



Quantum Theory in LQG constructed in two steps:

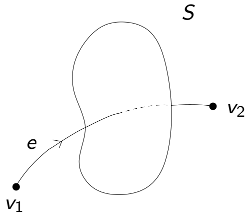
1. Replace fields (A_a^I, E_I^a) by holonomies and fluxes $h_e \in SU(2)$, $E_f \in \mathfrak{su}(2)$. (smooth fields smeared over $1-$ and $2-$ dim. submanifolds)
2. Build a quantum theory out of holonomy-flux algebra

$\Rightarrow$ Choice of coordinates on phase-space: h_e, E_f.

Phase-space of one graph

Thiemann '00

Choose graph $\gamma = \{e_1, \ldots, e_E\}$ and dual graph $\gamma^* = \{S_1, \ldots, S_E\}$



Variables are the h_e, E_e (one canonical pair per edge)

$$
\begin{aligned}
h_e &= \text{holonomy along edge } e \\
E_e &= \text{Flux integrated over } S \text{ (parallelly transported to } v_1\text{)}
\end{aligned}
$$

This defines a $6E$-dim sub-phase-space of the whole phase-space of GR ($\simeq (T^*SU(2))^E$).

Hall's complexifier coherent states

Hall '97, Sahlmann, Thiemann, Winkler '00

Hall introduced generalizations of Gaussian wave packets on $L^2(G, d\mu_H)$ for comapct, s.-s. Lie groups G, spheres,...

$$\psi^t_{g_e}(h_e) \quad := \quad \exp\left(\Delta\frac{t}{2}\right)\delta(h_e, h')\Big|_{h'\to g_e}$$

$$= \quad \sum_{j_e}(2j_e+1)\exp\left(-j_e(j_e+1)\frac{t}{2}\right)\ \chi_{j_e}(h_e^{-1}g_e)$$

χ_{j_e} = character or rep'n j_e, h_e holonomy along edge e, $g_e \in SL(2,\mathbb{C})$, $t > 0$

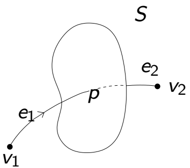
$$t \;=\; \frac{\ell_P^2}{a^2}$$

where a is a characteristic length scale. Semiclassical limit: $t \to 0$

Gauge-variant phase space

Complexifier procedure delivers correspondence between $T^*SU(2)$ and $SL(2,\mathbb{C})$ via *polar decomposition*:

$$\begin{aligned} g_e &= \sum_{n=0}^{\infty} \frac{i^n}{n!}\{C,\{C,\ldots,\{C,h_e\}\ldots\}\} \\ &= e^{tE_e}\, h_e \end{aligned}$$



Note: In SF context more convenient to parallelly transport E to p instead of v_1. (see talks by Rovelli and Perini) Then one has

$$e^{tE^{(v_1)}}\, h_{e_1 e_2} = h_{e_1}\, e^{tE^{(p)}}\, h_{e_2}$$

Properties of complexifier coherent states

Thiemann, Winkler '00

- Minimal uncertainty states, Gaussian peaked, Eigenstates of 'ladder operator'
- Approximate observables: Let f be a polynomial phase-space function (i.e. a polynomial function on holonomies and fluxes h_e, E_e), then

$$\frac{\langle \psi^t_{g_1,\ldots,g_E} | f(\hat{h}_e, \hat{E}_e) | \psi^t_{g_1,\ldots,g_E} \rangle}{\langle \psi^t_{g_1,\ldots,g_E} | \psi^t_{g_1,\ldots,g_E} \rangle} = f(h_e, E_e) + O(t)$$

where $g_e = \exp(tE_e)h_e$.

Properties of complexifier coherent states

- Resolution of identity:

$$\int_{SL(2,\mathbb{C})^E} d\nu \, |\psi^t_{g_1,\ldots,g_n}\rangle\langle\psi^t_{g_1,\ldots,g_n}| \;=\; \mathbb{1}_{L^2(SU(2)^E)}$$

  where ν is some measure on $SL(2,\mathbb{C})^E$ related to the heat kernel.
- Bargman-Segal representation: For a state ϕ, the function

$$\phi(g_1,\ldots,g_n) \;:=\; \langle\phi|\psi^t_{g_1,\ldots,g_n}\rangle$$

  is complex analytic in the g_e.

Gauge-invariant coherent states:

Thiemann, Winkler '00

Projection of complexifier coherent states:

$\Pi^{\mathrm{gauge}} : L^2(SU(2)^E) \to L^2(SU(2)^E/SU(2)^V) = \mathcal{H}_\gamma$

$$\begin{aligned} \Psi^t_{[g_1,\ldots,g_n]} &:= \Pi^{\mathrm{gauge}}\psi^t_{g_1,\ldots,g_n} \\ &= \sum_{\vec{j},\vec{\iota}} \left[e^{-\sum_e j_e(j_e+1)t/2} T_{\gamma,\vec{j},\vec{\iota}}(\{g_e^*\}) \right] T_{\gamma,\vec{j},\vec{\iota}} \end{aligned}$$

with $g^* := \epsilon g \epsilon^{-1}$.

Labels of gauge-invariant coherent states

BB, Thiemann '06-'08

Since for a gauge action $\alpha_{k_1,\ldots,k_V}$ with $k_v \in SU(2)$ one has

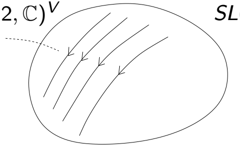
$$\alpha_{k_1,\ldots,k_V}\psi^t_{g_1,\ldots,g_E} = \psi^t_{k_{s(e_1)}g_1k^{-1}_{t(e_1)},\ldots,k_{s(e_E)}g_Ek^{-1}_{t(e_E)}}$$

one might think that $\Psi^t_{[g_1,\ldots,g_E]}$ are labelled by $[g_1,\ldots,g_E] \in SL(2,\mathbb{C})^E/SU(2)^V$, but this is *not* the case:

$$(k_1,\ldots,k_V) \longmapsto \Pi^{\mathrm{gauge}}\alpha_{k_v}\psi^t_{g_1,\ldots,g_E} \qquad (1)$$

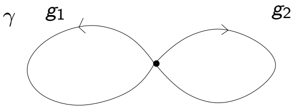
can be extended analytically to all of $SL(2,\mathbb{C})^V$. But then (1) is a complex analytic function which is constant on the 'real line' $SU(2)^V$, so it has to be constant on all of $SL(2,\mathbb{C})^V$.

$[g_1,\ldots,g_n] \in \frac{SL(2,\mathbb{C})^E}{SL(2,\mathbb{C})^V} =$ orbit of $(g_1,\ldots,g_n)$ under *complexified gauge transformation*.



Geometry of gauge-invariant phase space

The set of orbits $[g_1, \ldots, g_E]$ is not a manifold, but contains singular points:



$$(g_1,\, g_2) \;\sim\; (kg_1 k^{-1},\, kg_2 k^{-1}) \qquad k \in SL(2,\mathbb{C})$$

$$g_1 = g_2 = \pm\mathbb{1} \qquad \Rightarrow \qquad \dim \text{Orbit}(g_1, g_2) \;=\; 0$$

$$g_1 = \pm g_2 \neq \mathbb{1} \qquad \Rightarrow \qquad \dim \text{Orbit}(g_1, g_2) \;=\; 4$$

$$g_1 \neq \pm g_2 \neq \mathbb{1} \qquad \Rightarrow \qquad \dim \text{Orbit}(g_1, g_2) \;=\; 6$$

For generic points, the dimension of gauge-invariant phase space $SL(2,\mathbb{C})^E/SL(2,\mathbb{C})^V$ is

$$\dim SL(2,\mathbb{C})^E/SL(2,\mathbb{C})^V \;=\; 6(E-V) = 6(L-1)$$

where L is the number of loops in the graph γ.

Properties of the coherent states $\Psi^t_{[g_1,\ldots,g_E]}$:

BB, Thiemann '06-'08

- Approximation of *gauge-invariant* observables: Let f be a polynomial *gauge-invariant* phase space functions (e.g. $\hat{\mathrm{Ar}}_e^2$, $\mathrm{tr}_j(h_e)$), then one recovers (in lowest t-order) the classical expression:

$$\frac{\langle\psi^t_{g_1,\ldots,g_E}|f(\hat{h}_e,\hat{E}_e)|\psi^t_{g_1,\ldots,g_E}\rangle}{\langle\psi^t_{g_1,\ldots,g_E}|\psi^t_{g_1,\ldots,g_E}\rangle} = f(h_e,E_e) + O(t)$$

where $g_e = \exp(tE_e)h_e$.

Properties of the coherent states $\Psi^t_{[g_1,\dots,g_E]}$:

- Resolution of the identity:

$$\int_{\frac{SL(2,\mathbb{C})^E}{SL(2,\mathbb{C})^V}} dN\,\Delta_{\mathrm{FP}}\,|\Psi^t_{[g_1,\dots,g_E]}\rangle\langle\Psi^t_{[g_1,\dots,g_E]}| \;=\; \mathbb{1}_{L^2(SU(2)^E/SU(2)^V)}$$

  with the averaged measure

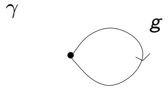
$$N([g_1,\dots,g_E]) \;=\; \int_{SL(2,\mathbb{C})^V} d\mu_H^{\otimes V}(\vec{k})\;\nu(\alpha_{\vec{k}}(g_1,\dots,g_E))$$

  and a Fadeev-Popov-determinant Δ_{FP} (see Bianchi, Magliaro, Perini '10)

- The states $\Psi^t_{[g_1,\dots,g_E]}$ are Gaussian peaked almost everywhere, apart from the points where the gauge-invariant phase space has singular points (degenerate gauge orbits).

Peakedness properties of the coherent states $\Psi^t_{[g]}$:

Example:



Assume $g \in SL(2, \mathbb{C})$

$$g = e^{iz_I \sigma^I}, \; z_I \in \mathbb{C}$$

with $z^2 := z_1^2 + z_2^2 + z_3^2 \neq 0$:

Then

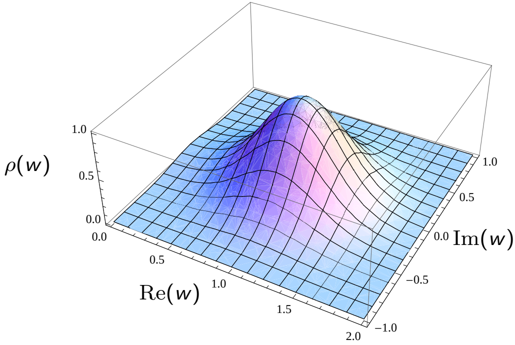
$$\Psi^t_{[g]} \;\equiv\; \Psi^t_z \;=\; \sum_j e^{-j(j+1)t/2} \frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}} \; T_{\gamma,j},$$

and

$$\rho(w) \;=\; \frac{|\langle \Psi^t_w | \Psi^t_z \rangle|^2}{\|\Psi^t_w\|^2 \|\Psi^t_z\|^2} \;=\; \frac{\sinh \frac{\overline{w}z}{2t} \sinh \frac{\overline{z}w}{2t}}{\sinh \frac{|w|^2}{2t} \sinh \frac{|z|^2}{2t}} (1 + O(t^\infty))$$

is the phase-space density of the state Ψ^t_z.

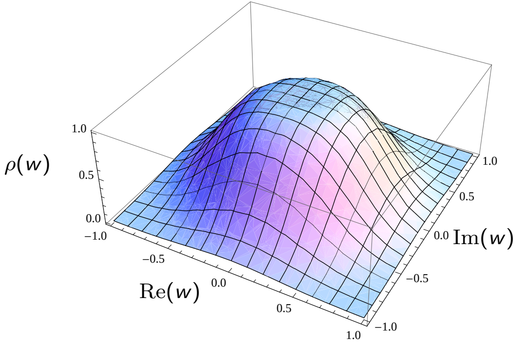
Peakedness properties of the coherent states $\Psi^t_{[g]}$:



Phase-space density $\rho(w)$ of the state Ψ^t_z with $z = 1$.

$\Rightarrow$ Gaussian peaked around $w = z = 1$.

Peakedness properties of the coherent states $\Psi^t_{[g]}$:



Phase-space density $\rho(w)$ of the state Ψ^t_z with $z = 0$ (corresponds to $g = \mathbb{1}$, i.e. degenerate gauge orbit.

$\Rightarrow$ Non-Gaussian peaked around $w = z = 0$ (rather $\exp(-|z|^4/t)$ - profile).

Semiclassical limit of Master constraint

The (non-graph changing) master constraint $\hat{M}$ as defined by Thiemann has the correct semiclassical limit, in the following sense: (Giesel, Thiemann '06)

- Choose classical fields A_0, E_0 in Σ
- Choose then a cubic graph γ (and dual graph γ^*) such that the fields A_0, E_0 do not vary much inbetween lattice sites.
- The classical fields A_0, E_0 induce, by smearing along the edges and surfaces of γ, γ^*, discrete coordinates $h_e \in SU(2)$, $E_e \in \mathfrak{su}(2)$.
- Consider the coherent state $\Psi^t_{[g_1,\ldots,g_E]}$ with $g_e = \exp(tE_e)h_e$.

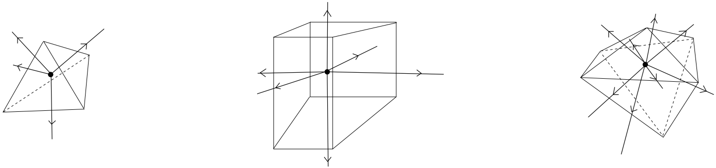
Then

$$\frac{\langle \Psi^t_{[g_1,\ldots,g_E]} | \hat{M} | \Psi^t_{[g_1,\ldots,g_E]} \rangle}{\langle \Psi^t_{[g_1,\ldots,g_E]} | \Psi^t_{[g_1,\ldots,g_E]} \rangle} = M(A_0, E_0) + O(t) + O(\epsilon)$$

where t is the semiclassicality parameter, and ϵ measures the variation of the fields A_0, E_0 inbetween lattice sites.

Semiclassical limit of the (Ashtekar-Lewandowski-) volume operator

The (AL-) volume operator V has the correct semiclassical limit for 6-valent graphs *only* in the following sense:



- Choose *flat background* A_0, E_0 in Σ in the manifold Σ.
- Embed a tetrahedron, cuboid or octahedron into Σ
- Construct appropriate coherent state (on a graph dual to polyhedron) $\Psi^t_{[g_1,\ldots,g_E]}$

Then

$$\frac{\langle\Psi^t_{[g_1,\ldots,g_E]}|\hat{V}|\Psi^t_{[g_1,\ldots,g_E]}\rangle}{\langle\Psi^t_{[g_1,\ldots,g_E]}|\Psi^t_{[g_1,\ldots,g_E]}\rangle} = \kappa_n V(E_0) + O(t)$$

($n = 4, 6, 8$), and $V(E_0)$ is the classical, flat volume of the embedded polyhedron. The numbers κ_n are given by

$$\kappa_4 = \frac{\sqrt{2}}{6}, \qquad \kappa_6 = 1, \qquad \kappa_8 = \frac{1}{2\sqrt{2}}$$

How to deal with this result?

- Change the states $\Psi^t_{[g_1,\ldots,g_E]}$
  - Change of complexifier from $\hat{C} = \Delta\frac{t}{2}$ so something else (e.g. $\hat{C} = \hat{V}$)? See Flori '08
  - Something different than complexifier procedure?
  - However: States work well on many, many other levels.
- Change the volume operator
  - Adjust factors κ_n
  - Different regularization procedure
  - However: "Triad test": Classical identity Giesel, Thiemann '06

$$E(S) \;=\; \int_S \det E\{A, V\} \wedge \{A, V\}$$

    shall also hold on quantum level
- Work only on six-valent graphs
  - Favoured by Grimstrup, Aastrup: Specrtal triple construction in LQG see talk by Jepser Møller
  - However: Not a representation space of the holonomy-flux algebra.

Summary:

- The complexifier coherent states $\psi^t_{g_1,\ldots,g_E}$ are good semiclassical states on $L^2(SU(2)^E)$ (approximate well fluxes *and* holonomies). $t =$ semiclassicality parameter, g_e obtain geometric interpretation in terms of polar decomposition:

$$g_e \;=\; \exp\left(tE_e\right)\, h_e$$

  $((g_1,\ldots,g_E)$=point in gauge-variant phase-space)
- Their gauge-invariant projections

$$\Psi^t_{[g_1,\ldots,g_E]} \;=\; \Pi^{\mathrm{gauge}}\psi^t_{g_1,\ldots,g_E}$$

  are good semiclassical states for gauge-invariant sector ($[g_1,\ldots,g_E] =$ point in gauge-invariant phase-space).

Summary:

- Gauge-invariant phase-space $SL(2,\mathbb{C})^E/SL(2,\mathbb{C})^V$ contains singular points (degenerate gauge orbits). There e.g. smooth structure, complex structure, etc. breaks down. Correspond to phase-space points with non-trivial symmetry (e.g. all g_e equal).
- On generic points however, the dimension of gauge-invariant phase-space is $6(L-1)$, where L is the number of 'loops' in the graph γ (generators of first fundamental group).

Summary:

- The coherent states $\Psi^t_{[g_1,\dots,g_E]}$ can be used to investigate semiclassical ($t \to 0$) limit of operators:
- It is possible to approximate a classical smooth field configuration (A_0, E_0) with a coherent state Ψ^t situated on a very fine graph.
- AL-volume operator and Master constraint in LQG have the correct semiclassical limit, if this graph is cubic.

  Nontrivial, since Master constraint is no poynomial in the fields.
- States can be used to write down and investigate coherent propagator for LQG

  Han, '09
- On non-cubic graphs (i.e. with valence different from $n = 6$), the (AL-) volume operator has not the correct semiclassical limit: In the sum

$$\hat{V}^2_{\mathrm{Al}} = \left| \frac{1}{48} \sum_{e,e',e''} \epsilon(e,e',e'')\epsilon_{IJK} \hat{E}^I_e \hat{E}^J_{e'} \hat{E}^K_{e''} \right|$$

  the coherent states seem to overcount triples of edges e, e', e''.

Outlook:

Recent development:

Works of Bianchi, Magliaro, Perini:

On gauge-invariant phase-space $SL(2,\mathbb{C})^E/SL(2,\mathbb{C})^V$ introduce coordinates given by Speziale, Freidel. $\Rightarrow$ For four-valent graphs, these have nice interpretation in terms of twisted geometries of simplicial complexes (see Claudio's talk)!