Graph knottings in the EPRL model

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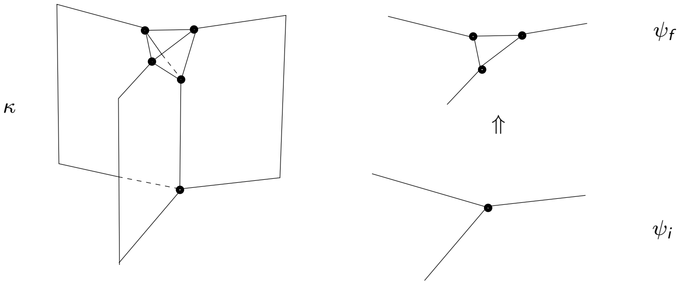
Outline:

- Motivation
- EPRL model for two-complexes
- Graph knottings and deformations of spin foams
- Summary

Spin Foams as histories of Spin Networks

In its original formulation Rovelli, Reisenberger, Baez '99, Spin foams were perceived as a possibility to describe *histories of spin networks*



Such a spin foam can (after taking into account the various labels, e.g. spins and intertwiners) be assigned its *spin foam amplitude*

$$Z[\kappa] = \underbrace{\prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \mathcal{A}_v}_{\text{interior}} \underbrace{\prod_e \mathcal{B}_e \prod_v \mathcal{B}_v}_{\text{boundary}}$$

The physical inner product

The spin foam κ mediates between an initial state ψ_i and a final state ψ_f. Write

$$\psi_i \xrightarrow{\kappa} \psi_f$$

The spin foam amplitudes should then be used to define the physical inner product between boundary states ψ_1, ψ_f via

$$\begin{aligned} \langle\, \psi_f \,|\, \psi_i \,\rangle_{\mathrm{phys}} \quad &= \quad \sum_{\kappa \,:\, \psi_i \xrightarrow{\kappa} \psi_f} Z[\kappa] \\ \text{"} = \quad &\int \mathcal{D}\omega \mathcal{D}E \; e^{iS_{\mathrm{Pl}}(\omega, E)} \quad \text{"} \end{aligned}$$

The sum is of course vastly infinite, ill-defined,... (see GFT approach, though)!

The projector on the physical Hilbert space

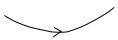
Two kinematical states ψ_1, ψ_2 are projected onto the same physical state, if one has

$$\langle\phi\,|\,\psi_1\rangle_{\rm phys} \;=\; \langle\phi\,|\,\psi_2\rangle_{\rm phys} \qquad \text{for all } \phi$$

Example: ψ_1, ψ_2 are diffeomorphic to each other.

Aim of this talk:

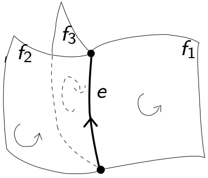
Although no explicit way of summing over κ is known, we will argue (by explicitly computing some SF amplitudes) that, if the $\sum_\kappa$ behaves 'reasonable', then the physical inner product (obtained by using the EPRL spin foam model) will not know about knottings of graphs anymore. I.e.

$\Big\langle \phi \,\Big|$  $\Big\rangle_{\text{phys}} = \Big\langle \phi \,\Big|$  $\Big\rangle_{\text{phys}}$

for all ϕ. Therefore the physical states will not contain any knotting information.

EPRL à la KKL

We use the EPRL spin foam amplitude Engle, Pereira, Rovelli, Livine '07 as defined for arbitrary two-complexes Kamiński, Kisielowski, Lewandowski. In this model of Euclidean quantum gravity one assigns $Spin(4)$ representations (j_f^+, j_f^-) to faces f, and intertwiners ι_e to the edges e.



$$\iota_e \;:\; V_{j_{f_1}^+, j_{f_1}^-} \otimes V_{j_{f_3}^+, j_{f_3}^-} \longrightarrow V_{j_{f_2}^+, j_{f_2}^-}$$

under the conditions that for each face $j_f^\pm = |1 \pm \gamma| k_f$ for some spin k_f. Furthermore the ι_e need to be of the form $\iota_e = \Phi(\hat{\iota}_e)$, where

$$\Phi : \; \mathrm{Inv}\left(V_{k_{f_1}} \otimes V_{k_{f_3}} \otimes V^*_{k_{f_2}}\right) \longrightarrow \mathrm{Inv}\left(V_{j_{f_1}^+, j_{f_1}^-} \otimes V_{j_{f_3}^+, j_{f_3}^-} \otimes V^*_{j_{f_2}^+, j_{f_2}^-}\right)$$

is given in terms of fusion coefficients.

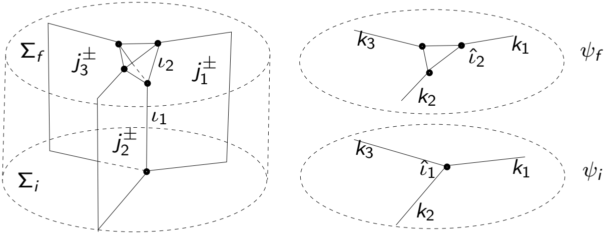
EPRL à la KKL

For a vertex v in the interior of κ, the vertex-amplitude $\mathcal{A}_v$ is given by contracting the intertwiners ι_e for all edges e that meet in v. Two indices of two intertwiners are contracted iff there is an face between the two according edges.

$$\mathcal{A}_v = \sum_{n,m} \left(\prod_{f \supset v} \delta^{n_{e_f}}_{m_{e_f}} \right) \left(\prod_{e \supset v} (\iota_e)^{m_{e_1} \ldots, m_{e_{M_e}}}_{n_{e_1} \ldots, n_{e_{N_e}}} \right)$$

Boundary Hilbert space: Spin networks

The two-complex κ induces two spin network states ψ_i, ψ_2, living in the initial and the final hypersurface Σ_i, Σ_f

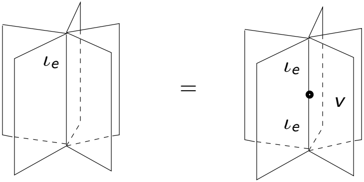


The labels of the graphs are induced by $j_f^\pm = |1 \pm \gamma| k_f$ and $\iota_e = \Phi(\hat{\iota}_e)$. We write

$$\psi_i \xrightarrow{\kappa} \psi_f$$

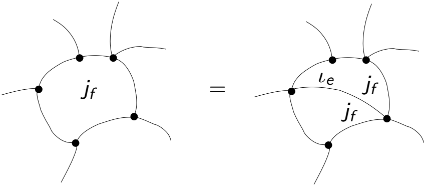
Observations about the EPRL amplitude:

Demanding invariance of $Z[\kappa]$ under trivial subdivisions fixes the edge- and face amplitudes in terms of the vertex amplitude:



$$\mathcal{A}_e \stackrel{!}{=} \mathcal{A}_e^2 \mathcal{A}_v$$

$$\Rightarrow \mathcal{A}_e = \frac{1}{\mathcal{A}_v} = \frac{1}{\mathrm{tr}(\iota_e \iota_e^\dagger)}$$

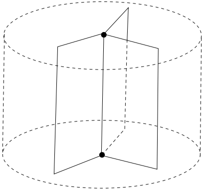


$$\iota_e = \mathrm{id}, \; \mathcal{A}_f \stackrel{!}{=} \mathcal{A}_f^2 \mathcal{A}_e$$

$$\Rightarrow \mathcal{A}_f = \frac{1}{\mathcal{A}_e} = (2j_f^+ + 1)(2j_f^- + 1)$$

Observations about the EPRL amplitude:

Demanding the trivial amplitude to evaluate to 1 fixes the boundary amplitudes:



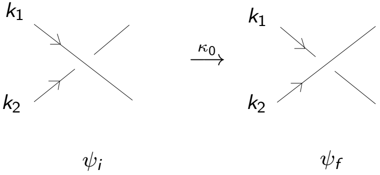
$\stackrel{!}{=} 1$

$$\mathcal{B}_v = \frac{1}{\sqrt{\mathcal{A}_e}} = \sqrt{\mathrm{tr}(\iota_e \iota_e^\dagger)}$$

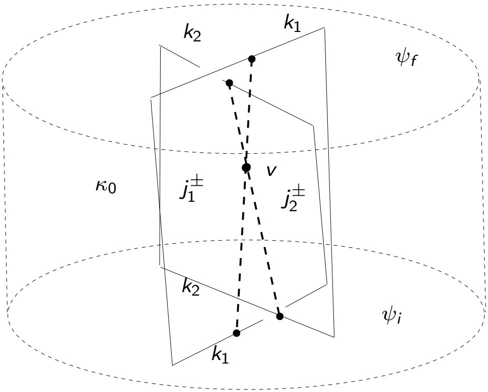
$$\mathcal{B}_e = \frac{1}{\sqrt{\mathcal{A}_f}} = \frac{1}{\sqrt{(2j_f^+ + 1)(2j_f^- + 1)}}$$

Unknotting knots

We consider two spin network states ψ_i and ψ_f (embedded in σ_i and Σ_f), which are the same combinatorially, but have a different knotting class.



Consider the transition $\psi_i \xrightarrow{\kappa_0} \psi_f$ between the two states, where the two edges slide across each other. To make life easier, introduce a trivial vertex in the two edges, at the point where they will meet.



The computation of the amplitude gives

$$
\begin{aligned}
Z[\kappa_0] &= \mathcal{A}_{f_1}^2 \mathcal{A}_{f_2}^2 \mathcal{A}_e^4 \mathcal{A}_v \, \mathcal{B}_{v_1}^2 \mathcal{B}_{v_2}^2 \mathcal{B}_{e_1}^4 \mathcal{B}_{e_2}^2 \\
&= \mathcal{A}_e^2 \mathcal{A}_v = 1
\end{aligned}
$$

Interlude: How to compute the vertex amplitude of κ_0:

Work in $\mathbb{R}^4$, an parametrise the faces of the two surface with

$$f_1(\sigma,\tau) = \Big(\tau,\sigma,0,0\Big)$$

$$f_2(\sigma,\tau) = \Big(\tau,0,\sigma,\tau\Big)$$

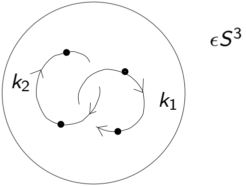
with $-1 \leq \tau, \sigma \leq 1$ The two faces meet for $\sigma = \tau = 0$ only. If one cuts a S^3 with radius ϵ with the two-complex, one gets:

$$f_1 \cap \epsilon S^3 = \big\{(\tau,\sigma,0,0) \mid \tau^2 + \sigma^2 = \epsilon^2\big\}$$

$$f_2 \cap \epsilon S^3 = \big\{(\tau,0,\sigma,\tau) \mid 2\tau^2 + \sigma^2 = \epsilon^2\big\}$$

These are two (knotted) circles embedded in ϵS^3:

Interlude: How to compute the vertex amplitude of κ_0



One can now easily compute the vertex amplitude to be

$$\mathcal{A}_v \;=\; (2j_1^+ + 1)(2j_1^- + 1)(2j_2^+ + 1)(2j_2^- + 1)$$

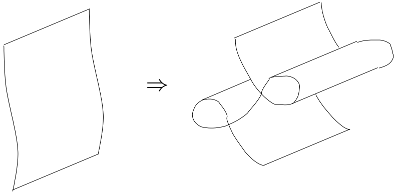
And hence:

$$Z[\kappa_0] \;=\; \mathcal{A}_e^2\,\mathcal{A}_v \;=\; 1$$

Consistent deformations

We will consider *consistent deformations* of a spin foam κ, and show that the amplitude $Z[\kappa]$ is invariant under them.

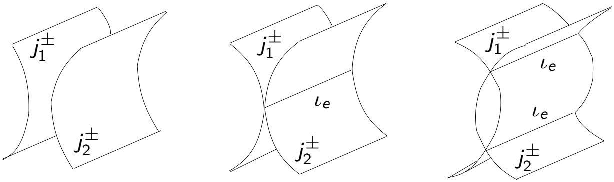
A consistent deformation of a spin foam $\kappa = (c, \rho, \iota)$ is a homotopy of the underlying two complex c, such that the faces, edges, etc. are allowed to intersect each other:



To ensure that the new object is in fact a two-complex, introduce new faces, edges, vertices by trivial subdivision (under which $Z[\kappa]$ is invariant).

Consistent deformations

Example:



For the intertwiner ι_e at the "new" edges

$$\iota_e \;:\; V_{j_1^+,j_1^-} \otimes V_{j_2^+,j_2^-} \longrightarrow V_{j_1^+,j_1^-} \otimes V_{j_2^+,j_2^-}$$

one chooses the identity intertwiner $\iota_e = \mathrm{id}_{V_{j_1^+,j_1^-}} \otimes \mathrm{id}_{V_{j_2^+,j_2^-}}$.

Consistent deformations

One can show: If κ_1 and κ_2 are consistent deformations of each other (write $\kappa_1 \sim \kappa_2$), then

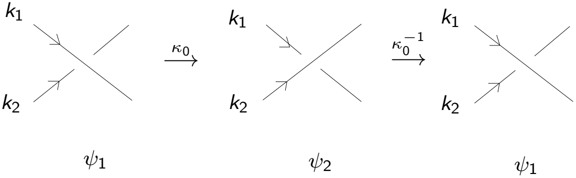
$$Z[\kappa_1] \;=\; Z[\kappa_2]$$

So in fact one can restrict the sum in the physical inner product over equivalence classes $[\kappa]$ instead of κ. We assume:

$$\langle\, \psi_f \,|\, \psi_i \,\rangle_{\rm phys} \quad = \quad \sum_{[\kappa]\,:\,\psi_i \xrightarrow{\kappa} \psi_f} Z[\kappa]$$

On κ_0:

Consider the two states ψ_1 and ψ_2 with a different knotting class, and the two-complex κ_0 inbetween them. We have shown already that $Z[\kappa_0] = 1$



So $\psi_1 \xrightarrow{\kappa_0} \psi_2$, hence $\psi_1 \xrightarrow{\kappa_0\kappa_0^{-1}} \psi_1$, where $\kappa_0\kappa_0^{-1}$ is the concatenation of the two spin foams κ_0 and κ_0^{-1}. One can show that:

$$\kappa_0\kappa_0^{-1} \sim \mathrm{id}_{\psi_1}$$

On the spin foam sum

It follows: Let ϕ be any spin network state. Then, there is a one-to-one correspondence between (equivalence classes of) spin foams $[\kappa_1]$ and $[\kappa_2]$, where

$$[\kappa_1] \,:\, \phi \xrightarrow{\kappa_1} \psi_1 \qquad [\kappa_2] \,:\, \phi \xrightarrow{\kappa_2} \psi_2$$

Define this by

$$[\kappa_2] \;:=\; [\kappa_1\kappa_0]$$

because of $\kappa_0\kappa_0^{-1} \;\sim\; \mathrm{id}_{\psi_1}$ this is actually a bijection between (equivalence classes of) spin foams!

We have already shown that

$$Z[\kappa_2] \;=\; Z[\kappa_1\kappa_0] \;=\; Z[\kappa_1]\,\underbrace{Z[\kappa_0]}_{=1} \;=\; Z[\kappa_1]$$

Hence:

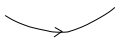
$$\begin{aligned}
\langle\, \phi \,|\, \psi_1 \,\rangle_{\rm phys} \quad &= \sum_{[\kappa_1]\,:\,\phi \xrightarrow{\kappa_1} \psi_1} Z[\kappa_1] \;=\; \sum_{[\kappa_1]\,:\,\phi \xrightarrow{\kappa_1} \psi_1} Z[\kappa_1\kappa_0] \\
&= \sum_{[\kappa_2]\,:\,\phi \xrightarrow{\kappa_2} \psi_2} Z[\kappa_2] \;=\; \langle\, \phi \,|\, \psi_2 \,\rangle_{\rm phys}
\end{aligned}$$

Summary

- Using the generalization of the EPRL vertex amplitudes to arbitrary two-complexes, we have considered the (ill-defined) state sum for the physical inner product

$$\langle \psi_f \,|\, \psi_i \rangle_{\mathrm{phys}} \quad = \quad \sum_{[\kappa]\,:\,\psi_f \xrightarrow{\kappa} \psi_i} \quad Z[\kappa]$$

and shown that

$\Big\langle \phi \,\Big|$  $\Big\rangle_{\mathrm{phys}} = \Big\langle \phi \,\Big|$  $\Big\rangle_{\mathrm{phys}}$

for all ϕ. Therefore the physical states will not contain any knotting information.

What we have assumed and what we have shown

- We have chosen a normalization of the $\mathcal{A}_e$, $\mathcal{A}_f$ such that $Z[\kappa_1\kappa_2] = Z[\kappa_1]Z[\kappa_2]$, and that $Z[\kappa]$ is invariant under trivial subdivisions ($\mathcal{A}_e$, $\mathcal{A}_f$ and the boundary amplitudes are fixed in terms of the vertex amplitude by this condition).
- We showed that the spin foam amplitude for an 'unknotting' $\psi_1 \stackrel{\kappa_0}{\to} \psi_2$ has $Z[\kappa_0] = 1$.
- We have shown that the spin foam amplitude $Z[\kappa]$ is unchanged under a consistent deformation of κ.
- We have assumed that by dividing out the action of the consistent deformations, no non-trivial measure factor arise:

$$\langle\, \psi_f \,|\, \psi_i \,\rangle_{\text{phys}} \quad = \quad \sum_{[\kappa]\,:\,\psi_i \xrightarrow{\kappa} \psi_f} \quad Z[\kappa]$$

What to change, in order to keep knotting classes of graphs:

The key point in the analysis is that for the 'unknotting' spin foam κ_0, one has $Z[\kappa_0] = 1$. This rests on the generalisation of the EPRL amplitude to more complicated vertices (in particular to ones with knotted neighbourhood SNF):