

Diffeomorphism symmetry and triangulation independence in discrete gravity

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with

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B.D. 0810.3594 [gr-qc]
B.Bahr, B.D. 0905.1670 [gr-qc]
B.Bahr, B.D. 0907.4323 [gr-qc]



B.Bahr, B.D. 0907.4325 [gr-qc]
B.D., P. Höhn 0912.1817 [gr-qc]
B.Bahr, B.D., S.He wip

OPLQG, Zakopane, 02.03. 2010

Questions

- ▶ Is there a notion of diffeomorphism symmetry in discrete models?
- ▶ Can it help us to address:
 - ▶ ambiguities and anomalies, lattice effects
 - ▶ path integral measure (for labels and triangulations)
 - ▶ sum over triangulations vs. continuum limit?
- ▶ Relation to triangulation independence? [Pfeiffer 04] How much do we have to sum over?
- ▶ Is the Regge action special?

Two Remarks

- ▶ We will apply a precise **dynamical** criterium for the existence/non-existence of symmetries. We give reasons to identify the (broken) symmetries we find in this way as discrete incarnations of (active) diffeomorphism symmetry. You might disagree and call these symmetries by some other name and then explain relationship to diffeomorphism symmetry in continuum limit.
- ▶ Although we explicitly work in a discrete setting, these considerations might be also useful for continuum theories where discrete features appear in the regularization of operators. [Perez & Pranzetti 10, Giesel & Thiemann AQG framework]

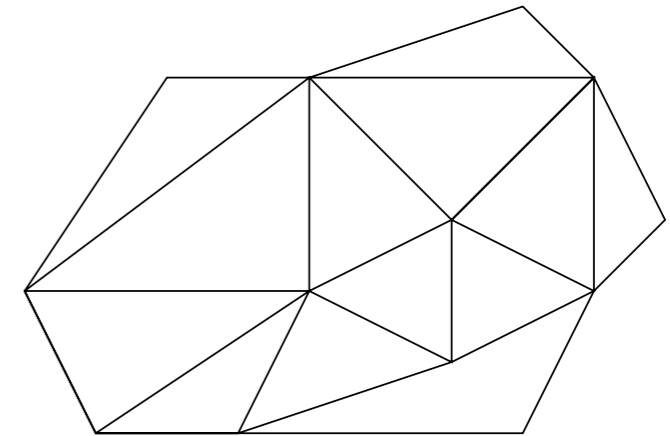
Overview

- A. Criterium for gauge symmetries
- B. Do we have gauge symmetries in discrete gravity?
- C. Why do we care?
- D. Improving the dynamics with renormalization
- E. Perturbative Expansion
- F. Repercussions for canonical formalism
- G. Conclusions

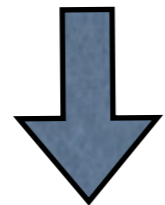
Set up: Regge calculus

(classical theory corresponding to spin foam models, lattice loop quantum gravity)

- approximate space time by piecewise flat triangulation
- length variables on edges fix geometry
- discrete action defines dynamics



$$S_{cont} = \int d^D x \sqrt{g} \left(\frac{1}{2} R - \Lambda \right)$$



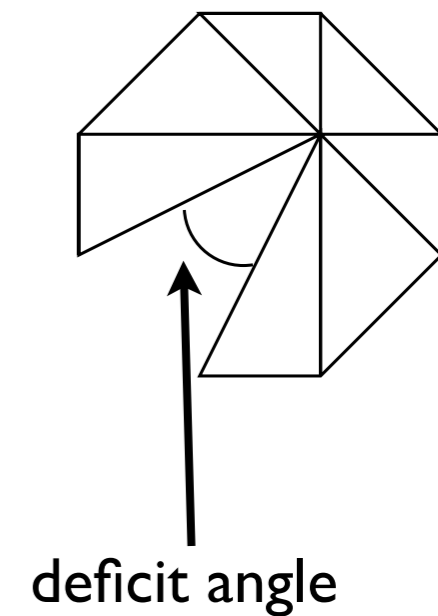
$$S_{discr} = \sum_{\text{hinges } h} F_h \epsilon_h - \Lambda \sum_{\text{simplices } \sigma} V_\sigma$$

4d: triangles
 3d: edges

volume of triangle/edge

deficit angle

volume of 4-simplex/
 tetrahedron



A. and B.

Is there a notion of diffeomorphism symmetry in discretized actions?

A. Criterium for gauge symmetries

- criterium: **non-uniqueness of solutions** for fixed boundary conditions

- ▶ $\det \left(\frac{\partial^2 S}{\partial x^i \partial x^j} \right) \Big|_{\text{solution}} = 0$

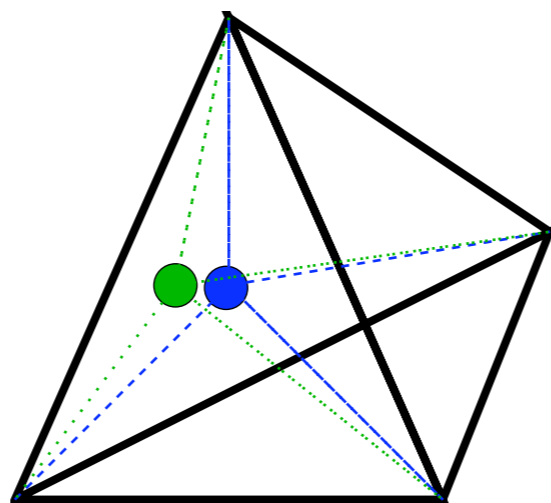
- existence of symmetries depends **on dynamics** (that is action)!
- different solutions might have gauge orbits of different size
- invariance of action not sufficient for gauge symmetry

- criterium relevant for
 - ▶ canonical analysis
 - ▶ perturbative expansion
 - ▶ counting of physical degrees of freedom

B. Gauge symmetries in Regge calculus?

- for boundary conditions leading to flat solutions: non-uniqueness of solutions!
⇒ there are gauge symmetries!
- 3d (vanishing cosmological constant): all boundary conditions lead to flat solutions
⇒ gauge symmetries for all configurations
- 4d (vanishing cosmological constant): some boundary conditions lead to flat solutions
⇒ gauge symmetries for these configurations
- gauge modes correspond to changing position of vertices on flat background
⇒ matched to continuum diffeomorphism symmetry in linearization [Rocek and Williams 81]

vertex translation acting on
flat solution

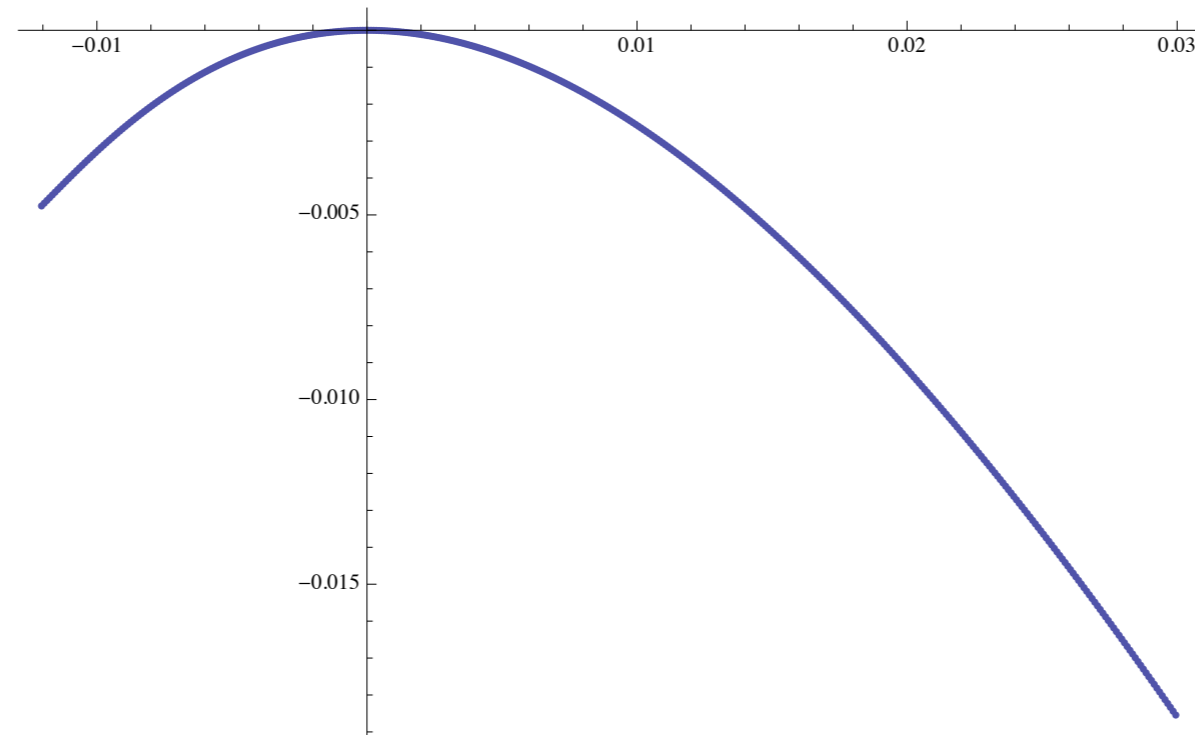


Hessian of action evaluated on
flat solutions has null modes

B. Gauge symmetries in Regge calculus?

For (a) curved solution: symmetries are broken.

[Bahr, BD 09]



lowest eigenvalues of Hessians as function of deviation parameter from 4d flat solution (curvature)

Symmetry is broken, effect quadratic in curvature.

Is there a notion of diffeomorphism symmetry in discretized actions?

Yes.

- 1) flat vertices allow for gauge modes corresponding to translations
- 2) invariance (slightly) broken at non-flat vertices for the Regge action; effect is quadratic in curvature

C. Why do we care?

- exact symmetries \Rightarrow exact (first class) constraints

[Gambini & Pullin et al 03-05, et al, Bahr & BD 09, BD & Hoehn 09]

- anomalies in quantization (by regularization) vs fixing of ambiguities

[for instance Perez & Pranzetti 10 in 3d with cosm. const.]

- perturbative expansion around flat geometries is very subtle if symmetries are broken

[related: Horava-Lifshitz gravity]

- path integral computation: no propagator for pseudo gauge modes

- strong condition on measure in path integral

- action with exact diffeomorphism symmetry hopefully related to triangulation independent Hamilton-Jacobi functional: **control sum over triangulation!**

D. Is there a discretization with exact symmetry?

Gauge symmetries are properties of the (discrete) action.
⇒ Improve the action.

Construct better actions

- remember: (broken) symmetries are properties of action
- idea: construct actions that capture better continuum dynamics
 - [Improved and Perfect actions: ..., Symanzik, Wilson, Hasenfratz et al in QCD: avoid Lorentz symmetry breaking!]
- by renormalization group transformation:
 - fine grain and integrate out fine grained degrees of freedom
 - obtain effective action on coarse grained lattice, capturing dynamics of fine grained lattice

Question: Do we regain local gauge symmetries from continuum?

Examples

1d discretized
systems,
perturbatively
and non-
perturbatively

[quantum: Bahr,
Steinhaus & BD wip]

It works!

3d Regge
calculus with
cosmological
constant

[Bahr & BD 09]

It works!

4d Regge
calculus,
perturbative
expansion

[BD & Hoehn 09]
[Bahr & BD & He wip]

3d Regge
calculus with
matter

[Banisch & BD wip]

3d Regge calculus

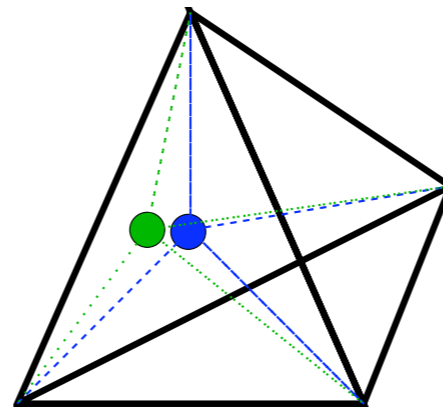
without cosmological constant

- any triangulation of flat space is a solution
- at every vertex 3dim translation symmetry
- triangulation independent
- zero physical degrees of freedom

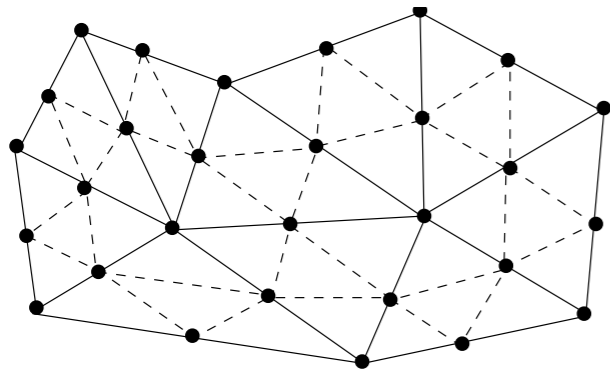
with cosmological constant

- unique solutions to equations of motion
- there is no translation symmetry acting
- not triangulation independent
- all degrees of freedom physical

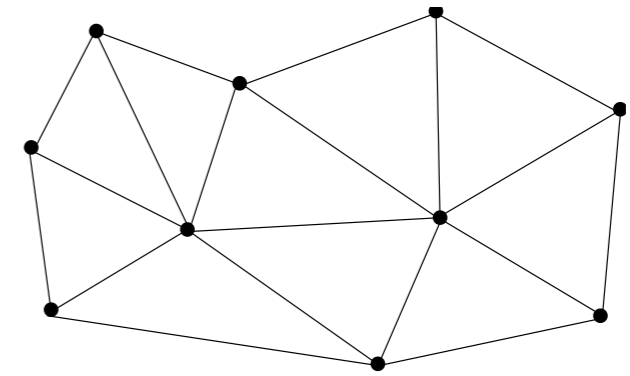
exact diffeo symmetry



approximate diffeo symmetry



→
integrate out small edge lengths



3d Regge with cosmological constant

3d Regge with curved simplices

[B.Bahr, BD 09]

$$S_{\mathcal{T}} = \sum_e l_e \epsilon_e - \Lambda \sum_{\sigma} V_{\sigma}$$

action for flat simplices

approximate
symmetries,
triang. dependent

$$S_{\mathcal{T}}^{\kappa} = \sum_E L_E \epsilon_E^{\kappa} + 2\kappa \sum_{\sigma} V_{\sigma}^{\kappa}$$

action for simplices with curvature

$$\kappa = \Lambda$$

exact
symmetries,
triang. independent

1d reparametrization invariant systems

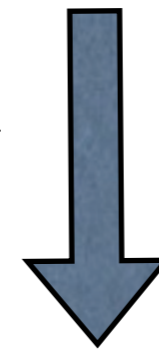
continuum:

- take q and t as variables
- use auxiliary parameter evolution parameter s
- solutions $t(s), q(s)$ invariant under reparametrizations in s

$$L = t' \left(\frac{m}{2} \frac{q'^2}{t'^2} - V(q) \right)$$

discretization

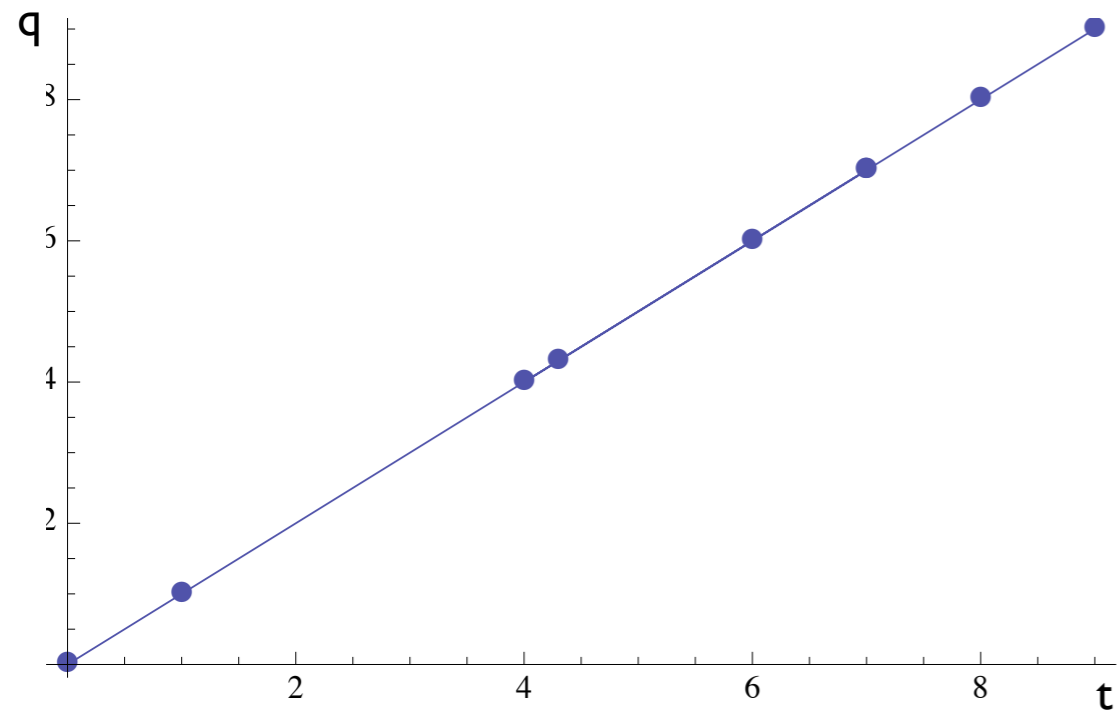
$s \rightarrow n$



$$L(n, n+1) = (t_{n+1} - t_n) \left(\frac{m}{2} \frac{(q_{n+1} - q_n)^2}{(t_{n+1} - t_n)^2} - V\left(\frac{1}{2}q_n + \frac{1}{2}q_{n+1}\right) \right)$$

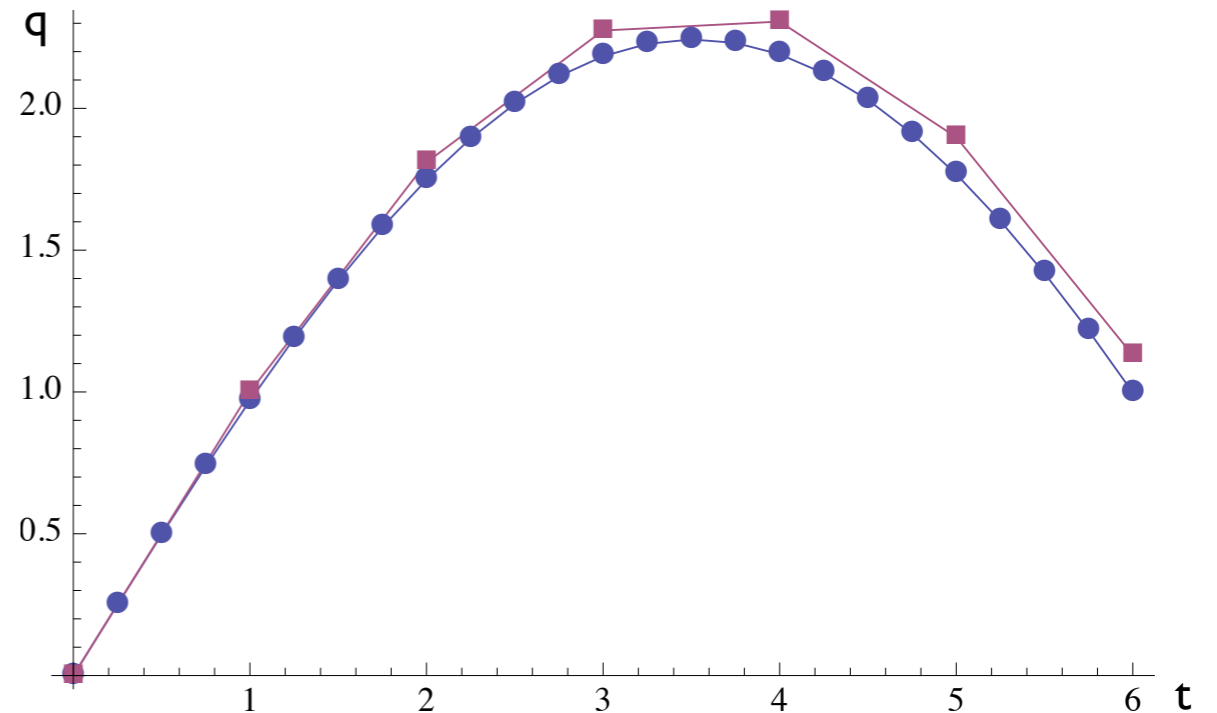
- vertex translations symmetry for $V = 0$
- symmetry broken for $V \neq 0$ [Gamini Pullin 03, Marsden West 01]

Examples



- vanishing potential
- position of vertices arbitrary
- one gauge mode
- refinement independent

Remark: piecewise linear approximation added by hand!



- quadratic potential
- position of vertices fixed
- one pseudo gauge mode
- refinement dependent

- linearization around solution: kinetic term of pseudo gauge mode vanishing
- gauge breaking in potential!

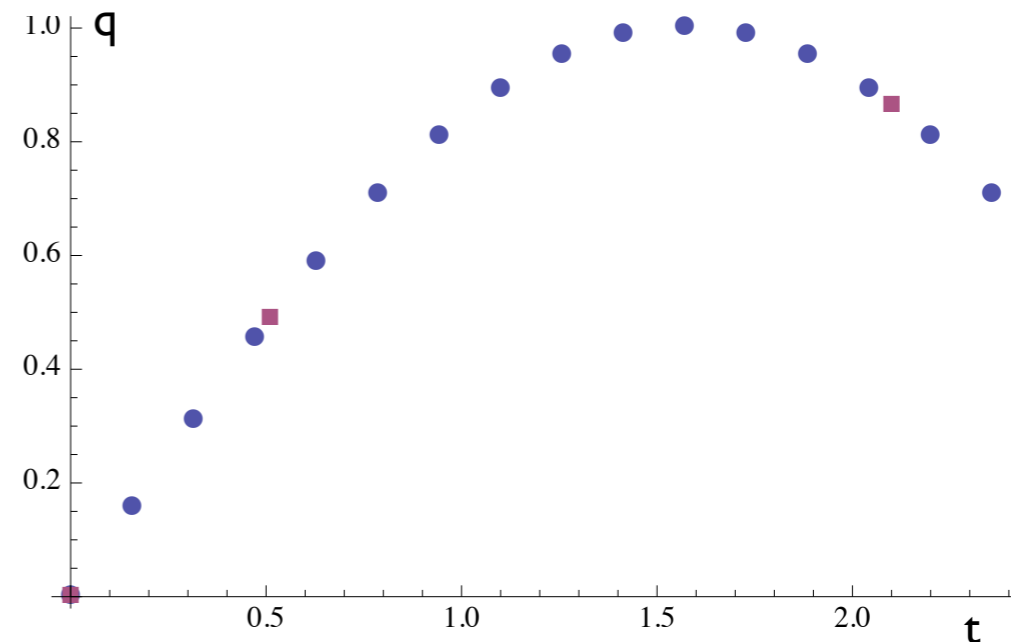
Id reparametrization invariant discrete systems

- There is always a discrete action with exact symmetries!
- trick: use the Hamilton-Jacobi functional of continuum theory as discrete action
- \Rightarrow discrete theory captures exactly continuums dynamic
- can be obtained by integrating out almost all variables (“renormalization group flow”) \longrightarrow refinement independent

$$S_e = \sum_{n=0}^{N-1} S_{HJ}^{s_n, s_{n+1}}(t_n, q_n, t_{n+1}, q_{n+1})$$

$$= \sum_{n=0}^{N-1} \int_{s_n}^{s_{n+1}} ds L(t(s), q(s)) \quad .$$

continuum solution \nearrow



Remark: piecewise linear approximation introduces errors

Existence of symmetries depends on the dynamics.

This dynamics can be improved by constructing actions that approximate continuum dynamics very well/perfectly.

Interpretation of discrete building blocks depends on dynamics.

Do not see them literally as (flat) blocks but as representing coarse grained quantities.

4d?

- action will be non-local, but might be triangulation independent [Bahr, BD, He wip]
- impossible to solve equation of motion non-perturbatively:
 - ⇒ expansion around flat space
- What are the properties of this expansion?
To which order are the gauge symmetries/ triangulation independence realized?

Regge calculus

- gauge symmetries for flat solutions
- background gauge parameters
position of vertices in flat background
- symmetries broken for curved solutions

Parametrized (an-)harmonic oscillator

- gauge symmetries for $q_n = 0$, t_n arbitrary
- background gauge parameters
 t_n
- symmetries broken for $q_n \neq 0$

E. Perturbative expansion

[BD, Höhn 09]

$$x^i = x_0^i + \varepsilon x_1^i + \varepsilon^2 x_2^i + \dots$$



$$S = \varepsilon^2 \frac{1}{2} S_{ij} x_1^i x_1^j + \varepsilon^3 S_{ij} x_2^i x_1^j + \varepsilon^3 \frac{1}{3!} S_{ijk} x_1^i x_1^j x_1^k + \dots$$



solutions not unique



solutions unique

?

We will see:

- Typically: consistent expansion only possible for specific choices of background gauge.
- For other choices: $x_1 \sim \varepsilon^{-1}$
- Precise relation with invariance properties of (truncated) Hamilton-Jacobi functional.

E. Perturbative expansion

[BD, Höhn 09]

$$S = \varepsilon^2 \frac{1}{2} S_{ij} x_1^i x_1^j + \varepsilon^3 S_{ij} x_2^i x_1^j + \varepsilon^3 \frac{1}{3!} S_{ijk} x_1^i x_1^j x_1^k + \dots$$

linear order:

$$S_{ij}(x_0) y_g^i(x_0) = 0, \quad y_g^i(x_0) \text{ null vectors with index } g$$

$$x_O^i = x_O^g y_g^i + x_O^p y_p^i \quad \text{coordinate transformation to gauge and physical modes}$$

$$\longrightarrow x_0^g \text{ and } x_1^g \text{ remain free}$$

$$\longrightarrow x_1^p \text{ determined}$$

first non-linear order:

$$\longrightarrow x_1^g \text{ and } x_2^g \text{ remain free}$$

$$\longrightarrow x_2^p \text{ determined}$$

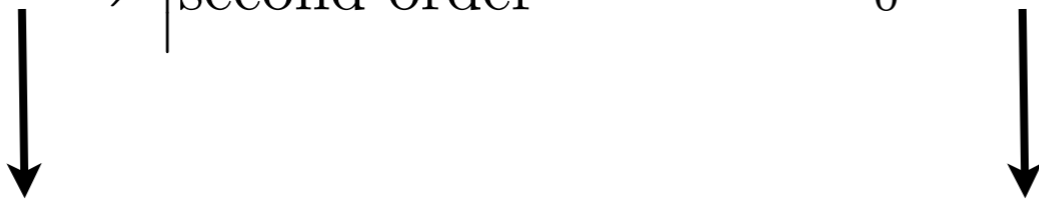
But there are still the EOM for the gauge modes!

E. Perturbative expansion

[BD, Höhn 09]

Theorem: After solving for the physical modes we have

$$\left(\frac{S}{\partial x^i} y_g^i \right) \Big|_{\text{second order}} = y_g^i \frac{\partial}{\partial x_0^i} (S_{HJ}) \Big|_{\text{second order}}$$



EOM in non-linear theory computed in linearized theory!

in particular:

- first order and second order gauge variables do not appear in EOM
- if EOM is not automatically zero: have to use it as a consistency condition for background gauge parameters
- EOM is automatically zero if Hamilton Jacobi functional of linearized theory does not depend on background gauge parameters

Interpretation: background parameters get fixed such that dependence of Hamilton-Jacobi functional on these parameters is minimal.

Hamilton-Jacobi functional for linearized Regge

Does the Hamilton-Jacobi functional of linearized Regge calculus depend on background gauge?

Yes! (for a specific example) [BD, Hoehn 09]

-also the case for the parametrized (an-)harmonic oscillator

Consistent perturbative expansion only possible around certain choices for positions of vertices.

Although linearized Regge has exact symmetries,
it is not triangulation independent.

Need to improve even the quadratic part of the Regge action.

[Bahr, BD, He wip]

Improving the action order by order

$$S = \varepsilon^2 \frac{1}{2} S_{ij} x_1^i x_1^j + \varepsilon^3 S_{ij} x_2^i x_1^j + \varepsilon^3 \frac{1}{3!} S_{ijk} x_1^i x_1^j x_1^k + \dots$$

↓ improve

$$S = \varepsilon^2 \frac{1}{2} S_{ij}^{IMP} x_1^i x_1^j + \varepsilon^3 S_{ij}^{IMP} x_2^i x_1^j + \varepsilon^3 \frac{1}{3!} S_{ijk} x_1^i x_1^j x_1^k + \dots$$

now background gauge arbitrary
to non-linear order

↓ improve

$$S = \varepsilon^2 \frac{1}{2} S_{ij}^{IMP} x_1^i x_1^j + \varepsilon^3 S_{ij}^{IMP} x_2^i x_1^j + \varepsilon^3 \frac{1}{3!} S_{ijk}^{IMP} x_1^i x_1^j x_1^k + \dots$$

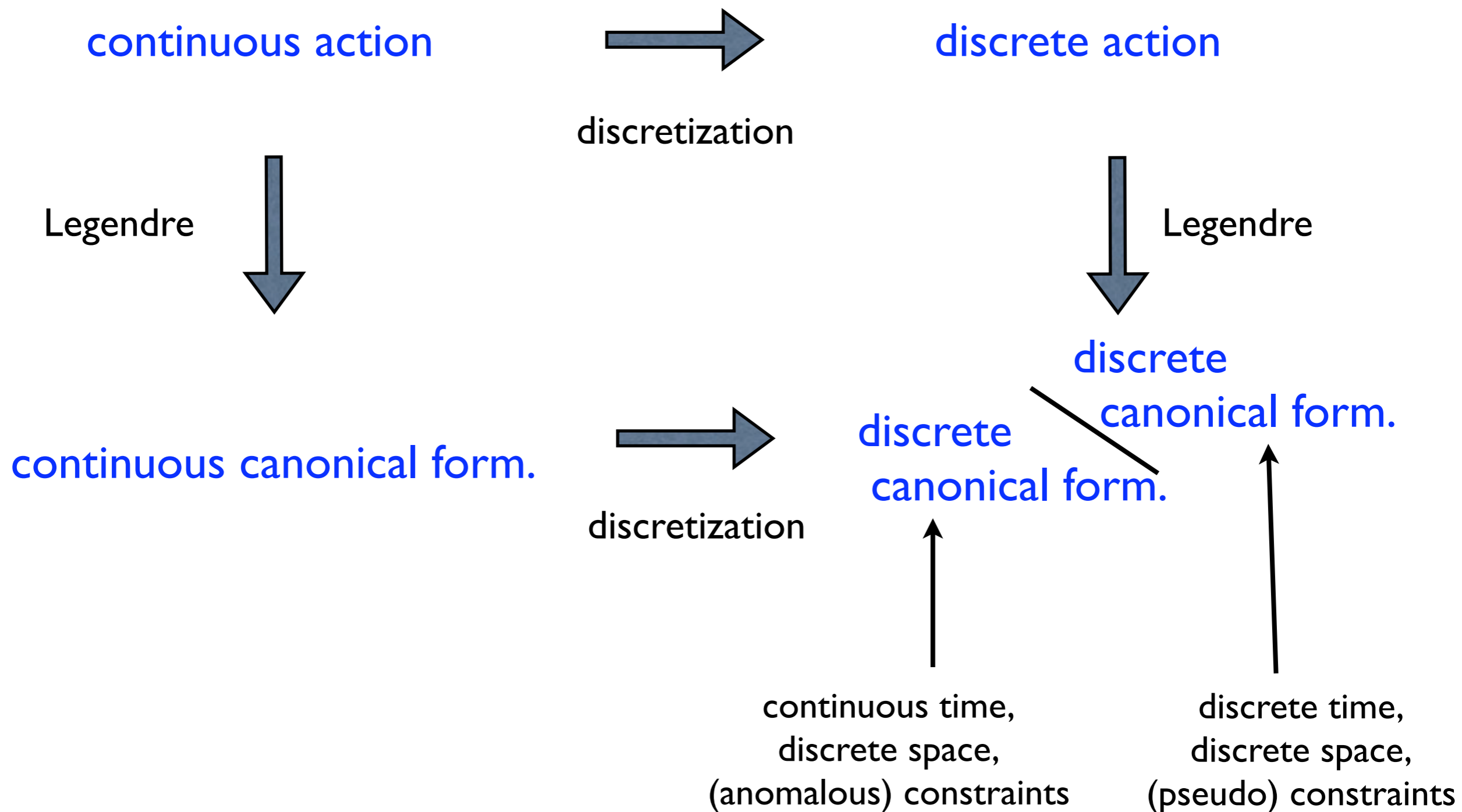
It works not only for the harmonic oscillator but also for the anharmonic one!

Repercussions for canonical framework and quantization?

typical problem of lattice approaches:
anomalous constraint algebra, inconsistent dynamics

- a) Canonical formalism reproducing exactly solutions and (broken) symmetries of discretized action?
- b) Constraints? Constraint algebra? Anomalies?

Canonical Framework



Canonical Framework

[Bahr, BD '09; BD, Höhn 09]

- evolve spatial triangulation locally by tent moves [Sorkin 75, Barrett et al 97]
- finite time steps
- use action as generating function for time evolution map
[consistent discretizations, Gambini & Pullin et al 03-05]

- reproduces (broken) symmetries exactly [Bahr, BD 09] :

symmetries exact \Rightarrow eom not independent \Rightarrow constraints (first class)

broken \Rightarrow eom almost not independ. \Rightarrow pseudo-constraints

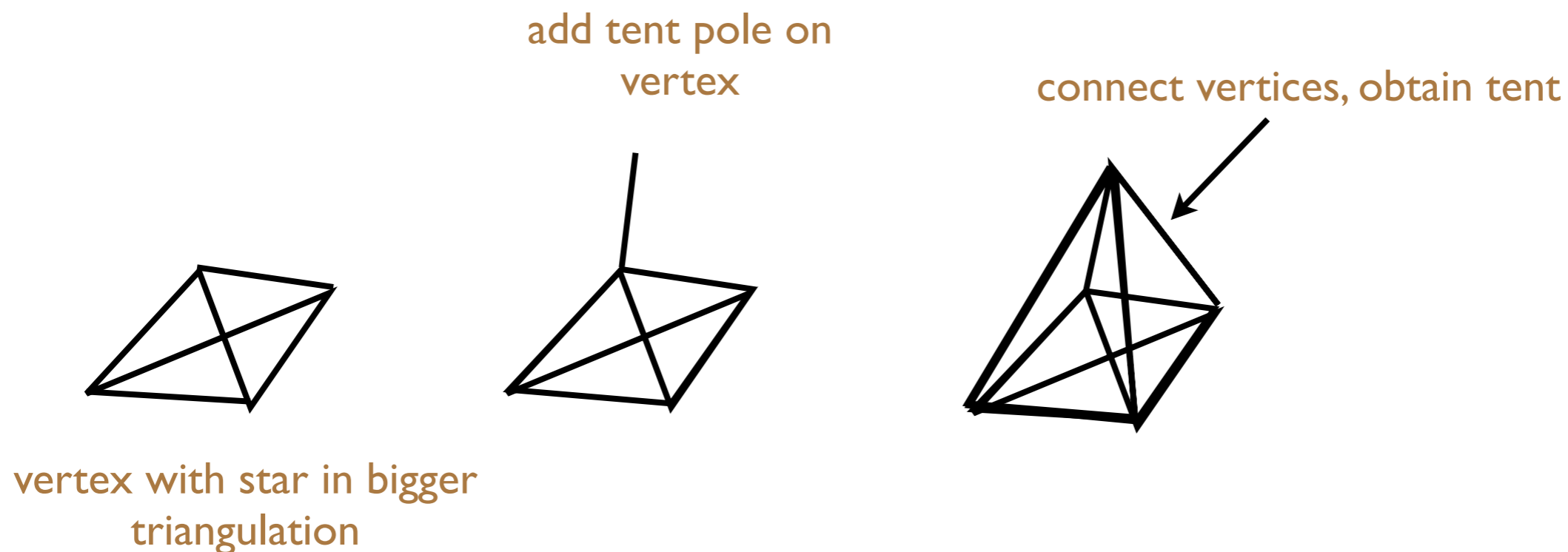
Obtaining anomaly free constraints is equivalent to constructing an action with exact symmetries.

Evolving spatial triangulations with tent moves

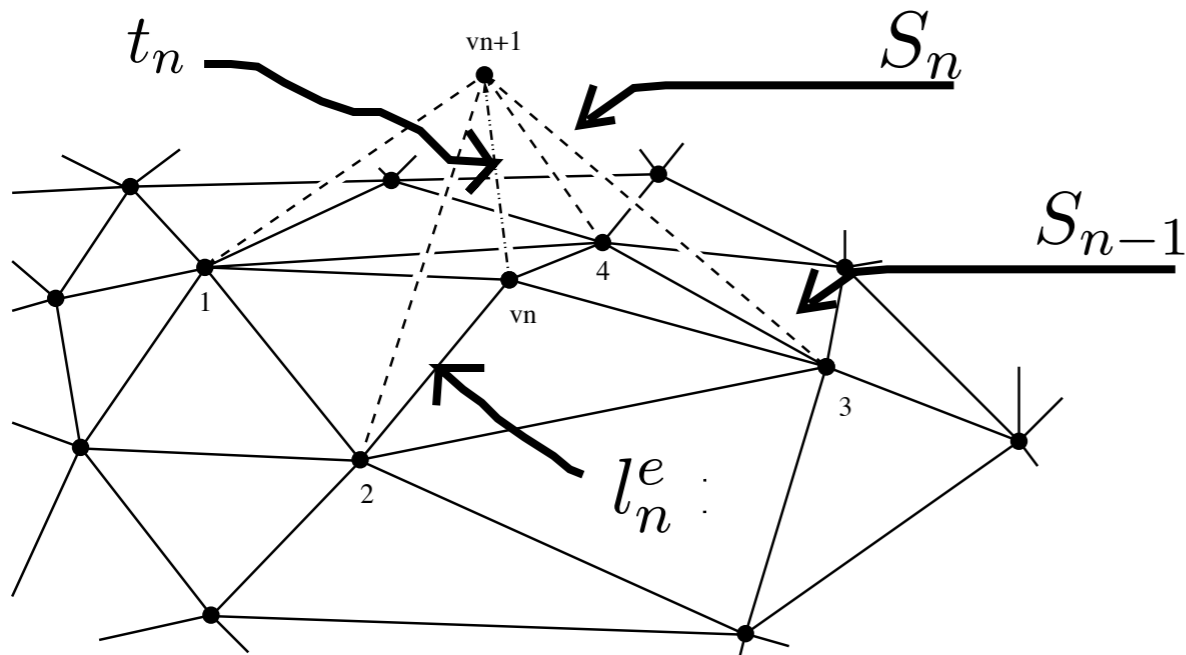
[Sorkin 75, Barrett et al 97]

time evolution moves:

- do not change spatial triangulation/ number of variables
- act local, involving only star of a vertex
- can obtain local (pseudo-) constraints based at vertices



Canonical Framework



equations of motion:

$$0 = \frac{\partial S_n}{\partial t_n} \longrightarrow -p_t^n$$

$$0 = \frac{\partial S_{n-1}}{\partial l_n^e} + \frac{\partial S_n}{\partial l_n^e}$$

$$\downarrow \qquad \qquad \downarrow$$

$$-p_e^n \qquad \qquad p_e^n$$

canonical (tent move)
transformation:

$$p_t^n := -\frac{\partial S_n}{\partial t_n} \qquad p_e^n := -\frac{\partial S_n}{\partial l_n^e}$$

$$p_t^{n+1} := \frac{\partial S_n}{\partial t_{n+1}} \qquad p_e^{n+1} := \frac{\partial S_n}{\partial l_{n+1}^e}$$

use S_n as generating
function for canonical
transformation

4-valent vertex: flat dynamics

equation for the tent pole

$$0 = p_t^n = -\frac{\partial S_n}{\partial t_n} = -\sum_{\Delta \supset t} \frac{\partial A_\Delta}{\partial t_n} \epsilon_\Delta$$

solution

$$\epsilon_\Delta = 0$$

momenta associated to edges

$$p_e^n = -\frac{\partial S_n}{\partial l_n^e} = -\sum_{\Delta \supset e} \frac{\partial A_\Delta}{\partial l_n^e} \psi_\Delta - \sum_{\Delta \supset e} \frac{\partial A_\Delta}{\partial l_n^e} \epsilon_\Delta$$

constraints

$$C_e = p_e^n + \sum_{\Delta \supset e} \frac{\partial A_\Delta}{\partial l_n^e} (l_{e'}^n) \psi_\Delta (l_{e'}^n)$$

Momenta do not depend on variables at next time step \Rightarrow **constraints.**

For higher valent vertices $\epsilon_\Delta \neq 0$, momenta depend (weakly) on variables at next time step \Rightarrow **pseudo constraints.**

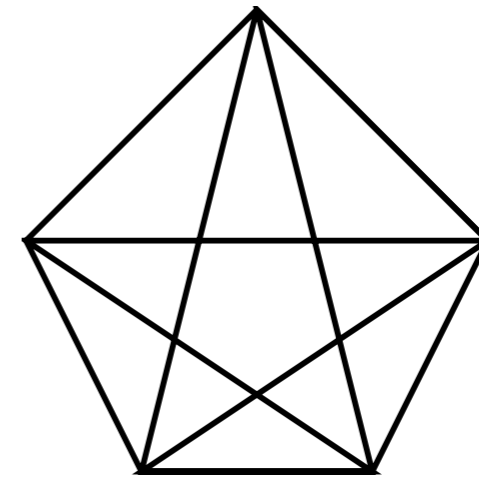
'Dynamics' for a 4-simplex [BD, Ryan 08, BD, Hoehn 09]

- 3d surface of a 4-simplex: five 4-valent vertices
- apply constraints to every vertex

$$C_e = p_e + \sum_{\Delta \supset e} \frac{\partial A^\Delta}{\partial l^e} \psi_\Delta(l)$$

dihedral angles

geometric meaning?



- symplectic coordinate transformation:

$$A_\Delta = A_\Delta(l), \quad p_\Delta = \frac{\partial l^e}{\partial A^\Delta} p_e \quad \longrightarrow \quad C_\Delta = p_\Delta + \psi_\Delta(l)$$

'Dynamics' for a 4-simplex [BD, Ryan 08, BD, Hoehn 09]

$$C_{\Delta} = p_{\Delta} + \psi_{\Delta}(l)$$

- constraints fix the momenta to agree with the dihedral angles as defined by lengths
- are first class! (despite very complicated form of dihedral angles)
- generate deformation of hypersurface (via vertex translations): Hamiltonian and diffeomorphism constraints
- 3d surface of a 4-simplex: **zero physical degrees of freedom**: no 4d curvature

Higher-valent vertex: (linearized) dynamics [BD, Hoehn 09]

For higher valent vertices $\epsilon_\Delta \neq 0$, momenta depend (weakly) on variables at next time step
 \Rightarrow **pseudo constraints.**

But for the linearized dynamics \Rightarrow **constraints.**

$$l = {}^0l + y, \quad p = {}^0p + \pi, \quad S = y^e \frac{\partial S}{\partial l^e \partial l^{e'}} y^{e'}$$

Hessian on flat space has null eigenvectors Y_I^e . \longrightarrow gauge symmetries

\longrightarrow **linearized constraints**

$$C_I = Y_I^{e'} \pi_e^n + Y_I^{e'} \left[\frac{\partial}{\partial l_n^{e'}} \sum_{\Delta \supset e} \frac{\partial A_\Delta}{\partial l_n^e} \psi_\Delta \right]_{|l={}^0l} y_n^e$$

Higher-valent vertex: (linearized) dynamics [BD, Hoehn 09]

$$C_I = Y_I^{e'} \pi_e^n + Y_I^{e'} \left[\frac{\partial}{\partial l_n^{e'}} \sum_{\Delta \supset e} \frac{\partial A_\Delta}{\partial l_n^e} \psi_\Delta \right]_{|l=0|} y_n^e$$

- constraints give relation between intrinsic and extrinsic geometry
- are first class! (despite very complicated form of dihedral angles)
- generate linearized deformation of hypersurface (via vertex translations): Hamiltonian and diffeomorphism constraints
- preserved by linearized tent move dynamics (analogous to quadratic Hamiltonian)
- split into gauge and physical variables (relation to linearized curvature on inner triangles)

Repercussions:

1) action with exact symmetries:

-proper first class constraints, gauge freedom

2) action with broken symmetries:

-pseudo constraints with weak dependence on lapse/shift

3) linearized theory inherits symmetries of solution

-exact constraints in linearized theory

-background gauge gets fixed at lowest non-linear order

Conclusions

- discrete actions generally break diffeomorphism symmetries
- regaining symmetries by coarse graining, renormalization
- canonical framework exactly mimics covariant symmetries: constraints and pseudo-constraints
- perturbative expansion subtle: background gauge fixed if symmetries are broken

Prospects

- understand triangulation (in-)dependence and investigate non-locality properties of improved actions
- generalize curved simplices: improved action adapted to background solution (application to cosmology)
- improved quantum action/ renormalization in spin foams
- canonical quantization: improve constraints

- Explore general mechanisms and conditions for regaining gauge symmetries.