Coherent loops and holomorphic foams

Claudio Perini

Centre de Physique Théorique de Luminy, Marseille
Università degli studi Roma Tre

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Coherent spin-networks (arXiv:0912.4054)
Holomorphic vertex amplitudes for quantum gravity (to appear very soon)

Collaboration with E. Bianchi, E. Magliaro

- Coherent spin-networks are introduced via a heat-kernel technique
- The labels are the ones used in Spin Foam semiclassical calculations
- The set of labels can be viewed as $SL(2,\mathbb{C})$ elements $\rightarrow$ coincide with the gauge invariant projection of Hall's coherent states for $SU(2)$
- We study the properties of semiclassicality an find that they reproduce a superposition over spins with nodes labeled by Livine-Speziale coherent intertwiners. The weight associated to spins is a Gaussian with phase, as originally proposed by Rovelli
- Associated to the holomorphic representation of Loop Quantum Gravity
- New: holomorphic representation of Spin Foams
- Conclusions and discussion

A key ingredient in the semiclassical calculations are the semiclassical states. They are peaked on a prescribed intrinsic and extrinsic geometry of space. The original idea of Rovelli was to take a superposition over spins of spin-network states, with a simple ansatz for the weight associated to each link:

$$c_j(j_0, \xi) = \exp\Big(- \frac{(j-j_0)^2}{2\sigma_0}\Big) \exp(-i\xi j) \tag{1}$$

The spin j_0 is the classical value of the area of the surface cut by the link. The angle ξ is the variable conjugate to the spin, the 4-dimensional dihedral angle coding the extrinsic curvature.

The dispersion is chosen to be given by $\sigma_0 \approx (j_0)^k$ (with $0 < k < 2$) so that, in the large j_0 limit, both variables have vanishing relative dispersions. Those kind of states where used for the calculations with the Barrett-Crane SFM.

On the other hand, Rovelli and Speziale introduced an ansatz for the semiclassical tetrahedron (superposition over virtual spins at each node):

$$c_k(k_0, \phi) = \exp\Big(- \frac{(k-k_0)^2}{2\tau_0}\Big) \exp(-i\phi k) \tag{2}$$

The virtual spin k_0 is the classical value of the 3-dimensional dihedral angle between two faces of the tetrahedron. The phase ϕ is needed to peak on the correct value all the dihedral angles.

Livine-Speziale coherent intertwiner

The Rovelli-Speziale ansatz can be introduced through the mathematical theory of coherent states for $SU(2)$. A coherent state is defined by:

$$\vec{J}\cdot\vec{n}\,|j,\vec{n}\rangle = j\,|j,\vec{n}\rangle \tag{3}$$

There is a phase ambiguity $|j,\vec{n}\rangle \to e^{i\alpha}|j,\vec{n}\rangle$. CS (3) minimize

$$\langle J^2\rangle - \langle J\rangle^2 = j \tag{4}$$

Livine-Speziale (N-valent) coherent intertwiner is

$$|j_a,\vec{n}_a\rangle = \int g \triangleright \otimes_{a=1}^{N} |j_a,\vec{n}_a\rangle \mathrm{d}g \tag{5}$$

Their components on the usual *virtual spin* basis are:

$$\Phi_i(\vec{n}_a) = v_i \cdot \Big(\otimes_{a=1}^{N}\,|j_a,\vec{n}_a\rangle\Big) \tag{6}$$

When $N=4$, the coefficients (6) reproduce the Rovelli-Speziale ansatz in the large spin limit. The states (5) have good semiclassical properties, e.g.

$$\langle j_a,\vec{n}_a|V(\vec{J})|j_a,\vec{n}_a\rangle = V(j\vec{n}) + corr. \tag{7}$$

The volume operator gives the classical volume of a tetrahedron, for $j \gg 1$.

Graviton propagator from the new Spin Foams

The new Spin Foam models give non trivial dynamics to the intertwiner d.o.f. To test their semiclassical limit, we can use a boundary state with a Gaussian weight associated to spins, and nodes labeled by Livine-Speziale intertwiners.
Our candidate semiclassical state (Bianchi-Magliaro-CP) was:

$$|\Psi\rangle = \sum_{j_{ab}, i_a} \exp\big(-\frac{(j_{ab}-j_0)^2}{2\sigma_0}\big) \exp(-i\xi j_{ab}) \Phi_{i_a}(\vec{n}_{ab}) |j_{ab}, i_a\rangle \tag{8}$$

Recall our definition of LQG graviton propagator. It is the connected 2-point function of electric-flux operators (indices omitted) acting at 2 different nodes a, b

$$G(a,b) = \langle E_a \cdot E_a E_b \cdot E_b \rangle - \langle E_a \cdot E_a \rangle \langle E_b \cdot E_b \rangle \tag{9}$$

With the state (8) for a 4-simplex, we found, in the large j_0 limit with γj_0 fixed:

$$\tilde{G}(a,b) = \frac{M}{l^2} + corr. \tag{10}$$

with M the tensorial structure of the standard propagator of perturbative quantum gravity. In other words, we showed at least in this simplified context that the new SFM, together with our ansatz for the boundary semiclassical state, overcome the problem of BC SFM.

1st question: can we find a top-down derivation of our states (8) ?

On the other hand, within the canonical framework, Thiemann and collaborators have strongly advocated the use of complexifier coherent states.

When restricted to a single graph, they are labeled by an $SL(2,\mathbb{C})$ element per each link. Their peakedness properties (Thiemann-Bahr) have been studied in detail. However the relation of $SL(2,\mathbb{C})$ labels with LS coherent states, and with the Rovelli ansatz remained unexplored.

In the literature there is confusion on which kind of semiclassical states are the correct ones: Thiemann's complexifier coherent states or the ones used in Spin Foams (e.g. graviton propagator) ?

2nd question: can we clarify this confusion?

1st answer = 2nd answer

Thiemann's coherent states coincide with our artigianal ansatz for the graviton propagator !

More precisely: they coincide in a particular limit, and with a particular choice of the heat-kernel time which is fixed in terms of $SL(2,\mathbb{C})$ labels.

Trick: write $SL(2,\mathbb{C})$ in terms of geometric labels of area, extrinsic curvature, and 3d unit vectors

Consider a general graph Γ. Coherent spin-networks are defined as follows: we consider the gauge-invariant projection of a product over links of Hall's heat kernels,

$$\Psi_{\Gamma,H_{ab}}(h_{ab}) = \int \big(\prod_a dg_a\big) \prod_{ab} K_{t_{ab}}(g_a\, h_{ab}\, g_b^{-1}, H_{ab}) \tag{11}$$

The notation refers to a complete graph, just for convenience. These are the coherent states associated to the Segal-Bargmann transform for theories of connections introduced by Ashtekar, Lewandowski, Marolf, Mourao, Thiemann in 1994.

We rediscovered them following a completely different path; the path we followed (not in the paper) came with a particular interpretation of the $SL(2,\mathbb{C})$ labels and with a particular way of taking the semiclassical large distance limit.

The main observation is the following: every $SL(2,\mathbb{C})$ element can written in terms of

$$(\eta, \vec{n}, \vec{n}', \xi) \qquad H = n\, e^{z\tau_3}\, \tilde{n}^{-1} \qquad z = \xi + i\eta \tag{12}$$

A positive real number η, two unit vectors $\vec{n}$, $\vec{n}'$, an angle ξ. $\vec{n}$ is the (unit-)flux of the electric field $\vec{E}$ through a surface intersected by the link, as viewed from the first node. $\vec{n}'$ the flux viewed from the second node. Finally, η is related to modulus of the electric field, namely to the area of the surface. Exactly the labels used in Spin foams!

The coherent spin-network can be expanded on the spin-network basis

$$\Psi_{\Gamma,H_{ab}}(h_{ab}) = \sum_{j_{ab}} \sum_{i_a} f_{j_{ab},i_a} \; \Psi_{\Gamma,j_{ab},i_a}(h_{ab}) \tag{13}$$

with components

$$f_{j_{ab},i_a} = \Big(\prod_{ab} (2j_{ab}+1) e^{-j_{ab}(j_{ab}+1)t_{ab}} \Pi^{(j_{ab})}(H_{ab}) \Big) \cdot \Big(\prod_a v_{i_a} \Big) \tag{14}$$

We are interested in its asymptotics for $\eta_{ab} \gg 1$. The crucial observation is that in this limit, we have the following asymptotic behavior

$$\overset{j_{ab}}{\Pi} (e^{-iz_{ab}\tau_3})_{mm'} = \delta_{mm'} e^{-imz_{ab}} \sim \delta_{mm'} \, e^{+\eta_{ab} j_{ab}} \, \delta_{m,j_{ab}} e^{-i\xi_{ab} j_{ab}} \tag{15}$$

Therefore, introducing the projector $P_+ = |j_{ab}, +j_{ab}\rangle\langle j_{ab}, +j_{ab}|$ on the highest magnetic number, we can write (15) as

$$\overset{j_{ab}}{\Pi} (e^{-iz_{ab}\tau_3}) \sim e^{-i\xi_{ab} j_{ab}} e^{+\eta_{ab} j_{ab}} P_+ \tag{16}$$

The projection on the highest magnetic number is the key for the link with coherent states of $SU(2)$, hence with the Livine-Speziale intertwiners: $SU(2)$ coherent states are defined as the (rotations of the) highest magnetic number states.

Next step: notice that

$$-j(j+1)t + j\,\eta = -\big(j - \frac{\eta - t}{2t}\big)^2\,t + \frac{(\eta - t)^2}{4t} \tag{17}$$

so defining

$$(2j^0_{ab} + 1) \equiv \frac{\eta_{ab}}{t_{ab}} \quad \text{and} \qquad \sigma^0_{ab} \equiv \frac{1}{2t_{ab}} \tag{18}$$

we find the following asymptotics for coherent spin-networks:

$$f_{j_{ab},i_a} \sim \Big(\prod_{ab} \exp\big(-\frac{(j_{ab} - j^0_{ab})^2}{2\sigma^0_{ab}}\big)\, e^{-i\xi_{ab}j_{ab}}\Big)\,\Big(\prod_{a}\Phi_{i_a}(n_{ab})\Big) \tag{19}$$

These are exactly the semiclassical states we considered for the graviton propagator. In particular, their intertwiners are the Livine-Speziale ones. The parameters ξ were chosen so to reproduce the extrinsic curvature, so also in this more general context ξ must be interpreted an an extrinsic angle, as originally proposed by Rovelli.

This result confirms the geometric interpretation of our variables and extends the validity of the semiclassical states used in Spin Foams well beyond the simplicial setting: coherent spin-networks are defined in full LQG.

In the large η limit, the expectation value of the area operator is easily computed

$$\langle A \rangle = \frac{(\Psi_{\gamma,\xi+i\eta},\, \hat{A}\,\Psi_{\gamma,\xi+i\eta})}{(\Psi_{\gamma,\xi+i\eta},\, \Psi_{\gamma,\xi+i\eta})} = \gamma L_P^2 \sqrt{j_0(j_0+1)} \tag{20}$$

and confirms the interpretation of η as the quantity that prescribes the expectation value of the area. The Wilson loop operator acts on basis vectors as

$$\hat{W}_\gamma\, \chi^{(j)}(h) = \chi^{(\frac{1}{2})}(h)\, \chi^{(j)}(h) = \chi^{(j+\frac{1}{2})}(h) + \chi^{(j-\frac{1}{2})}(h) \tag{21}$$

As a result, we find

$$\langle W_\gamma \rangle = 2\, \cos(\xi/2)\; e^{-\frac{t}{8}} \tag{22}$$

Therefore, in the limit $t \longrightarrow 0$ compatible with η large, the parameter ξ can be interpreted as the conjugacy class of the group element h_0 where the Ashtekar-Barbero connection is peaked on. Similarly

$$\Delta A \equiv \sqrt{\langle A^2 \rangle - \langle A \rangle^2} = \frac{1}{2} \gamma L_P^2 \sqrt{2\sigma_0} \tag{23}$$

$$\Delta W_\gamma \equiv \sqrt{\langle W_\gamma^2 \rangle - \langle W_\gamma \rangle^2} = \sin(\xi/2)\, \frac{1}{\sqrt{2\sigma_0}} \tag{24}$$

If we require that the relative dispersions vanish in the large j_0 limit, this fixes the scaling

$$t \sim (j_0)^{-k} \qquad 0 < k < 1 \tag{25}$$

Coherent spin-networks satisfy the following coherence properties

1. Are eigenstates of the annihilation operator associated to a link e

$$\hat{H}_e = e^{-t\hat{A}_e^2}\,\hat{h}_e\,e^{t\hat{A}_e^2} \tag{26}$$

2. Saturate the associated Heisenberg relations
3. Form an overcomplete basis

Point 3 is very important, because it means that every LQG state can be expressed as a superposition of states with semiclassical labels. Of course, coherent spin-networks do not have in general a Regge-like interpretation unless some constraints on their labels are satisfied (i.e. closure condition at each node, and gluing constraints).

The resolution of identity (we give it for a loop state but is general) is

$$\int \overline{\Psi_t^H(h)}\Psi_t^H(h')\mathrm{d}\nu_t(H) = \delta(h,h') \tag{27}$$

The measure $\mathrm{d}\nu_t$ is related to the Haar measure $\mathrm{d}H$ on $SL(2,\mathbb{C})$

$$d\nu_t = \Omega_{2t}(H)\mathrm{d}H \tag{28}$$

and Ω_t is the $SU(2)$-averaged heat kernel of $SL(2,\mathbb{C})$ (not the an. cont. of the $SU(2)$ one).

Coherent s.n.'s lead naturally to an holomorphic representation for LQG. Consider a single copy of $SU(2)$, to simplify the notation. The scalar product in $\mathcal{H} = L^2(SU(2), \mathrm{d}g)$ defines a correspondence between a state $\Psi \in \mathcal{H}$ and a holomorphic function Ξ, defined

$$\Xi : H \longmapsto \langle \Psi_t^H, \Psi \rangle \tag{29}$$

There is more. The correspondence is a unitary map (isometric, onto)

$$L^2(SU(2), \mathrm{d}g) \longmapsto \mathscr{H} L^2(SL(2,\mathbb{C}), \Omega_t \mathrm{d}H) \tag{30}$$

To be more explicit, the $SU(2)$-averaged kernel is

$$\Omega_t(H) = \int_{SU(2)} F_t(Hg)\, dg \tag{31}$$

where F_t is the heat kernel over $SL(2,\mathbb{C})$. Ω_t can be viewed as the heat kernel on $SL(2,\mathbb{C})/SU(2)$.

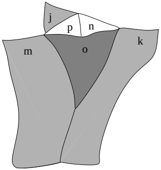
What is now available is a representation (Ashtekar et al. 1994) for Loop Quantum Gravity where states are holomorphic functions of classical variables H_{ab} that admit a geometric interpretation in terms of areas, extrinsic angles and unit-fluxes,

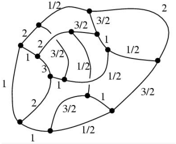
$$(\eta_{ab}, n_{ab}, n_{ba}, \xi_{ab}) \tag{32}$$

the variables generally used in the Spin Foam setting.

Spin Foams: candidate models for nonperturbative quantum gravity. Provide quantum amplitudes for a gravitational process happening inside a region of spacetime.
Spin foams are built over a cellular decomposition of the spacetime manifold.

$$Z[\Gamma] = \sum_{j_f, i_e} \prod_f A_f(j_f) \prod_e A_e(j_f, i_e) \prod_v A_v(j_f, i_e) \tag{33}$$





The vertex amplitude A_v is the elementary amplitude associated to a single spacetime cell.
Typical vertex amplitude: combinatorial symbol of the recoupling theory of angular momentum
$\Rightarrow$ function of spins labelling the spin-network (colored graph) on the boundary of the cell.

Example: 3D Ponzano-Regge model

$$A_v = \{6j\} \tag{34}$$

amplitude for a tetrahedron

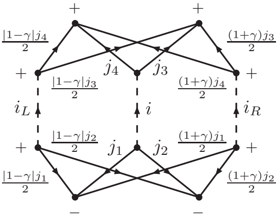
Physical models are based on some 4-dimensional generalization of the $6j$-symbol. In particular, the Euclidean models are built with the "square" of the Wigner $15j$-symbol and suitable fusion coefficients.

The fusion coefficients implement the constraints on the quantum numbers that turn the topological BF theory into general relativity.

Example: 4D EPRL

$$A_{v\ \mathsf{EPRL}} = \sum_{i_a^+, i_a^-} 15j(j^+, i^+) 15j(j^-, i^-) f(i_1, i_1^+, i_1^-) \ldots f(i_5, i_5^+, i_5^-) \tag{35}$$

$$j^+ = \frac{1+\gamma}{2} j \qquad j^- = \frac{|1-\gamma|}{2} j \tag{36}$$



We propose a general abstract definition of vertex amplitude independent of the basis. Define the propagation kernel as the following state in $\mathcal{K}_\Gamma = L^2(SU(2)^l/SU(2)^n, d^n g)$ supported on the tetrahedron, or 4-simplex graph:

- PR:

$$W_{PR}(h_{ab}) = \int \prod_a \mathrm{d}g_a \prod_{ab} P(h_{ab}, g_a g_b^{-1}) \qquad g \in SU(2) \tag{37}$$

where

$$P(h,g) = \delta(hg^{-1}) \tag{38}$$

- EPRL:

$$W_{EPRL}(h_{ab}) = \int \prod_a \mathrm{d}G_a \prod_{ab} P(h_{ab}, G_a G_b^{-1}) \qquad G \in SO(4) \tag{39}$$

where

$$P(h,G) = \sum_j (2j+1)\mathrm{Tr}\Big(\overset{j}{\Pi}(h)\; Y \overset{j^\pm}{\Pi}(G) Y^\dagger\Big) \tag{40}$$

and Y is the embedding map $\mathcal{H}_j \hookrightarrow \mathcal{H}_{(j^+,j^-)}$ with components (CG coeff.)

$$Y_m^{m^+m^-} = \langle j^+, m^+; j^-, m^- | j, m\rangle \tag{41}$$

Abstract vertex amplitude (2)

We define the abstract vertex amplitude as the linear functional associated to $W(h_{ab})$ (Riesz representation):

$$W(\Psi) = \langle W, \Psi \rangle, \quad \Psi \in \mathcal{K}_\Gamma \tag{42}$$

The $6j$-symbol and the standard EPRL vertex amplitudes are recovered evaluating the functional on a spin-network basis:

$$\{6j\} = W_{PR}(\psi_{j_{ab}}) = \langle W_{PR}, \Psi_{j_{ab}} \rangle \tag{43}$$

$$V_{EPRL}(j_{ab}, i_a) = W_{EPRL}(\psi_{j_{ab},i_a}) = \langle W_{EPRL}, \Psi_{j_{ab},i_a} \rangle \tag{44}$$

$$\Psi_{j_{ab},i_a}(h_{ab}) = \bigotimes_{ab} \overset{j_{ab}}{\Pi}(h_{ab}) \cdot \bigotimes_a v_{i_a} \tag{45}$$

In 3d $v_{i_a} = v_a$ are the unique normalized intertwiners associated to 3-valent nodes. In 4d the spin i_a labels the virtual spin of the 4-valent node a.

This perspective opens the possibility to define other useful representations of spin foam vertex amplitudes. Before introducing the new representation, we review very briefly some asymptotic formulae.

In the large spin limit

- PR model

$$\{6j\} \sim A\cos(S_R + \frac{\pi}{4}) \tag{46}$$

where $S_R(j_e)$ is the boundary Regge action for an oriented tetrahedron with edge lengths j_e, that is:

$$S(j_e) = \sum_e j_e\theta_e. \tag{47}$$

The angle θ_e is the exterior 3d dihedral angle between the two triangles sharing the edge e. The formula (46) sums over the two possible orientations and this is the origin of the cosine.

- BC model

$$\{10j\} \sim A\cos(S_R) + \frac{\pi}{4} + D \tag{48}$$

where S_R is the (area) Regge action for a 4-simplex and D are the contributions of degenerate configurations which dominate in the asymptotic regime.

- EPRL with LS intertwiners

$$V_{3d}(j,\vec{n}) \sim Ae^{i\gamma S_R} + Ae^{-i\gamma S_R} + C_1e^{iS_R} + C_2e^{-iS_R} \tag{49}$$

The first two form the usual $\cos$ term; the other two correspond to vector geometries.

Holomorphic representation

We argue that the undesired terms in the asymptotic formulae are an *artifact* of the basis which diagonalizes the intrinsic curvature and not the extrinsic curvature, so it permits different classical solutions for fixed boundary spins.

We introduce a new holomorphic representation (Bianchi-Magliaro-CP)

The new representation is simply defined as the evaluation of the vertex functional on the coherent spin-network (overcomplete) basis:

$$W(H_{ab}) = W(\Psi_{\Gamma,H_{ab}}) \tag{50}$$

The label $H_{ab} \in SL(2,\mathbb{C})$ contains all the physical information: intrinsic + extrinsic curvature $\Rightarrow$ not only areas and normals but also extrinsic angles!

The holomorphic vertex amplitude has information about the intrinsic *and* extrinsic geometry of the boundary of the elementary cell.

The $6j$-symbol, the $10j$-symbol and the generalized $15j$-symbols of the new spin foam models are the evaluation of the vertex functional on states which are sharply peaked on spins, and consequently *maximally spread* in the extrinsic curvature.

The uncertainty of extrinsic curvature is the reason of the cosine and degenerate terms in the asymptotic formulae.

Main result

The asymptotics of the new holomorphic representation is a single "$\exp iS$" term

What we can compute?

There is a precise sense in which the vertex expansion of QG transition amplitudes could be interpreted: we have integrated out small wave-lengths, small with respect to the typical scale of the boundary states (this scale is fixed by their peakedness properties). This picture makes sense if some properties of renormalizability hold in some regime, and this is an important research direction (Freidel, CP-Rovelli-Speziale, Magnen-Noui-Rivasseau-Smerlak).

What we can compute *now*?

As soon as we introduced the new holomorphic representation, it was applied to a simple model in spin-foam cosmology (Bianchi-Rovelli-Vidotto).

At lowest order, the EPRL holomorphic amplitude for the transition between two homogeneous dipoles is

$$W(z, z') \sim zz' e^{-z^2 - z'^2}$$

Despite the simplicity of this expression, is turns out that it is annihilated by the Hamiltonian constraint!

We believe that other results will come soon.

Conclusions

- We have shown that coherent spin-networks for Loop Quantum Gravity reproduce the semiclassical states used in the Spin Foam framework.
- Coherent spin-networks coincide with Thiemann's complexifier coherent states with a natural choice of complexifier operator.
- It is possible that coherent spin-networks can be obtained via geometric quantization (cft. Freidel-Speziale, to appear), and that the two coincide on a subset. This would be an instance of Guillemin-Sternberg's 'quantization commutes with reduction'.
- Our work brings together many (sometimes conflicting) ideas that have been proposed in the search for semiclassical states in Loop Quantum Gravity.
- Given a space-time metric (e.g. Minkowski or Schwarzschild), we can smear the Ashtekar-Barbero connection on links of the graph and the electric field on surfaces dual to links. This finite amount of data can be used as labels for the coherent states.
- The fact that in the large spin limit they are 'effectively' labeled by Livine-Speziale coherent intertwiners guaranties that they are actually peaked on a classical expectation value of non-commuting geometric operators, e.g. the volume operator.

- We introduced the propagation kernel and the associated abstract vertex functional for spin foam models in 2+1 and 3+1 dimensions. This basis-independent approach permits to introduce other useful representations of spin foam vertex amplitudes
- We defined the holomorphic representation as the vertex functional in the basis of coherent spin-networks.
- In the semiclassical limit, the new holomorphic representation selects a single "$\exp iS$" term, as opposed to the standard vertex amplitudes
- The "problem of the cosine" and of degenerate terms is an artifact of the basis which diagonalizes the extrinsic curvature, with maximal spread in the extrinsic curvature. The undesired terms are there because of this uncertainty
- The holomorphic representation contains all the classical geometric information of the boundary of the spin foam elementary cell
- Outlook: extend these concepts to the full state sum of quantum gravity models... work in progress
- Application in loop cosmology: dipole transition amplitude

Thanks !