Continuum approximation of microscopic quantum dynamics: lessons from condensed matter systems ~~and analogue gravity models~~

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Plan of the tour

- General discussion/motivation
- Crystals
- Bose-Einstein condensation
- Large N and Phase Transitions
- Matrix models
- Conclusion

First of all...

I will always have in mind LQG, spinfoams, simplicial gravity, etc.

I will try to explore the implications that the analysis of CM systems can have for these scenarios.

Background

- QG scenarios based generally on discrete microscopic constituents (even just as tools, e.g. regulators)
- The microscopic dynamics is assigned
- The case of small number of constituents is understood (e.g. asymptotics & single 4-simplex)

Ref.: all the other talks

General problem

- Continuum semiclassical gravity is the endpoint of the analysis.
- What is the macroscopic/continuum/classical limit of QG?
- Context in which the same problems arise: Condensed Matter (CM)

The Problem in CM

- In CM one is asking a very basic question: *Given a set of a certain number of quantum particles in a given region of space, what is the ground state of the system.*
- As a side issue: how to pass from discrete to continuum
- The microscopic dynamics (atomic physics) is known exactly (at least in principle).

$$\hat{H} = \sum_I -\frac{\hbar^2 \nabla_I^2}{2M_I} + \sum_{i_e} -\frac{\hbar^2 \nabla_{i_e}^2}{2m_e} + \frac{1}{2} \sum_{i_e \neq j_e} V_{ee}(\mathbf{x}_{i_e} - \mathbf{x}_{j_e}) + \frac{1}{2} \sum_{I \neq J} V_{IJ}(\mathbf{x}_I - \mathbf{x}_J) + \sum_{i_e, I} V_{Ie}(\mathbf{x}_{i_e} - \mathbf{x}_J)$$

How to solve the problem?

- No general rules, few analytical techniques
- Some methods (mean field approx, symmetries...) variously applied and improved
- Sometimes they turn out to be wrong
- Experiments and numerical approaches are providing new insights and challenges for the development of accurate theoretical models

I: warm up with crystalline solids

Crystals

Landau-Lifshitz #7
Kittel (solid state)

- A very simple example (but already complicated enough) of CM system is represented by crystals
- Described by regular lattices with atoms/ions sitting on the nodes
- Classification of all the possible lattices based on symmetries (crystallographic groups)

Continuum limit: Elasticity

- Natural problem: elasticity properties of certain macroscopic bodies. Other possibilities (optics, thermal, etc.)
- On large scales, the discrete lattice is replaced with a continuum density function (coarse graining).
- All the microphysics is encoded and summarized into some macroscopic parameters. Ideally, they could be computed from Van der Waals forces.

Elasticity/2 : collective fields

$$\sigma^{ij} \rightarrow F_i(V) = \int_V dV \partial_j \sigma^{ij}$$

Stress tensor

Displacement from equilibrium configuration

$$x^i \rightarrow x^i + \xi^i(x) \qquad dx^i \rightarrow dx^i + \frac{\partial \xi^i(x)}{\partial x^j} dx^j$$

$$ds^2 \rightarrow ds^2 + 2\frac{\partial \xi^i(x)}{\partial x^j} dx^j dx^k \delta_{ik} + O(\xi^2)$$

$$u_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x^j} + \frac{\partial \xi_j}{\partial x^i}\right)$$

Strain tensor

Mechanical properties encoded into the free energy

$$F = F_0 + \int_V \sigma_{ij} u^{ij} \, d^3x$$

Constitutive relations

$$\sigma_{ij} = \lambda_{ijhk} u^{hk}$$

$$\boxed{F = F_0 + \int_V \lambda_{ijhk} u^{ij} u^{hk} \, d^3x}$$

Elasticity/3

In general the structure of the elastic modulus tensor will depend on the microscopic structure (symmetries of the lattice)

Isotropic case $$\lambda_{ijhk} = a\delta_{ij}\delta_{hk} + b\left(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}\right)$$

This tensor encodes all the macroscopic mechanical properties of the continuum (e.g. sound waves).

Comments

- There are no theorems that prove that the ground state of the system should be a crystal (plausibility arguments)
- Some intuition come from atomic physics (He vs Li, but see Hg)
- This intuition is not enough!
- Allotropic forms: for C you have, for instance, both diamond (insulator) and graphite (conductor)

Coarse graining

- Crystals, but also fluids, are inhomogeneous on scales small enough.
- To go for the continuous representation one should perform a coarse graining
- Find a scale L, such that any quantity, averaged over a cell of size L, has small fluctuations.
- This scale is statistical in nature, not necessarily dynamical

In practice

$$\rho(\mathbf{x}) \approx \sum_i m_i \delta^3(\mathbf{x} - \mathbf{x}_i)$$

$$\bar{\rho}_L(\mathbf{x_i}) = \frac{1}{L^3} \int_{V_i} d^3x \, \rho(x)$$

Constant in each cell in which the body is partitioned

$$\Delta\rho = \langle \rho^2 \rangle_{C_i} - \bar{\rho}_L^2(C_i)$$

$$\frac{\Delta\rho}{\bar{\rho}^2} \ll 1$$

The **effective** continuum theory (e.g. differential equations for continuous density) is a theory for the coarse grained density, not for the microscopic one.

Comments

- Microscopic symmetries and/vs macroscopic ones:
- Quantum mechanics not really relevant.
- Renormalization group: how the effective theory changes when we change the coarse graining
- Concrete example: take Regge calculus (but other discrete approaches as well) and try to address the problem of coarse graining.
- What is the effective action after coarse graining?
- What is the fate of symmetries?

Bahr and Dittrich (Regge calculus) 2009
Bombelli, Corichi, Winkler (How to reconstruct a manifold out of a graph) 2004,2009

II: Inclusion of Quantum Mechanics

Bose Einstein Condensates

- Macroscopic system in which quantum mechanics is crucial
- At sufficiently low temperature, a system of bosons condenses.
- Macroscopic occupation number of the ground state (for single particle)

Pethick and Smith
Fetter and Walecka
Abrikosov,Gorkov, Dzyaloshinski

General framework

General setting: dilute, weakly interacting Bose gas

Formalism: second quantization

$$\hat{\Psi} = \frac{1}{\sqrt{V}} \sum_k \hat{a}_k e^{ik\cdot x}$$

Operators creating/destroying bosons (atoms)

$$\hat{H}_0 = \int \hat{\Psi}^\dagger(x) \left(-\frac{\hbar}{2m}\nabla^2 - \mu + \frac{\kappa}{2}|\hat{\Psi}|^2 \right) \hat{\Psi}(x) d^3x$$

Condensation via mean field

$$\hat{\Psi} \approx \psi\mathbb{I} + \hat{\chi} \qquad \langle\Omega|\hat{\Psi}|\Omega\rangle = \psi$$

condensate wavefunction

Gross-Pitaevski equation for the condensate (neglecting the fluctuations)

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - \mu\psi + \kappa|\psi|^2\psi$$

Note that:

$$\hat{\Psi}\boxed{|0\rangle} = 0 \qquad \hat{\Psi}\boxed{|\Omega\rangle} \neq 0$$

atomic Fock vacuum

ground state

More about MFA

- MFA out of coherent states, peakedness around a given classical complex field.

$$|\Omega\rangle \approx \exp(z\hat{a}_0^\dagger)|0\rangle \qquad |z|^2 = N_0$$

$$\hat{\Psi}(x)|\Omega\rangle = \sqrt{N_0}u_0(x)|\Omega\rangle = \psi(x)|\Omega\rangle$$

- Example of how the microscopic theory goes onto a continuum fluid description

$$\psi = \sqrt{n_c}\exp(-i\theta/\hbar)$$ *Madelung representation*

- Gross-Pitaevski goes onto continuity and Euler equation

$$|\psi(x)|^2 = n_c(x) \qquad \vec{v} \propto \vec{\nabla}\theta$$

Comments

- The mean field approximation can be seen as the **guess** that the ground state is a coherent state.
- The continuum limit follows immediately
- There is no obvious coarse graining scale
- Coherent state just an approximation (presence of interactions). Beyond mean field methods required for certain experimental conditions
- Coarse graining is needed for BEC (failure related to large fluctuations).

What about QG?

- Coherent state for QG as a route to classical spacetime?
- There is no coarse graining scale: is the use of semiclassical states enough to get continuum GR?
- LQG, spinfoam & semiclassical states
- What about GFT? The sum over the discretizations already implemented: what about coherent states?

Oriti 2007

- What are the physical consequences for excitations around semiclassical states? (i.e. are there collective d.o.f.?)

III: Large N

Thermodynamic limit...

- Thermodynamics: one ignores the microstates of the system, and tries to describe it via a small number of global variables (N,p,V,T,S...).
- Thermodynamic limit: one takes the limit of infinitely large system (e.g. N,V formally becoming infinite) keeping fixed intensive quantities (e.g. number density)

... and phase transitions

- The state of a macroscopic system is characterized by certain state functions.
- Phase transition: discontinuities of these functions
- Critical exponents: scaling laws describing the critical behavior

$$f(t) \sim t^{\alpha} \qquad t = \frac{T_c - T}{T_c}$$

Comments

- Phase transitions are strictly related to the thermodynamic limit. Absent for systems with finite number of d.o.f.
- Clear case in which the macroscopic regime is qualitatively different from the few-body case
- Universality: at critical point many microscopically different systems have the same critical exponents (insensitivity to microphysics.)

Matrix models

- Hermitian NxN matrices $M = M^\dagger$

- Potential $$V(M) = \sum_{k=0}^{\infty} \frac{g_k}{k} \mathrm{tr}(M^k)$$

- "Classical theory": gauge field theory in zero dimensions.

$$\left.\begin{array}{c} U^\dagger U = \mathbb{I} \\ M \to U^\dagger M U \end{array}\right\} \Longrightarrow V(M) \to V(M)$$

$$V'(M) = 0$$ Classical EOM

Matrix models/2

- Statistical mechanics

$$Z = \int dM \exp(-NV(M))$$

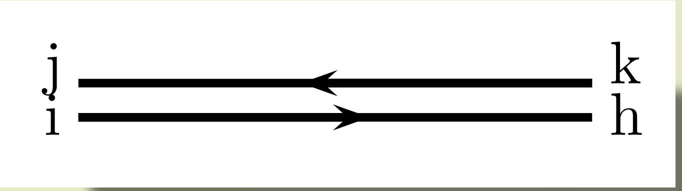
$$N \sim \beta = \frac{1}{k_B T} \qquad N \sim \frac{1}{\hbar}$$

- *Feynman diagrams: perturbative expansion is linked to two dimensional compact surfaces (sum over all finite polygonulations)*
- Proposal: use it as the definition of the partition function for two dimensional quantum gravity

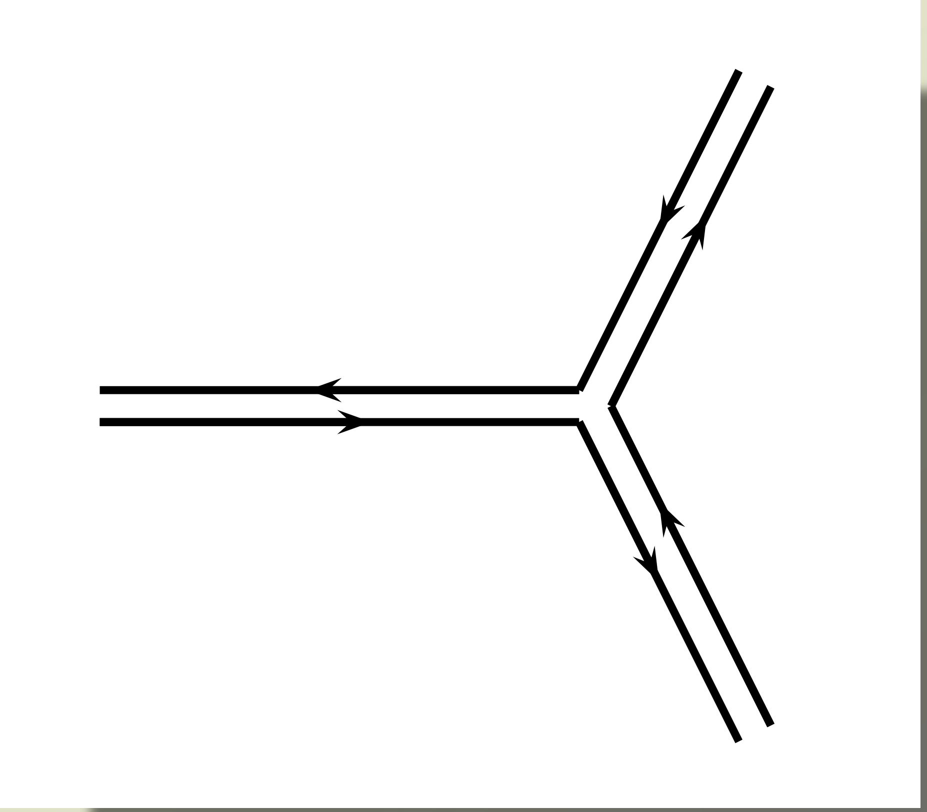
Di Francesco, Ginsparg, Zinn-Justin
hep-th/9306153

Diagrammatics

Propagator



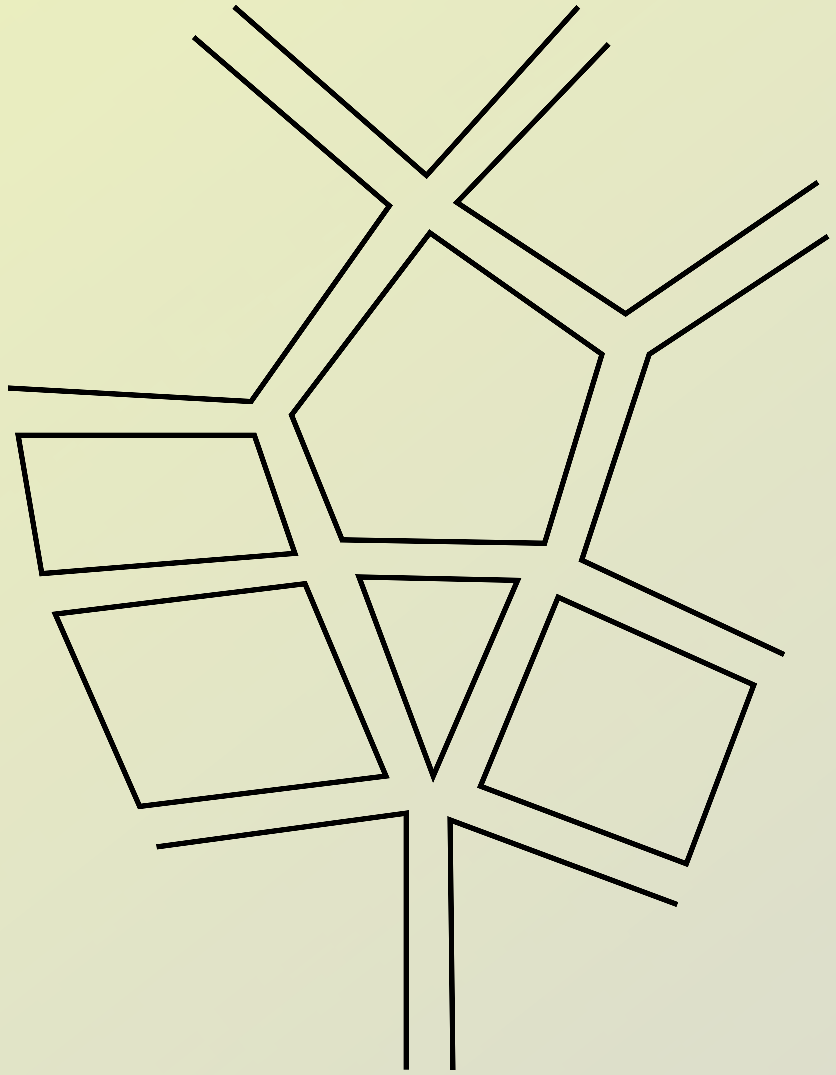
Cubic vertex



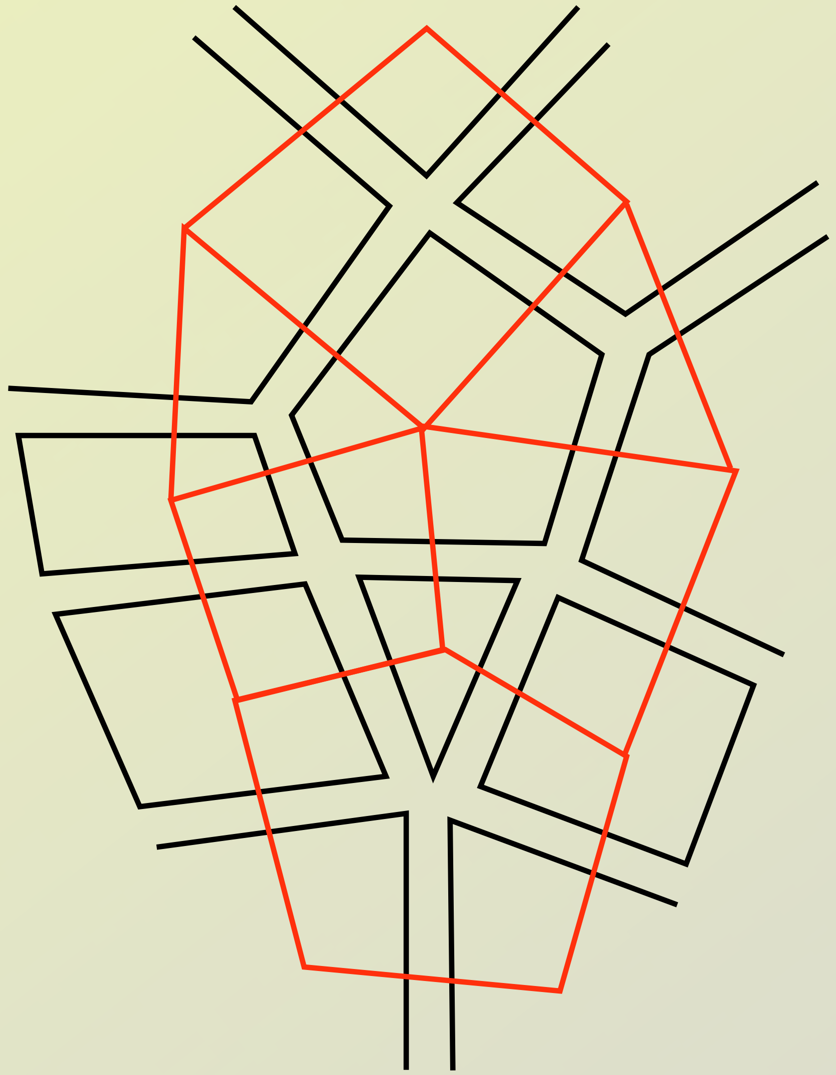
't Hooft NPB 1974

Diagrammatics/2

Diagrammatics/2



Diagrammatics/2



Matrix models/3

- The graphs have a weight which depends on the Euler characteristic of the corresponding dual triangulation
$$A(\Gamma) = N^{2-2h} a(g)$$
- Large N limit: only planar surfaces are contributing!
- But for the continuum limit one has to go for the double scaling limit
- Case in which the sum over the discretizations is not enough to get a very refined continuum limit.
- Besides the large N, also tuning of the coupling constants to some critical value.

Saddle point evaluation

- One can always diagonalize an Hermitian matrix

$$M = U^\dagger \Lambda U \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$$
$$V'(\lambda_i) = 0 \;\forall i = 1, \ldots, N$$

- Large N/semiclassical limit of the partition function: saddle point approximation

$$Z = \int dM \exp(-NV(M))$$

- Crucial feature: effect of the measure of the path integral (genuinely "quantum" in nature)

$$Z = \int \left(\prod_{i=1}^{N} d\lambda_i \right) \boxed{\Delta^2(\lambda)} \exp\left[-N \sum_{j=1}^{N} V(\lambda_j) \right]$$

Vandermonde determinant (Faddeev-Popov)

“Hydrodynamics” of the eigenvalues

- Equations for the saddle point
$$\frac{1}{N}\sum_{j,j\neq i}\frac{1}{\lambda_i-\lambda_j}=V'(\lambda_i)$$
- Large N: the density of eigenvalues (collective field)
$$\rho(x)=\frac{1}{N}\sum_{k=1}^{N}\delta(x-\lambda_k) \qquad \int dx\,\rho(x)=1$$
- The large N limit (the saddle point) as an equation for <u>continuous</u> $\rho(\lambda)$
$$2\int\frac{\rho(\lambda')}{\lambda-\lambda'}d\lambda'=V'(\lambda)$$
- The density of eigenvalues encodes the properties of the thermodynamical limit (large N expansion) and all the critical properties

Sakita, Jevicki 1980,1981

Comments

- Matrix models: an example in which the "quantum" theory is qualitatively different from the classical case even in the semiclassical regime (single matrix vs 2D QG)
- Obvious relevance for GFT (generalization to Boulatov and Ooguri).

  Livine, Perez and Rovelli 2003

  Oriti 2007
- What is the technical point that turns a GFT into a theory for gravity on macroscopic continuum spacetime?
- Ongoing programs in perturbative renormalization (Freidel, Gurau, Oriti), condensation in GFT (work in progress)?
- What are the implications for spinfoams/LQG? Is the continuum limit a sort of phase transition (thermodynamic limit involved!)?

Wrapping up...

- Not covered topics like kinetic theory, quantum phase transitions, superconductivity, etc.
- Bottomline: "More is different". P.W. Anderson, 1972
- Knowing the microscopic d.o.f. and their dynamics is <u>not enough</u> even in the "simple" case of CM
- Even the fate of symmetries is not completely clear (crystal, QCD, diffeomorphism inv.).
- One has to use several strategies (numerical, heuristics, experiments) to circumvent the impossibility of doing analytical calculations

The pessimistic slide

- In fact, besides the big problem of obtaining the continuum limit, we should face an additional challenge.
- The inverse problem: given the macroscopic dynamics (GR and modifications), what are the specific signatures of the underlying dynamics?
- Typical situation: the microscopic dynamics is (partially) washed away when we go for the macroscopic limit.

The optimistic slide

- After all, you can work on CM and you can understand a lot of the physical properties of the systems considered.
- Perhaps among the machineries elaborated for CM there is the right tool to attack the continuum semiclassical limit of QG.

Yes, we can!